An interface between physics and number theory

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Abstract. We extend the Hopf algebra description of a simple quantum system given previously, to a more elaborate Hopf algebra, which is rich enough to encompass that related to a description of perturbative quantum field theory (pQFT). This provides a *mathematical* route from an algebraic description of non-relativistic, non-field theoretic quantum statistical mechanics to one of relativistic quantum field theory.

Such a description necessarily involves treating the algebra of polyzeta functions, extensions of the Riemann Zeta function, since these occur naturally in pQFT. This provides a link between physics, algebra and number theory. As a by-product of this approach, we are led to indicate *inter alia* a basis for concluding that the Euler gamma constant γ may be rational.

1. Introduction

In an introductory paper delivered at this Conference[1], *"From Quantum Mechanics to Quantum Field Theory: The Hopf route"*, Allan I. Solomon [1, 2] *et al.* start their exposition with the Bell numbers $B(n)$ which count the number of *set partitions* within a given set of n elements. It is shown there that these very elementary combinatorial ideas are in a sense generic within quantum statistical mechanics. Further, it turns out that this approach also leads to a Hopf algebraic description of the physical system. Since recent work on relativistic field theory (perturbative quantum field theory or pQFT) also leads to a Hopf algebra description, the question naturally arise as to whether the algebraic structure, implicit in essentially the non-relativistic commutation relations, may be extended to the more complicated, relativistic system. Instead of extending the physical picture, we have chosen to follow a mathematical route. Doing so, it turns out that the basic structure, as exemplified in *"The Hopf route"* for the special case of Bell number diagrams (par. 3) provides the building blocks of a non-commutative parametric Hopf algebra which specializes on the one hand to a Hopf algebra of noncommutative symmetric functions, and on the other hand to a proper Hopf sub-algebra of that inherent in perturbative quantum field theory (pQFT). It well known that analysis of pQFT reveals that polyzeta functions, extensions of the Riemann zeta function of number theory, play an important role. It is not surprising therefore that the algebraic analysis embarked upon here should reveal properties of the polyzetas. And indeed it is the case that our algebra admits, as a homomorphic

[1] Talk delivered at the 28th International Colloquium on Group Theoretical Methods in Physics, Northumbria University, Newcastle, July 2010.

image, the algebra of polyzeta functions of number theory. This foray into number theory leads us to consider Euler's gamma constant γ, and conclude, on the basis of Lemmas given in the text, that γ may in fact be a rational number.

2. One-parameter groups and the exponential Hadamard product

As often in physics, one has to consider one-parameter groups with infinitesimal generator H,

$$G(t) := e^{tH} \ . \tag{1}$$

When we have two infinitesimal generators which commute $[H_1, H_2] = 0$ (for example, with hamiltonians constructed from two distinct boson modes), one can obtain the group $G_{12}(t) := e^{tH_1H_2}$ with operation

$$G_{12}(t) := G_1(t\frac{\partial}{\partial x})[G_2(x)]|_{x=0}) \ . \tag{2}$$

where $G_i(t) := e^{tH_i},\ i = 1, 2$. Up to constant terms, this operation is a *Hadamard product* (i.e. the componentwise product, which we denote below by $\mathcal{H}$) but performed on the exponential generating series [3]. Specifically with

$$F(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!},\ G(z) = \sum_{n\geq 0} b_n \frac{z^n}{n!},$$
$$\mathcal{H}(F, G) := \sum_{n\geq 0} a_n b_n \frac{z^n}{n!} \ . \tag{3}$$

This formula is known as the *product formula* [4]. Since one-parameter groups are exponentials, we can try to find a universal formula for (3). To this end we set

$$F(z) = \exp\left(\sum_{n=1}^{\infty} L_n \frac{z^n}{n!}\right),\ G(z) = \exp\left(\sum_{n=1}^{\infty} V_n \frac{z^n}{n!}\right). \tag{4}$$

and remark that F, G can simultaneously be expressed in terms of Bell[2] polynomials [5] B_n as

$$\exp\left(\sum_{n=1}^{\infty} X_n \frac{z^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{z^n}{n!} B_n(X_1, X_2, \cdots X_n) \ . \tag{5}$$

The coefficients of the polynomials B_n are positive integers which can be interpreted as the number of *unordered set partitions* which are collections of mutually disjoint non-empty subsets (blocks) of a (finite) set F whose union is F. For example, the *set partitions* of $F = \{1, 2, 3, 4\}$ in two blocks of size two are

$$\{\{1, 2\}, \{3, 4\}\}\ \{\{1, 3\}, \{2, 4\}\}\ \{\{1, 4\}, \{2, 3\}\} \ . \tag{6}$$

The type of a set partition $P = \{A_1, A_2, \cdots A_k\}$ of F is the sequence of numbers

$$Type(P) := (\#\{l\ s.t.\ |A_l| = k\})_{k\geq 1} \ .$$

For example, the type of the set partition $P = \{\{2, 4, 6, 7\}, \{1, 5\}, \{3, 8\}\}$ is $(0, 2, 0, 1, 0, 0, 0, \cdots)$. We have the formula

$$B_n(X_1, X_2, \cdots X_n) = \sum_{P\in UP_n} \mathbb{X}^{Type(P)} \tag{7}$$

[2] Named after the mathematician and science fiction author Eric Temple Bell (1883, Scotland, to 1960, US).

(where UP_n is the set of unordered partitions of $[1\cdots n]$) which can be proved (following the argument of Faà di Bruno) using the exponential formula [6]. Now from (4), (5), (7), one concludes that

$$\mathcal{H}(F,G)=\sum_{n\geq 0}\frac{z^n}{n!}\sum_{P_1,P_2\in UP_n}\mathbb{L}^{Type(P_1)}\mathbb{V}^{Type(P_2)}\ . \tag{8}$$

This formula can be rewritten as a diagrammatic expansion using the incidence matrix of the pairs of set partitions P_1, P_2 involved in (8) with respect to intersection numbers, i.e.

$$\text{Matrix of } (P_1,P_2)=\Big(\#B_1\cap B_2\Big)_{B_1\in P_1,\ B_2\in P_2}\ . \tag{9}$$

These matrices, with unordered rows and columns, can be represented by a diagram with the following rule. Draw two rows of spots (white spots on the upper row and black on the lower row) label each black spot by a block of P_1 and each white spot by a block of P_2 and join two spots (one of each kind) by the number of common elements in the corresponding blocks[3]. the result is the diagram of the following figure.

