Optimality of entanglement witnesses - a general formulation

Gniewomir Sarbicki

8 września 2009

Outline

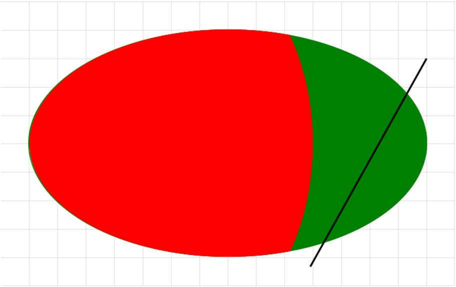
- Introduction - What are entanglement witnesses?
- The world of proper cones
- The proper cones most interesting for us
- The general concept of detection and optimality
- The geometrical characterization of optimality

An observable W is an entanglement witness, iff:

- $\forall \psi \otimes \phi \quad \langle \psi \otimes \phi | W | \psi \otimes \phi \rangle \geqslant 0 \quad (W \in \mathcal{W}_1)$,
- $\exists \Psi : \quad \langle \Psi | W | \Psi \rangle < 0 \quad (W \notin \mathcal{B}_+)$
  - we say, that Ψ is detected by W.

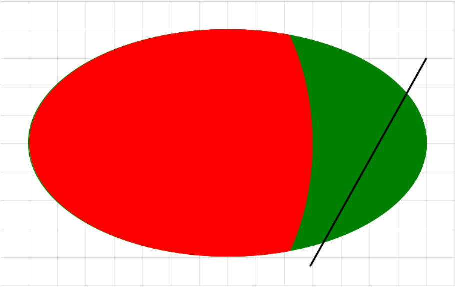
An observable W is an entanglement witness, iff:

- $\forall \psi \otimes \phi \quad \langle \psi \otimes \phi | W | \psi \otimes \phi \rangle \geqslant 0 \quad (W \in \mathcal{W}_1)$,
- $\exists \Psi : \quad \langle \Psi | W | \Psi \rangle < 0 \quad (W \notin \mathcal{B}_+)$
  - we say, that Ψ is detected by W.



An observable W is an entanglement witness, iff:

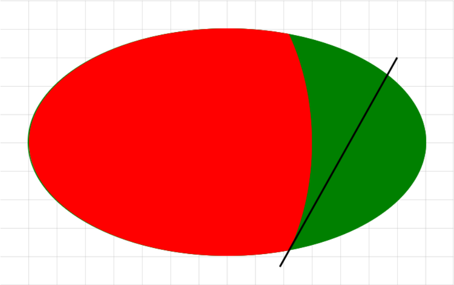
- $\forall \psi \otimes \phi \quad \langle \psi \otimes \phi | W | \psi \otimes \phi \rangle \geqslant 0 \quad (W \in \mathcal{W}_1)$,
- $\exists \Psi : \quad \langle \Psi | W | \Psi \rangle < 0 \quad (W \notin \mathcal{B}_+)$
  - we say, that Ψ is detected by W.



Optimal witness - a witness detecting the most of the states.

An observable W is an entanglement witness, iff:

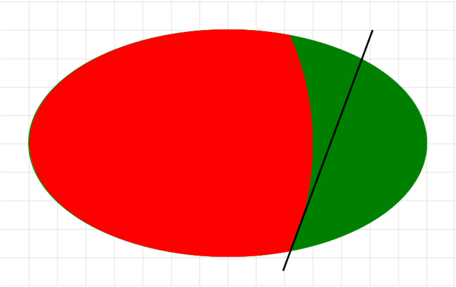
- $\forall \psi \otimes \phi \quad \langle \psi \otimes \phi | W | \psi \otimes \phi \rangle \geqslant 0 \quad (W \in \mathcal{W}_1)$,
- $\exists \Psi : \quad \langle \Psi | W | \Psi \rangle < 0 \quad (W \notin \mathcal{B}_+)$
  - we say, that Ψ is detected by W.



Optimal witness - a witness detecting the most of the states.

An observable W is an entanglement witness, iff:

- $\forall \psi \otimes \phi \quad \langle \psi \otimes \phi | W | \psi \otimes \phi \rangle \geqslant 0 \quad (W \in \mathcal{W}_1)$,
- $\exists \Psi : \quad \langle \Psi | W | \Psi \rangle < 0 \quad (W \notin \mathcal{B}_+)$
  - we say, that Ψ is detected by W.



Optimal witness - a witness detecting the most of the states.

A proper cone

Def: A closed subset K of a real vector space X is called a proper cone, if:

A proper cone

Def: A closed subset K of a real vector space X is called a proper cone, if:

- $x \in K \;\Rightarrow\; \forall\lambda \in \mathbb{R}_+ \; \lambda x \in K$ (any point of the cone belongs to the cone with the whole **halfray** generated by this point)

A proper cone

Def: A closed subset K of a real vector space X is called a proper cone, if:

- $x \in K \Rightarrow \forall\lambda \in \mathbb{R}_+ \ \lambda x \in K$ (any point of the cone belongs to the cone with the whole **halfray** generated by this point)
- $x, y \in K \Rightarrow x + y \in K$ (convexity)

A proper cone

Def: A closed subset K of a real vector space X is called a proper cone, if:

- $x \in K \;\Rightarrow\; \forall\lambda \in \mathbb{R}_+ \; \lambda x \in K$ (any point of the cone belongs to the cone with the whole **halfray** generated by this point)
- $x, y \in K \;\Rightarrow\; x + y \in K$ (convexity)
- $x \in K \;\wedge\; -x \in K \;\Rightarrow\; x = 0$ (pointedness)

