Independance of hyperlogarithms over function fields via algebraic combinatorics.

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Abstract

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1 Introduction

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the method of Weshung consists in a recurrence based on the total degree. However this method cannot be used with variable coefficients. Another proof was given in [Joris and al. 1998] based on monodromy. Here, we give a general theorem on differential algebra and show that, at the cost of using variable domains (which is the realm of germ spaces) and replace the recurrence on total degree by a recursion on the words (with the graded lexicographic ordering), one can encompass the previous results mentionned above and get much larger rings of coefficients

2 Non commutative differential equations (abstract setting).

The ground field k is supposed commutative and of characteristic zero. We suppose given a commutative differential k-algebra $(\mathcal{A}, d)$ that is a k-algebra (associative and commutative with unit) $\mathcal{A}$ endowed with an element $d \in \mathfrak{Der}(\mathcal{A})$. We will suppose that the ring of constants $ker(d)$ is exactly k.

An alphabet X being given, one can at once extend the derivation d to a derivation of the algebra $\mathcal{A}\langle\langle X\rangle\rangle$ by

$$\mathbf{d}(S) = \sum_{w\in X^*} d(\langle S|w\rangle)w \ . \tag{1}$$

Theorem 2.1 *Let $(\mathcal{A}, d)$ be a k-commutative associative differential algebra with unit ($ch(k) = 0$) and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $d(\mathcal{C}) \subset \mathcal{C}$). We suppose that $S \in \mathcal{A}\langle\langle X\rangle\rangle$ is a solution of the differential equation*

$$\mathbf{d}(S) = MS \ ; \ \langle S|1\rangle = 1 \tag{2}$$

where the multiplier $M = \sum_{x\in X} u_x x \in \mathcal{C}\langle\langle X\rangle\rangle$ is an homogeneus series (a polynomial in case X is finite) of degree 1.
The following condition are equivalent :
i) The family $(\langle S|w\rangle)_{w\in X^*}$ of coefficients of S is free over $\mathcal{C}$.
ii) The family of coefficients $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ is free over $\mathcal{C}$.
iii) The family $(u_x)_{x\in X}$ is such that, for $f \in \mathcal{C}$ and $\alpha_x \in k$

$$d(f) = \sum_{x\in X} \alpha_x u_x \Longrightarrow (\forall x \in X)(\alpha_x = 0) \ . \tag{3}$$

iv) The family $(u_x)_{x\in X}$ is free over k and

$$d(\mathcal{C}) \cap span_k\Big((u_x)_{x\in X}\Big) = \{0\} \ . \tag{4}$$

Proof — (i)$\Longrightarrow$(ii) Obvious.
(ii)$\Longrightarrow$(iii)

Suppose that the family $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ (coefficients taken at letters and the emty word) of coefficients of S is free over $\mathcal{C}$ and let us consider a relation as eq. (32)

$$d(f)=\sum_{x\in X}\alpha_x u_x \ . \tag{5}$$

We form the polynomial $P=-f1_{X^*}+\sum_{x\in X}\alpha_x x$. One has $\mathbf{d}(P)=-d(f)1_{X^*}$ and

$$d(\langle S|P\rangle)=\langle \mathbf{d}(S)|P\rangle+\langle S|\mathbf{d}(P)\rangle=\langle MS|P\rangle-d(f)\langle S|1_{X^*}\rangle=(\sum_{x\in X}\alpha_x u_x)-d(f)=0 \tag{6}$$

and, then $\langle S|P\rangle$ must be a constant, say $\lambda\in k$. For $Q=P-\lambda.1_{X^*}$, we have

$$supp(Q)\subset X\cup\{1_{X^*}\} \text{ and } \langle S|Q\rangle=\langle S|P\rangle-\lambda\langle S|1_{X^*}\rangle=\langle S|P\rangle-\lambda=0\ .$$