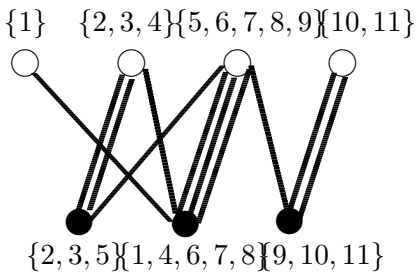

Fig 1. — *Diagram from P_1, P_2 (set partitions of $[1\cdots 11]$).*
$P_1=\{\{2,3,5\},\{1,4,6,7,8\},\{9,10,11\}\}$ *and* $P_2=\{\{1\},\{2,3,4\},\{5,6,7,8,9\},\{10,11\}\}$ *(respectively black spots for P_1 and white spots for P_2).*

The incidence matrix corresponding to the diagram (as drawn) or these partitions is $\begin{pmatrix}0&2&1&0\\1&1&3&0\\0&0&1&2\end{pmatrix}$ *but, due to the fact that the defining partitions are unordered, one can permute the spots (black and white) among themselves and so the lines and columns of this matrix can be permuted; the diagram could be represented by the matrix* $\begin{pmatrix}0&0&1&2\\0&2&1&0\\1&0&3&1\end{pmatrix}$ *as well.*

The set of such diagrams will be denoted by **diag**. This is precisely the set of bipartite graphs with multiple edges and no isolated vertex (and no order between the spots) and integer multiplicities. It is straightforward to verify that concatenation is associative with unit (the empty graph) within the set **diag**. We therefore denote the result of concatenation by $[d_1|d_2]_D$ (arbitrarily putting d_2 on the right of d_1 because the law is commutative). The monoid $(\mathbf{diag},[-|-]_D,1_{\mathbf{diag}})$ is a *free commutative monoid*. Its generators are the connected (non-empty) diagrams $(\mathbf{diag}_c)$. The algebra of **diag** is the set of formal (finitely supported) sums

$$\{\sum_{d\in\mathbf{diag}}\alpha(d)d\}_{\alpha\in\mathbb{C}^{(\mathbf{diag})}} \tag{10}$$

The freeness of **diag** (as a commutative monoid) implies that this algebra is isomorphic to the algebra of polynomials $\mathbb{C}[\mathbf{diag}_c]$. We shall denote it differently (i. e. **DIAG**) in the sequel (see

[3] Compare this diagrammatic approach with the white and black spots of the preceding presentation[1].

[3] for details) as we wish to endow it with the structure of a Hopf algebra.
Indeed, the diagrams allow us to rewrite formula (8) as

$$\mathcal{H}(F,G) = \sum_{n\geq 0} \frac{z^n}{n!} \sum_{\substack{d\in diag \\ |d|=n}} mult(d)\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)} \tag{11}$$

where $\alpha(d)$ (resp. $\beta(d)$) is the "white spot type" (resp. the "black spot type") i.e. the multi-index $(\alpha_i)_{i\in\mathbb{N}^+}$ (resp. $(\beta_i)_{i\in\mathbb{N}^+}$) such that α_i (resp. β_i) is the number of white spots (resp. black spots) of degree i (i lines connected to the spot) and $mult(d)$ is the number of pairs of unordered partitions of $[1\cdots|d|]$ (here $|d| = |\alpha(d)| = |\beta(d)|$ is the number of lines of d) with associated diagram d.

Now one may naturally ask
Q1) "Is there a (graphically) natural multiplicative structure on **diag** *such that the arrow*

$$d \mapsto mon(d) = \mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)} \tag{12}$$

is a morphism ?"

The answer is "yes" concerning the algebraic structure of **LDIAG** as it is easily seen that

$$\mathrm{mon}([d_1|d_2]_D) = \mathrm{mon}(d_1)\mathrm{mon}(d_2)\ . \tag{13}$$

But the algebra of polynomials is also endowed with the structure of a Hopf algebra (decomposition property [1, 2]). So it is natural to ask whether the coalgebra structure of the polynomials could be lifted to **DIAG** and, further, if this allows enriching the structure of **DIAG** to a Hopf algebra. Again, the answer is "yes" and the appropriate comultiplication is denoted by Δ_{BS} ("black spot" coproduct, see [3] for details). Computing $\Delta_{BS}(d)$ is immediate; one simply divides the black spots into two subsets, then obtain the tensors and sum all the results. More formally; noting the set of black spots of the diagram d by $BS(d)$ and denoting by $d[I]$ the sub-diagram whose black spots are in I and the white connected to I, one has

$$\Delta_{BS}(d) = \sum_{I+J=BS(d)} d[I]\otimes d[J] \tag{14}$$

For example, one labels by $\{a,b,c\}$ the three black spots of the diagram d of **Fig 1**. In this case, the copoduct reads

$$\begin{aligned}
&\Delta_{BS}(d) = d\otimes[\ ] + [\ ]\otimes d + \\
&d[\{a\}]\otimes d[\{b,c\}] + d[\{b\}]\otimes d[\{a,c\}] + \\
&d[\{c\}]\otimes d[\{a,b\}] + +d[\{a,b\}]\otimes d[\{c\}] + \\
&d[\{a,c\}]\otimes d[\{b\}] + d[\{b,c\}]\otimes d[\{a\}]\ .
\end{aligned}$$

2.1. Labeling the nodes and the noncommutative analogue **LDIAG**

If one needs to label the black spots in order to compute the comultiplication, it is obvious that the precise form of labelling is ultimately irrelevant. It is more important to label the nodes endowing their sets with a linear order (one for the black spots between themselves and one for the white spots). This leads to **LDIAG**, the (non-deformed) noncommutative analogue of **DIAG** [7].

The solution (of this non-deformed problem [7]) is simple and consists in labeling the black (resp. white) spots from left to right and from "1" to p (resp. q); thus one obtains the *labelled*

diagrams. Again concatenation is associative and endows the set of the labelled diagrams (here denoted by **ldiag**) with the structure of a monoid (the unit is the empty diagram characterized by $p = q = 0$).
Again, and with the same formula (14) (but not applied to the same objects), the algebra of this monoid, **LDIAG** and the counit which consists in taking the coefficient of the empty diagram, is a Hopf algebra [4].

2.2. The deformed case.
The preceding coding is particularly well adapted to the deformation we want to construct here. The philosophy of the deformed product is expressed by the descriptive formula

$$[d_1|d_2]_{L(q_c,q_s)} = \sum_{CS(d_1,d_2)} q_c^{w1(CS)} q_s^{w2(CS)} CS([d_1|d_2]_{L(q_c,q_s)}) \tag{15}$$

where

- $q_c, q_s \in \mathbb{C}$ or q_c, q_s formal. These and other cases may be unified by considering the set of coefficients as belonging to a ring K
- the exponent of $w1(CS)$ is the number of crossings of "what crosses" times its weight
- the exponent of $w2(CS)$ is the product of the weights of "what is overlapped"
- $CS(d_1, d_2)$ are the diagrams obtained from $[d_1|d_2]_L$ by the process of crossing and superposing the black spots of d_2 on to those of d_1, the order and distinguishability of the black spots of d_1 (i.e. d_2) being preserved.

What is striking is that this law is associative. This fact is by no means trivial and three proofs have been given (one of them can be found in [3]). It can be shown that this new algebra (associative with unit denoted $\mathbf{LDIAG}(q_c, q_s)$) can be endowed with two comultiplications, Δ_0 and Δ_1, such that the result is a Hopf algebra which we denote by $\mathbf{LDIAG}(q_c, q_s, q_t)$.
It can be shown that this deformed Hopf algebra has two interesting specializations

$$\mathbf{LDIAG}(0,0,0) \simeq \mathbf{LDIAG} \quad \text{and} \quad \mathbf{LDIAG}(1,1,1) \simeq \mathbf{MQSym} \ ;$$

$\mathbf{LDIAG}(0,0,0)$ (no crossing and no superposition) is the undeformed case whereas $\mathbf{LDIAG}(1,1,1)$ is **MQSym**, the Hopf algebra of Noncommutative Matrix Quasisymmetric functions [8].