A proper cone

Def: A closed subset K of a real vector space X is called a proper cone, if:

- $x \in K \;\Rightarrow\; \forall \lambda \in \mathbb{R}_+ \; \lambda x \in K$ (any point of the cone belongs to the cone with the whole **halfray** generated by this point)
- $x, y \in K \;\Rightarrow\; x + y \in K$ (convexity)
- $x \in K \;\wedge\; -x \in K \;\Rightarrow\; x = 0$ (pointedness)
- $\mathrm{span} K = X$ (fullness)

A proper cone

Def: A closed subset K of a real vector space X is called a proper cone, if:

- $x \in K \;\Rightarrow\; \forall\lambda \in \mathbb{R}_+ \; \lambda x \in K$ (any point of the cone belongs to the cone with the whole **halfray** generated by this point)
- $x, y \in K \;\Rightarrow\; x + y \in K$ (convexity)
- $x \in K \;\wedge\; -x \in K \;\Rightarrow\; x = 0$ (pointedness)
- $\mathrm{span} K = X$ (fullness)

Fact: A proper cone is a convex hull of its extreme rays.

Duality of cones

Def: A dual cone $K^* \subset X^*$

$$K^* = \{y : \forall x \in K \ \langle y|x\rangle \geqslant 0\}$$

Such defined set is called a dual cone.

Duality of cones

Def: A dual cone $K^* \subset X^*$

$$K^* = \{y : \forall x \in K \ \langle y|x\rangle \geqslant 0\}$$

Such defined set is called a dual cone.

Remark: It sufficies to check the condition in defintion only for points generating the extremal halfrays of cone K.

Duality of cones

Def: A dual cone $K^* \subset X^*$

$$K^* = \{y : \forall x \in K \ \langle y|x\rangle \geqslant 0\}$$

Such defined set is called a dual cone.

Remark: It sufficies to check the condition in defintion only for points generating the extremal halfrays of cone K.

Properties of duality:

- $F \subset G \ \Leftrightarrow \ F^* \supset G^*$
- $(F \cap G)^* = \mathrm{conv}(F^* \cup G^*)$
- $(\mathrm{conv}(F \cup G))^* = F^* \cap G^*$

Duality of cones

Def: A dual cone $K^* \subset X^*$

$$K^* = \{y : \forall x \in K \ \langle y|x\rangle \geqslant 0\}$$

Such defined set is called a dual cone.

Remark: It sufficies to check the condition in defintion only for points generating the extremal halfrays of cone K.

Properties of duality:

- $F \subset G \ \Leftrightarrow \ F^* \supset G^*$
- $(F \cap G)^* = \mathrm{conv}(F^* \cup G^*)$
- $(\mathrm{conv}(F \cup G))^* = F^* \cap G^*$

Def: If $X = X^*$ (for example we have a scalar product), then a cone $K = K^*$ is called **self-dual**.

Face of a cone and the ordering relation between them

Def: We call a set F a face of the cone K, if it is an intersection of the cone and a kernel of a functional which is non-negative on the cone (intersection of the cone and a hipersurphace tangent to it). One denotes it as: $F \lhd K$.

Face of a cone and the ordering relation between them

Def: We call a set F a face of the cone K, if it is an intersection of the cone and a kernel of a functional which is non-negative on the cone (intersection of the cone and a hipersurphace tangent to it). One denotes it as: $F \lhd K$.

Special faces: $\{0\}$, K.

Face of a cone and the ordering relation between them

Def: We call a set F a face of the cone K, if it is an intersection of the cone and a kernel of a functional which is non-negative on the cone (intersection of the cone and a hipersurphace tangent to it). One denotes it as: $F \lhd K$.

Special faces: $\{0\}$, K.

Fact: A face $F \lhd K$ is a proper cone in vector space $\mathrm{span} F$.

Face of a cone and the ordering relation between them

Def: We call a set F a face of the cone K, if it is an intersection of the cone and a kernel of a functional which is non-negative on the cone (intersection of the cone and a hipersurphace tangent to it). One denotes it as: $F \lhd K$.

Special faces: $\{0\}$, K.

Fact: A face $F \lhd K$ is a proper cone in vector space $\mathrm{span} F$.

For this proper cone one can again find faces: $G \lhd F \lhd K$.

Face of a cone and the ordering relation between them

Def: We call a set F a face of the cone K, if it is an intersection of the cone and a kernel of a functional which is non-negative on the cone (intersection of the cone and a hipersurphace tangent to it). One denotes it as: $F \lhd K$.

Special faces: $\{0\}$, K.

Fact: A face $F \lhd K$ is a proper cone in vector space $\mathrm{span} F$.

For this proper cone one can again find faces: $G \lhd F \lhd K$.

We have then a partial order in the set of faces of the cone K. The minimal element is $\{0\}$, and the maximal one is K.

The face generated by an element of cone

Intersection of two faces is a face $\Rightarrow$ there exists a smallest face containing a given element $x \in K$.

The face generated by an element of cone

Intersection of two faces is a face $\Rightarrow$ there exists a smallest face containing a given element $x \in K$.

Such a face is called **a face generated by the element** x.

Complementarity between faces of cones K i K^*

Def: The face complementary to a face F is the following set:

$$\Phi(F) = \{y \in K^* : \ \forall x \in F \ \langle y|x\rangle = 0\}$$

This set is a face of the cone K^*.

Complementarity between faces of cones K i K^*

Def: The face complementary to a face F is the following set:

$$\Phi(F) = \{y \in K^* : \ \forall x \in F \ \langle y|x \rangle = 0\}$$

This set is a face of the cone K^*.

