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From Local Perturbation Theory to Hopf and Lie Algebras of Feynman Graphs

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Abstract. We review the algebraic structures imposed on the renormalization procedure in terms of Hopf and Lie algebras of Feynman graphs, and exhibit the connection to diffeomorphisms of physical observables.

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1. Introduction

Perturbative renormalization is by far the most successful technique for computing physical quantities in quantum field theory, allowing for instance the unique achievement of being able to accurately predict the first ten decimal places of the anomalous magnetic moment of the electron.

Renormalization has been settled as a self-consistent approach to the treatment of short-distance singularities in the perturbative expansion of quantum field theories thanks to the work of Bogoliubov, Parasuik, Hepp, Zimmermann, and followers. Nevertheless, its intricate combinatorics went unrecognized for a long time. In this short review we want to describe the results in a recent series of papers devoted to the Hopf algebra structure of quantum field theory (QFT).

These results [1–5, 8, 13] culminate in a few remarkable facts which give mathematical structure to perturbative renormalization.

- For each perturbative QFT, with its Green functions expanded in terms of Feynman diagrams, one finds a Lie algebra of Feynman graphs obtained from anti-symmetrizing the pre-Lie algebra operation of plugging graphs into each other in all possible ways.
- The corresponding Lie group is a group of characters of a Hopf algebra of Feynman graphs.
- The dual of the universal enveloping algebra of this Lie algebra gives rise to this commutative Hopf algebra of Feynman graphs whose structure maps (counit,

coproduct, and coinverse) determine the forest formulas underlying the Bogoliubov recursion.

- Resolving Feynman graphs into their divergent sectors shows that the universal Hopf algebra providing the role model for all such Lie algebras is a Hopf algebra of rooted trees, as it should not surprise the practitioner of QFT: the short-distance singularities are located along diagonals of configuration spaces known to be stratified by such trees.
- The most successful approach in practice, minimally subtracted dimensional regularization, corresponds to a Riemann–Hilbert decomposition of the character given by the Feynman rules in this regularization.
- Locality allows to use this decomposition to construct a homomorphism from the above Lie group to the group of diffeomorphisms of physical parameters, culminating in a geometrical understanding of the β-function and the renormalization group.

It is the last two points on which we will focus here, but let us start gently by reviewing the spirit of renormalization.

The physical motivation behind the renormalization technique is quite clear and goes back to the concept of effective mass in nineteen century hydrodynamics. To appreciate it, one should dive under water with a ping-pong ball and start applying Newton's law

$$F = m\,a \tag{1}$$

to compute the initial acceleration of the ball B when we let it loose (at zero speed relative to the water). If one naively applies (1), one finds (see the QFT course by Sidney Coleman) an unrealistic initial acceleration of about 20 g! In fact as explained in *loc. cit.* due to the interaction of B with the surrounding field of water, the inertial mass m involved in 1 is not the bare mass m_0 of B but is modified to

$$m = m_0 + \tfrac{1}{2}M, \tag{2}$$

where M is the mass of the water occupied by B.

It follows, for instance, that the initial acceleration a of B is given, using the Archimedean law, by

$$-(M - m_0)g = \big(m_0 + \tfrac{1}{2}M\big)a \tag{3}$$

and is always of magnitude less than $2g$.

The additional inertial mass $\delta m = m - m_0$ is due to the interaction of B with the surounding field of water and if this interaction could not be turned off (which is the case if we deal with an electron instead of a ping-pong ball) there would be no way to measure the bare mass m_0.

The analogy between hydrodynamics and electromagnetism led (through the work of Thomson, Lorentz, Kramers' etc. [10]) to the crucial distinction between the bare parameters, such as m_0, which enter the field theoretic equations, and the observed parameters, such as the inertial mass m.

A quantum field theory in $D = 4$ dimensions, is given by a classical action functional

$$S(A) = \int \mathcal{L}(A)\, \mathrm{d}^4 x \tag{4}$$

where A is a classical field and the Lagrangian is of the form

$$\mathcal{L}(A) = (\partial A)^2/2 - \frac{m^2}{2} A^2 + \mathcal{L}_{\mathrm{int}}(A), \tag{5}$$

and where $\mathcal{L}_{\mathrm{int}}(A)$ is usually a polynomial in A and possibly its derivatives.