2.3. The arrow $\mathbf{LDIAG}(1,1) \to$ *polyzetas.*
We now explain how a refinement of the "black spot type" $\beta(d)$ of formula (11) evaluates the statistics $(\beta(d))$ by "number of nodes with outgoing degree k". We take advantage of the labelling and count these degrees node by node. Thus, for a labelled diagram d, $I(d,k)$ is simply the outgoing degree of the node with label k. It turns out, and is straightforward to show, that

$$I(d) := [I(d,1), I(d,2) \cdots I(d,p)]$$

(d is a diagram with p black spots) is a composition of the integer $|d|$ (number of edges of d).
Now, we define two structures which will be ubiquitous in the next section: *the shuffle and stuffle algebras.* Let X be any alphabet (i.e. set of variables). The shuffle algebra is defined

[4] At this stage, it is only a graded bialgebra, but, as it is graded and of finite dimension, a general theorem states that an antipode exists; (see [3] formula (19) for an explicit formula of this antimorphism).

on $\mathbb{C}\langle X\rangle := \mathbb{C}[X^*]$ by its values on the monomials (here the strings or words) by the following recursion

$$\begin{array}{rcl} 1_{X^*} ⧢ w & = & w ⧢ 1_{X^*} = w \text{ and} \\ xu ⧢ yv & = & x(u ⧢ yv) + y(xu ⧢ v) \ . \end{array} \tag{16}$$

Likewise, let $Y = y_{i i\geq 1}$ be a countable alphabet indexed by integers ≥ 1. The stuffle algebra [9] is defined on $\mathbb{C}\langle Y\rangle := \mathbb{C}[Y^*]$ by a recursion on the words as

$$\begin{array}{rcl} 1_{Y^*} \uplus w & = & w \uplus 1_{Y^*} = w \text{ and} \\ y_i u \uplus y_j v & = & y_i(u \uplus y_j v) + y_j(y_i u \uplus v) + y_{i+j}(u \uplus v) \ . \end{array} \tag{17}$$

The shuffle (resp. stuffle) endow $\mathbb{C}\langle X\rangle := \mathbb{C}[X^*]$ (resp. $\mathbb{C}\langle Y\rangle := \mathbb{C}[Y^*]$) with the structure of an AAU[5]. We define mappings

$$\begin{array}{rcl} \phi_{X2LDIAG} & : & \mathbf{LDIAG}(1,0) \to \mathbb{C}\langle X\rangle \\ \phi_{Y2LDIAG} & : & \mathbf{LDIAG}(1,1) \to \mathbb{C}\langle Y\rangle \end{array} \tag{18}$$

(here $X = \{x_i\}_{i\geq 1}$) by

$$\phi_{X2LDIAG}(d) = x_{I(d,1)} x_{I(d,2)} \cdots x_{I(d,p)}$$
$$\phi_{Y2LDIAG}(d) = y_{I(d,1)} y_{I(d,2)} \cdots y_{I(d,p)})$$

and have the following result.

Proposition 1 *The mappings*

$$\begin{array}{rcl} \phi_{X2LDIAG} & : & (\mathbf{LDIAG}(1,0), [\ .|\ .]_{LD(1,0)} \to (\mathbb{C}\langle X\rangle,\ ⧢\ ) \\ \phi_{Y2LDIAG} & : & (\mathbf{LDIAG}(1,1), [\ .|\ .]_{LD(1,1)} \to (\mathbb{C}\langle Y\rangle,\ ⧢\ ) \end{array}$$

are epimorphisms of AAU's.

This provides the opening to the next section.

3. Knizhnik-Zamolodchikov differential system

In 1986, in order to study the linear representation of the braid group B_n [10] coming from the monodromy of the Knizhnik-Zamolodchikov differential equations over $\mathbb{C}^n_* = \{(z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ for } i \neq j\}$:

$$dF(z) = \Omega_n F(z), \tag{19}$$

where

$$\Omega_n = \frac{1}{2i\pi} \sum_{1\leq i<j\leq n} t_{i,j} \frac{d(z_i - z_j)}{z_i - z_j}, \tag{20}$$

Drinfel'd introduced a class of formal power series Φ on noncommutative variables over the finite alphabet $X = \{x_0, x_1\}$. Such a power series Φ is called an *associator*.

Since the system (19) is completely integrable then

$$d\Omega_n - \Omega_n \wedge \Omega_n = 0. \tag{21}$$

This is equivalent to the following braid relations

$$[t_{i,j}, t_{i,k} + t_{j,k}] = 0, \quad \text{for} \quad i \neq j \neq k, \tag{22}$$
$$[t_{i,j}, t_{k,l}] = 0, \quad \text{for} \quad i \neq j \neq k \neq l. \tag{23}$$

[5] Associative Algebra with Unit: Here they are, moreover, Hopf algebras [9] but we will not use this here.

- For $n=2$, $\mathcal{T}_2=\{t_{1,2}\}$, one has

$$\Omega_2(z)=\frac{t_{1,2}}{2\mathrm{i}\pi}\frac{d(z_1-z_2)}{z_1-z_2} \tag{24}$$

and a solution of (19) is given by

$$F(z_1,z_2)=(z_1-z_2)^{\frac{t_{1,2}}{2\mathrm{i}\pi}}. \tag{25}$$

- For $n=3$, $\mathcal{T}_3=\{t_{1,2},t_{1,3},t_{2,3}\}$, there are two relations :

$$[t_{1,3},t_{1,2}+t_{2,3}]=0 \quad \text{and} \quad [t_{2,3},t_{1,2}+t_{1,3}]=0. \tag{26}$$

One has

$$\Omega_3(z)=\frac{1}{2\mathrm{i}\pi}\left[t_{1,2}\frac{d(z_1-z_2)}{z_1-z_2}+t_{1,3}\frac{d(z_1-z_3)}{z_1-z_3}+t_{2,3}\frac{d(z_2-z_3)}{z_2-z_3}\right] \tag{27}$$

and a solution of (19) is given by

$$F(z_1,z_2,z_3)=G\left(\frac{z_1-z_2}{z_1-z_3}\right)(z_1-z_3)^{\frac{t_{1,2}+t_{1,3}+t_{2,3}}{2\mathrm{i}\pi}} \tag{28}$$

where G satisfies the following Fuchs differential equation with three regular singularities at $0,1$ and ∞ :

$$dG(z)=[x_0\omega_0(z)+x_1\omega_1(z)]G(z), \tag{29}$$

with

$$\omega_0(z):=\frac{dz}{z} \quad \text{and} \quad \omega_1(z):=\frac{dz}{1-z}, \tag{30}$$

$$x_0:=\frac{t_{1,2}}{2\mathrm{i}\pi} \quad \text{and} \quad x_1:=-\frac{t_{2,3}}{2\mathrm{i}\pi}. \tag{31}$$

In the sequel, we set $X=\{x_0,x_1\}$; X^* denotes the set of words defined over X. The empty word is denoted by ε.

