General theory of detection and optimality

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Abstract. A general formulation of the problem of detection for a pair of two cones is presented. The special case is the detection of entangled states by entanglement witnesses. Having defined what means "to detect", one can identify the subset of elements, which detect optimally. I will present the properties of this set for a general pair of cones.

In particular, I prove the generalization of the theorem of Lewenstein, Krauss, Cirac, Horodecki. The entanglement witness W is optimall iff the set of product wectors $\{\phi\otimes\psi : \langle\phi\otimes\psi|W|\phi\otimes\psi\rangle = 0\}$ spans the whole Hilbert space of a system.

1. Introduction

In the set of mixed states of a bipartite quantum system one can define the subset of separable states [1]. The state ρ is called separable, when there exists a decomposition $\rho = \sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)}$. In such states of a system, its subsystems can be correlated only clasically.

There is no general method to determine, whether a given state is separable or not. One of the most important tools are entanglement witnesses [2], [3]. A hermitian observable is called entanglement witness, when its mean value in any separable state is positive, but the observable is not semi-positive.

For an entanglement witness W we can define the set of entangled states, which are *detected* by this entanglement witness, i.e. the states in which the mean value of the entanglement witness is negative. We denote this set by $\mathcal{D}(W)$. Now, we say that entanglement witness W_1 is *finer* than an entanglement witness W_2, when it detects more states, i.e. when $\mathcal{D}(W_1) \supset \mathcal{D}(W_2)$. Entanglement witness, for which there exists no finer witness, is called *optimal* [4].

In the set of quantum states we can define another set of states, the set of PPT states. A states is a PPT state, when its partial transposition $\rho^\Gamma = (I\otimes T)\rho$ is positive. The set of PPT states is a superset of the set of separable states, and the equality holds only, when the dimensions of subsystems are 2×2 and 2×3 [2]. All entangled PPT states are bound entangled [5]. Entanglement witness, which cannot detect PPT entangled states, is called *decomposable* [4]. Any decomposable entanglement witness can be written as $W = A + B^\Gamma$, where A and B are semi-positive. Entanglement witnesses, which are not decomposable, are called *non-decomposable.*

A nondecomposable entanglement witness W_1 is called *nd-finer* (non-decomposable finer) than another non-decomposable entanglement witness W_2, when W_1 detects more PPT enatangled states than W_2. A non-decomposable entanglement witness W is called *nd-optimal* (non-decomposable optimal), when there is no other entanglement witness detecting more PPT entangled states than W [4].

There are two theorems characterizing optimality [4]:

Theorem 1. *W_1 is finer than W_2 $\iff$ $W_2 = \lambda W_1 + P$, for a positive scalar λ and a semi-positive observable P.*

Theorem 2. *W_1 is nd-finer than W_2 $\iff$ $W_2 = \lambda W_1 + D$, for a positive scalar λ and $D = A + B^\Gamma$ for semi-positive observables A and B.*

I will present later a generalization of these two theorems. To do that, it is necessary to remind some basic concepts and facts about the geometry of proper cones.

2. Geometry of proper cones

This section presents basic definitions and facts of theory of proper cones. For a more detailed discussion see [6], [7].

Definition 3. *A set $K \subset \mathbb{R}^N$ is called a proper cone, iff:*

(i) $\forall \mu, \nu \geq 0 \quad \forall x, y \in K \quad \mu x + \nu y \in K$

(ii) K is closed in $\mathbb{R}^N$

(iii) $\mathrm{span} K = \mathbb{R}^N$ *(fullness)*

(iv) There exists no subspace of $\mathbb{R}^N$ contained in K (pointedness)

A set of points of a cone differing by a positive scalar is called a *ray* of the cone. Any ray can be written as $\{k \cdot x : \; k \in \mathbb{R}_+\}$, and then we say, that it is generated by an element x. A ray is called *extremal*, when a point of the ray cannot be decomposed as a convex combination of points out of the ray.

