Spectral Properties of Positive Maps

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Main message

Quantum physicists love

spectral problems

There are important problems in QM

which are NOT spectral

Quantum entanglement – pure states

Composite system

$$\mathcal{H}_{\text{total}} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$\psi \in \mathcal{H}_{\text{total}}$ is **separable** iff

$$\psi = \psi_A \otimes \psi_B$$

$$\psi_A \in \mathcal{H}_A \quad \& \quad \psi_B \in \mathcal{H}_B$$

entangled = not separable

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Quantum entanglement – mixed states

ρ mixed state in $\mathcal{H}_A \otimes \mathcal{H}_B$

$$\rho \geq 0 \quad (\Longrightarrow \quad \rho^* = \rho) \quad \& \quad \mathrm{Tr}\, \rho = 1$$

ρ separable $\Longleftrightarrow$ $\rho = \sum_\alpha p_\alpha \, \rho_\alpha^{(A)} \otimes \rho_\alpha^{(B)}$

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To be or not to be a separable state

Is it spectral property ?

YES if it is pure (ψ)

NO if it is mixed $(\rho \neq |\psi\rangle\langle\psi|)$

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Pure states

$$\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$e_1,\ e_2,\ \ldots,\ e_{d_1} \ \in\ \mathcal{H}_1$$

$$f_1,\ f_2,\ \ldots,\ f_{d_2} \ \in\ \mathcal{H}_2$$

$$\psi \ = \sum_{i=1}^{d_1} \sum_{\alpha=1}^{d_2} \psi_{i\alpha}\, e_i \otimes f_\alpha$$

$$F \ :\ \mathcal{H}_1 \ \longrightarrow\ \mathcal{H}_2\ ; \qquad F e_i \ = \sum_{\alpha=1}^{d_2} \psi_{i\alpha}\, f_\alpha$$

$$\psi \ = \ \sum_i e_i \otimes F e_i$$

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Schmidt rank

$$\psi = \sum_{\alpha} e_\alpha \otimes F e_\alpha$$

$$\psi \quad \longleftrightarrow \quad F$$

$$\mathrm{SR}(\psi) := \mathrm{rank}\, F$$

$$1 \;\leq\; \mathrm{SR}(\psi) \;\leq\; d := \min\{d_1, d_2\}$$

$$\psi \;\; \text{separable} \quad \Longleftrightarrow \quad \mathrm{SR}(\psi) = 1$$

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Warning: it does not need to be a spectral decomposition !!!

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Positive linear maps

M_n — algebra of $n \times n$ complex matrices

A linear map

$$\Phi \; : \; M_{d_1} \longrightarrow M_{d_2}$$

is positive iff

$$X \geq 0 \implies \Phi(X) \geq 0$$

It is unital iff

$$\Phi(\mathbb{I}_1) = \mathbb{I}_2$$

It is trace preserving iff

$$\mathrm{Tr}\, \Phi(X) = \mathrm{Tr}\, X$$

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Example: transposition

$$\Phi(X) = X^T$$

$$\mathbb{I}^T = \mathbb{I}$$

$$\operatorname{Tr} X^T = \operatorname{Tr} X$$

Positivity of matrices

$X \in M_n$ is positive iff

$$\langle \psi | X | \psi \rangle \geq 0 \ , \quad \psi \in \mathbb{C}^n$$

spectral condition

$$\langle \psi | X | \psi \rangle \geq 0 \implies X^* = X$$

$\lambda_1, \ldots, \lambda_n$ – eigenvalues of X

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non – spectral condition

k–positive maps

$$\mathrm{id}_k \ : \ M_k \longrightarrow M_k \ ; \quad \mathrm{id}_k(X) = X$$

A positive map $\Phi : M_{d_1} \longrightarrow M_{d_2}$ is k–positive iff

$$\mathrm{id}_k \otimes \Phi \ : \ M_k \otimes M_{d_1} \longrightarrow M_k \otimes M_{d_2}$$

is positive

Φ is completely positive (CP) iff it is k–positive for $k = 1, 2, \ldots$

(Actually, it is enough to check for d-positivity ($d = \min\{d_1, d_2\}$)).

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k–positive maps

$\mathcal{P}_k$ — convex cone of k–positive maps

positive maps $= \mathcal{P}_1 \supset \mathcal{P}_2 \supset \ldots \supset \mathcal{P}_d =$ CP maps

$$d \ := \min\{d_1, d_2\}$$

unital CP $=$ quantum channel

CP maps – fully characterized

Stinespring, Krauss, Choi

$$\Phi \ : \ M_{d_1} \ \longrightarrow \ M_{d_2}$$

$$\Phi(X) \ = \ \sum_{\alpha} V_\alpha X V_\alpha^*$$

$$V_\alpha \ : \ \mathbb{C}^{d_1} \ \longrightarrow \ \mathbb{C}^{d_2}$$

Φ is unital $\iff \sum_{\alpha} V_\alpha V_\alpha^* = \mathbb{I}_2$

Φ trace preserving $\iff \sum_{\alpha} V_\alpha^* V_\alpha = \mathbb{I}_1$

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$$d_1 = d_2 = d \; , \qquad V = U \in U(d)$$

$\Phi(X) = UXU^{*}$ Heisenberg unitary evolution!

Any quantum operation (evolution, measurement, ...) can be described by some CP map.

