Radford theorem for generalized deformations of Hoffmann type

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Abstract. Résumé.

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1 Introduction

With the advent of Quantum theories emerged the need for deforming not only parameters or formulas but structures and laws (see [3]). On the other hand, computer science provided us with laws the revealed later to be better understood as dual as the shuffle product recursively defined by

$$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v) \tag{1}$$

and the q-infiltration product [5] by

$$au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v) + q\delta_{a,b}a(u \uparrow v). \tag{2}$$

In fact all these laws are particular cases of diagonal deformations [7] and such deformations can be applied to stuffles.

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Dual laws of theoretical computer science : shuffle, infiltration, q-infiltration (work with Luque), $LDIAG(q_{c,q}\, s)$ and explanationn of the different natures of the parameters.

2 Background

Twisted products of graded algebras, diagonal deformations.

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3 Colour factors and products

Colours factors were introduced by [10] and the theory was developped or used in [6, 4, 9, 11].

Let $\mathcal{A} = \oplus_{\alpha\in\mathcal{D}}\mathcal{A}_\alpha$ and $\mathcal{B} = \oplus_{\beta\in\mathcal{D}}\mathcal{B}_\beta$ be two $\mathcal{D}$-graded associative algebras ($\mathcal{D}$ is a commutative semigroup whose law is denoted additively). Readers that are not familiar with graded algebras can think of $\mathcal{D} = \mathbb{N}^{(X)}$, the free commutative monoid over X and $\mathcal{A}_\alpha = K[X]_\alpha$, the space of homogeneous polynomials of multidegree α.

We suppose given a mapping $\chi : \mathcal{D}\times\mathcal{D} \longrightarrow K$ and define a law of algebra on $\mathcal{A}\otimes\mathcal{B}$ by

$$(x_1\otimes y_1)(x_2\otimes y_2) \quad = \quad \chi(\beta_1,\alpha_2)(x_1x_2\otimes y_1y_2) \tag{3}$$

for $(x_i)\in\mathcal{A}_{\alpha_i}$ and $(y_i)\in\mathcal{B}_{\beta_i}$ $(i=1,2)$.

The computations of $((x_1\otimes y_1)(x_2\otimes y_2))\,(x_3\otimes y_3)$ and $(x_1\otimes y_1)\,((x_2\otimes y_2)(x_3\otimes y_3))$ using (3) both lead to the following proposition the second part – converse – of which relies on the existence of free elements.

Proposition 1 *[11] Let $\chi : \mathcal{D}\times\mathcal{D} \longrightarrow K$. The following are equivalent*
i) For $\mathcal{A}$, $\mathcal{B}$ $\mathcal{D}$-graded associative algebras, the product defined by (3) is associative.
ii) $(\forall\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)\in\mathcal{D}$

$$\chi(\beta_1,\alpha_2)\chi(\beta_1+\beta_2,\alpha_3) = \chi(\beta_2,\alpha_3)\chi(\beta_1,\alpha_2+\alpha_3) \tag{4}$$

Definition 2 *Every mapping $\chi : \mathcal{D}\times\mathcal{D} \longrightarrow K$ which fulfills the equivalent conditions of proposition (1) will be called a colour factor.*

Remark 3 *i) If χ is bilinear, which means in this context that the following equations are satisfied (for all $\alpha,\alpha',\beta,\beta'\in\mathcal{D}$)*

$$\begin{aligned}\chi(\alpha+\alpha',\beta) &= \chi(\alpha,\beta)\chi(\alpha',\beta)\\ \chi(\alpha,\beta+\beta') &= \chi(\alpha,\beta)\chi(\alpha,\beta')\end{aligned} \tag{5}$$

then, the two members of (3) amount to

$$\chi(\beta_1,\alpha_2)\chi(\beta_1,\alpha_3)\chi(\beta_2,\alpha_3) \quad = \prod_{1\le i<j\le 3}\chi(\beta_i,\alpha_j) \tag{6}$$

and hence χ is a colour factor. But the full class of colour factors is much larger than solutions of Eq. (5). Just observe that Eq.(4) is homogeneous in the classical sense i.e. *for all $\lambda\in k$, if χ fulfills (4) then $\lambda\chi$ still does. Hence, for example, any constant function on $\mathcal{D}\times\mathcal{D}$ is a colour factor. This shows the existence of colour factors that are not bilinear.*
ii) It may seem that one could generalize (5) to the case when $\mathcal{D}$ is noncommutative but, in fact, there is no gain of generality because, as K is commutative, the bicharacter factorizes through $\mathcal{D}^{ab}$ (the quotient of $\mathcal{D}$ by the finest congruence $\equiv$ such that $\mathcal{D}/\equiv$ is abelian).