Example 4 (Examples of proper cones)**.** *The set of unnormalized quantum states (positive matrices) of d-level system is a proper cone in the real vector space of hermitian matrices $\mathcal{B}(\mathbb{C}^d)$. It's extreme rays are generated by projectors of rank one. This cone will be denoted by $\mathcal{B}_+(\mathbb{C}^d)$ or simply by $\mathcal{B}_+$.*

A matrix $\rho \in \mathcal{B}_+(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ is called unnormalized separable state of two subsystems of dimensions d_1 and d_2, when it can be decomposed as

$$\rho = \sum_i A_i \otimes B_i,$$

where $A_i \in \mathcal{B}_+(\mathbb{C}^{d_1})$ and $B_i \in \mathcal{B}_+(\mathbb{C}^{d_2})$. It's easy to check, that the set of unnormalized separable quantum states is a proper cone in the space of hermitian matrices $\mathcal{B}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$. An extreme ray of this cone is generated by a tensor product of rank-one semi-positive matrices, so one can write an alternative definition of unnormalize separable state as:

$$\rho = \sum_i |\phi_i \otimes \psi_i\rangle\langle\phi_i \otimes \psi_i|, \tag{1}$$

where vectors ψ_i and ϕ_i need not to be normalized. This cone will be denoted by $\mathcal{S}_1(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ or simply by $\mathcal{S}_1$.

A bipartite quantum state ρ is called PPT state, when $\rho^\Gamma \geq 0$. The set of unnormalized PPT states is an intersection of cones $\mathcal{B}_+(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ and $\mathcal{B}_+^\Gamma(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$.

The set of its extreme rays is not known in general. This cone will be denoted by $\mathcal{S}_{PPT}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ or simply by $\mathcal{S}_{PPT}$.

The set of positive matrices and witnesses detecting entanglement in $d_1 \times d_2$-level quantum system (a set of matrices positive on product vectors) is a proper cone in $\mathcal{B}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$. The set of its extreme rays is not known in general. This cone will be denoted by $\mathcal{W}_1(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ or simply by $\mathcal{W}_1$.

The set of positive matrices and decomposable witnesses in $d_1 \times d_2$-level quantum system is a proper cone, and its extreme rays are generated by matrices of the form P or P^Γ, where P is a projector of rank one. This cone will be denoted by $\mathcal{W}_D(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ or simply by $\mathcal{W}_D$.

2.1. Duality

For a cone K in a real vector space X one defines a proper cone K^* in X^*.

Definition 5. *A set K^* defined as*

$$K^* = \{y \in X^* : \quad \forall x \in K \ \langle y|x\rangle > 0\}$$

is called a dual cone of a proper cone K.

One can restrict the quantified set in definition to points of extreme rays of K.

The set K^* is a proper cone. One can consider then the proper cone $(K^*)^*$. Using the reflexivity of a finite-dimensional real vector space, one can easily prove, that $(K^*)^* = K$. The duality of proper cones has the following properties:

Fact 6. *The properties of duality of cones:*

- $K \subset L \ \Rightarrow \ K^* \supset L^*$.
- $(K \cap L)^* = \mathrm{conv}(K^* \cup L^*)$.
- $\mathrm{conv}(K \cup L)^* = K^* \cap L^*$

An inner product in X constitutes isomorfism between X and X^*. One can then consider K and K^* as elements of the same space. When for a cone K holds $K = K^*$, one calls K *self dual*. In spaces of hermitian matrices, which we are interested in, such an inner product is Hilbert-Schmidt product.

Example 7 (Quantum states)**.** *The cone $\mathcal{B}_+$ is self dual. Indeed, a matrix $\rho \in \mathcal{B}_+$ is semi-positive, iff $\forall\psi \ \langle\psi|\rho|\psi\rangle \geq 0$, what can be rewritten as $\forall\psi \ \mathrm{Tr}(|\psi\rangle\langle\psi|\rho) \geq 0$. The matrix ρ is then positive on all extreme rays of $\mathcal{B}_+$, so $\rho \in \mathcal{B}_+^*$.*

Example 8 (Separable states and entanglement witnesses)**.** *By definition, the matrix W is an element of the proper cone $\mathcal{W}_1$ iff $\forall\rho \in \mathcal{S}_1 \ \langle\rho|W\rangle_{HS} \geq 0$, so $W \in \mathcal{S}_1^*$. These proper cones are dual to each other.*