This implies that $Q=0$ and, as $Q=-(f+\lambda)1_{X^*}+\sum_{x\in X}\alpha_x x$, one has, in particular, all the $\alpha_x=0$.
(iii)$\Longleftrightarrow$(iv)
Obvious, (iv) being a geometric reformulation of (iii).
(iii)$\Longleftrightarrow$(i)
Let $\mathcal{K}$ be the kernel of $P\mapsto\langle S|P\rangle$ (a linear form $\mathcal{C}\langle X\rangle\to\mathcal{C}$) i.e.

$$\mathcal{K}=\{P\in\mathcal{C}\langle X\rangle|\langle S|P\rangle=0\}\ . \tag{7}$$

If $\mathcal{K}=\{0\}$, we are done. Otherwise, let us adop the following strategy.
First, we order X by some well-ordering $<$ ([3] III.2.1) and X^* by the graded lexicographic ordering $\prec$ defined by

$$u\prec v\Longleftrightarrow |u|<|v| \text{ or } (u=pxs_1\ ,\ v=pys_2 \text{ and } x<y) \tag{8}$$

it is easy to check that $\prec$ is also a well-ordering relation. For each nonzero polynomial P, we note $lead(P)$ its leading monomial i.e. the greatest element of its support $supp(P)$ (for $\prec$).
Now, as $\mathcal{R}=\mathcal{K}-\{0\}$ is not empty, let w_0 be the minimmal element of $lead(\mathcal{R})$ and choose a $P\in\mathcal{R}$ such that $lead(P)=w_0$. We write

$$P=fw_0+\sum_{u\prec w_0}\langle P|u\rangle u\ ;\ f\in\mathcal{C}-\{0\}\ . \tag{9}$$

the polynomial $Q=\frac{1}{f}P$ is also in $\mathcal{R}$ with the same leading monomial, but the leading coefficient is now 1 and Q reads

$$Q=w_0+\sum_{u\prec w_0}\langle Q|u\rangle u\ . \tag{10}$$

Differentiating $\langle S|Q\rangle=0$, one gets

$$\begin{aligned}0=\langle \mathbf{d}(S)|Q\rangle+\langle S|\mathbf{d}(Q)\rangle=\langle MS|Q\rangle+\langle S|\mathbf{d}(Q)\rangle=\\ \langle S|M^\dagger Q\rangle+\langle S|\mathbf{d}(Q)\rangle=\langle S|M^\dagger Q+\mathbf{d}(Q)\rangle\end{aligned} \tag{11}$$

with

$$M^\dagger Q+\mathbf{d}(Q)=\sum_{x\in X}u_x(x^\dagger Q)+\sum_{u\prec w_0}d(\langle Q|u\rangle)u\in\mathcal{C}\langle X\rangle\ . \tag{12}$$

It is impossible that $M^\dagger Q + \mathbf{d}(Q) \in \mathcal{R}$ because it would be of leading monomial strictly less than w_0, hence $M^\dagger Q + \mathbf{d}(Q) = 0$. This is equivalent to the recursion

$$d(\langle Q|u\rangle) = -\sum_{x\in X} u_x \langle Q|xu\rangle \ ; \ \text{for } x \in X \ , \ v \in X^* \tag{13}$$

From this last relation, we derive that $\langle Q|w\rangle \in k$ for every w of length $deg(Q)$ and, because $\langle S|1\rangle = 1$, one must have $deg(Q) > 0$. Then, write $w_0 = x_0 v$ and compute the coefficient at v

$$d(\langle Q|v\rangle) = -\sum_{x\in X} u_x \langle Q|xv\rangle = \sum_{x\in X} \alpha_x u_x \tag{14}$$

with coefficients $\alpha_x = -\langle Q|xv\rangle \in k$ as $|xv| = \deg(Q)$ for all $x \in X$. Condition **PI** implies that all coefficients $\langle Q|xu\rangle$ are zero, in particular, as $\langle Q|x_0 u\rangle = 1$, we get a contradiction. This proves that $\mathcal{K} = \{0\}$.
□