Properties of complementarity:

- $F \lhd G \ \Leftrightarrow \ \Phi(F) \rhd \Phi(G)$
- $\Phi(\{0\}) = K^*$
- $\Phi(K) = \{0\}$

Cone $\mathcal{B}_+$

$\mathcal{B}_+ \subset \mathcal{B}_H$ the cone of semipositive observables in the space of hermitian observables.

Cone $\mathcal{B}_+$

$\mathcal{B}_+ \subset \mathcal{B}_H$ the cone of semipositive observables in the space of hermitian observables.

Its extremal halfrays are generated by projectors of rank one.

Cone $\mathcal{B}_+$

$\mathcal{B}_+ \subset \mathcal{B}_H$ the cone of semipositive observables in the space of hermitian observables.

Its extremal halfrays are generated by projectors of rank one.

$\mathcal{S}_+ \subset \mathcal{B}_H^*$ the cone of (unnormalized) density matrices

A matrix ρ is density matrix, iff $\forall \Psi \quad \langle \Psi | \rho | \Psi \rangle \quad \geqslant 0$,

Cone $\mathcal{B}_+$

$\mathcal{B}_+ \subset \mathcal{B}_H$ the cone of semipositive observables in the space of hermitian observables.

Its extremal halfrays are generated by projectors of rank one.

$\mathcal{S}_+ \subset \mathcal{B}_H^*$ the cone of (unnormalized) density matrices

A matrix ρ is density matrix, iff $\forall \Psi \quad \mathrm{Tr}(P_\Psi \rho) \geqslant 0$,

Cone $\mathcal{B}_+$

$\mathcal{B}_+ \subset \mathcal{B}_H$ the cone of semipositive observables in the space of hermitian observables.

Its extremal halfrays are generated by projectors of rank one.

$\mathcal{S}_+ \subset \mathcal{B}_H^*$ the cone of (unnormalized) density matrices

A matrix ρ is density matrix, iff $\forall \Psi \quad \langle P_\Psi | \rho \rangle_{HS} \geqslant 0$,

Cone $\mathcal{B}_+$

$\mathcal{B}_+ \subset \mathcal{B}_H$ the cone of semipositive observables in the space of hermitian observables.

Its extremal halfrays are generated by projectors of rank one.

$\mathcal{S}_+ \subset \mathcal{B}_H^*$ the cone of (unnormalized) density matrices

A matrix ρ is density matrix, iff $\forall\Psi \quad \langle P_\Psi|\rho\rangle_{HS} \geqslant 0$,
i.e. iff it is positive on all extremal halfrays of $\mathcal{B}_+ \;\Rightarrow\; \mathcal{S}_+ = \mathcal{B}_+^*$

Cone $\mathcal{B}_+$

$\mathcal{B}_+ \subset \mathcal{B}_H$ the cone of semipositive observables in the space of hermitian observables.

Its extremal halfrays are generated by projectors of rank one.

$\mathcal{S}_+ \subset \mathcal{B}_H^*$ the cone of (unnormalized) density matrices

A matrix ρ is density matrix, iff $\forall \Psi \quad \langle P_\Psi | \rho \rangle_{HS} \geqslant 0$,
i.e. iff it is positive on all extremal halfrays of $\mathcal{B}_+ \;\Rightarrow\; \mathcal{S}_+ = \mathcal{B}_+^*$

Conclusion: Taking $\mathcal{B}_H = \mathcal{B}_H^*$, a cone $\mathcal{B}_+$ is self dual.

Faces of the cone $\mathcal{B}_+$

Faces of the cone $\mathcal{B}_+$

A face of the cone $\mathcal{B}_+$ is a set of semipositive matrices with image contained in a subspace V. We denote it by F_V.

Faces of the cone $\mathcal{B}_+$

A face of the cone $\mathcal{B}_+$ is a set of semipositive matrices with image contained in a subspace V. We denote it by F_V.

A set of faces is isomorphic to a lattice of subspaces of Hilbert space (a connection with quantum logic).

Faces of the cone $\mathcal{B}_+$

A face of the cone $\mathcal{B}_+$ is a set of semipositive matrices with image contained in a subspace V. We denote it by F_V.

A set of faces is isomorphic to a lattice of subspaces of Hilbert space (a connection with quantum logic).

A face generated by an element ρ is $F_{\mathrm{Im}\rho}$.

Faces of the cone $\mathcal{B}_+$

A face of the cone $\mathcal{B}_+$ is a set of semipositive matrices with image contained in a subspace V. We denote it by F_V.

A set of faces is isomorphic to a lattice of subspaces of Hilbert space (a connection with quantum logic).

A face generated by an element ρ is $F_{\mathrm{Im}\rho}$.

A face complementary to F_V is $F_{V^\perp}$.

Cones $\mathcal{S}_1$ and $\mathcal{W}_1$

Cones $\mathcal{S}_1$ and $\mathcal{W}_1$

$\mathcal{S}_1(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \subset \mathcal{B}_H^*(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ - the cone of unnormalized separable quantum states.

Cones $\mathcal{S}_1$ and $\mathcal{W}_1$

$\mathcal{S}_1(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \subset \mathcal{B}^*_H(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ - the cone of unnormalized separable quantum states.

$$\rho \in \mathcal{S}_1 \overset{df}{\Leftrightarrow} \rho = \sum_i \lambda_i P_{\psi_i \otimes \phi_i}$$

Cones $\mathcal{S}_1$ and $\mathcal{W}_1$

$\mathcal{S}_1(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \subset \mathcal{B}_H^*(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ - the cone of unnormalized separable quantum states.

$$\rho \in \mathcal{S}_1 \overset{df}{\Leftrightarrow} \rho = \sum_i \lambda_i P_{\psi_i \otimes \phi_i}$$

Proposition: The extremal rays of $\mathcal{S}_1$ are generated by rank-one projectors onto product vectors.

