Polyzêtas

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INFINITESIMAL BRAID RELATIONS
&
STRUCTURE OF POLYLOGARITHMS

Infinitesimal braid relation & Knizhnik-Zamolodchikov syst.

In 1986, Drinfel'd introduced the differential system associated to the Lie algebra of pure braid groups $\mathcal{T}_n$:

$$\Omega_n(z_1,\ldots,z_n)=\sum_{1\le i<j\le n}\frac{t_{i,j}}{2\mathrm{i}\pi}\,\omega_{i,j}(z_1,\ldots,z_n)=\sum_{1\le i<j\le n}\frac{t_{i,j}}{2\mathrm{i}\pi}\frac{d(z_i-z_j)}{z_i-z_j}.$$

The flatness condition, $d\Omega_n-\Omega_n\wedge\Omega_n=0$, is equivalent to the infinitesimal braid relation generated by $\{t_{i,j}\}_{1\le i,j\le n}$:

$$\begin{aligned}
t_{i,j}&=0 &&\text{for} && i=j,\\
t_{i,j}&=t_{j,i} &&\text{for} && i\neq j,\\
[t_{i,j},t_{i,k}+t_{j,k}]&=0 &&\text{for distinct} && i,j,k,\\
[t_{i,j},t_{k,l}]&=0 &&\text{for distinct} && i,j,k,l.
\end{aligned}$$

Hence, the differential system KZ_n

$$dF(z_1,\ldots,z_n)=\Omega_n(z_1,\ldots,z_n)F(z_1,\ldots,z_n)$$

is completely integrable over $\{(z_1,\ldots,z_n)\in\mathbb{C}^n|z_i\neq z_j \text{ for } i\neq j\}$.

Examples

- $\mathcal{T}_2 = \{t_{1,2}\}$.

$$\Omega_2(z) = \frac{t_{1,2}}{2\mathrm{i}\pi} \frac{d(z_1 - z_2)}{z_1 - z_2}, \qquad F(z_1, z_2) = (z_1 - z_2)^{t_{1,2}/2\mathrm{i}\pi}.$$

- $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\},\ [t_{1,3}, t_{1,2} + t_{2,3}] = [t_{2,3}, t_{1,2} + t_{1,3}] = 0.$

$$\Omega_3(z) = \frac{1}{2\mathrm{i}\pi}\left[t_{1,2}\frac{d(z_1 - z_2)}{z_1 - z_2} + t_{1,3}\frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3}\frac{d(z_2 - z_3)}{z_2 - z_3}\right].$$

$$F(z_1, z_2, z_3) = G\left(\frac{z_1 - z_2}{z_1 - z_3}\right)(z_1 - z_3)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi},$$

where G satisfies the following differential equation

$$(DE) \qquad dG = (x_0\, \omega_0(z) + x_1\, \omega_1)G,$$

$$\text{with} \qquad x_0 := \frac{t_{1,2}}{2\mathrm{i}\pi}, \qquad \omega_0(z) := \frac{dz}{z}$$

$$\text{and} \qquad x_1 := -\frac{t_{2,3}}{2\mathrm{i}\pi}, \qquad \omega_1(z) := \frac{dz}{1 - z}.$$

Iterated integral along a path and dilogarithms

Let $\omega_0(z) = dz/z$ et $\omega_1(z) = dz/(1-z)$. The iterated integral over ω_0, ω_1 associated to $w = x_{i_1} \cdots x_{i_k} \in X^*$ is defined by

$$\alpha_{z_0}^{z}(\varepsilon) = 1 \quad \text{and} \quad \alpha_{z_0}^{z}(x_{i_1} \ldots x_{i_k}) = \int_{z_0}^{z} \ldots \int_{z_0}^{z_{k-1}} \omega_{i_1}(z_1) \ldots \omega_{i_k}(z_k).$$

Example

$$\begin{aligned}
\alpha_0^z(x_0x_1) &= \int_0^z \int_0^s \omega_0(s)\omega_1(t) \\
&= \int_0^z \int_0^s \frac{ds}{s}\frac{dt}{1-t} \\
&= \int_0^z \frac{ds}{s} \int_0^s dt \sum_{k\geq 0} t^k \\
&= \sum_{k\geq 1} \int_0^z ds \frac{s^{k-1}}{k} \\
&= \sum_{k\geq 1} \frac{z^k}{k^2} \\
&= \mathrm{Li}_2(z).
\end{aligned}$$

Polylogarithms, harmonic sums and polyzêtas

Classic cases ($N \in \mathbb{N}_+, r > 0$ and $|z| < 1$) :

$$\mathrm{Li}_r(z) = \alpha_0^z(x_0^{n-1}x_1) = \sum_{n\geq 1} \frac{z^n}{n^r} \quad \text{and} \quad \mathrm{H}_r(N) = \sum_{n=1}^{N} \frac{1}{n^r}.$$

Generalization on multi-indices $\mathbf{s} = (s_1, \ldots, s_r)$:

$$\alpha_0^z(x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1) = \mathrm{Li}_{\mathbf{s}}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}},$$

$$\mathrm{P}_{\mathbf{s}}(z) = \frac{\mathrm{Li}_{\mathbf{s}}(z)}{1-z} = \sum_{N\geq 0} \mathrm{H}_{\mathbf{s}}(N)\, z^N, \text{ where } \mathrm{H}_{\mathbf{s}}(N) = \sum_{n_1 > \ldots > n_r = 1}^{N} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}.$$

If $s_1 > 1$, by an Abel's theorem, then one has

$$\lim_{z\to 1} \mathrm{Li}_{\mathbf{s}}(z) = \lim_{N\to\infty} \mathrm{H}_{\mathbf{s}}(N) = \zeta(\mathbf{s}) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}$$

else ?

Euler-Maclaurin sommation formula

$$\sum_{n=1}^{N} \frac{1}{n} = \log N + \gamma - \sum_{j=1}^{k-1} \frac{B_j}{j} \frac{1}{N^j} + O\left(\frac{1}{N^k}\right),$$

$$\sum_{n=1}^{N} \frac{1}{n^r} = \zeta(r) - \frac{N^{1-r}}{(r-1)} - \sum_{j=r}^{k-1} \frac{B_{j-r+1}}{j-r+1} \binom{k-1}{j-1} \frac{1}{N^j} + O\left(\frac{1}{N^k}\right).$$

Are γ and $\zeta(r)$ algebraically independent ?

