ANR programme blanc édition 2008 "Physique Combinatoire"
Acronyme: PhysComb

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1 Technical and Scientific description of the proposal

1.1 Rationale

In the past years, the use of (more or less) sophisticated tools from classical analysis to discover nontrivial properties of discrete structures such as graphs, turned out to be particularly appealing. On the other hand, the success of the use of Combinatorial methods to solve problems of Physics (as, for example, the use of rook numbers to give an efficient computation of Wick's theorem), is the cause of the advent of a new domain called "Combinatorial Physics".
The proposed program consists in achieving "as much as possible" the dictionary

QUANTUM OPERATORS $\leftrightarrow$ COMBINATORICS

This program has been specialized here and divided into three sub-programs which are actually tightly intertwinned:
(A) One-parameter groups and semigroups
(B) Deformations, discrete structures and low degree operators
(C) Probability and graph models

(A) This set of problems is part of the same problematic :
How to get new correspondences/algorithms/formulas from differentiable paths drawn on spaces defined by Quantum Mechanics ?

Sheffer-type sequences are well-known and are part of a long-standing combinatorial tradition. They often come from the bivariate enumeration of some classes of graphs. Exponential formula (due a gaz theoretist [46]) states that, in some "good" cases, the bivariate mixed generating series (exponential by vertices and ordinary by number of connected

components) is an exponential (the same series restricted to connected graphs). On the other hand, it is of the first importance, for physicists, to harness exactly evolution operators arising from creation/annihilation operators.
A significant part of the dictionnary can be read through the Generalized Stirling Matrix (GSM) attached to a homogeneous operator and is already achieved [19]. A promising track (induced by computing necessities) is to examine (theoretically and experimentally) appropriate truncations of this group called "approximate substitutions" [12]. Other investigations are planned namely :

- the non-analytic and non-unipotent cases
- combinatorial description of algebraic groups of approximate substitutions
- vector fields coming from combinatorial matrices (in particular, the Stirling and Burnside fields).

The second part of this problem is linked to the thermalisation of squeezed states. Our hope is to derive the Lindblad equations from the Kraus model proposed by Solomon [49]. The challenge here is to understand the semigroup of transformations defined in [49] and to derive from it a correct differential model.

(B) There exists a classical deformation which allows to interpolate between boson and fermion statistics. The trick consists in replacing the bracket by the q-bracket

$$[a,a^+]_q = aa^+ - qa^+a = 1 \ . \tag{1}$$

There is a lot of combinatorial features which look like the classical case (often at the cost of replacing numbers and factorials and Rook numbers [54] by their q-analogues). Here, in spite of the existence of the q-derivative which, with the multiplication by x satisfies the (12) relation, the richness of the correspondence with vector fields is still lacking. We have some evidences for two fruitful ways. The first consists in restoring the existence of "true" one parameter groups, the second is to generalize the deformation in order to be able to use the correspondence with orthogonal polynomials [29] and we will take this opportunity to exploit a preceding work on continued fractions relating to multiparameter deformations of the Heisenberg-Weyl's algebra [38].
The road to deformations leads to the consideration of duality which is very familiar to quantum physicists by means of tools like "Bra" and "Ket", introduced by Dirac. In this way, close to our program, we have to elucidate two problems

- the nature of the parameters in the deformed case of the Hopf algebras of Feynman-Bender diagrams [20, 24] (which comes from the coupling of two evolution exponentials). This is hoped to be clarified by the notion of "dual laws" [23].
- The setting, by means of automata theory, of an efficient calculus for Sweedlers duals of free Hopf algebras such as noncommutative Connes-Kreimer Hopf algebra (the algebra of planar rooted trees introduced by Loic Foissy) or the deformed case of the Hopf algebras of Feynman-Bender diagrams [22].

We can remark that it is essentially the use of operators of low degree (creation is $+1$, annihilation is -1 and the number operator is 0) that suffices to generate the richness of Quantum Combinatorics. In spite of their apparent simplicity, the combinatorics of these operators is sufficiently powerful to be able to solve problems like the structure constants of the algebra of rational functions (with no pole at 0) or to express exactly weights of orthogonal polynomials by means of Motzkin paths [29].