Note(s) 4 *i) The colour product of two algebras $\mathcal{A} = \oplus_{\alpha\in\mathcal{D}}\mathcal{A}_\alpha$ and $\mathcal{B} = \oplus_{\beta\in\mathcal{D}}\mathcal{B}_\beta$ comes also as a graded algebra by*

$$(\mathcal{A}\otimes\mathcal{B})_\gamma = \oplus_{\alpha+\beta=\gamma}\,\mathcal{A}_\alpha\otimes\mathcal{B}_\beta. \tag{7}$$

The usual identification

$$(\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C}\simeq\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C}) \tag{8}$$

holds for the coloured products.
ii) Moreover, if $\mathcal{A}\xrightarrow{f}\mathcal{A}'$ *(resp.* $\mathcal{B}\xrightarrow{g}\mathcal{B}'$*) are two morphisms of (graded) algebras, then* $\mathcal{A}\otimes\mathcal{B}\xrightarrow{f\otimes g}\mathcal{B}\otimes\mathcal{B}'$ *is a morphism of algebras (colour products).*

4 Special classes of laws

4.1 Dual laws

4.1.1 Algebras and coalgebras in duality

An algebra $(\mathcal{A},\mu)$ and a coalgebra (C,Δ) are called in duality iff there is a non-degenerate pairing $\langle -|-\rangle$ such that for all $x,y\in\mathcal{A},\ z\in C$

$$\langle\mu(x,y)|z\rangle=\langle x\otimes y|\Delta(z)\rangle^{\otimes 2} \tag{9}$$

In the following, we will call *dual law* a law $K\langle A\rangle\otimes K\langle A\rangle\xrightarrow{*}K\langle A\rangle$ on the free algebra which is the dual of a comultiplication, the pairing being given on the basis of words by $\langle u|v\rangle=\delta_{u,v}$.

Our first examples are essential in modern and not-so-modern research ([**?**, **?**]). Firstly, we have the dual of the Cauchy product

$$\Delta_{Cauchy}(w)=\sum_{uv=w}u\otimes v\ . \tag{10}$$

Contrary to this one (10), which is not a morphism of algebras[(i)]

$$K\langle A\rangle\longrightarrow K\langle A\rangle\otimes K\langle A\rangle\ , \tag{11}$$

one has three very well-known examples being so, namely duals of the shuffle $\sqcup\!\sqcup$, the Hadamard $\odot$ and the infiltration product $\uparrow$. As they are morphisms between the algebras (11), they are well defined by their values on the letters. Respectively

$$\Delta_{\sqcup\!\sqcup}(x)=x\otimes 1+1\otimes x\ ;\ \Delta_{\odot}(x)=x\otimes x\ ;\ \Delta_{\uparrow}(x)=x\otimes 1+1\otimes x+x\otimes x\ . \tag{12}$$

One can prove that the deformations $\Delta_q=\Delta_{\sqcup\!\sqcup}(x)+q\Delta_{\odot}(x)$ are also co-associative and that they are the unique solutions of the problem of bialgebra comultiplications on $K\langle A\rangle$ that are compatible with subalphabets [5].

In the sequel, we will make use several time of the following lemma the proof of which is left to the reader.

Lemma 5 *Let* $\mathcal{A}$ *be an algebra and* $\mathcal{C}$ *be a coalgebra in (non-degenerate) duality, then* $\mathcal{A}$ *is associative iff* $\mathcal{C}$ *is coassociative.*

(i) Unless $A=\emptyset$.

4.1.2 Duality between grouplike elements and unities

Let $(\mathcal{C}, \Delta)$ be a coalgebra with counit ϵ. We call *group-like* an element u such that

$$\epsilon(u) = 1 \; ; \; \Delta(u) = u \otimes u \; . \tag{13}$$

One then has $\mathcal{C} = ker(\epsilon) \oplus K.u$ and

$$\Delta(y) = \Delta^+(y) + y \otimes u + u \otimes y - \epsilon(y) u \otimes u \; . \tag{14}$$

where Δ^+ is a comultiplication on $\mathcal{C}$ for which $ker(\epsilon) = \mathcal{C}^+$ is a subcoalgebra (i. e. $\Delta^+(\mathcal{C}^+) \subset \mathcal{C}^+ \otimes \mathcal{C}^+$) [**?**].