Let $\mathcal{Z}$ be the $\mathbb{Q}$-algebra generated by convergent polyzêtas.

Let $\mathcal{Z}'$ be the $\mathbb{Q}[\gamma]$-algebra generated by convergent polyzêtas.

How to determine the asymptotic expansion and the constants associated to the divergents harmonic sums of the form

$$\mathrm{H}_{\{1\}^r}(N) = \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1 \ldots n_r}?$$

$$\mathrm{H}_{\{1\}^k, \underbrace{s_{k+1}}_{>1}, \ldots, s_r}(N) = \sum_{N \geq n_1 > \ldots > n_r > 0}^{N} \frac{1}{n_1 \ldots n_k n_{k+1}^{s_{k+1}} \ldots n_r^{s_r}}?$$

Do there exists a generalization of γ ?

Encoding the multi-indices by words

$Y = \{y_k | k \in \mathbb{N}_+\}$ $(y_1 < y_2 < \ldots)$ and $X = \{x_0, x_1\}$ $(x_0 < x_1)$.
Y^* (resp. X^*) : monoide generated by Y (resp. X).

$$\mathbf{s} = (s_1, \ldots, s_r) \leftrightarrow w = y_{s_1} \ldots y_{s_r} \leftrightarrow w = x_0^{s_1 - 1} x_1 \ldots x_0^{s_r - 1} x_1.$$

u and v are convergent if $s_1 > 1$. A word divergent is of the form

$$(\{1\}^k, s_{k+1}, \ldots, s_r) \leftrightarrow y_1^k y_{s_{k+1}} \ldots y_{s_r} \leftrightarrow x_1^k x_0^{s_{k+1}-1} x_1 \ldots x_0^{s_r - 1} x_1, \quad \text{for} \quad k \geq 1.$$

$$\begin{aligned}
\mathrm{Li}_w : w &\mapsto \mathrm{Li}_w(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}}, \\
\zeta_w : w &\mapsto \zeta(w) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}, \\
\mathrm{H}_w : w &\mapsto \mathrm{H}_w(N) = \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}, \\
\mathrm{P}_w : w &\mapsto \mathrm{P}_w(z) = \sum_{N \geq 0} \mathrm{H}_w(N) z^N = \frac{\mathrm{Li}_w(z)}{1 - z}.
\end{aligned}$$

Let $\Pi_X : \mathbb{C}\langle\langle Y \rangle\rangle \longrightarrow \mathbb{C}\langle\langle X \rangle\rangle$ and $\Pi_Y : \mathbb{C}\langle\langle X \rangle\rangle \longrightarrow \mathbb{C}\langle\langle Y \rangle\rangle$ denote the "change" of alphabets over $\mathbb{C}\langle\langle X \rangle\rangle$ and $\mathbb{C}\langle\langle Y \rangle\rangle$ respectively.

Structure of polylogarithms

Let $\mathcal{C} = \mathbb{C}[z, z^{-1}, (1-z)^{-1}]$

Theorem (HNM, van der Hoeven & Petitot, 1998)

Putting $\mathrm{Li}_{x_0}(z) = \log z$, $\mathrm{Li} : w \mapsto \mathrm{Li}_w$ *becomes an isomorphism from* $(\mathbb{C}\langle X \rangle, ш)$ *to* $(\mathbb{C}\{\mathrm{Li}_w\}_{w \in X^*}, .)$.

- $\mathrm{Li}_w, w \in X^*$, are $\mathcal{C}$-linearly independent.
  Then $\{\mathrm{Li}_w\}_{w \in X^*}$ is universal Picard-Vessiot extension of fuchsian differential equations with three regular singularities.
- $\mathrm{Li}_l, l \in \mathcal{L}ynX$, are $\mathcal{C}$-algebraically independent.
- $\zeta(l), l \in \mathcal{L}ynX \setminus \{x_0, x_1\}$, are generators of the algebra $\mathcal{Z}$.

More about structure of polylogarithms

Hoffman proved that $\forall u, v \in Y^*, \mathrm{H}_u \mathrm{H}_v = \mathrm{H}_{u \,⧢\, v}$.
Therefore, $\forall u, v \in Y^* \setminus y_1 Y^*$, $\zeta(u)\zeta(v) = \zeta(u \,⧢\, v)$.

$$\mathrm{P}_u(z) \odot \mathrm{P}_v(z) = \sum_{n\geq 0} \mathrm{H}_u(n)\mathrm{H}_v(n) z^n = \sum_{n \geq 0} \mathrm{H}_{u \,⧢\, v}(n) z^n = \mathrm{P}_{u \,⧢\, v}(z).$$

Theorem (HNM, 2003)

$(\mathbb{C}\{\mathrm{P}_w\}_{w \in Y^*}, \odot) \cong (\mathbb{C}\langle Y \rangle, ⧢) \cong (\mathbb{C}\{\mathrm{H}_w\}_{w \in Y^*}, .)$.

- $\mathrm{P}_w, w \in Y^*$, are $\mathcal{C}$-linearly independent.
  Then $\mathrm{H}_w, w \in Y^*$, are linearly independent.
- $\mathrm{P}_l, l \in \mathcal{L}ynY$, are $\mathcal{C}$-algebraically independent.
  Then $\mathrm{H}_l, l \in \mathcal{L}ynY$, algebraically independent.
- $\zeta(l), l \in \mathcal{L}ynY \setminus \{y_1\}$, are generators of the algebra $\mathcal{Z}$.