3 Series with variable coefficients.

3.1 Motivations.

In this section, we implement an abstract setting which is intended to apply on function spaces. As a motivation, let us illustrate this by an example.
Let V be a connected and simply connected analytic variety of dimension one (for example, the doubly cut plane $\mathbb{C} - (]-\infty, 0[\cup]1, +\infty[)$, or the universal covering of $\mathbb{C} - \{0, 1\}$), $\mathcal{H} = C^\omega(V; \mathbb{C})$ be the space of all analytic fonctions on V. This space is a differential algebra with the derivative $\frac{d}{dz}$. One extends at once this derivative (then denoted by $\mathbf{d}$) to $\mathcal{H}\langle\langle X\rangle\rangle$ by

$$\mathbf{d}(S) = \sum_{w\in X^*} \frac{d}{dz}(\langle S|w\rangle) w \tag{15}$$

it is easy to check (proof in the general case below) that $\mathbf{d}$ is a derivation of the algebra $\mathcal{H}\langle\langle X\rangle\rangle$. Differential equations of the type

$$\mathbf{d}(S) = MS \tag{16}$$

where $M = \sum_{x\in X} u_x(z) x$ were widely considered in the domains of (`à faire` automatique, Drinfel'd, Weshung, etc...) and provide, through integrators build by iterated integrals, spaces of special functions. An immediate application of theorem (`à faire` ??) below provides the result that, for any solution of (16) (with $\langle S|1\rangle = 1$), the family of functions $(\langle S|w\rangle)_{w\in X^*}$ is free over the field of rational functions on V .

3.2 General setting

Let $\mathcal{M}$, be a locally finite monoid [**?**] and $\mathcal{H}\langle\langle \mathcal{M}\rangle\rangle$, be the large algebra [4] of $\mathcal{M}$ with coefficients in $\mathcal{H}$). Let $\mathcal{M}$ be a locally finite monoid [**?**] and V be a set (where the variable z stands) and $\mathcal{H} \subset C^V$, an algebra of fonctions. Every series $S \in \mathcal{H}\langle\langle \mathcal{M}\rangle\rangle$ can be written

$$S := \sum_{m\in\mathcal{M}} \langle S|m\rangle m \tag{17}$$

(as the family $(\langle S|m\rangle m)_{m\in\mathcal{M}}$ is summable). Thus, one can consider the series of $\mathcal{H}\langle\langle\mathcal{M}\rangle\rangle$ as functions on V (with values in $\mathbb{C}\langle\langle\mathcal{M}\rangle\rangle$) and specialize them by

$$S(z_0) := \sum_{m\in\mathcal{M}} \Big(\langle S|m\rangle\Big)\Big|_{z=z_0} m \ . \tag{18}$$

Moreover, if d is a derivation in $\mathcal{H}$, its extension to $\mathcal{H}\langle\langle\mathcal{M}\rangle\rangle$ "coefficient by coefficient" given as

$$\mathbf{d}(S) := \sum_{m\in\mathcal{M}} d(\langle S|m\rangle)m \tag{19}$$

is a derivation of $\mathcal{H}\langle\langle\mathcal{M}\rangle\rangle$. Let us show, on an example, how the proof of theorem `à faire` ?? below works on an example.
The data are

1. $X = \{x_0, x_1\}$
2. V is a connected and simply connected subset of $\mathbb{C}$
3. $\mathcal{H} = C^\omega(V,\mathbb{C})$ which does not contain $\{0,1\}$, eendowed with the derivation $\frac{d}{dz}$
4. $M = \frac{x_0}{z} + \frac{x_1}{1-z}$
5. **Statement** : If S is any solution of

$$\mathbf{d}(S) = MS \ ; \ \langle S|1\rangle = 1 \tag{20}$$

then the functions $(\langle S|w\rangle)_{w\in X^*}$ are linearly independant over the field of rational functions i.e. if

$$f_i = \frac{p_i}{q_i},\ i = 1\cdots N \ ; \ p_i, q_i \in \mathbb{C}[z] \tag{21}$$