One way to describe the quantum fields $\phi(x)$, is by means of the time-ordered Green's functions

$$G_N(x_1, \ldots, x_N) = \langle\, 0 \,|T\, \phi(x_1) \ldots \phi(x_N)|\, 0\, \rangle \tag{6}$$

where the time-ordering symbol T means that the $\phi(x_j)$'s are written in order of increasing time from right to left.

The probability amplitude of a classical field configuration A is given by

$$\mathrm{e}^{i \frac{S(A)}{\hbar}} \tag{7}$$

and if one could ignore the renormalization problem, the Green's functions would be computed as

$$G_N(x_1, \ldots, x_N) = \mathcal{N} \int \mathrm{e}^{i \frac{S(A)}{\hbar}}\, A(x_1) \ldots A(x_N)\, [\mathrm{d}A], \tag{8}$$

where $\mathcal{N}$ is a normalization factor required to ensure the normalization of the vacuum state

$$\langle\, 0 \mid 0\, \rangle = 1\,. \tag{9}$$

If one could ignore renormalization, the functional integral (8) would be easy to compute in perturbation theory, i.e. by treating the term $\mathcal{L}_{\mathrm{int}}$ in (5) as a perturbation of

$$\mathcal{L}_0(A) = (\partial A)^2/2 - \frac{m^2}{2} A^2\,. \tag{10}$$

With obvious notations, the action functional splits as

$$S(A) = S_0(A) + S_{\mathrm{int}}(A), \tag{11}$$

where the free action S_0 generates a Gaussian measure $\exp(i\, S_0(A))\,[\mathrm{d}A] = \mathrm{d}\mu$.

The series expansion of the Green's functions is then given in terms of Gaussian integrals of polynomials as

$$G_N(x_1, \dots, x_N) = \left(\sum_{n=0}^{\infty} i^n/n! \int A(x_1) \dots A(x_N) \, (S_{\text{int}}(A))^n \, \mathrm{d}\mu \right) \times \\ \times \left(\sum_{n=0}^{\infty} i^n/n! \int S_{\text{int}}(A)^n \, \mathrm{d}\mu \right)^{-1}.$$

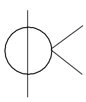
The various terms of this expansion are computed using integration by parts under the Gaussian measure μ. This generates a large number of terms $\mathrm{U}(\Gamma)$, each being labelled by a Feynman graph Γ, and having a numerical value $\mathrm{U}(\Gamma)$ obtained as a multiple integral in a finite number of space-time variables. As a rule the unrenormalized values $\mathrm{U}(\Gamma)$ are given by nonsensical divergent integrals.

The conceptually really nasty divergences are called ultraviolet and are associated to the presence of arbitrarily large frequencies or equivalently to the unboundedness of momentum space on which integration has to be carried out. Equivalently, when one attempts to integrate in coordinate space, one confronts divergences along diagonals, reflecting the fact that products of field operators are defined only on the configuration space of distinct spacetime points.

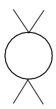
The physics resolution of this problem is obtained by first introducing a cut-off in momentum space (or any suitable regularization procedure) and then by cleverly making use of the unobservability of the bare parameters, such as the bare mass m_0. By adjusting, term by term of the perturbative expansion, the dependence of the bare parameters on the cut-off parameter, it is possible for a large class of theories, called renormalizable, to eliminate the unwanted ultraviolet divergences.

The main calculational complication of this subtraction procedure occurs for diagrams which possess nontrivial subdivergences, i.e. subdiagrams which are already divergent. In that situation the procedure becomes very involved since it is no longer a simple subtraction, and this for two obvious reasons: (i) the divergences are no longer given by local terms, and (ii) the previous corrections (those for the subdivergences) have to be taken into account.

To have an example for the combinatorial burden imposed by these difficulties consider the problem below of the renormalization of a two-loop four-point function in massless scalar ϕ^4 theory in four dimensions, given by the following Feynman graph:

$\Gamma^{(2)} =$  .