Towards the structure of polyzêtas

Corollary

$\forall u, v \in X^*, \mathrm{Li}_u \mathrm{Li}_v = \mathrm{Li}_{u ⧢ v} \Rightarrow \forall u, v \in x_0 X^* x_1, \zeta(u)\zeta(v) = \zeta(u ⧢ v).$

Example

$$\begin{aligned} x_0x_1 ⧢ x_0^2x_1 &= x_0x_1x_0^2x_1 + 3x_0^2x_1x_0x_1 + 6x_0^3x_1^2, \\ \mathrm{Li}_2 \mathrm{Li}_3 &= \mathrm{Li}_{2,3} + 3\,\mathrm{Li}_{3,2} + 6\,\mathrm{Li}_{4,1}, \\ \zeta(2)\zeta(3) &= \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1). \end{aligned}$$

Corollary

$\forall u, v \in Y^*, \mathrm{H}_u \mathrm{H}_v = \mathrm{H}_{u \uplus v} \Rightarrow \forall u, v \in Y^* \setminus y_1 Y^*, \zeta(u)\zeta(v) = \zeta(u \uplus v).$

Example

$$\begin{aligned} y_2 \uplus y_3 &= y_2y_3 + y_3y_2 + y_5, \\ \mathrm{P}_{y_2} \odot \mathrm{P}_{y_3} &= P_{y_2y_3} + \mathrm{P}_{y_3y_2} + \mathrm{P}_{y_5}, \\ \mathrm{H}_2\mathrm{H}_3 &= \mathrm{H}_{2,3} + \mathrm{H}_{3,2} + \mathrm{H}_5, \\ \zeta(2)\zeta(3) &= \zeta(2,3) + \zeta(3,2) + \zeta(5). \end{aligned}$$

$$\left.\begin{aligned} \zeta(2)\zeta(3) &= \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1) \\ \zeta(2)\zeta(3) &= \zeta(2,3) + \zeta(3,2) + \zeta(5) \end{aligned}\right\} \Rightarrow \zeta(5) = 2\zeta(3,2) + 6\zeta(4,1).$$

Polynomial relations among $\{\zeta(l)\}_{l\in\mathcal{L}ynX\setminus\{x_0,x_1\}}$

$$
\begin{aligned}
\zeta(2,1) &= \zeta(3)\\
\zeta(4) &= \frac{2}{5}\zeta(2)^2\\
\zeta(3,1) &= \frac{1}{10}\zeta(2)^2\\
\zeta(2,1,1) &= \frac{2}{5}\zeta(2)^2\\
\zeta(4,1) &= 2\zeta(5)-\zeta(2)\zeta(3)\\
\zeta(3,2) &= -\frac{11}{2}\zeta(5)+3\zeta(2)\zeta(3)\\
\zeta(3,1,1) &= 2\zeta(5)-\zeta(2)\zeta(3)\\
\zeta(2,2,1) &= -\frac{11}{2}\zeta(5)+3\zeta(2)\zeta(3)\\
\zeta(2,1,1,1) &= \zeta(5)\\
\zeta(6) &= \frac{8}{35}\zeta(2)^3\\
\zeta(5,1) &= -\frac{1}{2}\zeta(3)^2+\frac{6}{35}\zeta(2)^3\\
\zeta(4,2) &= \zeta(3)^2-\frac{32}{105}\zeta(2)^3\\
\zeta(4,1,1) &= -\zeta(3)^2+\frac{23}{70}\zeta(2)^3\\
\zeta(3,2,1) &= 3\zeta(3)^2-\frac{29}{30}\zeta(2)^3\\
\zeta(3,1,2) &= -\frac{3}{2}\zeta(3)^2+\frac{53}{105}\zeta(2)^3\\
\zeta(3,1,1,1) &= -\frac{1}{2}\zeta(3)^2+\frac{6}{35}\zeta(2)^3\\
\zeta(2,2,1,1) &= \zeta(3)^2-\frac{32}{105}\zeta(2)^3\\
\zeta(2,1,1,1,1) &= \frac{8}{35}\zeta(2)^3
\end{aligned}
$$

Irreducible polyzêtas by computer

in Axiom at weight **10** (avec Petitot),
in Maple at weight **12** (with Bigotte, Jacob, Oussous, Petitot),
in C++ at weight **16** (with El Wardi, Jacob, Oussous, Petitot).

r / n	1	2	3	4	5
2	$\zeta(2)$				
3	$\zeta(3)$				
5	$\zeta(5)$				
7	$\zeta(7)$				
8		$\zeta(6,2)$			
9	$\zeta(9)$				
10		$\zeta(8,2)$			
11	$\zeta(11)$		$\zeta(8,2,1)$		
12		$\zeta(10,2)$		$\zeta(8,2,1,1)$	
13	$\zeta(13)$		$\zeta(9,3,1)$ $\zeta 10,2,1)$		
14		$\zeta(10,4)$ $\zeta(12,2)$		$\zeta(10,2,1,1)$	
15	$\zeta(15)$		$\zeta(11,3,1)$ $\zeta(12,2,1)$		$\zeta(10,2,1,1,1)$
16		$\zeta(12,4)$ $\zeta(14,2)$		$\zeta(10,4,1,1)$ $\zeta(11,3,1,1)$ $\zeta(12,2,1,1)$	

NONCOMMUTATIVE GENERATING SERIES TECHNOLOGY

Noncommutative generating series

Definition

$$\mathrm{L}(z) := \sum_{w \in X^*} \mathrm{Li}_w(z)\, w \quad \text{and} \quad \mathrm{H}(N) := \sum_{w \in Y^*} \mathrm{H}_w(N)\, w.$$

Let $\mathcal{L}ynX$ and $l \in \mathcal{L}ynX$ (resp. $\{\hat{l}\}_{l \in \mathcal{L}ynY}$ and $\{\hat{l}\}_{l \in \mathcal{L}ynY}$) be the transcendence basis of $(\mathbb{C}\langle X \rangle, ш)$ (resp. $(\mathbb{C}\langle Y \rangle, \uplus)$) and its dual basis respectively. Then

Theorem

L and H are group-like and

$$\mathrm{L}(z) = e^{x_1 \log \frac{1}{1-z}} \mathrm{L}_{\mathrm{reg}}(z) e^{x_0 \log z} \quad \text{and} \quad \mathrm{H}(N) = e^{\mathrm{H}_1(N)\, y_1} \mathrm{H}_{\mathrm{reg}}(N),$$

$$\text{where} \quad \mathrm{L}_{\mathrm{reg}}(z) = \prod_{\substack{l \in \mathcal{L}ynX \\ l \neq x_0, x_1}}^{\searrow} e^{\mathrm{Li}_l(z)\, \hat{l}} \quad \text{and} \quad \mathrm{H}_{\mathrm{reg}}(N) = \prod_{\substack{l \in \mathcal{L}ynY \\ l \neq y_1}}^{\nearrow} e^{\mathrm{H}_l(N)\, \hat{l}}.$$