(C) There are many instances where a couple (raising/lowering) operator is operating. For example, one can consider the fact that there is one way to add a ball in an Urn (addition operator X) whereas there are n ways (provided that the Urn contain n balls) of withdrawing a ball (deletion operator D). So the urn histories are governed by the equation

$$DX - XD = 1 \tag{2}$$

which is the realm of creation/annihilation combinatorics. Other examples were developped for graph theory [30]. We want to interpret with these models the amount of "Quantum Combinatorics" developped from creation/annihilation operators.

In conclusion, we believe that the presented program is well-coordinated. The growing interests for this new domain called *Combinatorial Physics* [1] and also, the support obtained recently (2007 and 2008) for connected projects but with different teams (POLONIUM and PICS) is, for us, the sign that the community of Physicists is utterly awaiting to push forward a set of theories relying on the expertise and computational power provided by Combinatorics, Computer Algebra, Algebra, Analysis, Geometry and Probability.

1.2 Background, objective, issues and hypothesis

One-Parameter Groups. —
The first systematic study of the Combinatorics of Normal Orders goes back to Navon [43] and q-deformed versions can be found for in Katriel and Kibler [39] and, more recently with q-rook numbers, in [54]. Studies of normal orders of powers of $\Omega = (a^+)^r a^s$ in relation with new Stirling Matrices, Dobiński formula and the moment problem can be traced back to Solomon et al. [4, 5, 6]. The link between "one annihilation", substitutions and vector fields can be found in [19].

Thermalization. —
The differential model (Lindblad equations) is in [42] and the algebraic model can be found in [49].

Deformations of the Heisenberg-Weyl algebra and of the Fock space. —
Deformations of Fock space in connection with the problem of relizability can be found in [58]. the link with orthogonal polynomials can be found in Bożejko [8, 9].

Direct and dual laws. —
Hadamard [36], Shuffle and Infiltration [44] products are discussed in the light of Dual laws with alphabetic constraints in [23]. Later, a rather simple non-alphabetic dual law occured linked to the arithmetic of Euler-Zagier sums. It is the Hoffman shuffle, sometimes called Stuffle [11, 31]. Now, it seems that the three-parameter deformation of the Hopf algebra of Feynman-Bender Diagrams $LDIAG(q_c,q_s,q_t)$, see [24] could have its algebra structure enlightened by the examples above.

Partially asymmetric exclusion process and Askey-Wilson polynomials. —
Exclusion processes with open boundaries have attracted much attention as simple models of an open non-equilibrium system in contact with two reservoirs having different chemical potentials. Despite their simplicity (a Markov process evolving acoording to simple

1. Many conferences were organized on the subject recently. Our was especially devoted to the interface Physics-Combinatorics-Computer Science and entitled : "Combinatorial Physics" `http://www.ifj.edu.pl/conf/combphys/.`

dynamical rules), these models exhibit properties believed to be characteristic of realistic non-equilibrium systems, such as long range correlations and phase transitions in one dimension.
A number of exact results have been obtained for the one dimensional exclusion process with open boundaries, using the fact that the weights of the microscopic configurations in the stationary state can be calculated exactly. In the case of the partially or totally symmetric exclusion processes (PASEP and TASEP), these results have involved many combinatorial invariants, such as orthogonal polynomials and permutation tableaux.

q-analogs of orthogonal polynomials. —
Macdonald mentionned many examples of the deep connection between the theory of symmetric functions, and the world of combinatorial identities and q-series. Our objective in the present part of the project can bee seen as a few steps towards a theory of hypergeometric series (ordinary, basic and elliptic) at the level of symmetric functions, by showing that many well-known results in the theory of basic and elliptic hypergeometric series have symmetric function analogues.

Low degree operators. —
Creation/annihilation operators have a rich literature. The general study of the combinatorics of a raising (of dedree 1) an lowering (of degree -1) operator applied to graph theory can be found in [30] and for a general study of the algebra generated by such two operators linked to continued fraction techniques see [38].