Proposition 2 ([11]) *If $G(z)$ and $H(z)$ are exponential solutions of (29) then there exists a Lie series $C\in\mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle$ such that $G(z)=H(z)\exp(C)$.*

Proof – Since $H(z)H(z)^{-1}=1$ then by differentiating, we have $d[H(z)]H(z)^{-1}=-H(z)d[H(z)^{-1}]$. Therefore if $H(z)$ is solution of the Drinfel'd equation then

$$\begin{aligned}
d[H(z)^{-1}] &= -H(z)^{-1}[dH(z)]H(z)^{-1}\\
&= -H(z)^{-1}[x_0\omega_0(z)+x_1\omega_1(z)],\\
d[H(z)^{-1}G(z)] &= H(z)^{-1}[dG(z)]+[dH(z)^{-1}]G(z)\\
&= H(z)^{-1}[x_0\omega_0(z)+x_1\omega_1(z)]G(z)\\
&- H(z)^{-1}[x_0\omega_0(z)+x_1\omega_1(z)]G(z).
\end{aligned}$$

By simplification, we deduce that $H(z)^{-1}G(z)$ is a constant formal power series. Since the inverse and the product of group-like elements is group-like then we get the expected result. □

4. Iterated integral and Chen generating series

The iterated integral associated with $w = x_{i_1} \cdots x_{i_k} \in X^*$, over ω_0 and ω_1 and along the path $z_0 \rightsquigarrow z$, is defined by the following multiple integral

$$\int_{z_0}^{z} \cdots \int_{z_0}^{z_{k-1}} \omega_{i_1}(t_1) \ldots \omega_{i_k}(t_k), \tag{32}$$

where $t_1 \cdots t_{r-1}$ is a subdivision of the path $z_0 \rightsquigarrow z$. In an abbreviated notation, we denote this integral by $\alpha_{z_0}^{z}(w)$ and $\alpha_{z_0}^{z}(\varepsilon) = 1$.

Example 1

$$\begin{aligned}
\alpha_0^z(x_0x_1) &= \int_0^z \int_0^s \omega_0(s)\omega_1(t) \\
&= \int_0^z \int_0^s \frac{ds}{s}\frac{dt}{1-t} \\
&= \int_0^z \frac{ds}{s} \int_0^s dt \sum_{k\geq 0} t^k \\
&= \sum_{k\geq 1} \int_0^z ds \frac{s^{k-1}}{k} \\
&= \sum_{k\geq 1} \frac{z^k}{k^2}.
\end{aligned}$$

The last sum is nothing other than the Taylor expansion of the dilogarithm $\mathrm{Li}_2(z)$.

Example 2 *In the same way the classical polylogarithm of order $n \geq 1$ is the iterated integral associated with $x_0^{n-1}x_1$, over ω_0 and ω_1 and along the path $0 \rightsquigarrow z$:*

$$\mathrm{Li}_n(z) = \sum_{k\geq 1} \frac{z^k}{k^n} = \alpha_0^z(x_0^{n-1}x_1).$$

Generalizing to multi-indices $(n_1, \ldots, n_r)$, one has :

$$\mathrm{Li}_{n_1,\ldots,n_r}(z) = \sum_{k_1>\ldots>k_r>0} \frac{z^{k_1}}{k_1^{n_1} \ldots k_r^{n_r}} = \alpha_0^z(x_0^{n_1-1}x_1 \ldots x_0^{n_r-1}x_1).$$

This provides an analytic prolongation of $\mathrm{Li}_{n_1,\ldots,n_r}$ *over the Riemann surface of* $\mathbb{C} \setminus \{0, 1\}$.

The Chen generating series along the path $z_0 \rightsquigarrow z$ associated with ω_0, ω_1 is the following power series

$$S_{z_0 \rightsquigarrow z} = \sum_{w \in X^*} \alpha_{z_0}^{z}(w)\, w \tag{33}$$

and it is a solution of the differential equation (29) with the initial condition

$$S_{z_0 \rightsquigarrow z}(z_0) = 1. \tag{34}$$

Any Chen generating series $S_{z_0 \rightsquigarrow z}$ is group-like [12] and depends only on the homotopy class of $z_0 \rightsquigarrow z$ [13]. The product of two Chen generating series $S_{z_1 \rightsquigarrow z_2}$ and $S_{z_0 \rightsquigarrow z_1}$ is the Chen generating series

$$S_{z_0 \rightsquigarrow z_2} = S_{z_1 \rightsquigarrow z_2} S_{z_0 \rightsquigarrow z_1}. \tag{35}$$

5. Polylogarithm, harmonic sum and polyzetas

To the polylogarithm $\mathrm{Li}_{n_1,\ldots,n_r}(z)$ we associate the following ordinary generating series

$$\mathrm{P}_{n_1,\ldots,n_r}(z) = \frac{\mathrm{Li}_{n_1,\ldots,n_r}(z)}{1-z} = \sum_{N\geq 0} \mathrm{H}_{n_1,\ldots,n_r}(N)\, z^N, \tag{36}$$

where

$$\mathrm{H}_{n_1,\ldots,n_r}(N) = \sum_{N\geq k_1>\ldots>k_r>0} \frac{1}{k_1^{n_1}\ldots k_r^{n_r}} \tag{37}$$

For $n_1 > 1$, the limit of $\mathrm{Li}_{n_1,\ldots,n_r}(z)$ and of $\mathrm{H}_{n_1,\ldots,n_r}(N)$, for $z \to 1$ and $N \to \infty$ exist and, by Abel's Theorem, are equal :

$$\lim_{z\to 1} \mathrm{Li}_{n_1,\ldots,n_r}(z) = \lim_{N\to\infty} \mathrm{H}_{n_1,\ldots,n_r}(N) = \zeta(n_1,\ldots,n_r), \tag{38}$$

where $\zeta(n_1,\ldots,n_r)$ is the convergent polyzeta

$$\zeta(n_1,\ldots,n_r) = \sum_{k_1>\ldots>k_r>0} \frac{1}{k_1^{n_1}\ldots k_r^{n_r}}. \tag{39}$$

Definition 1 *Let $\mathcal{Z}$ be the $\mathbb{Q}$-algebra generated by convergent polyzetas and let $\mathcal{Z}'$ be the $\mathbb{Q}[\gamma]$-algebra*[6] *generated by $\mathcal{Z}$.*

To any multi-index $\mathbf{n} = (n_1,\ldots,n_r)$ corresponds the word $v = y_{n_1}\ldots y_{n_r}$ over the infinite alphabet $Y = \{y_k\}_{k\geq 1}$. The word v itself corresponds to the word ending by the letter x_1, $u = x_0^{n_1-1}x_1\ldots x_0^{n_r-1}x_1 \in X^*x_1$. Then it is usual to index the polylogarithms, harmonic sums and polyzetas by words (over X or Y) :

$$\mathrm{Li}_{\mathbf{n}} = \mathrm{Li}_v(z) = \mathrm{Li}_u, \tag{40}$$
$$\mathrm{P}_{\mathbf{n}} = \mathrm{P}_v(z) = \mathrm{P}_u, \tag{41}$$
$$\mathrm{H}_{\mathbf{n}} = \mathrm{H}_v(N) = \mathrm{H}_u, \tag{42}$$
$$\zeta(\mathbf{n}) = \zeta(v) = \zeta(u). \tag{43}$$

Let us extend, over X^*, the definition of $\{\mathrm{Li}_w\}_{w\in X^*x_1}$ and $\{\mathrm{P}_w\}_{w\in X^*x_1}$ by putting

$$\forall k\geq 0,\quad \mathrm{Li}_{x_0^k}(z) = \frac{\log^k(z)}{k},\quad \mathrm{P}_{x_0^k}(z) = \frac{\mathrm{Li}_{x_0^k}(z)}{1-z}. \tag{44}$$

We get the following structures :

Theorem 1 ([15]) $(\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}, .) \cong (\mathbb{C}\langle X\rangle, ⧢)$.