and $w_i \in X^*$ are such that

$$\sum_{i=1}^{N} f_i(z)\langle S|w_i\rangle = 0 \tag{22}$$

on some open (non void) set (which does not contain the opes of the f_i) then

$$(\forall i \in [1\cdots N])(f_i \equiv 0) \ . \tag{23}$$

6. **An example for the recursion** Order X by $x_0 < x_1$ and use $\prec_{glex}$, the graded lexicographic ordering on X^*. For each non-trivial relation (Rel) (if there are such)

$$\sum_{i=1}^{N} f_i(z)\langle S|w_i\rangle = 0 \tag{24}$$

we consider the leading monomial $lead(\text{Rel}) = \sup\{w_i | f_i \not\equiv 0\}$. If there were nontrivial relatons, we could take one with the least possible leading monomial and the relation itself.
Assumption 1 Suppose that the set of its monomials be $\{x_0, x_1, x_1x_0, x_0^2x_1\}$ with $lead(\text{Rel}) = x_0^2x_1$. One has

$$f_{x_0}(z)\langle S|x_0\rangle + f_{x_1}(z)\langle S|x_1\rangle + f_{x_1x_0}(z)\langle S|x_1x_0\rangle + f_{x_0^2x_1}(z)\langle S|x_0^2x_1\rangle = 0 \ . \tag{25}$$

which is defined on U intersection of the domains $U_0 = dom(f_{w_i})$.
At the cost of restricting the relation to $U_1 = U_0 \setminus \mathcal{O}_{f_{x_0^2x_1}}$ one has also

$$g_{x_0}(z)\langle S|x_0\rangle + g_{x_1}(z)\langle S|x_1\rangle + g_{x_1x_0}(z)\langle S|x_1x_0\rangle + \langle S|x_0^2x_1\rangle = 0 \tag{26}$$

with $g_i = \frac{f_i}{f_{x_0^2x_1}}$. Differentiating (26), we get

$$\begin{aligned} &g'_{x_0}(z)\langle S|x_0\rangle + g_{x_0}(z)\langle S'|x_0\rangle + g'_{x_1}(z)\langle S|x_1\rangle + g_{x_1}(z)\langle S'|x_1\rangle + \\ &\quad g'_{x_1x_0}(z)\langle S|x_1x_0\rangle + +g_{x_1x_0}(z)\langle S'|x_1x_0\rangle + \langle S'|x_0^2x_1\rangle = 0\ . \end{aligned} \tag{27}$$

As $S' = MS$, we have

$$\begin{aligned} \langle S'|x_0u\rangle = \langle MS|x_0u\rangle &= \frac{1}{z}\langle S|u\rangle \text{ and} \\ \langle S'|x_1u\rangle = \langle MS|x_1u\rangle &= \frac{1}{1-z}\langle S|u\rangle\ . \end{aligned} \tag{28}$$

With this in hand (29) becomes

$$\begin{aligned} &g'_{x_0}(z)\langle S|x_0\rangle + g_{x_0}(z)\frac{1}{z}\langle S|1\rangle + g'_{x_1}(z)\langle S|x_1\rangle + g_{x_1}(z)\frac{1}{1-z}\langle S|1\rangle + \\ &\quad g'_{x_1x_0}(z)\langle S|x_1x_0\rangle + +g_{x_1x_0}(z)\frac{1}{1-z}\langle S|x_0\rangle + \frac{1}{z}\langle S|x_0x_1\rangle = 0\ . \end{aligned} \tag{29}$$

which is of rank strictly less than (25) and then should be trivial. A contradiction as $\frac{1}{z}$ is zero in no non-void open subset ; "**Assumption 1**" must be false and we are done.
□

3.3 Non commutative differential equations (abstract setting).

The ground field k is supposed commutative and of characteristic zero. We suppose given a commutative differential k-algebra $(\mathcal{A}, d)$ that is a k-algebra (associative and commutative with unit) $\mathcal{A}$ endowed with an element $d \in \mathfrak{Der}(\mathcal{A})$. We will suppose that the ring of constants $ker(d)$ is exactly k.