Probability and graph models. —
There are many instances where a couple (raising/lowering) operator, for example, one can consider the fact that there is one way to add a ball in an Urn (addition operator X) whereas there are n ways (provided that the Urn contain n balls) of withdrawing a ball (deletion operator D). So the urn histories are governed by the equation

$$DX - XD = 1 \tag{3}$$

which is the realm of creation/annihilation combinatorics. Other examples were developped for graph theory [30]. We want to interpret with these models the amount of "Quantum Combinatorics" developped from creation/annihilation operators. There is a rich literature (for a reference on these models linked to analysis, see [27]). The presence of two operators D,X such that $[D,X] = 1$ can be seen in [28].

1.3 Specific aims, highlight of the originality and novelty of the project

Looking at the history of Combinatorics, one can tell the originality of our project.
As a matter of fact, Classical and Enumerative Combinatorics could be considered as ripe with the publication of the treatise of D. Knuth in Computer Science ("The Art of Computer Programming") and of R. Stanley in Enumerative Combinatorics ("Enumerative Combinatorics"). Then came the advent of Algebraic Combinatorics with, for a significant example, the series of Colloquiums FPSAC - founded, with others, by one of the participants of the project - and the development of methods relating to Combinatorics, Computer Science and Algebra.
Now, the interaction with Physicists allows for the use of operator theory, analysis (classical and modern) and Theoretical Computer Science to exactly solve models of (Quantum)

Physics. In return, these methods can enrich the corpus of Combinatorics.
The originality of the project presented here is to be placed at the time of growing interest for Combinatorial Physics and develop a fragment of it with a very well coordinated program and a synergy of teams for doing it.

1.4 Detailed description of the work plan

(A) One parameter groups and semigroups

As mentionned previously, this set of problems splits into two parts:
- One parameter groups of operators
- Thermalization of squeezed states

One parameter groups of operators. —

As metionned previously, Sheffer-type sequences often come from the bivariate enumeration of some classes of graphs

$$M[n,k] \;=\; \text{number of graphs labelled as } \{1,2,\cdots,n\} \text{ with } k \text{ connected components} \tag{4}$$

Exponential formula (due a gaz theoretist [46]) states that, in some "good" cases, the following mixed generating series (LHS) is an exponential (RHS).

$$\sum_{n,k=0}^{\infty} M[n,k]\frac{x^n}{n!}y^k = e^{y\left(\sum_{n=1}^{\infty} M[n,1]\frac{x^n}{n!}\right)} \tag{5}$$

On the other hand, it is of the first importance, for physicists, to harness exactly the evolution operators of the type

$$U_\lambda(f) = e^{\lambda\Omega}[f] \tag{6}$$

where

$$\Omega = \sum_{i,j} \alpha(i,j)(a^+)^i a^j \tag{7}$$

and (a^+,a) are, respectively, the creation/annihilation operators of Quantum Mechanics (which satisfy $[a^+,a] = 1$) and f belongs to a space of well suited test-functions.
A significant part of the dictionnary can be read through the Generalized Stirling Matrix (GSM). When Ω is homogeneous operator i. e. of the form

$$\Omega = \sum_{i-j=e} \alpha(i,j)(a^+)^i a^j \tag{8}$$

the normal forms[2] of the powers of Ω can be written, for example when $e \geq 0$

$$\mathcal{N}(\Omega^n) = (a^+)^{ne} \sum_{k=0}^{\infty} S_\Omega(n,k)(a^+)^k a^k \tag{9}$$

2. That is to say expressions of the type given by Eq. (7), they will be denoted by the result of the operator $\mathcal{N}$.

and the GSM is just the doubly indexed family $(S_\Omega(n,k))_{(n,k)\in\mathbb{N}^2}$.
In the special case when there is only one annihilation, we can compute completely the GSM by means of one-parameter groups of substitution with prefunctions, i.e. of operators of the type

$$f \to \hat{f}(z) = g(z)f(\phi(z)) \text{ with } g(z) = 1 + \cdots \text{ and } \phi(z) = z + \cdots . \tag{10}$$

The one-parameter groups of these transformations have infinitesimal generators that are vector fields and their conjugates [19].
On the other hand, computation necessities forced us to initiate this program by computing on Taylor series. This led us to the evidence that the group of substitution with prefunctions is the projective limit of a sequence of algebraic unipotent groups of "approximate substitutions" [12]. As a consequence, every non-unit element pertains to a unique one-parameter group. Other investigations are planned namely

a) the non-analytic and non-unipotent cases
b) combinatorial description of algebraic groups of approximate substitutions
c) vector fields coming from combinatorial matrices (in particular, the Stirling and Burnside fields).