It contains a divergent subgraph:

$\Gamma^{(1)} =$  .

We work in the Euclidean framework and introduce a cut-off λ which we assume to be always much bigger than the square of any external momentum p_i. Analytic expressions for these Feynman graphs are obtained by utilizing a map Γ_λ which assigns integrals to them according to the Feynman rules and employs the cut-off λ to the momentum integrations. Then $\Gamma_\lambda[\Gamma^{[1]}]$ and $\Gamma_\lambda[\Gamma^{[2]}]$ are given by

$$\Gamma_\lambda[\Gamma^{[1]}](p_i) = \int \mathrm{d}^4 k \frac{\Theta(\lambda^2 - k^2)}{k^2} \frac{1}{(k + p_1 + p_2)^2},$$

and

$$\Gamma_\lambda[\Gamma^{[2]}](p_i) = \int \mathrm{d}^4 l \frac{\Theta(\lambda^2 - l^2)}{l^2 (l + p_1 + p_2)^2} \Gamma_\lambda[\Gamma^{[1]}(p_1, l, p_2, l)].$$

It is easy to see that $\Gamma_\lambda[\Gamma^{[1]}]$ decomposes into the form $b \log \lambda$ (where b is a real number) plus terms which remain finite for $\lambda \to \infty$, and hence will produce a divergence in $\Gamma^{[2]}$ which is a nonlocal function of external momenta

$$\sim \log \lambda \int \mathrm{d}^4 l \frac{\Theta(\lambda^2 - l^2)}{l^2 (l + p_1 + p_2)^2} \sim \log \lambda \ \log(p_1 + p_2)^2.$$

Fortunately, the counterterm $\mathcal{L}_{\Gamma^{[1]}} \sim \log \lambda$ generated to subtract the divergence in $\Gamma_\lambda[\Gamma^{[1]}]$ will precisely cancel this nonlocal divergence in $\Gamma^{[2]}$, so that the remaining divergence is local.

2 The Hopf Algebra of Graphs and the Riemann–Hilbert Problem

That this type of cancellation occurs at any order of perturbation theory, i.e. that the two diseases above actually cure each other in general is a very nontrivial fact that took decades to prove [9].

The detailed combinatorics is governed by the $\bar{R}$ operation of Bogoliubov and Parasiuk (for a 1PI graph Γ)

$$\bar{R}(\Gamma) = \mathrm{U}(\Gamma) + \sum_{\gamma \subsetneq \Gamma} C(\gamma)\, \mathrm{U}(\Gamma/\gamma) \tag{12}$$

which prepares a given graph with unrenormalized value $\mathrm{U}(\Gamma)$ by adding the counterterms $C(\gamma)$, eliminating the subdivergences as in the above example. These counterterms are constructed by induction using

$$C(\Gamma) = -T\left(\mathrm{U}(\Gamma) + \sum_{\gamma \subsetneq \Gamma} C(\gamma)\, \mathrm{U}(\Gamma/\gamma) \right) \tag{13}$$

where, for example using dimensional regularization and minimal subtraction, T is just the extraction of the pole part in $D = 4 - \varepsilon$. The renormalized graph is then given by

$$R(\Gamma) = \mathrm{U}(\Gamma) + C(\Gamma) + \sum_{\substack{\gamma \subset \Gamma \\ \neq}} C(\gamma)\, \mathrm{U}(\Gamma/\gamma)\,. \tag{14}$$

For a mathematician the intricacies of the detailed combinatorics and the lack of any obvious mathematical structure underlying it make it totally inaccessible, in spite of the existence of a satisfactory formal approach to the problem [11].

This situation was drastically changed by the discovery of one of us ([1]) that the formula for the $\bar{R}$ operation in fact dictates a Hopf algebra coproduct on the free commutative algebra $\mathcal{H}$ generated by the 1PI graphs Γ

$$\Delta\,\Gamma = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\substack{\gamma \subset \Gamma \\ \neq}} \gamma \otimes \Gamma/\gamma\,, \tag{15}$$

This Hopf algebra is commutative as an algebra and we showed in [3, 5] that it is the dual Hopf algebra of the enveloping algebra of a Lie algebra $\underline{G}$ whose basis is labelled by the one particle irreducible Feynman graphs. The Lie bracket of two such graphs is computed from insertions of one graph in the other and vice versa. The corresponding Lie group G is the group of characters of $\mathcal{H}$.