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$$\mathbf{d}(S) = \sum_{w\in X^*} d(\langle S|w\rangle)w\ . \tag{30}$$

Theorem 3.1 *Let $(\mathcal{A}, d)$ be a k-commutative associative differential algebra with unit ($ch(k) = 0$) and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $d(\mathcal{C}) \subset \mathcal{C}$). We suppose that $S \in \mathcal{A}\langle\langle X\rangle\rangle$ is a solution of the differential equation*

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iii) The family $(u_x)_{x\in X}$ *is such that, for* $f\in\mathcal{C}$ *and* $\alpha_x\in k$

$$d(f)=\sum_{x\in X}\alpha_x u_x \Longrightarrow (\forall x\in X)(\alpha_x=0)\ . \tag{32}$$

iv) The family $(u_x)_{x\in X}$ *is free over* k *and*

$$d(\mathcal{C})\cap span_k\Big((u_x)_{x\in X}\Big)=\{0\}\ . \tag{33}$$

Proof — (i)$\Longrightarrow$(ii) Obvious.
(ii)$\Longrightarrow$(iii)
Suppose that the family $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ (coefficients taken at letters and the emty word) of coefficients of S is free over $\mathcal{C}$ and let us consider a relation as eq. (32)

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We form the polynomial $P=-f1_{X^*}+\sum_{x\in X}\alpha_x x$. One has $\mathbf{d}(P)=-d(f)1_{X^*}$ and

$$d(\langle S|P\rangle)=\langle \mathbf{d}(S)|P\rangle+\langle S|\mathbf{d}(P)\rangle=\langle MS|P\rangle-d(f)\langle S|1_{X^*}\rangle=(\sum_{x\in X}\alpha_x u_x)-d(f)=0 \tag{35}$$

and, then $\langle S|P\rangle$ must be a constant, say $\lambda\in k$. For $Q=P-\lambda.1_{X^*}$, we have

$$supp(Q)\subset X\cup\{1_{X^*}\}\text{ and }\langle S|Q\rangle=\langle S|P\rangle-\lambda\langle S|1_{X^*}\rangle=\langle S|P\rangle-\lambda=0\ .$$

This implies that $Q=0$ and, as $Q=-(f+\lambda)1_{X^*}+\sum_{x\in X}\alpha_x x$, one has, in particular, all the $\alpha_x=0$.
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Obvious, (iv) being a geometric reformulation of (iii).
(iii)$\Longleftrightarrow$(i)
Let $\mathcal{K}$ be the kernel of $P\mapsto\langle S|P\rangle$ (a linear form $\mathcal{C}\langle X\rangle\to\mathcal{C}$) i.e.

$$\mathcal{K}=\{P\in\mathcal{C}\langle X\rangle|\langle S|P\rangle=0\}\ . \tag{36}$$

If $\mathcal{K}=\{0\}$, we are done. Otherwise, let us adop the following strategy.
First, we order X by some well-ordering $<$ ([3] III.2.1) and X^* by the graded lexicographic ordering $\prec$ defined by

$$u\prec v\Longleftrightarrow |u|<|v|\text{ or }(u=pxs_1\ ,\ v=pys_2\text{ and }x<y) \tag{37}$$

it is easy to check that $\prec$ is also a well-ordering relation. For each nonzero polynomial P, we note $lead(P)$ its leading monomial i.e. the greatest element of its support $supp(P)$ (for $\prec$).
Now, as $\mathcal{R}=\mathcal{K}-\{0\}$ is not empty, let w_0 be the minimmal element of $lead(\mathcal{R})$ and choose a $P\in\mathcal{R}$ such that $lead(P)=w_0$. We write

$$P=fw_0+\sum_{u\prec w_0}\langle P|u\rangle u\ ;\ f\in\mathcal{C}-\{0\}\ . \tag{38}$$