Theorem 2 ([16]) $(\mathbb{C}\{\mathrm{P}_w\}_{w\in Y^*}, \odot) \cong (\mathbb{C}\langle Y\rangle, \text{⧢}\!\!\!\!+)$.

Extended to $\mathcal{C} = \mathbb{C}[z, z^{-1}, (1-z)^{-1}]$, we also get as a consequence

[6] Here, γ stands for the Euler constant

$$\gamma = .57721566490153286060651209008240243104215933593992359880576723488486 77\ldots$$

- The polylogarithms $\{\mathrm{Li}_w\}_{w\in X^*}$ (resp. $\{\mathrm{P}_w\}_{w\in Y^*}$) are $\mathcal{C}$-linearly independent Then the harmonic sums $\{\mathrm{H}_w\}_{w\in Y^*}$ are linearly independent.
- The polylogarithms $\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}$ (resp. $\{\mathrm{P}_l\}_{l\in\mathcal{L}ynY}$) are $\mathcal{C}$-algebraically independent Then the harmonic sums $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$ are algebraically independent.
- The polyzetas $\{\zeta(l)\}_{l\in\mathcal{L}ynX\setminus\{x_0,x_1\}}$(resp. $_{\mathcal{L}ynY\setminus\{y_1\}}$), are generators of $\mathcal{Z}$.

Definition 2 *We put*

$$\mathrm{L}(z) := \sum_{w\in X^*} \mathrm{Li}_w(z)\, w \quad \textit{and} \quad \mathrm{H}(N) := \sum_{w\in Y^*} \mathrm{H}_w(N)\, w.$$

The noncommuting generating series of the polylogarithms is a solution of the differential equation (29) with the following boundary condition

$$\mathrm{L}(z) \underset{z\to 0}{\widetilde{\quad}} \exp(x_0 \log z). \tag{45}$$

It follows that L is group-like and then H is also group-like, *i.e.*

$$\Delta_{ш}(\mathrm{L}) = \mathrm{L}\otimes\mathrm{L} \quad \text{and} \quad \Delta_{⧢}(\mathrm{H}) = \mathrm{H}\otimes\mathrm{H}, \tag{46}$$

and we have

Theorem 3

$$\mathrm{L}(z) = e^{x_1 \log\frac{1}{1-z}}\mathrm{L}_{\mathrm{reg}}(z)e^{x_0\log z} \quad \textit{and} \quad \mathrm{H}(N) = e^{\mathrm{H}_1(N)\, y_1}\mathrm{H}_{\mathrm{reg}}(N),$$

where, denoting by $\{\hat{l}\}_{\mathcal{L}ynX}$ *(resp.* $\{\hat{l}\}_{\mathcal{L}ynY}$*) the dual basis of* $\mathcal{L}ynX$ *(resp.* $\mathcal{L}ynY$*),*

$$\mathrm{L}_{\mathrm{reg}}(z) := \prod_{l\in\mathcal{L}ynX, l\neq x_0,x_1}^{\searrow} e^{\mathrm{Li}_l(z)\,\hat{l}} \quad \textit{and} \quad \mathrm{H}_{\mathrm{reg}}(N) := \prod_{l\in\mathcal{L}ynY, l\neq y_1}^{\nearrow} e^{\mathrm{H}_l(N)\,\hat{l}}.$$

Corollary 1 *Let* $z_0 \rightsquigarrow z$ *be a differentiable path on* $\mathbb{C}-\{0,1\}$ *such that* L *admits an analytic continuation along this path. We have* $S_{z_0\rightsquigarrow z} = \mathrm{L}(z)\mathrm{L}(z_0)^{-1}$.

Proof – By Theorem 3, $\mathrm{L}(z)$ is group-like. Hence $\mathrm{L}(z_0)$ is also group-like as is $\mathrm{L}(z_0)^{-1}$. Since the power series $\mathrm{L}(z)\mathrm{L}(z_0)^{-1}$ and $S_{z_0\rightsquigarrow z}$ satisfy (29) taking the same value at z_0 then we get the expected result. □

Definition 3 *We put*

$$Z_{ш} := \mathrm{L}_{\mathrm{reg}}(1) \quad \textit{and} \quad Z_{⧢} := \mathrm{H}_{\mathrm{reg}}(\infty).$$

These two noncommuting generating series $Z_{ш}$ and $Z_{⧢}$ induce the two following regularization morphisms respectively :

Theorem 4 ([17]) *Let* $\zeta_{⧢} : (\mathbb{C}\langle\langle Y\rangle\rangle, ⧢) \to (\mathbb{R}, .)$ *be the morphism satisfying*

- *for* $u, v \in Y^*, \zeta_{⧢}(u ⧢ v) = \zeta_{⧢}(u)\zeta_{⧢}(v)$,
- *for all convergent words* $w \in Y^* - y_1Y^*, \zeta_{⧢}(w) = \zeta(w)$,
- $\zeta_{⧢}(y_1) = 0$.

Then

$$\sum_{w\in X^*} \zeta_{⧢}(w)\, w = Z_{⧢}.$$

Corollary 2 ([17]) *For any $w \in X^*, \zeta_{\uplus}(w)$ belongs to the algebra $\mathcal{Z}$.*

Theorem 5 ([17]) *Let $\zeta_{\sqcup\!\sqcup} : (\mathbb{C}\langle\langle X\rangle\rangle, \sqcup\!\sqcup) \to (\mathbb{R}, .)$ be the morphism verifying*

- *for $u, v \in X^*, \zeta_{\sqcup\!\sqcup}(u \sqcup\!\sqcup v) = \zeta_{\sqcup\!\sqcup}(u)\zeta_{\sqcup\!\sqcup}(v)$,*
- *for all convergent word $w \in x_0X^*x_1, \zeta_{\sqcup\!\sqcup}(w) = \zeta(w)$,*
- *$\zeta_{\sqcup\!\sqcup}(x_0) = \zeta_{\sqcup\!\sqcup}(x_1) = 0$.*

Then

$$\sum_{w\in X^*} \zeta_{\sqcup\!\sqcup}(w)\, w = Z_{\sqcup\!\sqcup}.$$

Corollary 3 ([17]) *For any $w \in Y^*, \zeta_{\sqcup\!\sqcup}(w)$ belongs to the algebra $\mathcal{Z}$.*

6. Group of associators theorem

Drinfel'd proved that Eqn.(29) admits two particular solutions on the domain $\mathbb{C}\backslash]-\infty, 0] \cup [1, +\infty[$,

$$G_0(z) \underset{z\rightsquigarrow 0}{\widetilde{\quad}} \exp[x_0 \log(z)] \text{ and } G_1(z) \underset{z\rightsquigarrow 1}{\widetilde{\quad}} \exp[-x_1 \log(1-z)]. \tag{47}$$

He also proved that there exists Φ_{KZ} such that

$$G_0(z) = G_1(z)\Phi_{KZ}. \tag{48}$$

Lê and Murakami expressed the coefficients of the Drinfel'd associator Φ_{KZ} in terms of convergent polyzetas [18].