Cones $\mathcal{S}_1$ and $\mathcal{W}_1$

$\mathcal{S}_1(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \subset \mathcal{B}_H^*(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ - the cone of unnormalized separable quantum states.

$$\rho \in \mathcal{S}_1 \overset{df}{\Leftrightarrow} \rho = \sum_i \lambda_i P_{\psi_i \otimes \phi_i}$$

Proposition: The extremal rays of $\mathcal{S}_1$ are generated by rank-one projectors onto product vectors.

In general, we know nothing about the structure of faces.

Cones $\mathcal{S}_1$ and $\mathcal{W}_1$

$\mathcal{S}_1(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \subset \mathcal{B}_H^*(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ - the cone of unnormalized separable quantum states.

$$\rho \in \mathcal{S}_1 \overset{df}{\Leftrightarrow} \rho = \sum_i \lambda_i P_{\psi_i \otimes \phi_i}$$

Proposition: The extremal rays of $\mathcal{S}_1$ are generated by rank-one projectors onto product vectors.

In general, we know nothing about the structure of faces.

Dual cone: $W \in \mathcal{S}_1^* \;\Leftrightarrow\; \forall\psi, \forall\phi \; \langle P_{\psi\otimes\phi} | W \rangle_{HS} \geqslant 0$

Cones $\mathcal{S}_1$ and $\mathcal{W}_1$

$\mathcal{S}_1(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \subset \mathcal{B}^*_H(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ - the cone of unnormalized separable quantum states.

$$\rho \in \mathcal{S}_1 \overset{df}{\Leftrightarrow} \rho = \sum_i \lambda_i P_{\psi_i \otimes \phi_i}$$

Proposition: The extremal rays of $\mathcal{S}_1$ are generated by rank-one projectors onto product vectors.

In general, we know nothing about the structure of faces.

Dual cone: $W \in \mathcal{S}_1^* \;\Leftrightarrow\; \forall \psi, \forall \phi \; \langle P_{\psi \otimes \phi} | W \rangle_{HS} \geqslant 0$

$\mathcal{S}_1^* \overset{den}{=} \mathcal{W}_1$ - a set of observables, which are semipositive on product vectors.

Cones $\mathcal{S}_1$ and $\mathcal{W}_1$

$\mathcal{S}_1(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \subset \mathcal{B}^*_H(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ - the cone of unnormalized separable quantum states.

$$\rho \in \mathcal{S}_1 \overset{df}{\Leftrightarrow} \rho = \sum_i \lambda_i P_{\psi_i \otimes \phi_i}$$

Proposition: The extremal rays of $\mathcal{S}_1$ are generated by rank-one projectors onto product vectors.

In general, we know nothing about the structure of faces.

Dual cone: $W \in \mathcal{S}_1^* \;\Leftrightarrow\; \forall \psi, \forall \phi \; \langle P_{\psi \otimes \phi} | W \rangle_{HS} \geqslant 0$

$\mathcal{S}_1^* \overset{den}{=} \mathcal{W}_1$ - a set of observables, which are semipositive on product vectors.

- In general, we know nothing about the structure of faces.

Cones $\mathcal{S}_1$ and $\mathcal{W}_1$

$\mathcal{S}_1(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \subset \mathcal{B}_H^*(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ - the cone of unnormalized separable quantum states.

$$\rho \in \mathcal{S}_1 \overset{df}{\Leftrightarrow} \rho = \sum_i \lambda_i P_{\psi_i \otimes \phi_i}$$

Proposition: The extremal rays of $\mathcal{S}_1$ are generated by rank-one projectors onto product vectors.

In general, we know nothing about the structure of faces.

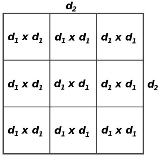
Dual cone: $W \in \mathcal{S}_1^* \;\Leftrightarrow\; \forall \psi, \forall \phi \; \langle P_{\psi \otimes \phi} | W \rangle_{HS} \geqslant 0$

$\mathcal{S}_1^* \overset{den}{=} \mathcal{W}_1$ - a set of observables, which are semipositive on product vectors.

- In general, we know nothing about the structure of faces.
- In general, we know nothing about the extremal halfrays.

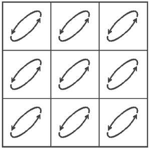
Cones $\mathcal{S}_{PPT}$ i $\mathcal{W}_D$

Partial transposition



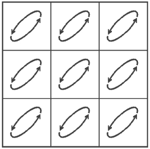
Cones $\mathcal{S}_{PPT}$ i $\mathcal{W}_D$

Partial transposition



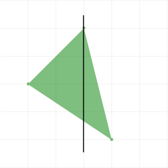
Cones $\mathcal{S}_{PPT}$ i $\mathcal{W}_D$

Partial transposition



Inversion in a subspace of $\mathcal{B}_H^*$

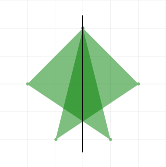
Cones $\mathcal{S}_{PPT}$ i $\mathcal{W}_D$



Partial transposition

Inversion in a subspace of $\mathcal{B}_H^*$

Cones $\mathcal{S}_{PPT}$ i $\mathcal{W}_D$

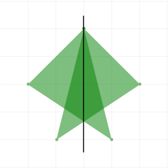


Partial transposition

Inversion in a subspace of $\mathcal{B}_H^*$

$\mathcal{S}_{PPT} = \mathcal{S}_+ \cap \mathcal{S}_+^\Gamma$
- states with positive partial transposition

Cones $\mathcal{S}_{PPT}$ i $\mathcal{W}_D$



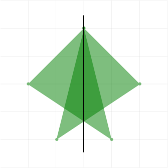
Partial transposition

Inversion in a subspace of $\mathcal{B}_H^*$

$\mathcal{S}_{PPT} = \mathcal{S}_+ \cap \mathcal{S}_+^\Gamma$
- states with positive partial transposition

We do not know extremal halfrays (edge states).