a) It remains to make precise (mathematically and algorithmically) the limit between the analytic (i. e. where the functions g and ϕ are defined by series with non-zero radius of convergence, or are entire) case and the non-analytic (the remaining, i. e. formal) case and, for both, make precise the computation of infinitesimal generators.
b) We can produce the algebraic equations defining the groups of approximate substitutions but the coefficients of these equations must be studied combinatorially. In particular, we should investigate why these matrices compose.
c) From the algebricity of the defining equations of the group of approximate substitutions, we can tell the fact that any matrix of substitution with prefunction which is not the identity is a member of a unique one-parameter group. Especially, the matrices defined in (5), when they are unipotent, can be interpreted in terms of flows of vector fields on the line. Many questions arise and among them: What are the fields for classical substitutions such as Stirling (i. e. $f \to f(e^z - 1)$) or idempotent (i. e. $f \to f(ze^z)$) or, more generally of any of the Burnside class? [19].

Paris XIII, with aid of Paris VI will be in charge of the task of this part. The schedule of the work is as follows:
1) "First year" make explicit the groups of "approximate substitutions" (algebraic equations and their combinatorics), (HC + GHED).
2) "Second year" examine and get a "calculus" with the infinitesimal generators of the group of substitutions with prefunctions, (LP + KAP + GHED).
3) "Third year" propose and develop ideas for higher order derivatives, (KAP + GHED + AS).
4) "Fourth year" develop the idea of "Combinatorial Vector Field" (beginning with Stirling and Borside ones) (AS + KAP + GHED).

Thermalization of squeezed states. —

It concerns the problem of deriving Lindblad formulas from the model of thermalization proposed by Kraus and Solomon [49].

We say that a finite family $(k_i)_{i\in I}$ (I is finite) of elements in $\mathbb{C}^{n\times n}$ fulfils the KS (Kraus-Solomon) condition iff $\sum_{i\in I} k_i^* k_i = I_n$ (I_n is the unity matrix) [21].
A consequence of the cyclic invariance of the trace function is that if $F = (k_i)_{i\in I}$ satisfies the KS condition, then the linear mapping $\phi_F : \mathbb{C}^{n\times n} \mapsto \mathbb{C}^{n\times n}$

$$\phi_F : X \to \sum_{i\in I} k_i X k_i^* \tag{11}$$

is trace and positivity preserving (and then state preserving). But, unlike the unitary transformations (case when $|I| = 1$), the spectrum is not in general preserved but "dissipated" which means that one can use the preceding transformation as a model of "Thermalization". These transformations form a (convex) semigroup. Another model, the one of Lindblad [42] exhibits infinitesimal generators. It remains to make the connection between the two in solving the following questions

1. is the "KS" semigroup closed, if not, can we characterize the closure?
2. how to draw paths (or, better, one-parameter semigroups) on this semigroup?
3. do the infinitesimal generators of these semigroups look like what shows Lindblad model? in particular, how to separate the dissipative and reversible (Hamiltonian) parts?

Paris VI, with aid of Paris XIII will be in charge of the task of this part. The schedule of the work is as follows:
1) "First year" complete description of the semigroup (AS, KAP, GHED, CT).
2) "Second year" description of the adherence of the semigroup (AS, GHED, CT).
3) "Third year" one-parameter-semigroups and Lindblad equations (AS, KAP, GHED).
4) "Fourth year" link of one-parameter semigroups with comultiplications and Hopf algebras (GHED, AS, KAP).