Example 9 (PPT states and nd-witnesses)**.** *The proper cone $\mathcal{S}_{PPT}$ is an intersection of two proper cones: $\mathcal{B}_+$ and $\mathcal{B}_+^\Gamma$. Due to the second property in Fact 6, the proper cone $\mathcal{B}_{PPT}^*$ is a convex hull of the sum of proper cones: $(\mathcal{B}_+)^* \cup (\mathcal{B}_+^\Gamma)^* = \mathcal{B}_+ \cup \mathcal{B}_+^\Gamma$. Such a sum is spanned by the sum of sets of extreme rays of both cones, so its extreme rays are generated by matrices P and P^Γ, where P is a projector of rank one. The proper cone spanned by the set of such extreme points is $\mathcal{W}_D$. The cones $\mathcal{S}_{PPT}$ and $\mathcal{W}_D$ are dual to each other.*

2.2. Faces of a cone

A subset F of a proper cone $K \subset \mathbb{R}^N$ is called a *face* of a cone, if it is an intersection of the cone and a kernel of a linear functional which is non-negative on the cone. Geometrically, a face is an intersection of the cone and a hipersurface tangent to the cone. The fact that F is a face of a cone K is denoted by $F \lhd K$.

A face F of a proper cone K is a proper cone in subspace $\mathrm{span}F$. A face G of this cone is also a face of K. It allows us to define a relation in the set of faces of the cone K:

Definition 10 (Subface)**.** *Having given two faces $F, G \lhd K$, we call a face G a subface of a face F, iff the face G is a face of the proper cone F in the subspace* $\mathrm{span}F$*.*

The relation of beeing subface constitutes a partial order in the set of faces of a proper cone. The maximal element due to this partial order is the whole cone K, and the minimal element is the face $\{0\}$.

Intersection of two faces is a face. It allows us to define for a given x the minimal face containing x as the intersection of all faces containing ρ. Such a face is said to be generated by an element ρ. From now, the face of a cone K generated by an element x will be denoted by $F_K(x)$.

Consider a face $F_\Phi \lhd K$, which is an intersection of the cone K and the kernel of functional Φ. Consider two elements $x_0, x_1 \in K$. Let $x_0 \in F_\Phi \;\Leftrightarrow\; \Phi(x_0) = 0$ and let $x_1 \notin F_\Phi \;\Leftrightarrow\; \Phi(x_0) > 0$. For arbitraty $\alpha > 0$ an element $x_0 - \alpha x_1 \notin K$, because $\Phi(x_0 - \alpha x_1) = \Phi(x_0) - \alpha\Phi(x_1) < 0$. It means, that

$$x_1 \in F_\Phi \;\Leftrightarrow\; \exists \alpha > 0 :\; x_0 - \alpha x_1 \in K,$$

but this condition holds for any F_Φ containing x_0, so also for the intersection of such faces:

$$x_1 \in F_K(x_0) \;\Leftrightarrow\; \exists \alpha > 0 :\; x_0 - \alpha x_1 \in K. \tag{2}$$

There exists one-to-one correspondece between faces of K and faces of K^*. For any face F of K one defines a face $\Phi(F)$ called the *complementary* face of F:

Definition 11 (Complementary face)**.** *A face $\Phi(F) \lhd K^*$ defined by the formula*

$$\Phi(F) = \{y \in K^* :\; \forall x \in F \;\langle y|x\rangle = 0\}$$

is called a complementary face of the face F.

Further, we will need some properties of complementarity:

Proposition 12. *Properties of complementarity:*

(i) $F \lhd G \;\Leftrightarrow\; \Phi(G) \rhd \Phi(F)$.

(ii) $\Phi(\{0\}) = K$.

(iii) $\Phi(K) = \{0\}$.

Example 13 (Faces of a cone of positive matrices)**.** *The structure of the cone $\mathcal{B}_+(\mathbb{C}^d)$ is exactly known [8]. A face generated by a given element $\rho \in \mathcal{B}_+(\mathbb{C}^d)$ is the set of all positive matrices with the image contained in the image of ρ. The dimension of $F_{\mathcal{B}_+}(\rho)$ is equal* $(\mathrm{rank}\rho)^2$*. Faces are then in one-to-one correspondence with the lattice of subspaces of $\mathbb{C}^d$. Denote the face of matrices with the image contained in a subspace V as F_V. It's quite easy to find the face complementary to F_V. We have $\Phi(F_V) = F_{V^\perp}$.*

3. Main results

From now on, we will consider two proper cones $K \subset L$ in real vector space X.

At the beginning, let us define a relation between an element $w \in L \setminus K \subset X$ and an element $\rho \in K^* \setminus L^* \subset X^*$.