Cones $\mathcal{S}_{PPT}$ i $\mathcal{W}_D$



Partial transposition

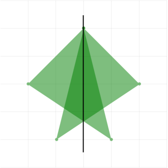
Inversion in a subspace of $\mathcal{B}_H^*$

$\mathcal{S}_{PPT} = \mathcal{S}_+ \cap \mathcal{S}_+^\Gamma$
- states with positive partial transposition

We do not know extremal halfrays (edge states).

A cone $\mathcal{S}_{PPT}^* \stackrel{den}{=} \mathcal{W}_D$ contains observables which can be written as $A + B^\Gamma$, $A, B \in \mathcal{B}_+$. Its extremal halfrays are generated by P_Ψ, P_Ψ^Γ. $\mathcal{W}_D \subset \mathcal{W}_1$.

Cones $\mathcal{S}_{PPT}$ i $\mathcal{W}_D$



Partial transposition

Inversion in a subspace of $\mathcal{B}_H^*$

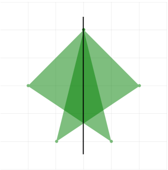
$\mathcal{S}_{PPT} = \mathcal{S}_+ \cap \mathcal{S}_+^\Gamma$
- states with positive partial transposition

We do not know extremal halfrays (edge states).

A cone $\mathcal{S}_{PPT}^* \stackrel{den}{=} \mathcal{W}_D$ contains observables which can be written as $A + B^\Gamma$, $A, B \in \mathcal{B}_+$. Its extremal halfrays are generated by P_Ψ, P_Ψ^Γ. $\mathcal{W}_D \subset \mathcal{W}_1$.

- We do not know the structure of faces.

Cones $\mathcal{S}_{PPT}$ i $\mathcal{W}_D$



Partial transposition

Inversion in a subspace of $\mathcal{B}_H^*$

$\mathcal{S}_{PPT} = \mathcal{S}_+ \cap \mathcal{S}_+^\Gamma$
- states with positive partial transposition

We do not know extremal halfrays (edge states).

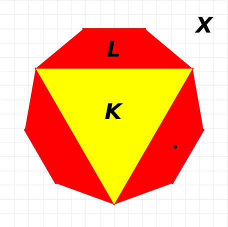
A cone $\mathcal{S}_{PPT}^* \stackrel{den}{=} \mathcal{W}_D$ contains observables which can be written as $A + B^\Gamma$, $A, B \in \mathcal{B}_+$. Its extremal halfrays are generated by P_Ψ, P_Ψ^Γ. $\mathcal{W}_D \subset \mathcal{W}_1$.

- We do not know the structure of faces.
- We do not have an effective criterion of decomposability of witness.

Details about important cones in Quantum Mechanics and the introduction to the cone geometry from physicial point of view can be found in:
arXiv:0902.4877 (K. Życzkowski, Ł. Skowronek, E. Størmer)

Detection

Let $K \subset L$ proper cones in X.

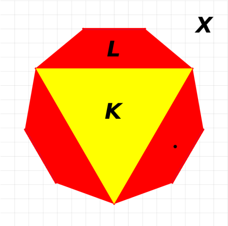


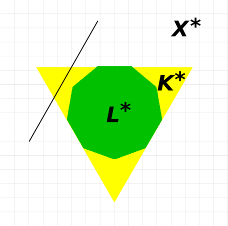
Detection

Let $K \subset L$ proper cones in X.

$\forall \rho \in K^* \setminus L^* \quad \exists w \in L \setminus K : \quad \langle \rho | w \rangle < 0$

we say then, that w detects ρ.





Detection

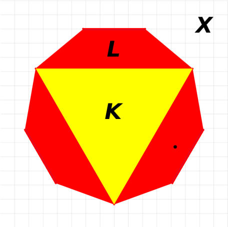
Let $K \subset L$ proper cones in X.

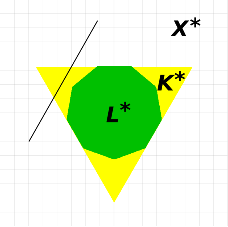
$\forall \rho \in K^* \setminus L^* \quad \exists w \in L \setminus K : \quad \langle \rho | w \rangle < 0$

we say then, that w detects ρ.

The Banach separation theorem implies, that

- for any w there exist detected ρ's
- for any ρ there exist w's detecting ρ





Detection

Let $K \subset L$ proper cones in X.

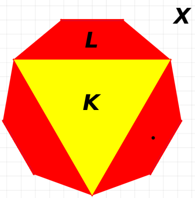
$\forall \rho \in K^* \setminus L^* \quad \exists w \in L \setminus K : \quad \langle \rho | w \rangle < 0$

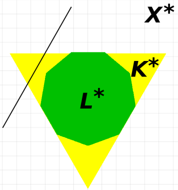
we say then, that w detects ρ.