(B) Deformations, discrete structures and low degree operators

This problem is divided in three parts:

a) Deformations of the Heisenberg-Weyl algebra, of the Fock space and low degree operators
b) Direct and dual laws
c) Partially asymmetric exclusion process and Askey-Wilson polynomials
d) q-analogs of orthogonal polynomials

Deformations of the Heisenberg-Weyl algebra and the Fock space. —
There exists a classical deformation which allows to interpolate between boson and fermion statistics. The trick consists in replacing the bracket by the q-bracket

$$[a,a^+]_q = aa^+ - qa^+a = 1 \ . \tag{12}$$

There is a lot of combinatorial features which look like the classical case (often at the cost of replacing numbers and factorials and Rook numbers [54] by their q-analogues). Here, in spite of the existence of the q-derivative which, with the multiplication by x satisfies the (12) relation, the richness of the correpondence with vector fields is still lacking. We have some evidences for two fruitful ways. The first consists in restoring the existence of "true" one parameter groups, the second is to generalize the deformation in order to be able to use the correpondence with orthogonal polynomials [29] and we will take this opportunity to exploit a preceding work on continued fractions relating to multiparameter deformations of the Heisenberg-Weyl algebra [38].

The quantum decomposition of a classical random variable is a technique which dates back to the representation of Gaussian and Poisson measures on $\mathbb{R}^d$ in terms of creation and annihilation operators. Such a decomposition is routinely used in quantum optics, and more advanced examples of its application appeared in the 1990's in the work of Bożejko and Speicher on generalized brownian motion [8, 9].

However, it is only through the notion of an interacting Fock space that the quantum decomposition technique reached its full strength. Interacting Fock spaces are infinite-dimensional separable Hilbert spaces with a distinguished orthonormal basis, and endowed with generalized creation and annihilation operators defined through a given sequence of positive real numbers. They provide a natural framework for interpreting the Jacobi coefficients of a probability distribution (or the associated orthogonal polynomials) in terms of quantum operators. Among the benefits of that approach is the computation of the moments of all order in terms of paths [1].

More recently, the same ideas have revealed fruitful in graph theory. Indeed, by considering the adjacency matrix of a graph as a classical random variable, and using its quantum decomposition (once a stratification of the graph has been chosen), one is in a position to reduce problems related to the asymptotics of large graphs to the combinatorics of quantum operators [33]. Hopefully, that will shade some light on the phase transition phenomenon observed in the Erdös-Rényi model of graph evolution [41], and on our understanding of inductive limits of spaces through their branching graphs (Bratteli diagrams) [32].

It is known that the Hadamard product of two rational functions (Taylor developpable, i. e. without singularity at zero) is rational. Remains the problem of computing explicitely the structure constants of the algebra of these functions. Remarking that diagonal operators (i. e. operators with the same number of creation and annihilations) act as scalars for the Hadamard product, one can solve this problem by creation/annihilation operators through Bargmann-Fock representation.

Paris XIII, with aid of Lyon1 will be in charge of the task of this part. The schedule of the work is as follows:
1) First year: refine Hora's noncommutative version of Kerov-Vershik central-limite theorem in terms of generalized creation and annihilation operators.
2) Second and third year: generalize the approach to other (self-dual) infinite graphs and branching graphs.
3) Third and fourth year: establish the link between the spectral theory of branching graphs, infinite-dimensional diffusions and the theory of "Box functions" [7].

Direct and dual laws. —

The use of duality, so familiar to the Quantum Physicist since the introduction of the "Bra" and "Ket" denotation by P. Dirac has strongly influenced the development Combinatorial Physics.
On this ground, there are two main directions that we want to develop.
The first one starts from the remark that certain laws defined on the free algebra (often

by means of recursions on the words) can be better understood as "Dual laws" i. e. laws dual to some simple comuliplication [23, 24]. The first examples of such laws come from Computer Science (and knew a glorious fate in Quantum Groups theory). These are the Shuffle ($q = 0$ below in the q-infiltration $\uparrow_q$), Infiltration ($q = 1$) and Hadamard ($q = \infty$ renormalized) products. For example $\uparrow_q$ is defined on the words by

$$au \uparrow_q bv = a(u \uparrow_q bv) + b(au \uparrow_q v) + \delta_{a,b} q(u \uparrow_q v) \tag{13}$$

and the comultiplication is simply given on the letters by

$$\Delta_{\uparrow_q}(x) = x \otimes 1 + 1 \otimes x + qx \otimes x \ . \tag{14}$$