Definition

$$Z_ш := \mathrm{L}_{\mathrm{reg}}(1) \quad \text{and} \quad Z_{\uplus} := \mathrm{H}_{\mathrm{reg}}(\infty).$$

Double regularization to 0

Proposition

Let $\zeta_{ш} : \mathbb{C}\langle\langle X \rangle\rangle \longrightarrow \mathbb{C}$ *be the shuffle algebra morphism defined by*

- $\zeta_{ш}(x_0) = \zeta_{ш}(x_1) = 0$,
- *for any* $r_1 > 1, \zeta_{ш}(x_0^{r_1-1}x_1 \ldots x_0^{r_k-1}x_1) = \zeta(r_1, \ldots, r_k)$,
- *for any* $u, v \in X^*, \zeta_{ш}(u ш v) = \zeta_{ш}(u)\zeta_{ш}(v)$.

Then $\displaystyle\sum_{w \in X^*} \zeta_{ш}(w)\, w = Z_{ш}$.

Proposition

Let $\zeta_{⧢} : \mathbb{C}\langle\langle Y \rangle\rangle \longrightarrow \mathbb{C}$ *be the algebra morphism defined by*

- $\zeta_{⧢}(y_1) = 0$,
- *for any* $r_1 > 1, \zeta_{⧢}(y_{r_1} \ldots y_{r_k}) = \zeta(r_1, \ldots, r_k)$,
- *for any* $u, v \in Y^*, \zeta_{ш}(u ⧢ v) = \zeta_{⧢}(u)\zeta_{⧢}(v)$.

Then $\displaystyle\sum_{w \in Y^*} \zeta_{⧢}(w)\, w = Z_{⧢}$.

Results *à la* Abel

Theorem (HNM, 2005)

$$\lim_{z\to 1} e^{y_1 \log \frac{1}{1-z}} \Pi_Y \mathrm{L}(z) = \lim_{N\to\infty} \left[\sum_{k\geq 0} \mathrm{H}_{y_1^k}(N)(-y_1)^k\right] \mathrm{H}(N) = \Pi_Y Z_{\mathrm{III}}.$$

$$\Rightarrow \mathrm{H}(N) \underset{N\to\infty}{\widetilde{}} \exp\left[-\sum_{k\geq 1} \mathrm{H}_{y_k}(N) \frac{(-y_1)^k}{k}\right] \Pi_Y Z_{\mathrm{III}}.$$

Theorem (Costermans, Enjalbert & HNM, 2005)

There exists algorithmically computable coefficients $b_i \in \mathcal{Z}'$, the $\mathbb{Q}$-algebra generated by $\mathcal{Z}$ and by γ, $\kappa_i \in \mathbb{N}$ and $\eta_i \in \mathbb{Z}$ such that

$$\forall w \in Y^*, \mathrm{H}_w(N) \underset{N\to\infty}{\widetilde{}} \sum_{i=0}^{+\infty} b_i N^{\eta_i} \log^{\kappa_i}(N).$$

Example

$$\mathrm{H}_{4,2}(N) = \zeta(4,2) - \frac{\pi^2}{6}\frac{1}{N^3} + \left(\frac{\pi^2}{12} + \frac{1}{4}\right)\frac{1}{N^4} - \left(\frac{\pi^2}{18} + \frac{2}{5}\right)\frac{1}{N^5} + \mathrm{O}\left(\frac{1}{N^6}\right),$$

$$\begin{aligned}\mathrm{H}_{1,4}(N) &= \frac{\pi^4}{90}\ln(N) + \frac{\pi^4}{90}\gamma - \zeta(4,1) - \zeta(5) \\ &+ \frac{\pi^4}{180}\frac{1}{N} - \frac{\pi^4}{1080}\frac{1}{N^2} + \frac{1}{9}\frac{1}{N^3} + \left(\frac{\pi^4}{10800} - \frac{1}{24}\right)\frac{1}{N^4} + \mathrm{O}\left(\frac{1}{N^5}\right).\end{aligned}$$

Generalized Euler constants

Theorem (HNM, 2005)

For any $k \geq 0$ and for any $w \in Y^* \setminus \{y_1\}$, let $\gamma_{y_1^k w}$ be the constant associated to $\mathrm{H}_{y_1^k w}$. Let $G := \sum_{w \in Y^*} \gamma_w \, w$.

Then G is group-like and $G = \exp\left[\gamma y_1 - \sum_{k \geq 2} \zeta(k) \frac{(-y_1)^k}{k}\right] \Pi_Y Z_{\text{ш}}$.

Let $b_{n,k}(t_1, \ldots, t_k)$ be the Bell polynomials. By specializing at $t_1 = \gamma$ and for $l \geq 2$, $t_l = (-1)^{l-1}(l-1)!\zeta(l)$ and by using the identity, for any $u \in X^*$, $x_1^k x_0 u = \sum_{l=0}^{k} x_1^l \text{ш}(x_0[(-x_1)^{k-l} \text{ш} u])$, we get

Corollary

$$\gamma_{y_1^k w} = \sum_{i=0}^{k} \frac{\zeta(x_0[(-x_1)^{k-i} \text{ш} \Pi_X w])}{i!} \left[\sum_{j=1}^{i} b_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots)\right].$$

In particular,

$$\gamma_{y_1^k} = \sum_{\substack{s_1, \ldots, s_k > 0 \\ s_1 + \ldots + k s_k = k}} \frac{(-1)^k}{s_1! \ldots s_k!} (-\gamma)^{s_1} \left(-\frac{\zeta(2)}{2}\right)^{s_2} \ldots \left(-\frac{\zeta(k)}{k}\right)^{s_k}.$$