The Banach separation theorem implies, that

- for any w there exist detected ρ's
- for any ρ there exist w's detecting ρ

The set of elements of $K^* \setminus L^*$ detected by $w \in L \setminus K$ is denoted by $\mathcal{D}_{L|K}(w)$.





Detection

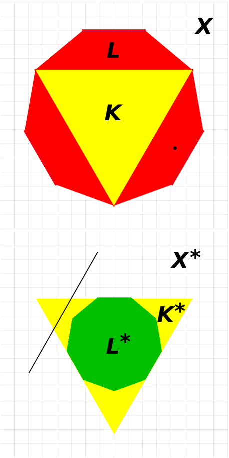
Let $K \subset L$ proper cones in X.
$\forall \rho \in K^* \setminus L^* \quad \exists w \in L \setminus K : \quad \langle \rho | w \rangle < 0$
we say then, that w detects ρ.

The Banach separation theorem implies, that

- for any w there exist detected ρ's
- for any ρ there exist w's detecting ρ

The set of elements of $K^* \setminus L^*$ detected by $w \in L \setminus K$ is denoted by $\mathcal{D}_{L|K}(w)$.

One can extend this definition to the whole L defining $\mathcal{D}_{L|K}(k) = \emptyset \; \forall k \in K$.



Optimality in general formulation

Optimality in general formulation

Def: Relation in L of **beeing finer** with respect to K:

$$w_1 \geqslant_{f(K)} w_2 \;\Leftrightarrow\; \mathcal{D}_{L|K}(w_1) \supset \mathcal{D}_{L|K}(w_2)$$

Optimality in general formulation

Def: Relation in L of **beeing finer** with respect to K:

$$w_1 \geqslant_{f(K)} w_2 \;\Leftrightarrow\; \mathcal{D}_{L|K}(w_1) \supset \mathcal{D}_{L|K}(w_2)$$

Such an element in L, that no other is finer is called an **optimal witness**.

Optimality in general formulation

Def: Relation in L of **beeing finer** with respect to K:

$$w_1 \geqslant_{f(K)} w_2 \;\Leftrightarrow\; \mathcal{D}_{L|K}(w_1) \supset \mathcal{D}_{L|K}(w_2)$$

Such an element in L, that no other is finer is called an **optimal witness**.

Second ordering relation: $w_1 \geqslant_K w_2 \;\overset{df}{\Leftrightarrow}\; \exists\lambda \in \mathbb{R}_+ : w_1 = \lambda w_2 + k$

Optimality in general formulation

Def: Relation in L of **beeing finer** with respect to K:

$$w_1 \geqslant_{f(K)} w_2 \;\Leftrightarrow\; \mathcal{D}_{L|K}(w_1) \supset \mathcal{D}_{L|K}(w_2)$$

Such an element in L, that no other is finer is called an **optimal witness**.

Second ordering relation: $w_1 \geqslant_K w_2 \overset{df}{\Leftrightarrow} \exists\lambda \in \mathbb{R}_+ : w_1 = \lambda w_2 + k$

Th: Both orderings are equivalent (generalization of the result of Phys. Rev. A **62** 052310 (2000) quant-ph/0005014).

Optimality in general formulation

Def: Relation in L of **beeing finer** with respect to K:

$$w_1 \geqslant_{f(K)} w_2 \;\Leftrightarrow\; \mathcal{D}_{L|K}(w_1) \supset \mathcal{D}_{L|K}(w_2)$$

Such an element in L, that no other is finer is called an **optimal witness**.

Second ordering relation: $w_1 \geqslant_K w_2 \overset{df}{\Leftrightarrow} \exists\lambda \in \mathbb{R}_+ : w_1 = \lambda w_2 + k$

Th: Both orderings are equivalent (generalization of the result of Phys. Rev. A **62** 052310 (2000) quant-ph/0005014).

Conclusion: w is optimal $\Leftrightarrow \; \forall k \in K \;\; w - k \notin L$

Optimality in general formulation

Special cases:

Optimality in general formulation

Special cases:

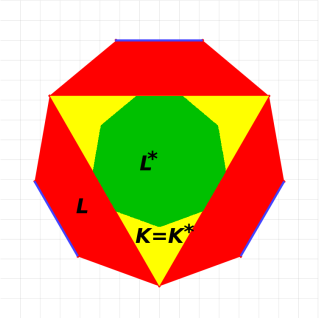
- for $\mathcal{B}_+ \subset \mathcal{W}_1$ (standart optimality)
  w is optimal $\Leftrightarrow \ \forall k \in \mathcal{B}_+ \ w - k \notin \mathcal{W}_1$ (subtracting any semipositive observable leads out of the set of witnesses)

Optimality in general formulation

Special cases:

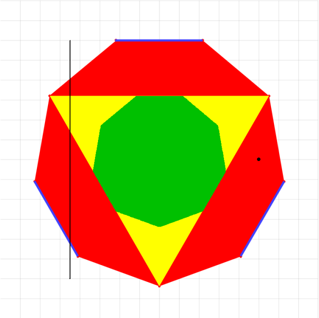
- for $\mathcal{B}_+ \subset \mathcal{W}_1$ (standart optimality)
  w is optimal $\Leftrightarrow \ \forall k \in \mathcal{B}_+ \ w - k \notin \mathcal{W}_1$ (subtracting any semipositive observable leads out of the set of witnesses)
- for $\mathcal{W}_D \subset \mathcal{W}_1$ (nd-optimality)
  w is nd-optimal $\Leftrightarrow \ \forall k \in \mathcal{W}_D \ w - k \notin \mathcal{W}_1$ (subtracting any decomposable witness leads out of the set of witnesses).