Definition 14 (Detection of an element in $K^* \setminus L^*$ by an element in $L \setminus K$)**.** *We say, that an element $w \in L \setminus K$ detects an element $\rho \in K^* \setminus L^*$, iff $\rho(w) < 0$. For an element $w \in L \setminus K$ we denote by $\mathcal{D}_{L|K}(w)$ the set of all states in $K^* \setminus L^*$ detected by w.*

The Banach separation theorem asures us, that for any element in $K^* \setminus L^*$ there exists an element in $L \setminus K$ detecting it, and that the dual fact holds, i.e. any element of the set $L \setminus K$ detects an element of a set $K^* \setminus L^*$. One can extend the definition 14 to the whole proper cone L fixing $\mathcal{D}_{L|K}(k) = \emptyset$ for all $k \in K \subset L$.

For two elements $w_1, w_2 \in L \setminus K$ one can define a relation of "being finer":

Definition 15 ("Being finer")**.** *We say, that an element $w_1 \in L \setminus K$ is finer than an element $w_2 \in L \setminus K$ with respect to the proper cone K, iff $\mathcal{D}_{L|K}(w_1) \supseteq \mathcal{D}_{L|K}(w_2)$ (w_1 detects more elements of $K^* \setminus L^*$ than w_2 in the sense of inclusion of sets). We denote this fact by $w_1 \geq_{f(K)} w_2$.*

An element which is maximall with respect to this order is called optimal:

Definition 16 (Optimality)**.** *An element $w_1 \in L \setminus K$ is called optimal with respect to the proper cone K, if there is no other element finer than w_1 in $L \setminus K$ (which detects more elements in $K^* \setminus L^*$).*

On the other hand, one can define a relation of order with respect to the cone K:

Definition 17. *An element $w_1 \in L$ is said to be greater than $w_2 \in L$ with respect to the cone K, iff*

$$\exists \lambda \in \mathbb{R}_+ : \; w_1 - \lambda w_2 \in K.$$

We will denote it by $w_1 \geq_K w_2$.

One can prove a theorem, that both following relations are equivalent. This is a generalization of Lemma 2 in [4] for arbitrary proper cones L and $K \subset L$ ‡ .

Theorem 18.
$$w_1 \geq_{f(K)} w_2 \;\Leftrightarrow\; w_1 \leq_K w_2$$

Proof: The proof bases on proof from [4].

"$\Leftarrow$": Assume, that $w_1 \leq_K w_2$. It means, that $w_1 = w_2 - k$ for an element of proper cone K. It means, that if only for an arbitrary $\rho \in K^*$ holds an inequality $\rho(w_2) < 0$, then also $\rho(w_1) < 0$ holds, so $\mathcal{D}_{L|K}(w_1) \supset \mathcal{D}_{L|K}(w_2)$, and then $w_1 \geq_{L|K} w_2$.

"$\Rightarrow$": In the other side, assume that $w_1 \geq_{L|K} w_2$, so that $\mathcal{D}_{L|K}(w_1) \supset \mathcal{D}_{L|K}(w_2)$. We will prove, that $\lambda w_2 - w_1 \in K$, when a parameter λ is chosen to be:

$$\lambda = \inf_{\rho \in \mathcal{D}_{L|K}(w_2)} \left| \frac{\rho(w_1)}{\rho(w_2)} \right|. \tag{3}$$

We will do it proving an inequality:

$$\forall \rho \in K^* \quad \lambda \rho(w_2) \geq \rho(w_1) \tag{4}$$

depending of the sign of the left-hand side.

‡ In [4] $L = \mathcal{W}_1$ and $K = \mathcal{B}_+$ or $K = \mathcal{S}_{PPT}$. Moreover, the work deals with normalized states and witnesses, what the reader should have in mind comparing results.

(i) $\rho(w_2) = 0 \;\Rightarrow\; \rho(w_1) \leq 0$.
Suppose, that for some $\rho \in K^*$ we have $\rho(w_2) = 0 \;\wedge\; \rho(w_1) > 0$. Then there exists such an $\epsilon > 0$, that $\forall \rho' \in B(\rho, \epsilon) \cap K^*\ \rho'(w_1) > 0$. This set contains unempty interior, so it have to consist states, for which $\rho'(w_2) < 0$, but it denies the assumption $w_1 \geq_{L|K} w_2$.