The next breakthrough came from our joint discovery [5] that identical formulas to equations (14–16) occur in the solution of the Riemann Hilbert problem for an arbitrary pronilpotent Lie group G, as demonstrated below!

This really unveils the true nature of this seemingly complicated combinatorics and shows that it is a special case of a general extraction of finite values based on the Riemann-Hilbert problem.

The Riemann-Hilbert problem comes from Hilbert's 21st problem which he formulated as follows: 'Prove that there always exists a Fuchsian linear differential equation with given singularities and given monodromy'. In this form it admits a positive answer due to Plemelj and Birkhoff (cf. [6] for a careful exposition). When formulated in terms of linear systems of the form

$$y'(z) = A(z)\, y(z)\,, \quad A(z) = \sum_{\alpha \in S} \frac{A_\alpha}{z - \alpha}\,, \tag{16}$$

(where S is the given finite set of singularities, $\infty \notin S$, the A_α are complex matrices such that

$$\sum A_\alpha = 0 \tag{19}$$

to avoid singularities at ∞), the answer is not always positive [7], but the solution exists when the monodromy matrices M_α are sufficiently close to 1. It can then be explicitly written as a series of polylogarithms [6].

Another formulation of the Riemann–Hilbert problem, intimately tied up to the classification of holomorphic vector bundles on the Riemann sphere $P_1(\mathbb{C})$, is in terms of the Birkhoff decomposition

$$\gamma(z) = \gamma_-(z)^{-1}\,\gamma_+(z), \quad z \in C, \tag{18}$$

where we let $C \subset P_1(\mathbb{C})$ be a smooth simple curve, C_- the component of the complement of C containing $\infty \notin C$ and C_+ the other component. Both γ and $\gamma_\pm$ are loops with values in $\mathrm{GL}_n(\mathbb{C})$

$$\gamma(z) \in G = \mathrm{GL}_n(\mathbb{C}), \quad \forall\, z \in \mathbb{C}, \tag{19}$$

and $\gamma_\pm$ are boundary values of holomorphic maps (still denoted by the same symbol)

$$\gamma_\pm : C_\pm \to \mathrm{GL}_n(\mathbb{C})\,. \tag{20}$$

The normalization condition $\gamma_-(\infty) = 1$ ensures that, if it exists, the decomposition (18) is unique (under suitable regularity conditions).

The existence of the Birkhoff decomposition (18) is equivalent to the vanishing

$$c_1\,(L_j) = 0 \tag{21}$$

of the Chern numbers $n_j = c_1\,(L_j)$ of the holomorphic line bundles of the Birkhoff–Grothendieck decomposition

$$E = \oplus\, L_j, \tag{22}$$

where E is the holomorphic vector bundle on $P_1(\mathbb{C})$ associated to γ, i.e. with total space:

$$(C_+ \times \mathbb{C}^n) \cup_\gamma (C_- \times \mathbb{C}^n)\,. \tag{23}$$

The above discussion for $G = \mathrm{GL}_n(\mathbb{C})$ extends to arbitrary complex Lie groups.

When G is a simply connected nilpotent complex Lie group the existence (and uniqueness) of the Birkhoff decomposition (18) is valid for any γ. When the loop $\gamma\colon C \to G$ extends to a holomorphic loop: $C_+ \to G$, the Birkhoff decomposition is given by $\gamma_+ = \gamma$, $\gamma_- = 1$. In general, for $z_0 \in C_+$ the evaluation

$$\gamma \to \gamma_+(z_0) \in G \tag{24}$$

is a natural principle to extract a finite value from the singular expression $\gamma(z_0)$. This extraction of finite values is a multiplicative removal of the pole part for a meromorphic loop γ when we let C be an infinitesimal circle centered at z_0.

Let G be a pronilpotent Lie group, $\mathcal{H}$ its graded Hopf algebra of coordinates. Let us recall the dictionary between the geometric and algebraic viewpoints.