Regularization to γ

$$\begin{aligned}
\gamma_{1,1} &= [\gamma^2 - \zeta(2)]/2, \\
\gamma_{1,1,1} &= [\gamma^3 - 3\zeta(2)\gamma + 2\zeta(3)]/6, \\
\gamma_{1,1,1,1} &= [80\zeta(3)\gamma - 60\zeta(2)\gamma^2 + 6\zeta(2)^2 + 10\gamma^4]/240, \\
\gamma_{1,7} &= \zeta(7)\gamma + \zeta(3)\zeta(5) - 54\zeta(2)^4/175, \\
\gamma_{1,1,6} &= 4\zeta(2)^3\gamma^2/35 + [\zeta(2)\zeta(5) + 2\zeta(3)\zeta(2)^2/5 - 4\zeta(7)]\gamma \\
&+ \quad \zeta(6,2) + 19\zeta(2)^4/35 + \zeta(2)\zeta(3)^2/2 - 4\zeta(3)\zeta(5), \\
\gamma_{1,1,1,5} &= \frac{3\zeta(6,2)}{4} - \frac{14\zeta(3)\zeta(5)}{3} + \frac{3\zeta(2)\zeta(3)^2}{4} + \frac{809\zeta(2)^4}{1400} + \frac{\zeta(5)\gamma^3}{6} \\
&+ \left[\frac{\zeta(3)^2}{4} - \frac{\zeta(2)^3}{5}\right]\gamma^2 - \left[2\zeta(7) - \frac{3\zeta(2)\zeta(5)}{2} + \frac{\zeta(3)\zeta(2)^2}{10}\right]\gamma.
\end{aligned}$$

Theorem

$\gamma_\bullet$ *realizes the morphism from* $(\mathbb{C}\langle\langle Y\rangle\rangle, ⧢)$ *to* $(\mathbb{C}, .)$ *verifying*

- *for any word* $u, v \in Y^*, \gamma_{u ⧢ v} = \gamma_u \gamma_v$,
- *for any convergent word* $w \in Y^* - y_1 Y^*, \gamma_w = \zeta(w)$,
- $\gamma_{y_1} = \gamma$.

Then $G = e^{\gamma y_1} Z_{⧢}$.

The meaning of the double regularization to 0

The constant $\gamma_{y_1} = \gamma$ is obtained as the finite part of the asymptotic expansion of $\mathrm{H}_1(n)$ in the scale $\{n^a \log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$.

In the same way, since for any $n \in \mathbb{N}$, n and $\mathrm{H}_1(n)$ are algebraically independent then $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$ constitutes a new scale for asymptotic expansions.

Let $C_1 = \mathbb{Q} \oplus x_0\mathbb{Q}\langle X\rangle x_1$ and $C_2 = \mathbb{Q} \oplus (Y \setminus \{y_1\})\mathbb{Q}\langle Y\rangle$. By the Radford theorem and its generalization over Y (due to Malvenuto & Reutenauer), one has respectively

$$(\mathbb{Q}\langle X\rangle, ш) \cong \mathbb{Q}[\mathcal{L}ynX] = C_1[x_0, x_1],$$
$$(\mathbb{Q}\langle Y\rangle, \uplus) \cong \mathbb{Q}[\mathcal{L}ynY] = C_2[y_1].$$

Thus, $\zeta_{ш}(x_1) = 0$ and $\zeta_{\uplus}(y_1) = 0$ can be interpreted as the finite part of the asymptotic expansions of Li_1 and H_1 in the scales $\{(1-z)^a \log(1-z)^b\}_{a\in\mathbb{Z},b\in\mathbb{N}}$ and $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$ respectively.

Differential Galois group of polylogarithms

$\mathrm{LI}_{\mathcal{C}}$ is the smallest algebra containing $\mathcal{C}$ closed by derivation, by integration w.r.t. ω_0 and ω_1. It is the $\mathcal{C}$-modulus generated by $\{\mathrm{Li}_w\}_{w\in X^*}$.

Let $\sigma \in \mathrm{Gal}(\mathrm{LI}_{\mathcal{C}})$. Then $\displaystyle\sum_{w\in X^*} \sigma\,\mathrm{Li}_w\ w = \prod_{l\in\mathcal{L}yn}^{\searrow} e^{\sigma\,\mathrm{Li}_{\check{S}_l} S_l}$.

Since $d\sigma\,\mathrm{Li}_{x_i} = \sigma d\,\mathrm{Li}_{x_i} = \omega_i$ then $\sigma\,\mathrm{Li}_{x_i} = \mathrm{Li}_{x_i} + c_{x_i}$.

More generally, $\displaystyle\sigma\,\mathrm{Li}_{\check{S}_l} = \int \omega_{x_i} \frac{\sigma\,\mathrm{Li}^{i_1}_{\check{S}_{l_1}}}{i_1!} \cdots \frac{\sigma\,\mathrm{Li}^{i_k}_{\check{S}_{l_k}}}{i_k!} + c_{\check{S}_l}$.

Consequently, $\displaystyle\sum_{w\in X^*} \sigma\,\mathrm{Li}_w\ w = \mathrm{L}\prod_{l\in\mathcal{L}yn}^{\searrow} e^{c_{\check{S}_l} S_l} = \mathrm{L}e^{C_\sigma}$.

The action of $\sigma \in \mathrm{Gal}(\mathrm{LI}_{\mathcal{C}})$ over $\{\mathrm{Li}_w\}_{w\in X^*}$ is equivalent to the action of $e^{C_\sigma} \in \mathrm{Gal}(DE)$ over the exponential solution L. So,

Theorem (HNM, 2003)

$\mathrm{Gal}(\mathrm{LI}_{\mathcal{C}}) \cong \mathrm{Gal}(DE) = \{e^C \mid C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle\}$.

Action of $\mathrm{Gal}(DE)$ on the asymptotic expansions

Theorem (Group of associators theorem)

For any commutative $\mathbb{Q}$-algebra A, let $\Phi \in A\langle\langle X\rangle\rangle$ and $\Psi \in A\langle\langle Y\rangle\rangle$ be group-like elements such that $\Psi = B(y_1)\Pi_Y\Phi$. There exists an unique $C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ such that $\Phi = Z_{\text{III}}e^C$ and $\Psi = G\Pi_Y e^C$.