(ii) $\rho(w_2) < 0 \;\Rightarrow\; \rho(w_1) \leq \rho(w_2)$.
We construct an element $\rho_1 \in K^*$ as $\rho_1 = \rho - \rho(w_2)I$, where I denotes arbitrary element of a proper cone K^*, for which $I(w_2) = I(w_1) = 1$. Such constructed ρ_1 fulfills the assumption from the previous case, so an equality $\rho_1(w_1) = \rho(w_1) - \rho(w_2) \leq 0$ holds, what proves the postulated inequality.
We will use it now to prove the inequality (4) for $\rho(w_2) < 0$. We know, that $\forall \rho \in \mathcal{D}_{L|K}(w_2)$ an inequality $\rho(w_1) < 0$ holds. It lets us to substitute the absolute value in the formula 3 with negation, what leads to:

$$\lambda = \inf_{\tilde{\rho} \in \mathcal{D}_{L|K}(w_2)} \frac{\tilde{\rho}(w_1)}{\tilde{\rho}(w_2)} \;\Rightarrow\; \frac{\rho(w_1)}{\rho(w_2)} \geq \lambda \;\Rightarrow\; \lambda\rho(w_2) \geq \rho(w_1),$$

what proves the inequality 4 in the case, when its left-hand side is negative.

(iii) $\rho(w_2) > 0 \;\Rightarrow\; \lambda\rho(w_2) \geq \rho(w_1)$
Let's take an arbitrary element $\rho_1 \in \mathcal{D}_{L|K}(w_2)$. Let's define by use of it new element of the proper cone K^* as $\rho_2 = \rho(w_2)\rho_1 - \rho_1(w_2)\rho$. An equality $\rho_2(w_2) = 0$ holds for it, so one can use to it the result of the first step an get $\rho_2(w_2) = \rho(w_2)\rho_1(w_1) - \rho_1(w_2)\rho(w_1) \leq 0$. We get in result an inequality $\rho(w_2)\rho_1(w_1) \leq \rho_1(w_2)\rho(w_1)$. Let's divide its sides by a negative number $\rho_1(w_2)\rho(w_2)$. We get then inequality:

$$\frac{\rho_1(w_1)}{\rho_1(w_2)} \geq \frac{\rho(w_1)}{\rho(w_2)}$$

The above inequality holds for an arbitrary $\rho_1 \in \mathcal{D}_{L|K}(w_2)$, so it holds also for the infimum of the right-hand side taken due to the set $\mathcal{D}(w_2)$. This infimum defines the λ. Multiplying both sides of such derived inequality by $\rho(w_1)$ one gets the inequality (4) for $\rho(w_2) > 0$.

We have shown in this way, that the inequality (4) is fulfilled independly on the sign of its left-hand side. □

Example 19. *Choosing $\mathcal{W}_1$ as L and $\mathcal{B}_+$ as K, one gets Theorem 1.*

Example 20. *Choosing $\mathcal{W}_1$ as L and $\mathcal{W}_D$ as K, one gets Theorem 2.*

Having proven the theorem, on has immediately the following:

Proposition 21. *The element $w \in L$ is optimal with respect to K iff $w - k \notin L$ for any $k \in K$.*

3.1. Geometrical properties of optimality

Lemma 22. *If an element $w \in \mathrm{opt}(L|K)$, then $F_L(w) \subset \mathrm{opt}(L|K)$.*

Proof: Suppose, that $y \in F_L(w)$, but y is not optimal. We will see, that then w cannot be optimal.

$$y \in F_L(w) \overset{def}{\Leftrightarrow} \exists \alpha > 0, \exists l_1 \in L:\; w - \alpha y = l_1,$$

but y is not optimal, so $\exists k \in K\ k \neq 0\ \exists l_2 \in L : y = l_2 + k$. Then $w = l_1 + \alpha l_2 + \alpha k$ so w is not optimal. □

Lemma 23. $F \subset \mathrm{opt}(L|K) \;\Leftrightarrow\; F \cap K = \{0\}$

Proof: "$\Rightarrow$" A non-zero element of K detects no elements in $K^* \setminus L^*$, so it cannot be optimal (any other set contains empty set - any other witness detects better) but $F \subset \mathrm{opt}(L|K)$ only if all elements of F are optimal, so when $F \cap K \neq \{0\}$, $F \not\subset \mathrm{opt}(L|K)$.