Proposition 6 *Let $(\mathcal{C}, \Delta, \epsilon)$ be a coalgebra with counit, u a group-like element in $\mathcal{C}$ and $(\mathcal{C}^+, \Delta^+)$ be as in (14). On the other hand, let $\mathcal{A}$ be an algebra and $\mathcal{A}^{(1)} = \mathcal{A} \oplus K.v$ be the algebra with unit constructed from $\mathcal{A}$ by adjunction of the unity v. Then, if $\mathcal{C}^+$ and $\mathcal{A}$ are in duality by $\langle \mid \rangle$, so are $\mathcal{C}$ and $\mathcal{A}^{(1)}$ by $\langle \mid \rangle_\bullet$ defined as follows*

$$\langle x + \alpha v | y + \beta u \rangle_\bullet = \langle x | y \rangle + \beta\alpha \; . \tag{15}$$

Proof — Let

$$\begin{aligned} \langle (x_1 + \alpha_1 v) \otimes (x_2 + \alpha_2 v) | \Delta(y + \beta u) \rangle_\bullet^{\otimes 2} = \\ \langle (x_1 + \alpha_1 v) \otimes (x_2 + \alpha_2 v) | \Delta^+(y) + y \otimes u + u \otimes y + \beta u \otimes u) \rangle_\bullet^{\otimes 2} \end{aligned} \tag{16}$$

but, according to the fact that

$$\langle x_i | u \rangle = \langle x_1 \otimes v | \Delta^+(y) \rangle = \langle v \otimes x_2 | \Delta^+(y) \rangle = \langle v \otimes v | \Delta^+(y) \rangle = \langle v | y \rangle = 0$$

one has from (16)

$$\begin{aligned} &\langle (x_1 + \alpha_1 v) \otimes (x_2 + \alpha_2 v) | \Delta(y + \beta u) \rangle_\bullet^{\otimes 2} = \\ &\langle x_1 \otimes x_2 | \Delta^+(y) \rangle^{\otimes 2} + \alpha_2 \langle x_1 | y \rangle + \alpha_1 \langle x_2 | y \rangle + \alpha_1 \alpha_2 \beta = \\ &\langle x_1 x_2 + \alpha_2 x_1 + \alpha_1 x_2 + \alpha_1 \alpha_2 v | y + \beta u \rangle_\bullet = \langle (x_1 + \alpha_1 v)(x_2 + \alpha_2 v) | y + \beta u \rangle_\bullet \end{aligned} \tag{17}$$

which proves the claim. □

4.2 Deformed laws

Let S be a semigroup graded on a semigroup of degrees $\mathcal{D}$ and $\mathcal{A} = K[S]$ its algebra. A colour factor $\chi : \mathcal{D} \times \mathcal{D} \to K$ being given, we endow the algebra $\mathcal{A} \otimes \mathcal{A}$ with the coloured tensor product structure. Notice that the diagonal subspace $D_S = \oplus_{x \in S} Kx \otimes x$ is a subalgebra as

$$(x \otimes x)(y \otimes y) = \chi(|x|, |y|) xy \otimes xy \; . \tag{18}$$

Carrying (18) back to $\mathcal{A}$ by means of the isomorphism of vector spaces, $\mathcal{A} \to D_S$, on sees immediately that the deformed product on $\mathcal{A}$ given by

$$x ._\chi y = \chi(|x|, |y|) xy \tag{19}$$

is associative.
If $\mathcal{A}$ is endowed with the scalar product for which the basis $(s)_{s\in S}$ is orthonormal, the pairing is non-degenerate and the dual comultiplucation is given by

$$\Delta(z)=\sum_{xy=z}\chi(|x|,|y|)x\otimes y\ . \tag{20}$$

The construction together with lemma (5) proves that this comultiplication on $\mathcal{A}$ is coassociative.

5 Generalized shuffles and stuffles

Discussion of the commutation factor

6 Conclusion

Grading polyzeta values

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