If $C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ then $\mathrm{L}' = \mathrm{L}e^C$ is group-like and $e^C \in \mathrm{Gal}(DE)$. Let $\mathrm{H}'(N)$ be the n.c.g.s. of the Taylor coefficients, belonging the harmonic algebra, of $\{(1-z)^{-1}\mathrm{L}'_w(z)\}_{w\in Y^*}$. Then $\mathrm{H}'(N)$ is group-like.

$$\frac{\mathrm{L}'(1-\varepsilon)}{\varepsilon} \widetilde{_{\varepsilon\to 0^+}} e^{-(1+x_1)\log\varepsilon}\Phi_{KZ}e^C \quad\Rightarrow\quad \mathrm{H}'(N) \widetilde{_{N\to\infty}} \mathrm{H}(N)\Pi_Y e^C.$$

Let κ_w be the constant part of $\mathrm{H}'_w(N)$. Then,

$$\sum_{w\in Y^*}\kappa_w\, w = \Psi_{KZ}\Pi_Y e^C, \quad \text{or eqivalenty} \quad \Pi_X\sum_{w\in Y^*}\kappa_w\, w = B^{-1}(x_1)\Phi_{KZ}e^C.$$

We put then $\Psi := Z_{\text{III}}\Pi_Y e^C$ and $\Phi := G\Pi_Y e^C$.

Examples (action of the monodromy group)

For $t\in]0,1[$, the monodromies around $0,1$ of L are given respectively by ($p=2i\pi$)

$$\mathcal{M}_0\mathrm{L}(t)=\mathrm{L}(t)e^{p\mathfrak{m}_0}\quad\text{and}\quad\mathcal{M}_1\mathrm{L}(t)=\mathrm{L}(t)\Phi_{KZ}^{-1}e^{-px_1}\Phi_{KZ}$$
$$=\mathrm{L}(t)e^{p\mathfrak{m}_1},$$

$$\text{where}\quad\mathfrak{m}_0=x_0\quad\text{and}\quad\mathfrak{m}_1=\prod_{l\in\mathcal{L}yn,l\neq x_0,x_1}^{\searrow}e^{-\zeta(\check{S}_l)\,\mathrm{ad}_{S_l}}(-x_1).$$

- If $C=p\mathfrak{m}_0$ then $\Phi=Z_{\mathrm{III}}e^{px_0}$ and
$$\Psi=\exp\left[\gamma y_1-\sum_{k\geq2}\zeta(k)\frac{(-y_1)^k}{k}\right]\Pi_Y Z_{\mathrm{III}}=G.$$
- If $C=p\mathfrak{m}_1$ then $\Phi=e^{-px_1}Z_{\mathrm{III}}$ and
$$\Psi=\exp\left[\underbrace{(\gamma-p)}_{=T}y_1-\sum_{k\geq2}\zeta(k)\frac{(-y_1)^k}{k}\right]\Pi_Y Z_{\mathrm{III}}=e^{-py_1}G.$$

Hence, the monodromies could not allow neither to introduce the factor $e^{\gamma x_1}$ on the left of Z_{III}, neither to eliminate the left factor $e^{\gamma y_1}$ in G (by putting $T=0$, for example).

Polynomial relations among generators of polyzêtas

$\{\zeta_{ш}(l)\}_{l\in\mathcal{L}ynX}$ and $\{\zeta_{\text{⧢}}(l)\}_{l\in\mathcal{L}ynY}$ are also generators of the algebras $\mathcal{Z}$ and $\mathcal{Z}'$. One also gets

Theorem

$$\prod_{l\in\mathcal{L}ynX, l\neq x_0,x_1}^{\searrow} e^{\zeta(l)\,\hat{l}} = e^{\sum\limits_{k\geq 2}\zeta(k)\frac{(-x_1)^k}{k}}\,\Pi_X \prod_{l\in\mathcal{L}ynY, l\neq y_1}^{\nearrow} e^{\zeta(l)\,\hat{l}}.$$

Since $\forall l \in \mathcal{L}ynY \iff \Pi_X l \in \mathcal{L}ynX \setminus \{x_0\}$ then identifying the local coordinates, in the Lyndon-PBW basis, we get polynomial relations among these generators which are algebraically independent on γ.

Corollary

For any $l \in \mathcal{L}ynY \setminus \{y_1\}$, let P_l be the decomposition of $\Pi_X \hat{l}$ in the Lyndon-PBW basis, over X, and let $\check{P}_l$ be its dual. Then $\Pi_X l - \check{P}_l \in \ker\zeta$. Moreover, $\Pi_X\lambda - \check{P}_\lambda$ is homogenous of degree $|\lambda|$ and if $\Pi_X l = \check{P}_l$ then $\zeta(l)$ is irreducible.

Structure of polyzêtas

Theorem (Structure of polyzêtas)

The $\mathbb{Q}$-algebra generated by convergent polyzêtas, $\mathcal{Z}$, is isomorphic to the graded algebra $(\mathbb{Q} \oplus (Y - y_1)\mathbb{Q}\langle Y\rangle / \ker\zeta, \,ш\,)$.

Proof.

Since $\ker\zeta$ is an ideal generated by the homogenous polynomials then the quotient $\mathbb{Q} \oplus (Y - y_1)\mathbb{Q}\langle Y\rangle / \ker\zeta$ is graded. □

Corollary

The $\mathbb{Q}$-algebra of polyzêtas is freely generated by irreducible polyzêtas.

Proof.

For any $\lambda \in \mathcal{L}ynY$, if $\lambda = \check{P}_\lambda$ then one gets the conclusion else $\Pi_X\lambda - \check{P}_\lambda \in \ker\zeta$. Since $\check{P}_\lambda \in \mathbb{Q}[\mathcal{L}ynX]$ then $\check{P}_\lambda$ is polynomial on Lyndon words of degree $\leq |\lambda|$. For each Lyndon word does appear in this decomposition of $\check{P}_\lambda$, after applying Π_Y, the same process goes on untill having irreducible polyzêtas. □

Towards the transcendence of γ over $\mathcal{Z}$

By considering the commutative indeterminates $t_1, t_2, \ldots$, then let $A = \mathbb{Q}[t_1, t_2, \ldots]$.