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CP property is spectral

$$\{e_1, \ldots, e_{d_1}\} \quad \longleftrightarrow \quad \text{ONB in } \mathcal{H}_1$$

$$e_{ij} := |e_i\rangle\langle e_j| \ \in \ M_{d_1}$$

Choi–Jamiołkowski matrix

$$\Phi \ \longrightarrow \ \widehat{\Phi} = \sum_{i,j=1}^{d_1} e_{ij} \otimes \Phi(e_{ij}) \ \in \ M_{d_1} \otimes M_{d_2}$$

$$\Phi \ \text{ is CP} \quad \Longleftrightarrow \quad \widehat{\Phi} \geq 0$$

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$$\Phi \ \text{positive but not CP} \quad \Longrightarrow \quad \widehat{\Phi} \ \ngeq \ 0$$

$$\langle \psi \otimes \phi \,|\, \widehat{\Phi} \,|\, \psi \otimes \phi \rangle \ \geq \ 0 \ , \quad \psi \in \mathcal{H}_1 \ , \ \phi \in \mathcal{H}_2 \qquad \textbf{(block positivity)}$$

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Entanglement Witness

A block positive (but not positive!) operator W is called an **Entanglement Witness**

A state ρ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is **entangled** iff there exists an **entanglement witness** W such that

$$\mathrm{Tr}\, \rho W < 0$$

see Gniewko Sarbicki talk tomorrow

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Connection with Professor Lauck

$$\Phi \; : \; M_d \; \longrightarrow \; M_d$$

positive, unital and trace preserving

$\{e_1, \ldots, e_d\}$ – ONB in $\mathbb{C}^d$

$$S_{ij} := \langle e_j \,|\, \Phi(|e_i\rangle\langle e_i|) \,|\, e_j\rangle \;\geq\; 0$$

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Φ unital

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$$\sum_{i=1}^{d} S_{ij} \;=\; ?$$

$$\sum_{i=1}^{d} S_{ij} = \left\langle e_j \,\middle|\, \Phi\left(\sum_{i=1}^{d} |e_i\rangle\langle e_i|\right) \,\middle|\, e_j \right\rangle$$

$$\sum_{i=1}^{d} S_{ij} = \langle e_j \,|\, \mathbb{I} \,|\, e_j\rangle = \langle e_j \,|\, e_j\rangle = 1$$

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Φ – positive $\implies$ $S_{ij} \geq 0$

Φ – unital $\implies$ $\sum_{i=1}^{d} S_{ij} = 1$

Φ – trace preserving $\implies$ $\sum_{j=1}^{d} S_{ij} = 1$

S_{ij} – doubly stochastic!!

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Connection with Professors Zak & Vourdas

$\{e_1, \ldots, e_d\}$ – ONB in $\mathbb{C}^d$

$\{f_1, \ldots, f_d\}$ – ONB in $\mathbb{C}^d$

$$Z_{ij} := \langle f_j \,|\, \Phi(|e_i\rangle\langle e_i|) \,|\, f_j\rangle$$

Z_{ij} – doubly stochastic!!

Does Z_{ij} possesses extra properties if

$$\{e_1, \ldots, e_d\} \ \& \ \{f_1, \ldots, f_d\} \text{ MUBs } ?$$

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$$\{e_1, \ldots, e_d\} \ \& \ \{f_1, \ldots, f_d\} \ \mathrm{MUBs} \ \ ?$$

Doubly Stochastic Matrices vs. CP maps

$A \in M_d$ – doubly stochastic

$$A = \sum_\pi a_\pi P_\pi \; ; \quad a_\pi \geq 0 \; ; \quad \sum_\pi a_\pi = 1$$

$$\pi \in S_d$$

$$\Phi \; : \; M_d \longrightarrow M_d$$

$$\Phi(X) = \sum_\pi a_\pi P_\pi X P_\pi^* = \sum_\pi V_\pi X V_\pi^*$$

$$V_\pi = \sqrt{a_\pi}\, P_\pi$$

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$$\Phi \quad \text{is unital} \quad \Longleftrightarrow \quad \sum_{\alpha} V_{\alpha} V_{\alpha}^{*} = \mathbb{I}$$

$$\Phi \quad \text{trace preserving} \quad \Longleftrightarrow \quad \sum_{\alpha} V_{\alpha}^{*} V_{\alpha} = \mathbb{I}$$

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Φ – CP, unital and trace preserving

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17th Hilbert problem

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The goal

I will construct a class of positive (but not CP!) maps using certain spectral conditions.

$$\langle \psi \otimes \phi \,|\, \widehat{\Phi} \,|\, \psi \otimes \phi \rangle \geq 0$$

$$\widehat{\Phi} \ngeq 0$$

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$$\widehat{\Phi}_- = -\sum_{\alpha=1}^{L} \lambda_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \; ; \qquad \lambda_\alpha \leq 0$$

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$$\widehat{\Phi} = \sum_{\alpha=L+1}^{D} \lambda_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| - \sum_{\alpha=1}^{L} \lambda_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$$

$$\psi_\alpha = \sum_{i=1}^{d_1} e_i \otimes F_\alpha e_i$$

$$\sum_{\alpha=1}^{L} ||F_\alpha||^2 \ < \ 1 \quad \Longrightarrow \quad \mu := \frac{\sum_{\alpha=1}^{L} \lambda_\alpha ||F_\alpha||^2}{1 - \sum_{\alpha=1}^{L} ||F_\alpha||^2}$$

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Example 1: transposition

$$\Phi \; : \; M_2 \longrightarrow M_2$$

$$\Phi(X) = X^T$$

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$$\lambda_1 = -1, \; \lambda_2 = \lambda_3 = \lambda_4 = 1$$

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$$\lambda_\alpha \geq \mu \; , \qquad \alpha = 2, 3, 4$$

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Conclusions

1. Similar spectral property holds for k-positive maps
2. Multipartite systems

$$\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$$

$$d_1 \ , \ \ldots \ , \ d_n$$

References

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