On the other hand, one knows a Three-parameter deformation of the Hopf algebra of Feynman-Bender Diagrams [20, 24]. These parameters are of different nature (the superposing parameter is of the q-infiltration nature). In our project, we aim to separate, using the notion of dual law, what comes from the deformation of the tensor structure and what is a perturbation of type Hoffman[3]. Moreover, this algebra specializes in the algebra of polyzeta funtions and the link and payoff of this specialization deserves to be elucidated. At last, many Hopf algebras coming from Physics are free (as algebras). For example the so-called "non commutative Connes-Kreimer Hopf Algebra" (algebra of planar rooted trees introduced by L. Foissy) or the three-parameter deformation of the Hopf algebra of Feynman-Bender Diagrams ($LDIAG(q_c,q_s,q_t)$, see [24]). As such, their Sweedler's duals are the algebra of representative functions on the free monoid (i. e. the functions that are recognizable by a finite-state automaton). In case of an infinite alphabet (which is the case of - many and - the two preceding algebras), Kleene-Schützenberger theorem does not apply as such but slightly modified [22]. Nevertheless, one can exploit the richness of the "calculus" on rational expressions initiated after the theory of languages [18, 22].

Paris XIII, with aid of Marne-la-Vallée will be in charge of the task of this part. The schedule of the work is as follows:
1) "First year" modify Kleene-Schützenberger theorem to adapt it to the new situation (GHED, CT).
2) "Second year" develop an efficient "rational calculus" and give explicit formulas for the rational Hadamard product (SG, GHED).
3) "Third year" set down a correct theory of Dual laws (JGL, GHED, HC).
4) "Fourth year" separate the rôle of the two parameters of $LDIAG(q_c,q_s)$, make the link with polyzeta functions and conventional Feynman Diagrams (GHED, CT, HC, LP).

Partially asymmetric exclusion process and Askey-Wilson polynomials. —

The partially asymmetric exclusion process (PASEP) is an important model from statistical mechanics : it describes a system of interacting particles hopping left and right on a one-dimensional lattice of n sites. The model is both simple and rich : it exhibits boundary-induced phase transitions and many other physically relevant features. Recently, its relations to combinatorics [16, 55, 25] and orthogonal polynomials [47] have been clarified ; in particular, the steady state of the model is intimately related to the Askey-Wilson polynomials [52].

The combinatorics of the Asymmetric Exclusion Process is far from being completely understood, and we believe that insightful results can be obtained through the study

3. Hoffman's shuffle is used in the theory of polyzeta functions [11, 31].

of Askey-Wilson polynomials: the main combinatorial ingredients will be permutation tableaux. They will hopefully give the key for carrying over to the Askey-Wilson polynomials the interpretation of the Al-Salam-Chihara polynomials in terms of the generating function of some Ferrers diagrams with respect to suitable weights.

Permutation tableaux are new objects that come from the enumeration of the totally positive Grassmannian cells [45, 57]. Permutation tableaux [51] are Ferrers diagrams filled with 0's and 1's and are in bijective correspondence [51, 15, 10]. Different statistics on permutation tableaux were defined in [17, 51].

It was shown in [16] that the probability of finding the PASEP in configuration tau in the steady state is the probability generating function of the tableau of shape tau, according to the number of superfluous ones, the number of ones in the first row and the number of unrestricted rows minus one. Moreover, the partition function Z_n for the PASEP is equal to the weight-generating function for all permutation tableaux of length n + 1. In [17], Corteel and Williams defined a Markov chain on the permutation tableaux that gave a combinatorial proof of the latter result.

Therefore we want to use these tableaux to construct some theory of orthogonal polynomials as done for certain polynomials by Viennot and others [55]. For example we know that the moments of q-Laguerre polynomials are related to the tableaux counted by their length and their number of superfluous ones. Finally the generating function of the tableaux can be written by continued fractions [14] or hypergeometric series [57] and we want to link those families of generating functions.