Geometrical characterization of optimality



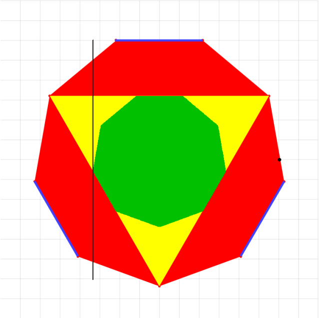
We denote the set of optimal elements as $\mathrm{opt}(\mathrm{L}|\mathrm{K})$.

Geometrical characterization of optimality



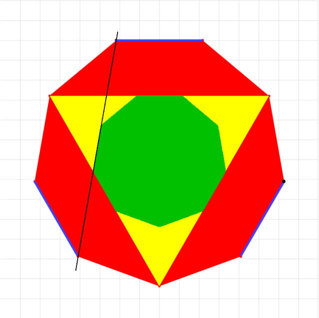
We denote the set of optimal elements as $\mathrm{opt}(\mathrm{L}|\mathrm{K})$.

Geometrical characterization of optimality



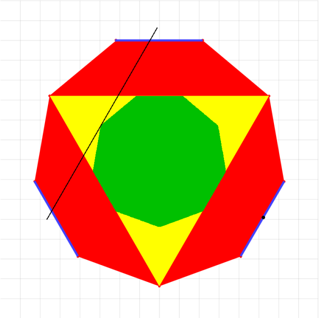
We denote the set of optimal elements as $\mathrm{opt}(\mathrm{L}|\mathrm{K})$.

Geometrical characterization of optimality



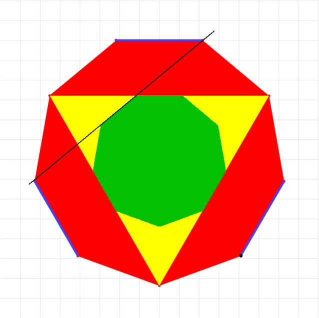
We denote the set of optimal elements as $\mathrm{opt}(\mathrm{L}|\mathrm{K})$.

Geometrical characterization of optimality



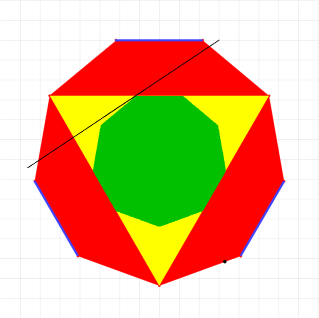
We denote the set of optimal elements as $\mathrm{opt}(\mathrm{L}|\mathrm{K})$.

Geometrical characterization of optimality



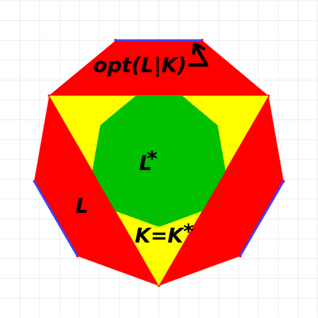
We denote the set of optimal elements as $\mathrm{opt}(\mathrm{L}|\mathrm{K})$.

Geometrical characterization of optimality



We denote the set of optimal elements as $\mathrm{opt}(\mathrm{L}|\mathrm{K})$.

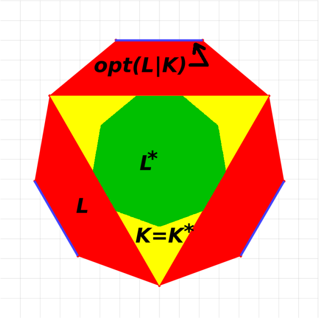
Geometrical characterization of optimality



We denote the set of optimal elements as $\mathrm{opt}(\mathrm{L}|\mathrm{K})$.

Fact 1: If $w \in \mathrm{opt}(L|K)$ then $F_L(w) \subset \mathrm{opt}(L|K)$ (any element $w \in L$ belongs to $\mathrm{opt}(\mathrm{L}|\mathrm{K})$ with the face generated by this element).

Geometrical characterization of optimality

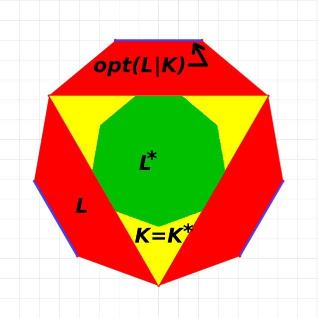


We denote the set of optimal elements as $\mathrm{opt}(\mathrm{L}|\mathrm{K})$.

Fact 1: If $w \in \mathrm{opt}(L|K)$ then $F_L(w) \subset \mathrm{opt}(L|K)$ (any element $w \in L$ belongs to $\mathrm{opt}(\mathrm{L}|\mathrm{K})$ with the face generated by this element).

Fact 2: Face $F_L \subset \mathrm{opt}(L|K) \quad \Leftrightarrow \quad F_L \cap K = \{0\}$.

Geometrical characterization of optimality



We denote the set of optimal elements as $\mathrm{opt}(\mathrm{L}|\mathrm{K})$.

Fact 1: If $w \in \mathrm{opt}(L|K)$ then $F_L(w) \subset \mathrm{opt}(L|K)$ (any element $w \in L$ belongs to $\mathrm{opt}(\mathrm{L}|\mathrm{K})$ with the face generated by this element).

Fact 2: Face $F_L \subset \mathrm{opt}(L|K) \Leftrightarrow F_L \cap K = \{0\}$.

Conclusion: The set $\mathrm{opt}(\mathrm{L}|\mathrm{K})$ is a sum of faces of the cone L which do not contain non-zero elements of K.