Let ρ be the monoid morphism verifying

$$\rho(x_0) = -x_1 \quad \text{and} \quad \rho(x_1) = -x_0. \tag{49}$$

We also have [19]

$$\mathrm{L}(z) = \rho[\mathrm{L}(1-z)]Z_{\sqcup\!\sqcup} = e^{x_0 \log z}\rho[\mathrm{L}_{\mathrm{reg}}(1-z)]e^{-x_1 \log(1-z)}Z_{\sqcup\!\sqcup}. \tag{50}$$

Thus,

$$\mathrm{L}(z) \underset{z\to 1}{\widetilde{\quad}} \exp(-x_1 \log(1-z))\, Z_{\sqcup\!\sqcup}. \tag{51}$$

It follows from Eqn.(45) and Eqn. (51), with reference [17], that

$$\Phi_{KZ} = Z_{\sqcup\!\sqcup}\ , \tag{52}$$

and it is group-like, *i.e.*

$$\Delta_{\sqcup\!\sqcup}(\Phi_{KZ}) = \Phi_{KZ} \otimes \Phi_{KZ}, \tag{53}$$

and it can be graded in the adjoint basis [11] as follows

$$\Phi_{KZ} = \sum_{\substack{k\geq 0\\ l_1,\cdots,l_k\geq 0}} \zeta_{\sqcup\!\sqcup}(x_1x_0^{l_1} \circ \cdots \circ x_1x_0^{l_k}) \prod_{i=0}^{k} \mathrm{ad}_{x_0}^{l_i} x_1, \tag{54}$$

where, for any $l \in \mathbb{N}$ and $P \in \mathbb{C}\langle X\rangle$, $\circ$ is defined by

$$x_1 x_0^l \circ P = x_1(x_0^l \sqcup\!\sqcup P), \tag{55}$$

and $\mathrm{ad}_{x_0}^l x_1$ is the iterated Lie bracket

$$\mathrm{ad}_{x_0}^0 x_1 = x_1 \quad \text{and} \quad \mathrm{ad}_{x_0}^l x_1 = [x_0, \mathrm{ad}_{x_0}^{l-1} x_1]. \tag{56}$$

Using the following expansion [20]

$$\mathrm{ad}_{x_0}^n x_1 = \sum_{i=0}^{n} \binom{i}{n} x_0^{n-i} x_1 x_0^i, \tag{57}$$

one then gets, via the regularization process of Theorem 5, the expression for the Drinfel'd associator Φ_{KZ} given by Lê and Murakami [18].

Finally, the asymptotic behaviour of L on (51) leads to

Proposition 3 ([15])

$$S_{\varepsilon \rightsquigarrow 1-\varepsilon} \underset{\varepsilon \to 0^+}{\widetilde{\quad}} e^{-x_1 \log \varepsilon}\, Z_{\sqcup\!\sqcup}\, e^{-x_0 \log \varepsilon}.$$

In other words, $Z_{\sqcup\!\sqcup}$ is the regularized Chen generating series $S_{\varepsilon \rightsquigarrow 1-\varepsilon}$ of differential forms ω_0 and ω_1 : $Z_{\sqcup\!\sqcup}$ is the noncommutative generating series of the finite parts of the Chen generating series $e^{x_1 \log \varepsilon}\, S_{\varepsilon \rightsquigarrow 1-\varepsilon}\, e^{x_0 \log \varepsilon}$, the concatenation of $e^{x_0 \log \varepsilon}$ and then $S_{\varepsilon \rightsquigarrow 1-\varepsilon}$ and finally, $S_{\varepsilon \rightsquigarrow 1-\varepsilon}$.

Let $\{b_{n,k}(t_1, \ldots, t_{n-k+1})\}_{\substack{n \geq k \geq 1 \\ n \in \mathbb{N}_+}}$ be the Bell polynomials given by the following exponential generating series[7]

$$\sum_{n,k=0}^{\infty} b_{n,k}(t_1, \ldots, t_{n-k+1}) \frac{v^n u^k}{n!} = \exp\left(u \sum_{l=0}^{\infty} t_l \frac{v^l}{l!} \right). \tag{58}$$

We specify the variables $\{t_l\}_{l \geq 0}$

$$t_1 = \gamma, \quad \text{for } l > 1, \quad t_l = (-1)^{l-1}(l-1)!\zeta(l), \tag{59}$$

then let $B(x_1)$ be the following power series

$$B(x_1) = 1 + \sum_{n \geq 1} \sum_{k=1}^{n} b_{n,k}(\gamma, -\zeta(2), 2\zeta(3), \ldots) \frac{(-x_1)^n}{n!} \tag{60}$$

and

$$B'(x_1) = e^{-\gamma\, x_1} B(x_1). \tag{61}$$

We get

[7] Compare this expression with the less general Eq.(5). Here $b(n,k)$ is a refinement by the number of blocks (*i.e.* k) of Eq.(5). Thus one has

$$B_n(X_1, X_2, \cdots X_n) = \sum_{k=0}^{n} b_{n,k}(X_1, \ldots, X_{n-k+1}) \ .$$

Proposition 4 *Let Ψ_{KZ} be an element of the quasi-shuffle algebra, $(\mathbb{C}\langle\langle Y\rangle\rangle, \uplus)$, such that*

$$\Pi_X \Psi_{KZ} = B(x_1)\Phi_{KZ},$$

where $\Pi_X \Psi_{KZ}$ is the projection of Ψ_{KZ} over X. Then Ψ_{KZ} is group-like, i.e.

$$\Delta_{\uplus}(\Psi_{KZ}) = \Psi_{KZ} \otimes \Psi_{KZ}, \tag{62}$$

and satisfies

- $\langle \Psi_{KZ} \mid \epsilon \rangle = 1$,
- $\langle \Psi_{KZ} \mid y_1 \rangle = \gamma$,
- *for any* $r_1 > 1$, $\langle \Psi_{KZ} \mid y_{r_1} \dots y_{r_k} \rangle = \zeta(r_1, \dots, r_k)$,
- *for any* $u, v \in Y^*$, $\langle \Psi_{KZ} | u \uplus v \rangle = \langle \Psi_{KZ} | u \rangle \langle \Psi_{KZ} | v \rangle$.

Therefore we obtain,

Proposition 5 *The noncommutative generating series Ψ_{KZ} can be factorized by Lyndon words as follows*

$$\Psi_{KZ} = e^{\gamma\, y_1} Z_{\uplus}.$$

More generally, we have

Theorem 6 *Let A be a commutative $\mathbb{Q}$-algebra. For any $\Phi \in A\langle\langle X\rangle\rangle$ and $\Psi \in A\langle\langle Y\rangle\rangle$ such that $\Pi_X \Psi = B(x_1)\Phi$, there exists a unique $C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ with coefficients in A such that*

$$\Phi = \Phi_{KZ} e^C \quad \textit{and} \quad \Psi = B(y_1)\Pi_Y(\Phi_{KZ} e^C),$$

where $\Pi_Y(\Phi_{KZ} e^C)$ is the projection of $\Phi_{KZ} e^C$ over Y.