"$\Leftarrow$" Suppose, that $F \not\subset \mathrm{opt}(L|K)$. It means, that there exists an element $x \in F$ which is not optimal:

$$\exists k \in K \; \exists l \in L : \; x = l + k \;\wedge\; k \neq 0,$$

but it implies, that $k = x - l$, so $k \in F_L(x)$. $\Box$

Theorem 24. *An element $w \in \mathrm{opt}(L|K)$, iff $\Phi(F_L(w)) \cap K \neq \{0\}$.*

Proof: If an element $w \in \mathrm{opt}(L|K)$, then the whole face generated by w is included in $\mathrm{opt}(L|K)$. Using the Lemma 23 we know that the only point of K contained in $F_L(w)$ is 0, in particular the only face of K contained in $F_L(w)$ is $\{0\}$:

$$\begin{aligned} & w \in \mathrm{opt}(L|K) \;\Leftrightarrow \sim \exists G \lhd K, G \neq \{0\} : G \lhd F_L(w) \\ & \Leftrightarrow \; \forall G \lhd K, G \neq \{0\} \; G \ntriangleleft F_L(w). \end{aligned}$$

Now one can use the complementarity relation between faces:

$$w \in \mathrm{opt}(L|K) \;\Leftrightarrow\; \forall H \lhd K^*, H \neq K^* \; H \ntriangleright \Phi[F_L(w)].$$

Any face of K^*, which is not the whole K^*, does not contain the face $\Phi[F_L(w)]$. It implies, that face in K^* generated by points of $\Phi[F_L(w)]$ is equal the whole K^*, so the set $\Phi[F_L(w)]$ must contain a point from $\mathrm{Int}K^*$. $\Box$

Definition 25. *We will denote by $\mathcal{P}^L(w)$ the set of points of extreme rays of $\Phi(F_L(w))$, i.e. the set of all points y in L^*, such that $\langle y|w\rangle = 0$.*

Proposition 26. *The element $w \in L \setminus K$ is optimal, iff there exists a convex combination of $\mathcal{P}^L(w)$, which gives an element of $\mathrm{Int}K^*$.*

Example 27. *Let $L = \mathcal{W}_1(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ and $K = \mathcal{B}_+(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$. Then $\mathcal{P}^L(w)$ is the set of projectors onto product vectors $\phi \otimes \psi$, for which the inequality $\langle \phi \otimes \psi | w | \phi \otimes \psi \rangle = 0$ holds. The elements of $\mathrm{Int}\mathcal{B}_+^* = \mathrm{Int}\mathcal{B}_+$ are hermitian matrices of the full rank. The combination of projectors from $\mathcal{P}^L(w)$ can be of the full rank, iff the vectors $\phi \otimes \psi$ spans the whole $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$.*

This fact was presented in [4], but only as a sufficient condition of optimality. Here it has been proved, that it is also a necessary condition.

Let's now consider the same condition for nd-optimality.

Example 28. *Let $L = \mathcal{W}_1(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ and $K = \mathcal{W}_D(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$. Then $\mathcal{P}^L(w)$ is again the set of projectors onto product vectors $\phi \otimes \psi$, for which the inequality $\langle \phi \otimes \psi | w | \phi \otimes \psi \rangle = 0$ holds. The cone $\mathcal{S}_{PPT} = \mathcal{B}_+ \cap \mathcal{B}_+^\Gamma$, so the interior of $\mathcal{S}_{PPT}$ is the intersection of the interiors of $\mathcal{B}_+$ and $\mathcal{B}_+^\Gamma$. One has:*

$$\rho \in \mathrm{Int}\mathcal{S}_{PPT} \;\Leftrightarrow\; \rho \in \mathrm{Int}\mathcal{B}_+ \cap \mathrm{Int}\mathcal{B}_+^\Gamma \;\Leftrightarrow \rho \in \mathrm{Int}\mathcal{B}_+ \;\wedge\; \rho^\Gamma \in \mathrm{Int}\mathcal{B}_+$$

It implies, that if $w \in \mathrm{opt}(\mathcal{W}_1, \mathcal{W}_D)$, then $w, w^\Gamma \in \mathrm{opt}(\mathcal{W}_1, \mathcal{B}_+)$. In the opposite direction the implication does not work, because one can not be sure, if one can choose the elements $\rho_1 \in \mathcal{B}_+$ and $\rho_2 \in \mathcal{B}_+^\Gamma$ to be equal.

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