Homomorphisms from $\mathcal{H} \to \mathcal{R}$	Loops from C to G
$\phi(\mathcal{H}) \subset \mathcal{R}_-$	γ extends to a holomorphic map from $\mathbb{C} \backslash \{z_0\} \to G$ with $\gamma(\infty) = 1$.
$\phi(\mathcal{H}) \subset \mathcal{R}_+$	γ extends to a holomorphic map defined at $z = z_0$.
$\phi = \phi_1 \star \phi_2$	$\gamma(z) = \gamma_1(z)\gamma_2(z), \ \forall\, z \in \mathbf{C}$.
$\phi \circ S$	$z \to \gamma(z)^{-1}$.

(25)

For elements $X \in \mathcal{H}$ we shall use the short-hand notation

$$\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$$

for the coproduct. The Birkhoff decomposition of a loop can the be captured by an inductive procedure given by the following theorem.

THEOREM 1. *For $\phi \colon \mathcal{H} \to \mathcal{R}$, the Birkhoff decomposition is given by*

$$\phi_-(X) = -T\Big(\phi(X) + \sum \phi_-(X')\phi(X'')\Big),$$
$$\phi_+(X) = \phi(X) + \phi_-(X) + \sum \phi_-(X')\phi(X'').$$

We are now ready to apply this procedure in quantum field theory. First, using dimensional regularization, the bare (unrenormalized) theory gives rise to a meromorphic loop

$$\gamma(z) \in G, \quad z \in \mathbb{C}. \tag{26}$$

Our main result [4, 5] is that the renormalized theory is just the evaluation at the integer dimension $z_0 = D$ of spacetime of the holomorphic part γ_+ of the Birkhoff decomposition of γ, a strikingly simple result.

3. The β-function and the Renormalization Group

In fact, the original loop $d \to \gamma(d)$ not only depends upon the parameters of the theory but also on the additional 'unit of mass' μ required by dimensional analysis. We showed in [8] that the mathematical concepts developed in our earlier papers provide very powerful tools to lift the usual concepts of the β-function and renormalization group from the space of coupling constants of the theory to the complex Lie group G.

We first observed that even though the loop $\gamma(d)$ does depend on the additional parameter μ,

$$\mu \to \gamma_\mu(d)\,, \tag{27}$$

the negative part, γ_{μ^-}, in the Birkhoff decomposition of γ_μ,

$$\gamma_\mu(d) = \gamma_{\mu^-}(d)^{-1}\,\gamma_{\mu^+}(d) \tag{28}$$

is actually independent of μ,

$$\frac{\partial}{\partial\mu}\,\gamma_{\mu^-}(d) = 0\,. \tag{29}$$

This is a restatement of a well known fact and follows immediately from dimensional analysis. Moreover, by construction, the Lie group G turns out to be graded, with grading

$$\theta_t \in \operatorname{Aut} G\,,\quad t \in \mathbb{R}\,, \tag{30}$$

inherited from the grading of the Hopf algebra $\mathcal{H}$ of Feynman graphs given by the loop number

$$L(\Gamma) = \text{loop number of } \Gamma \tag{31}$$

for any 1PI graph Γ.

The straightforward equality

$$\gamma_{e^t\mu}(d) = \theta_{t\varepsilon}(\gamma_\mu(d)) \qquad \forall\, t \in \mathbb{R}\,,\ \varepsilon = D - d \tag{32}$$

shows that the loops γ_μ associated to the unrenormalized theory satisfy the striking property that the negative part of their Birkhoff decomposition is unaltered by the operation

$$\gamma(\varepsilon) \to \theta_{t\varepsilon}(\gamma(\varepsilon))\,. \tag{33}$$

In other words, if we replace $\gamma(\varepsilon)$ by $\theta_{t\varepsilon}(\gamma(\varepsilon))$ we don't change the negative part of its Birkhoff decomposition. We settled now for the variable

$$\varepsilon = D - d \in \mathbb{C}\backslash\{0\}\,. \tag{34}$$

We give in [8] a complete characterization of the loops $\gamma(\varepsilon) \in G$ fulfilling the above striking invariance. This characterization only involves the negative part $\gamma_-(\varepsilon)$ of their Birkhoff decomposition which by hypothesis fulfills

$$\gamma_-(\varepsilon)\,\theta_{t\varepsilon}(\gamma_-(\varepsilon)^{-1}) \text{ is convergent for } \varepsilon \to 0\,. \tag{35}$$