Proof – Let $C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$. Then, by Proposition 2, $\mathrm{L}' = \mathrm{L}e^C$ is solution of (29). Let H' be the noncommutative generating series of the Taylor coefficients, belonging to the harmonic algebra, of $\{(1-z)^{-1}\langle \mathrm{L}' \mid w\rangle\}_{w \in Y^*}$. Then $\mathrm{H}'(N)$ is also group-like.

By the asymptotic expansion on (51), we get

$$\mathrm{L}'(z) \underset{\varepsilon \to 1}{\widetilde{\quad}} e^{-x_1 \log(1-z)} Z_{\sqcup\!\sqcup} e^C.$$

We then put $\Phi := Z_{\sqcup\!\sqcup} e^C$ and deduce that

$$\frac{\mathrm{L}'(z)}{1-z} \underset{z \to 1}{\widetilde{\quad}} \mathrm{Mono}(z)\Phi \quad \text{and} \quad \mathrm{H}'(N) \underset{N \to \infty}{\widetilde{\quad}} \mathrm{Const}(N)\Pi_Y \Phi,$$

where [21]

$$\mathrm{Mono}(z) = e^{-(x_1+1)\log(1-z)} = \sum_{k \geq 0} \mathrm{P}_{y_1^k}(z)\, y_1^k$$

$$\mathrm{Const} = \sum_{k \geq 0} \mathrm{H}_{y_1^k}\, y_1^k = \exp\left[-\sum_{k \geq 1} \mathrm{H}_{y_k} \frac{(-y_1)^k}{k} \right].$$

Let κ_w be the constant part of $\mathrm{H}'_w(N)$. Then,

$$\sum_{w \in Y^*} \kappa_w\, w = B(y_1)\Pi_Y \Phi.$$

We now put $\Psi := B(y_1)\Pi_Y(Z_{\sqcup\!\sqcup}e^C)$ (and also $\Psi' := B'(y_1)\Pi_Y(Z_{\sqcup\!\sqcup}e^C)$. $\square$

Thus, any associator Φ and its image Ψ, can be determined from the Drinfel'd associator Φ_{KZ} and its image Ψ_{KZ} respectively, by the action of the group of constant Lie exponential series over X. This group is nothing other than the differential Galois group of the differential equation (29) and contains in particular [11] the monodromy group of (29) given by [15]

$$\mathcal{M}_0 = e^{2i\pi x_0} \quad \text{and} \quad \mathcal{M}_1 = \Phi_{KZ}^{-1} e^{-2i\pi x_1}\Phi_{KZ}. \tag{63}$$

By Proposition 3, we already saw that $Z_{\sqcup\!\sqcup}$ is the concatenation of the Chen generating series [13] $e^{x_0\log\varepsilon}$ and then $S_{\varepsilon\rightsquigarrow 1-\varepsilon}$ and finally, $e^{x_1\log\varepsilon}$:

$$Z_{\sqcup\!\sqcup} \underset{\varepsilon\to 0^+}{\widetilde{\quad}} e^{x_1\log\varepsilon} S_{\varepsilon\rightsquigarrow 1-\varepsilon}\, e^{x_0\log\varepsilon}. \tag{64}$$

From (63), the action of the monodromy group gives

$$e^{x_1\, 2k_1 \mathrm{i}\pi} Z_{\sqcup\!\sqcup} e^{x_0\, 2k_0\mathrm{i}\pi} \underset{\varepsilon\to 0^+}{\widetilde{\quad}} e^{x_1(\log\varepsilon + 2k_1\mathrm{i}\pi)} S_{\varepsilon\rightsquigarrow 1-\varepsilon} e^{x_0(\log\varepsilon+2k_0\mathrm{i}\pi)} \tag{65}$$

as being the concatenation of the Chen generating series [13] $e^{x_0(\log\varepsilon+2k_0\mathrm{i}\pi)}$ (along a circular path turning k_0 times around 0), then the Chen generating series $S_{\varepsilon\rightsquigarrow 1-\varepsilon}$ and finally, the Chen generating series $e^{x_1(\log\varepsilon+2k_1\mathrm{i}\pi)}$ (along a circular path turning k_1 times around 1). More generally, the action of the Galois differential group of polylogarithms shows that, for any Lie series C, the associator $\Phi = Z_{\sqcup\!\sqcup}e^C$ is the concatenation of some Chen generating series e^C and $e^{x_0\log\varepsilon}$; then the Chen generating series $S_{\varepsilon\rightsquigarrow 1-\varepsilon}$ and finally, $e^{x_1\log\varepsilon}$:

$$\Phi \underset{\varepsilon\to 0^+}{\widetilde{\quad}} e^{x_1\log\varepsilon} S_{\varepsilon\rightsquigarrow 1-\varepsilon} e^{x_0\log\varepsilon}\, e^C. \tag{66}$$

As in the proof of Theorem 6, by construction the associator Φ is then the noncommutative generating series of the finite parts of the coefficients of the Chen generating series $S_{z_0\rightsquigarrow 1-z_0}e^C$, for $z_0 = \varepsilon \to 0^+$.

Lemma 1 *Let $\Phi \in dm(A) = \{\Phi_{KZ}e^C \mid C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ and $\langle e^C \mid \epsilon\rangle = 1, \langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0\}$. Let $\Pi_Y\Phi$ be the projection of Φ over Y. One has*

$$\Psi = B(y_1)\Pi_Y\Phi \quad \Longleftrightarrow \quad \Psi' = B'(y_1)\Pi_Y\Phi.$$

In particular,

$$\Psi_{KZ} = B(y_1)\Pi_Y\Phi_{KZ} \quad \Longleftrightarrow \quad \Psi'_{KZ} = B'(y_1)\Pi_Y\Phi_{KZ}.$$

7. The route to properties of Euler's γ constant

We have already mentioned that as a remarkable by-product of our extension of the algebraic considerations related to simple quantum mechanics, we are led to algebras involving polyzeta functions. This excursion is not surprising, as it is already well known that evaluations in quantum field theory involve these functions [14]. In this context, an important related number is the Euler constant γ. We are thus led to to consider properties of γ. In fact, in this note we arrive at a proposition concerning the rationality of γ. The basis of this result depends on the following lemma[8]

Lemma 2 *Let $\Phi \in dm(A)$ and $\Psi' = B'(y_1)\Pi_Y\Phi$. The local coordinates (of the second kind) of Φ (resp. Ψ'), in the Lyndon-PBW basis, are polynomials of generators $\{\zeta(l)\}_{l\in\mathcal{L}ynX} - \{x_0, x_1\}$ (resp. $\{\zeta(l)\}_{l\in\mathcal{L}ynY} - \{y_1\}$) of $\mathcal{Z}$. While C describes $\mathcal{L}ie_A\langle\langle X\rangle\rangle$, these local coordinates describe $A[\{\zeta(l)\}_{l\in\mathcal{L}ynX} - \{x_0, x_1\}]$ (resp. $A[\{\zeta(l)\}_{l\in\mathcal{L}ynY} - \{y_1\}]$).*

[8] We warn the reader that Lemma 2 is not universally accepted.