For a linear functional φ, and the monic polynomials $p_n(x)$'s orthogonal with respect to φ, the linearization coefficients are the values obtained when we apply φ to a product of the orthonal polynomials $p_n(x)$. In very rare cases, these coefficients are always non-negative integers, see [59], where this result is described for the Hermite, Charlier, Laguerre, Meixner, and Pollaczek polynomials. In 1987 Ismail, Stanton and Viennot found a q-analogue of these coefficients for Hermite polynomials. This is an important result because it gives a combinatorial evaluation of the fundamental Askey-Wilson integral. The similar extension of such a generalization to other polynomials did not happen until the recent work of Anshelevish [3], who found a q-analogue of these coefficients for Charlier polynomials. However, the proof in [3] was fairly sophisticated, involving combinatorial stochastic measures with respect to certain noncommutative stochastic processes. In [40], Kim, Stanton and Zeng gave a simpler and combinatorial proof of Anshelevishs theorem inspired by Viennot's combinatorial approach [55]. Recently Zeng found that the q-Laguerre polynomials encounted in the work of Williams and Corteel [57, 13] have nice linearization coefficients, which is a refinement of the linearization coefficient formula for Laguerre polynomials, which are a rescaled version of Al-Salam-Chihara polynomials.

Zeng will be in charge of the task of this part. The schedule of the work is as follows:

1) "First year" derive the Williams-Corteel moment formula from the classical theory of orthogonal polynomials and find the Touchard-Riordan type formula for the crossing numbers in partitions.

2) "Second year" prove the conjectured formula for the linearization coefficient formula for the new q-Laguerre formulas.

3) "Third year" generalize the above results for Al-Salam-Chihara polynomials.

4) "Fourth year" complete the picture for the Askey-Wilson polynomials.

q-analogs of orthogonal polynomials. —

We will also carry out another type of investigations about important families of orthogonal polynomials.

About 20 years ago, Andrews, Goulden and Jackson [2] found an extension of Cauchy formula for Schur functions, which, through the use of the characters of the symmetric group, gave a generalization of Mehlers formula for Charlier orthogonal polynomials. We are interested in extensions of this technique to find relations between Cauchy formula and other classical families of orthogonal polynomials, such as Hermite polynomials, which show up in the computation of the wave function associated the quantum harmonic oscillator.

The q-analogs of Hermite polynomials are particularly interesting as their combinatorial definition is given by the involutions of the symmetric group. Our idea is to prove Mehlers formulas for Hermite and q-Hermite polynomials, starting from the above Cauchy formula for Schur functions, or maybe a new extension of this formula, which has to be discovered. A tool we can use for finding such a formula is for example Stembridges SF Maple package on symmetric functions. It is natural to consider extensions of Schur functions, such as Hall-Littlewood or Macdonald polynomials, which are defined through orthogonality relations. Hall-Littlewood polynomials have a free parameter; say t, which can be specialized to 0 to recover Schur functions. Macdonald polynomials have two free parameters, say t and q, and Hall-Littlewood polynomials correspond to the case $q = 0$. There are Cauchy formulas for Hall-Littlewood and Macdonald polynomials, as well as many different summation formulas of these symmetric functions. In 1990 Stembridge [53] proved through some summations formulas of Hall-Littlewood polynomials the celebrated Rogers-Ramanujan identities, which are among the most beautiful and famous identities in q-series. The idea is to start from an infinite summation of Hall-Littlewood polynomials, to find a finite extension which gives, after specialization of the variables and the use of Jacobi triple product identity, a q-series summation formula. Jouhet and Zeng [37], and then Warnaar [56], have recently extended Stembridge's work on Hall-Littlewood polynomials, giving many finite summations of them with applications to q-series. We are interested in completing and gathering all these results, and we also want to extend them to the level of Macdonald polynomials. This would give interesting applications in terms of basic elliptic hypergeometric series.

In 1997, in the framework of symmetrizing operators, Kawanaka [46] gave as a conjecture a new infinite summation of Macdonald polynomials, which is still open although the special case of Hall-Littlewood polynomials can easily be proved through Pieri formula. This last case has been extended to a finite version by Ishikawa, Jouhet and Zeng, with some applications to q-series. We plan to use Pieri formula for Macdonald polynomials to reformulate and hopefully prove Kawanakas conjecture. As explained before, symmetric functions are strongly related to q-series and basic hypergeometric series. As a classical tool for proving q-series identities is given by Baileys lemma and its recent extensions by Andrews, Bressoud or Warnaar, it is natural to ask whether there could be a Bailey lemma at the level of Hall-Littlewood polynomials, or even the one of Macdonald polynomials.