Lemma

For any $\Phi \in \{\Phi_{KZ} e^{C} | C \in \mathcal{L}ie_A\langle\langle X \rangle\rangle\}$, *one get*

$$\Psi = B(y_1)\Pi_Y \Phi \quad \Longleftrightarrow \quad \Psi' = B'(y_1)\Pi_Y \Phi.$$

Theorem

For all $\Phi \in \{\Phi_{KZ} e^{C} | C \in \mathcal{L}ie_{\mathbb{Q}}\langle\langle X \rangle\rangle\}$, *the identies* $\Psi = B(y_1)\Pi_Y \Phi$ *yield all polynomial relations among convergent polyzêtas.*

Moreover, these relations are algebraically independent on γ. In other words, under the hypothesis $\gamma \notin \mathbb{Q}$, the constant γ does not verify any polynomial of coefficients in $\mathcal{Z}$.

Corollary

If $\gamma \notin \mathbb{Q}$ *then it is transcendental over* $\mathcal{Z}$.

COLOURED POLYLOGARITHMS, HARMONIC SUMS AND POLYZETAS

Sommes harmoniques et polylogarithmes colorés

Let

$$\begin{pmatrix} \mathbf{b} \\ \mathbf{s} \end{pmatrix} = \begin{pmatrix} b_1, \ldots, b_r \\ s_1, \ldots, s_r \end{pmatrix},$$

where $b_1, .., b_r \in \{1, q, \ldots, q^{n-1}\}$ and $q = e^{2i\pi/n}$.

$$\mathrm{H}_{\mathbf{b} \atop \mathbf{s}}(N) = \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{b_1^{n_1} \cdots b_r^{n_r}}{n_1^{s_1} \ldots n_r^{s_r}},$$

$$\mathrm{Li}_{\mathbf{b} \atop \mathbf{s}}(z) = \sum_{n_1 > \cdots > n_r > 0} \frac{b_1^{n_1} \cdots b_r^{n_r}}{n_1^{s_1} \cdots n_r^{s_r}} z^{n_1}.$$

If $\begin{pmatrix} b_1 \\ s_1 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then

$$\lim_{N \to \infty} \mathrm{H}_{\mathbf{b} \atop \mathbf{s}}(N) = \lim_{z \to 1} \mathrm{Li}_{\mathbf{b} \atop \mathbf{s}}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{b_1^{n_1} \cdots b_r^{n_r}}{n_1^{s_1} \ldots n_r^{s_r}} = \zeta \begin{pmatrix} \mathbf{b} \\ \mathbf{s} \end{pmatrix}.$$

$$\mathrm{P}_{\mathbf{b} \atop \mathbf{s}}(z) = \sum_{n \geq 0} \mathrm{H}_{\mathbf{b} \atop \mathbf{s}}(n) z^n = \frac{\mathrm{Li}_{\mathbf{b} \atop \mathbf{s}}(z)}{1 - z}.$$

Coloured alphabets

Let us consider the alphabets $X = \{x_0, x_q, \ldots, x_{q^n}\}$ and $Y = \{y_{\substack{b \\ k}}\}_{\substack{b \in \{1,\ldots,q^n-1\} \\ k \in \mathbb{N}_+}}$.

$$\begin{pmatrix} b_1, \ldots, b_r \\ s_1, \ldots, s_r \end{pmatrix} \leftrightarrow y_{\substack{b_1 \\ s_1}} \ldots y_{\substack{b_r \\ s_r}} \leftrightarrow x_0^{s_1-1} x_{\tau_1} \cdots x_0^{s_r-1} x_{\tau_r},$$

where $j = 1, .., r$, $\tau_j = \prod_{i=1}^{j} b_i$.

Example

$\begin{pmatrix} -1,1,-1 \\ 3,4,\ \ 5 \end{pmatrix} \leftrightarrow x_0^2 x_{-1} x_0^3 x_{-1} x_0^4 x_1$.

Let $\mathcal{Z}$ be the $\mathbb{Q}$-algebra generated by convergent coloured polyzêtas.

Let $\mathcal{Z}'$ be the $\mathbb{Q}[\gamma]$-algebra generated by convergent coloured polyzêtas.

Harmonic algebra

For $u, v \in Y^*$, we have $\mathrm{H}_u \mathrm{H}_v = \mathrm{H}_{u \star v}$, where $u \star v$ is defined by
$\varepsilon \star u = u \star \varepsilon = u$ and
$(y_{\frac{b}{i}} u) \star (y_{\frac{c}{j}} v) = y_{\frac{b}{i}} (u \star y_{\frac{c}{j}} v) + y_{\frac{c}{j}} (y_{\frac{b}{i}} u \star v) + y_{\frac{bc}{i+j}} (u \star v)$.
By consequent, $\mathrm{P}_u \odot \mathrm{P}_v = \mathrm{P}_{u \star v}$.

Example

$$\begin{aligned}
y_{\frac{-1}{2}} \star y_{\frac{-1}{3}} &= y_{\frac{-1}{2}} y_{\frac{-1}{3}} + y_{\frac{-1}{3}} y_{\frac{-1}{2}} + y_{\frac{1}{5}}, \\
\mathrm{H}_{\frac{-1}{2}} \mathrm{H}_{\frac{-1}{3}} &= \mathrm{H}_{\frac{-1,-1}{2,3}} + \mathrm{H}_{\frac{-1,-1}{3,2}} + \mathrm{H}_{\frac{1}{5}}, \\
\mathrm{P}_{\frac{-1}{2}} \odot \mathrm{P}_{\frac{-1}{3}} &= \mathrm{P}_{\frac{-1,-1}{2,3}} + \mathrm{P}_{\frac{-1,-1}{3,2}} + \mathrm{P}_{\frac{1}{5}},
\end{aligned}$$