Lemma 3 *For any $\Phi \in dm(A)$, by identifying the local coordinates (of the second kind) on two members of the identities $\Psi = B(y_1)\Pi_Y\Phi$, or equivalently on $\Psi' = B'(y_1)\Pi_Y\Phi$, we get polynomial relations with coefficients in A of the convergent polyzetas.*

Therefore,

Theorem 7 *While Φ describes $dm(A)$, the identities $\Psi = B(y_1)\Pi_Y\Phi$ describe the ideal of polynomial relations of coefficients in A of the convergent polyzetas.*

Moreover, if the Euler constant γ does not belong to A then these relations are algebraically independent of γ.

Corollary 4 *If $\gamma \notin A$ then $\gamma \notin \bar{A}$.*

With $A = \mathbb{Q}$, it follows immediately that

Corollary 5 *γ is not an algebraic irrational number.*

Corollary 6 *γ is a rational number.*

Proof – Since γ satisfies $t^2 - \gamma^2 = 0$ then γ is algebraic over $A = \mathbb{Q}(\gamma^2)$. If γ is transcendental then $\gamma \notin A = \mathbb{Q}(\gamma^2)$. Using Corollary 4, with $A = \mathbb{Q}(\gamma^2)$, γ is not algebraic over $A = \mathbb{Q}(\gamma^2)$. This contradicts the previous assertion. Thus, by Corollary 5, it is established that γ is rational over $\mathbb{Q}$. $\square$

8. Structure of polyzetas

Let $\Phi \in dm(A)$ and let $\Psi = B'(y_1)\Pi_Y\Phi$. We introduce two algebra morphisms

$$\begin{array}{rcl} \phi : (A\langle X\rangle, \sqcup\!\sqcup) & \longrightarrow & A, \\ u & \longmapsto & \langle \Phi \mid u\rangle, \end{array} \tag{67}$$

and

$$\begin{array}{rcl} \psi : (A\langle Y\rangle, \sqcup\!\sqcup) & \longrightarrow & A, \\ v & \longmapsto & \langle \Psi \mid v\rangle, \end{array} \tag{68}$$

verifying respectively

$$\phi(\epsilon) = 1 \quad \text{and} \quad \phi(x_0) = \phi(x_1) = 0, \tag{69}$$

$$\psi(\epsilon) = 1 \quad \text{and} \quad \psi(y_1) = 0. \tag{70}$$

Hence,

$$\Phi = \sum_{u \in X^*} \phi(u)\, u = \prod^{\searrow}_{\substack{l \in \mathcal{L}ynX \\ l \neq x_0, x_1}} e^{\phi(l)\, \hat{l}}, \tag{71}$$

$$\Psi = \sum_{v \in Y^*} \psi(u)\, u = \prod^{\nearrow}_{\substack{l \in \mathcal{L}ynY \\ l \neq y_1}} e^{\psi(l)\, \hat{l}}. \tag{72}$$

Using these factorizations of the monoids by Lyndon words, we get

Theorem 8 *For any $\Phi \in dm(A)$, let $\Psi = B'(y_1)\Pi_Y\Phi$. Then*

$$\prod_{\substack{l\in\mathcal{L}ynX\\ l\neq x_0,x_1}}^{\searrow} e^{\phi(l)\,\hat{l}} = e^{-\sum_{k\geq 2}\zeta(k)\frac{(-x_1)^k}{k}}\,\Pi_X \prod_{\substack{l\in\mathcal{L}ynY\\ l\neq y_1}}^{\nearrow} e^{\psi(l)\,\hat{l}}.$$

In particular, if $\Phi = Z_{ш}$ and $\Psi = Z_{⧢}$ then

$$\prod_{\substack{l\in\mathcal{L}ynX\\ l\neq x_0,x_1}}^{\searrow} e^{\zeta(l)\,\hat{l}} = e^{-\sum_{k\geq 2}\zeta(k)\frac{(-x_1)^k}{k}}\,\Pi_X \prod_{\substack{l\in\mathcal{L}ynY\\ l\neq y_1}}^{\nearrow} e^{\zeta(l)\,\hat{l}}.$$

Since

$$\forall l \in \mathcal{L}ynY \iff \Pi_X l \in \mathcal{L}ynX \setminus \{x_0\} \tag{73}$$

then identifying the local coordinates, we get polynomial relations among the generators which are algebraically independent of γ.

Corollary 7 *For $\ell \in \mathcal{L}ynY - \{y_1\}$ (resp. $\mathcal{L}ynX - \{x_0, x_1\}$), let $P_\ell \in \mathcal{L}ie_{\mathbb{Q}}\langle X\rangle$ (resp. $\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle$) be the decomposition of the polynomial $\Pi_X\hat{\ell} \in \mathbb{Q}\langle X\rangle$ (resp. $\Pi_Y\hat{\ell} \in \mathbb{Q}\langle Y\rangle$) in the Lyndon-PBW basis $\{\hat{l}\}_{l\in\mathcal{L}ynX}$ (resp. $\{\hat{l}\}_{l\in\mathcal{L}ynY}$) and let $\check{P}_\ell \in \mathbb{Q}[\mathcal{L}ynX - \{x_0, x_1\}]$ (resp. $\mathbb{Q}[\mathcal{L}ynY - \{y_1\}]$) be its dual. Then one obtains*

$$\Pi_X\ell - \check{P}_\ell \in \ker\phi \quad (\textit{resp.} \quad \Pi_Y\ell - \check{P}_\ell \in \ker\psi).$$

In particular, for $\phi = \zeta$ (resp. $\psi = \zeta$) then one also obtains

$$\Pi_X\ell - \check{P}_\ell \in \ker\zeta \quad (\textit{resp.} \quad \Pi_Y\ell - \check{P}_\ell \in \ker\zeta).$$

Moreover, for any $\ell \in \mathcal{L}ynY - \{y_1\}$ (resp. $\mathcal{L}ynX - \{x_0, x_1\}$), the polynomial $\Pi_Y\ell - \check{P}_\ell \in \mathbb{Q}\langle Y\rangle$ (resp. $\mathbb{Q}\langle X\rangle$) is homogenous of degree equal to $\mid \ell \mid > 1$.

Theorem 9 (Structure of polyzetas) *The $\mathbb{Q}$-algebra generated by convergent polyzetas is isomorphic to the* graded *algebra* $(\mathbb{Q} \oplus (Y - y_1)\mathbb{Q}\langle Y\rangle/\ker\zeta,\ ⧢)$.

Proof – Since $\ker\zeta$ is an ideal generated by the homogenous polynomials then the quotient $\mathbb{Q} \oplus (Y - y_1)\mathbb{Q}\langle Y\rangle/\ker\zeta$ is graded. □

Corollary 8 *The $\mathbb{Q}$-algebra of polyzetas is freely generated by* irreducible *polyzetas.*

Proof – For any $\lambda \in \mathcal{L}ynY$, if $\lambda = \check{P}_\lambda$ then one obtains the conclusion otherwise $\Pi_X\lambda - \check{P}_\lambda \in \ker\zeta$. Since $\check{P}_\lambda \in \mathbb{Q}[\mathcal{L}ynX]$ then $\check{P}_\lambda$ is polynomial in Lyndon words of degree less than or equal to $|\lambda|$. Since each Lyndon word does appear in this decomposition of $\check{P}_\lambda$, after applying Π_Y, we may repeat the same process until we obtain irreducible polyzetas. □

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