Geometrical characterization of optimality

Fact: A face $\Phi(F_L(w))$ is spanned by extremal elements of L^* which takes value zero on the element w. The set of such elements is denoted by $\mathcal{P}^L(w)$.

Geometrical characterization of optimality

Fact: A face $\Phi(F_L(w))$ is spanned by extremal elements of L^* which takes value zero on the element w. The set of such elements is denoted by $\mathcal{P}^L(w)$.

Theorem: $w \in L$ is optimal iff

$$\Phi(F_L(w)) \cap \mathrm{Int} K^* \neq \emptyset$$

(equivalently: $\mathrm{conv}(\mathcal{P}^L(w)) \cap \mathrm{Int} K^* \neq \emptyset$)

Geometrical characterization of optimality

Proof: $w \in \mathrm{opt}(L|K) \;\Leftrightarrow \sim \exists F_K \neq \{0\} : F_K \lhd F_L(w)$

Geometrical characterization of optimality

Proof: $w \in \mathrm{opt}(L|K) \ \Leftrightarrow \sim \exists F_K \neq \{0\} : F_K \lhd F_L(w)$
$\Leftrightarrow \quad \forall F_K \neq \{0\} : F_K \ntriangleleft F_L(w)$

Geometrical characterization of optimality

Proof: $w \in \mathrm{opt}(L|K) \;\Leftrightarrow \sim \exists F_K \neq \{0\} : F_K \lhd F_L(w)$

$\Leftrightarrow \quad \forall F_K \neq \{0\} : F_K \ntriangleleft F_L(w)$

$\Leftrightarrow \quad \forall F_{K^*} \neq K^* : F_{K^*} \ntriangleright \Phi(F_L(w))$

Geometrical characterization of optimality

Proof: $w \in \mathrm{opt}(L|K) \;\Leftrightarrow \sim \exists F_K \neq \{0\} : F_K \lhd F_L(w)$

$\Leftrightarrow \quad \forall F_K \neq \{0\} : F_K \ntriangleleft F_L(w)$

$\Leftrightarrow \quad \forall F_{K^*} \neq K^* : F_{K^*} \ntriangleright \Phi(F_L(w))$

it means, that the only face of K^*, containing the set $\Phi(F_L(w))$ is the whole cone K^*

Geometrical characterization of optimality

Proof: $w \in \mathrm{opt}(L|K) \;\; \Leftrightarrow \sim \exists F_K \neq \{0\} : F_K \lhd F_L(w)$

$\Leftrightarrow \quad \forall F_K \neq \{0\} : F_K \ntriangleleft F_L(w)$

$\Leftrightarrow \quad \forall F_{K^*} \neq K^* : F_{K^*} \ntriangleright \Phi(F_L(w))$

it means, that the only face of K^*, containing the set $\Phi(F_L(w))$ is the whole cone K^*

$\Rightarrow$ a face in K^* generated by $\Phi(F_L(w))$ is K^*

Geometrical characterization of optimality

Proof: $w \in \mathrm{opt}(L|K) \;\Leftrightarrow \sim \exists F_K \neq \{0\} : F_K \lhd F_L(w)$

$\Leftrightarrow \quad \forall F_K \neq \{0\} : F_K \ntriangleleft F_L(w)$

$\Leftrightarrow \quad \forall F_{K^*} \neq K^* : F_{K^*} \ntriangleright \Phi(F_L(w))$

it means, that the only face of K^*, containing the set $\Phi(F_L(w))$ is the whole cone K^*

$\Rightarrow$ a face in K^* generated by $\Phi(F_L(w))$ is K^*

$\Rightarrow$ there exists an element of $\mathrm{Int} K^*$ which belongs to $\Phi(F_L(w))$.

Geometrical characterization of optimality

Example: Let's consider $\mathcal{B}_+(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \subset \mathcal{W}_1(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ (standart optimality)

Geometrical characterization of optimality

Example: Let's consider $\mathcal{B}_+(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \subset \mathcal{W}_1(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ (standart optimality)

$\mathcal{P}^L(w)$ - the set of projectors $P_{\psi\otimes\phi}$, on which $\langle P_{\psi\otimes\phi}|w\rangle = \langle\psi\otimes\phi|w|\psi\otimes\phi\rangle = 0$.

Geometrical characterization of optimality

Example: Let's consider $\mathcal{B}_+(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \subset \mathcal{W}_1(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ (standart optimality)

$\mathcal{P}^L(w)$ - the set of projectors $P_{\psi\otimes\phi}$, on which $\langle P_{\psi\otimes\phi}|w\rangle = \langle\psi\otimes\phi|w|\psi\otimes\phi\rangle = 0$.

Element w is optimal iff, when by use of the convex combination of these projectors one can build an element from the interior of $\mathcal{B}_+$ (full-rank matrix).

Geometrical characterization of optimality

Example: Let's consider $\mathcal{B}_+(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \subset \mathcal{W}_1(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ (standart optimality)

$\mathcal{P}^L(w)$ - the set of projectors $P_{\psi\otimes\phi}$, on which $\langle P_{\psi\otimes\phi}|w\rangle = \langle\psi\otimes\phi|w|\psi\otimes\phi\rangle = 0$.

Element w is optimal iff, when by use of the convex combination of these projectors one can build an element from the interior of $\mathcal{B}_+$ (full-rank matrix).

$$\Leftrightarrow \ \mathrm{span}\{\psi\otimes\phi : \langle\psi\otimes\phi|W|\psi\otimes\phi\rangle = 0\} = \mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}.$$

Details: arXiv:0905.0778

Thank You for Your attention