It is easy to see that the limit of (33) for $\varepsilon \to 0$ defines a one parameter subgroup

$$F_t \in G\,,\quad t \in \mathbb{R} \tag{36}$$

and that the generator $\beta = (\partial/\partial t\,F_t)_{t=0}$ of this one parameter group is related to the *residue* of γ

$$\operatorname*{Res}_{\varepsilon=0} \gamma = -\left(\frac{\partial}{\partial u}\, \gamma_-\left(\frac{1}{u}\right)\right)_{u=0} \tag{37}$$

by the simple equation

$$\beta = Y \operatorname{Res} \gamma\,, \tag{38}$$

where $Y = (\partial/\partial t\, \theta_t)_{t=0}$ is the grading.

This is straightforward but our result is the following formula (40) which gives $\gamma_-(\varepsilon)$ in closed form as a function of β. We shall for convenience introduce an additional generator in the Lie algebra of G (i.e. primitive elements of $\mathcal{H}^*$) such that,

$$[Z_0, X] = Y(X), \quad \forall\, X \in \text{Lie}\; G\,. \tag{39}$$

The scattering formula for $\gamma_-(\varepsilon)$ is then

$$\gamma_-(\varepsilon) = \lim_{t\to\infty} \mathrm{e}^{-t(\beta\varepsilon + Z_0)}\, \mathrm{e}^{tZ_0}\,. \tag{40}$$

Both factors in the right hand side belong to the semi-direct product

$$\widetilde{G} = G \ltimes \mathbb{R} \tag{41}$$

of the group G by the grading, but of course the ratio (39) belongs to the group G.

This shows ([8]) that the higher pole structure of the divergences is uniquely determined by the residue and gives a strong form of the 't Hooft relations, which will come as an immediate corollary.

The main new result of [8], specializing to the massless case and taking φ_6^3 as an illustrative example to fix ideas and notations, is that the formula for the bare coupling constant

$$g_0 = g\, Z_1\, Z_3^{-3/2}, \tag{42}$$

where both $g\,Z_1 = g + \delta g$ and the field strength renormalization constant Z_3 are thought of as power series (in g) of elements of the Hopf algebra $\mathcal{H}$, does define a Hopf algebra homomorphism

$$\mathcal{H}_{CM} \xrightarrow{g_0} \mathcal{H}_K \tag{43}$$

from the Hopf algebra $\mathcal{H}_{CM}$ of coordinates on the group of formal diffeomorphisms of $\mathbb{C}$ such that

$$\varphi(0) = 0\,, \;\; \varphi'(0) = \text{id} \tag{44}$$

to the Hopf algebra $\mathcal{H}_K$ of the massless theory. We had already constructed in [5] a Hopf algebra homomorphism from $\mathcal{H}_{\text{CM}}$ to the Hopf algebra of rooted trees, but the physical significance of this construction was unclear.

The homomorphism (42) is quite different in that for instance the transposed group homomorphism

$$G \stackrel{\rho}{\longrightarrow} \mathrm{Diff}(\mathbb{C}) \tag{45}$$

lands, for ϕ^3 theory, in the subgroup of *odd* diffeomorphisms

$$\varphi(-z) = -\varphi(z), \quad \forall z\,, \tag{46}$$

due to the fact that each loop order in a radiative correction to $g_0 = g(1 + \cdots)$ increases the degree by g^2. Moreover, its physical significance is transparent. In particular, the image by ρ of $\beta = Y\,\mathrm{Res}\,\gamma$ is the usual β-function of the coupling constant g.

We discovered the homomorphism (44) by lengthy concrete computations which were an excellent test for the explicit ways of handling the coproduct, co-associativity, symmetry factors... that underly the theory.