Finally, we will be interested in finding relations between matroids and q-series. Matroid theory provides a natural context to study many questions in graph theory, (linear) algebra, lattice theory, and coding theory, in particular with respect to problems in operations research. We will study the role of generating functions and Lie algebras in this context. The noncommutative multiplication is given by an appropriate decomposition, and the variable q keeps track of the rank of the minors. It would be interesting to put the work of Brylawski and Varchenko in this matroid context.

Lyon 1 (B. Lass and F. Jouhet) with the aid of Marne-la-Vallée (JGL) will be in charge of this task and try to keep the following schedule:
-First year write a computer program to check further cases of Kawanaka's conjecture and the equivalent formulation found by Ishikawa and Zeng.
-Second year try to find similar fomulas for Jack polynomials and study the combinatorial interpretations.
-Third year and fourth years find the connection with the work of Brylawski and Varchenko.

About 20 years ago, Andrews, Goulden and Jackson [2] found an extension of Cauchy formula for Schur functions, which, through the use of the characters of the symmetric group, gave a generalization of Mehlers formula for Charlier orthogonal polynomials. We are interested in extensions of this technique to find relations between Cauchy formula and other classical families of orthogonal polynomials, such as Hermite polynomials, which show up in the computation of the wave function associated the quantum harmonic oscillator.

The q-analogs of Hermite polynomials are particularly interesting as their combinatorial definition is given by the involutions of the symmetric group. Our idea is to prove Mehlers formulas for Hermite and q-Hermite polynomials, starting from the above Cauchy formula for Schur functions, or maybe a new extension of this formula, which has to be discovered. A tool we can use for finding such a formula is for example Stembridges SF Maple package on symmetric functions. It is natural to consider extensions of Schur functions, such as Hall-Littlewood or Macdonald polynomials, which are defined through orthogonality relations. Hall-Littlewood polynomials have a free parameter; say t, which can be specialized to 0 to recover Schur functions. Macdonald polynomials have two free parameters, say t and q, and Hall-Littlewood polynomials correspond to the case $q = 0$. There are Cauchy formulas for Hall-Littlewood and Macdonald polynomials, as well as many different summation formulas of these symmetric functions. In 1990 Stembridge [50] proved through some summations formulas of Hall-Littlewood polynomials the celebrated Rogers-Ramanujan identities, which are among the most beautiful and famous identities in q-series. The idea is to start from an infinite summation of Hall-Littlewood polynomials, to find a finite extension which gives, after specialization of the variables and the use of Jacobi triple product identity, a q-series summation formula. Jouhet and Zeng [37], and then Warnaar [56], have recently extended Stembridges work on Hall-Littlewood polynomials, giving many finite summations of them with applications to q-series. We are interested in completing and gathering all these results, and we also want to extend them to the level of Macdonald polynomials. This would give interesting applications in terms of basic elliptic hypergeometric series.

Probability and graph models

There are many instances where a couple (raising/lowering) operator is acting. For example, one can consider the fact that there is one way to add a ball in an Urn (addition operator X) whereas there are n ways (provided that the Urn contain n balls) of withdrawing a ball (deletion operator D). So the urn histories are governed by the equation

$$DX - XD = 1 \tag{15}$$

which is the realm of creation/annihilation combinatorics. Other examples were developped for graph theory [30]. We want to interpret, with these models, the amount of "Quantum Combinatorics" developped from creation/annihilation operators. There is a rich literature (for a reference on these models linked to analysis, see [27]). The presence of two operators D,X such that $[D,X] = 1$ can be seen in [28].

Paris VI, with aid of Paris XIII will be in charge of the task of this part. The schedule of the work is as follows:
1) "First year" make explicit the variety of Unr models, of Graph models (transition graphs) and revisit it with creation/annihilation operators (KAP, AS, PB, GHED, CT, HC).
2) "Second year" explicit the rerurn of Quantum Combinatorics to Urn and Graph models in particular continued fractions developments (KAP, PB, GHED, CT, HC).
3) "Third year" develop the return of q-analog (KAP, PB, GHED, CT).
4) "Fourth year" through the experience of Urn models, develop the return of creation/annihilation operators to other purely combinatorial features like graphs (CT, GHED).

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