$$\begin{aligned}
y_{\frac{-1}{2}} y_{\frac{-1}{5}} \star y_{\frac{1}{4}} &= y_{\frac{-1}{2}} (y_{\frac{-1}{5}} \star y_{\frac{1}{4}}) + y_{\frac{1}{4}} (y_{\frac{-1}{2}} y_{\frac{-1}{5}} \star \varepsilon) + y_{\frac{-1}{6}} (y_{\frac{-1}{5}} \star \varepsilon) \\
&= y_{\frac{-1}{2}} (y_{\frac{-1}{5}} y_{\frac{1}{4}} + y_{\frac{1}{4}} y_{\frac{-1}{5}} + y_{\frac{-1}{9}}) + y_{\frac{1}{4}} y_{\frac{-1}{2}} y_{\frac{-1}{5}} + y_{\frac{-1}{6}} y_{\frac{-1}{5}} \\
&= y_{\frac{-1}{2}} y_{\frac{-1}{5}} y_{\frac{1}{4}} + y_{\frac{-1}{2}} y_{\frac{1}{4}} y_{\frac{-1}{5}} + y_{\frac{1}{4}} y_{\frac{-1}{2}} y_{\frac{-1}{5}} + y_{\frac{-1}{2}} y_{\frac{-1}{9}} + y_{\frac{-1}{6}} y_{\frac{-1}{5}}, \\
\mathrm{H}_{\frac{-1,-1}{2,5}} \mathrm{H}_{\frac{1}{4}} &= \mathrm{H}_{\frac{-1,-1,1}{2,5,4}} + \mathrm{H}_{\frac{-1,1,-1}{2,4,5}} + \mathrm{H}_{\frac{1,-1,-1}{4,2,5}} + \mathrm{H}_{\frac{-1,-1}{2,9}} + \mathrm{H}_{\frac{-1,-1}{6,5}}, \\
\mathrm{P}_{\frac{-1,-1}{2,5}} \odot \mathrm{P}_{\frac{1}{4}} &= \mathrm{P}_{\frac{-1,-1,1}{2,5,4}} + \mathrm{P}_{\frac{-1,1,-1}{2,4,5}} + \mathrm{P}_{\frac{1,-1,-1}{4,2,5}} + \mathrm{P}_{\frac{-1,-1}{2,9}} + \mathrm{P}_{\frac{-1,-1}{6,5}}.
\end{aligned}$$

Structure theorems

Let $\mathcal{O} = \{0\} \cup \{q^n = 1\}$ and let $\mathcal{C} = \mathbb{C}[z, \{a_i(z)\}_{i\in\mathcal{O}}]$ where $a_0(z) = z^{-1}$ and, for $\rho \in \mathcal{O} \setminus \{0\}$, $a_\rho(z) = \rho(1-\rho z)^{-1}$.

Theorem (Bigotte, Jacob, Oussous, Petitot, 2000)

$(\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}, .) \simeq (\mathcal{C}[\mathrm{Li}_{\mathcal{L}ynX}], .)$.

Proposition

$(\mathbb{C}\{\mathrm{P}_w\}_{w\in Y^*}, \odot) \simeq (\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, .) \simeq (\mathbb{C}\langle Y\rangle, \star)$.

Theorem (HNM, 2003)

$(\mathbb{C}\{\mathrm{P}_w\}_{w\in Y^*}, \odot) \simeq (\mathbb{C}[\mathrm{P}_{\mathcal{L}ynY}], \odot)$,
$(\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, .) \simeq (\mathbb{C}[\mathrm{H}_{\mathcal{L}ynY}], .)$.

Distribution relation

Proposition

Let q be a m-th root of unity. Let $\mathbf{s}$ and $\mathbf{i}$ be two compositions. Assume that $0 \leq i_1 < \ldots < i_r \leq m-1$. Then,

$$\zeta\binom{\mathbf{s}}{q^{\mathbf{i}}} = \frac{1}{m^r} \sum_{b_1,\ldots,b_r=1}^{m} \sum_{j_1,\ldots,j_r=0}^{m-1} \frac{q^{\sum_{l=1}^{r}(i_1+\ldots+i_l)b_l}}{q^{\sum_{l=1}^{r}(j_1+\ldots+j_l)b_l}} \zeta\binom{\mathbf{s}}{q^{\mathbf{j}}},$$

or equivalently

$$\sum_{b_1,\ldots,b_r=1}^{m} \sum_{\substack{j_1,\ldots,j_r=0 \\ j_1\neq i_1,\ldots,j_r\neq i_r}}^{m-1} \frac{\prod_{l=1}^{r} q^{\sum_{l=1}^{r}(i_1+\ldots+i_l)b_l}}{q^{\sum_{l=1}^{r}(j_1+\ldots+j_l)b_l}} \zeta\binom{\mathbf{s}}{q^{\mathbf{j}}} = 0.$$

Asymptotic expansions

Theorem (HNM, 2005)

$(\mathcal{C}\{\mathrm{P}_w\}_{w\in Y^*},\odot) \simeq (\mathcal{C}[\mathrm{P}_{Lyn(Y)}],\odot)$ *and for any* $g \in \mathcal{C}\{\mathrm{P}_w\}_{w\in Y^*}$, *there exists algorithmically computable coefficients* $b_i \in \mathbb{C}$, *the* $\mathbb{Q}$*-algebra generated by* $\mathcal{Z}$ *and by* γ, $\kappa_i \in \mathbb{N}$ *and* $\eta_i \in \mathbb{Z}$ *such that*

$$g(z) \sim \sum_{j=0}^{+\infty} c_j (1-z)^{\alpha_j} \mathrm{P}_{y_1^{\beta_j}}(z) \text{ for } z \to 1.$$

By consequence, there exists algorithmically computable coefficients $b_i \in \mathbb{C}, \eta_i \in \mathbb{Z}$ *and* $\kappa_i \in \mathbb{N}$ *such that*

$$[z^n]g(z) \sim \sum_{i=0}^{+\infty} b_i n^{\eta_i} \log^{\kappa_i}(n) \text{ for } n \to \infty.$$

Corollary

There exists algorithmically computable coefficients $b_i \in \mathcal{Z}', \eta_i \in \mathbb{Z}$ *and* $\kappa_i \in \mathbb{N}$ *such that*

$$\forall w \in Y^*, \mathrm{H}_w(N) \sim \sum_{i=0}^{+\infty} b_i N^{\eta_i} \log^{\kappa_i}(N) \text{ for } N \to +\infty.$$

THANK YOU FOR YOUR ATTENTION