As a corollary of the construction of ρ we get an *action* by (formal) diffeomorphisms of the group G on the space X of (dimensionless) coupling constants of the theory. We can then in particular formulate the Birkhoff decomposition *directly* in the group,

$$\mathrm{Diff}\,(X) \tag{47}$$

of formal diffeomorphisms of the space of coupling constants.

THEOREM 2 ([8]). *Let the unrenormalized effective coupling constant $g_{\mathrm{eff}}(\varepsilon)$ be viewed as a formal power series in g and let $g_{\mathrm{eff}}(\varepsilon) = g_{\mathrm{eff}_+}(\varepsilon)\,(g_{\mathrm{eff}_-}(\varepsilon))^{-1}$ be its (opposite) Birkhoff decomposition in the group of formal diffeomorphisms. Then the loop $g_{\mathrm{eff}_-}(\varepsilon)$ is the bare coupling constant and $g_{\mathrm{eff}_+}(0)$ is the renormalized effective coupling.*

This result is now, in its statement, no longer depending upon our group G or the Hopf algebra $\mathcal{H}$. But of course the proof makes heavy use of the above ingredients. It is a challenge to physicists to find a direct proof of this result.

Finally the Birkhoff decomposition of a loop

$$\delta(\varepsilon) \in \mathrm{Diff}\,(X)\,, \tag{48}$$

admits a beautiful geometric interpretation. If we let X be a complex manifold and pass from formal diffeomorphisms to actual ones, the data (47) is the initial data to perform, by the clutching operation, the construction of a complex bundle

$$P = (S^+ \times X) \cup_\delta (S^- \times X) \tag{49}$$

over the sphere $S = P_1(\mathbb{C}) = S^+ \cup S^-$, and with fiber X

$$X \longrightarrow P \stackrel{\pi}{\longrightarrow} S\,. \tag{50}$$

The meaning of the Birkhoff decomposition

$$\delta(\varepsilon) = \delta_-(\varepsilon)^{-1}\,\delta_+(\varepsilon) \tag{51}$$

is then exactly captured by an isomorphism of the bundle P with the trivial bundle

$$S \times X\,. \tag{52}$$

This shows the existence of a two-dimensional complex fiber bundle whose geometry encodes the renormalization by local counterterms.

4. Conclusions

While these results were explicitly worked out for the case of ϕ^3-theory in six dimensions, as an example of a renormalizable theory, the restriction to this case was essentially pedagogical. The Hopf and Lie algebra structures can in fact be determined for any perturbative expansion [13]. Nevertheless, it will not merely be a notational exercise. Obviously, in theories with several coupling constants, one confronts homomorphisms between the group of characters of the Hopf algebra to diffeomorphisms of a higher-dimensional space, with all the intricacies of such higher dimensional diffeomorphisms coming into account. Also, the switch to other renormalization schemes, elegantly provided by the group structure of characters, will add to these intricacies. It is a straightforward but interesting task to bring the powerful techniques described above to fruition in such circumstances, with first results emerging [14].

Equally striking is the connection between algebraic structures on Feynman graphs as revealed by these Lie and Hopf algebras and their transcendence. It is a well-established fact that the residues of Feynman graphs, providing the building blocks for the counterterm by the scattering type formula, evaluate in Euler–Zagier sums up to the six loop level. Afterwards, one might expect a more general class of numbers to appear. But these numbers themselves are in many ways related to configuration space integrals, and are themselves governed by Hopf and Lie algebra structures which are simpler, but still similar to the Hopf and Lie algebras obtained from Feynman graphs [12, 13].

With the proper understanding of the algebraic structure which locality imposes on the perturbative expansion being achieved with the discovery of these Hopf and Lie algebra structures, and with the principle of multiplicative subtraction so elegantly realized in the Birkhoff decomposition as described above, one can now dispense with any concern that the short-distance singularities render the perturbative expansion useless. Perturbative quantum field theory was always the workhorse for the practitioner of high-energy physics providing predictions in striking agreement with observed phenomena. One now has reason to hope that the firm conceptual ground on which this workhorse stands can finally be unraveled. A tempting challenge in this respect is the transition from formal to actual diffeomorphisms in the Birkhoff decomposition, translating to a transition from the perturbative to the non-perturbative regime in a quantum field theory.

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