the polynomial $Q = \frac{1}{f}P$ is also in $\mathcal{R}$ with the same leading monomial, but the leading coefficient is now 1 and Q reads

$$Q = w_0 + \sum_{u \prec w_0} \langle Q|u\rangle u \ . \tag{39}$$

Differentiating $\langle S|Q\rangle = 0$, one gets

$$\begin{aligned} 0 = \langle \mathbf{d}(S)|Q\rangle + \langle S|\mathbf{d}(Q)\rangle = \langle MS|Q\rangle + \langle S|\mathbf{d}(Q)\rangle = \\ \langle S|M^\dagger Q\rangle + \langle S|\mathbf{d}(Q)\rangle = \langle S|M^\dagger Q + \mathbf{d}(Q)\rangle \end{aligned} \tag{40}$$

with

$$M^\dagger Q + \mathbf{d}(Q) = \sum_{x\in X} u_x(x^\dagger Q) + \sum_{u \prec w_0} d(\langle Q|u\rangle)u \in \mathcal{C}\langle X\rangle \ . \tag{41}$$

It is impossible that $M^\dagger Q + \mathbf{d}(Q) \in \mathcal{R}$ because it would be of leading monomial strictly less than w_0, hence $M^\dagger Q + \mathbf{d}(Q) = 0$. This is equivalent to the recursion

$$d(\langle Q|u\rangle) = -\sum_{x\in X} u_x \langle Q|xu\rangle \ ; \ \text{for } x \in X \ , \ v \in X^* \tag{42}$$

From this last relation, we derive that $\langle Q|w\rangle \in k$ for every w of length $deg(Q)$ and, because $\langle S|1\rangle = 1$, one must have $deg(Q) > 0$. Then, write $w_0 = x_0 v$ and compute the coefficient at v

$$d(\langle Q|v\rangle) = -\sum_{x\in X} u_x \langle Q|xv\rangle = \sum_{x\in X} \alpha_x u_x \tag{43}$$

with coefficients $\alpha_x = -\langle Q|xv\rangle \in k$ as $|xv| = \deg(Q)$ for all $x \in X$. Condition **PI** implies that all coefficients $\langle Q|xu\rangle$ are zero, in particular, as $\langle Q|x_0 u\rangle = 1$, we get a contradiction. This proves that $\mathcal{K} = \{0\}$.
□

4 Applications of the main theorem.

4.1 Independance of the polylogarithms.

Let V be a connected and simply connected analytic variety of dimension 1 (for example, the doubly cut plane $\mathbb{C} - (]-\infty, 0] \cup [1, +\infty[)$, or the universal covering of $\mathbb{C} - \{0, 1\}$), $\mathcal{A} = C^\omega(V; \mathbb{C})$ be the space of analytic fonctions on V endowed with the derivative $d = \frac{d}{dz}$. Let $X = \{x_0, x_1\}$ and X^* be the free monoid on X. It is locally finite [**?**] and we note $\mathcal{A}\langle\langle X\rangle\rangle$ its large algebra [4] (with coefficients in $\mathcal{A}$).

5 Réserve à ordonner

We will use three types of differential equations.
a) **Left-sided equation**

$$\frac{d}{dz}S(z) = M(z)S(z) \tag{44}$$

b) **Right-sided equation**

$$\frac{d}{dz}S(z) = S(z)M(z) \tag{45}$$

c) **Two-sided equation**

$$\frac{d}{dz}S(z) = M_1(z)S(z) + S(z)M_2(z) \tag{46}$$

with $M, M_i \in \mathcal{H}_{\geq 1}\langle\langle \mathcal{M} \rangle\rangle$. One first give the resolution of equations of type (46) as their properties specialize, with $M_2 = 0$ (resp. $M_1 = 0$) to the type (44) (resp. (45)).

Theorem 5.1 *With the preceding assumptions.*
i) Equation (46) has solutions all of the form

$$S = (H_{z_0}^z)^* S_0 \tag{47}$$

where $H_{z_0}^z$ is the operator

$$G \mapsto \int_{z_0}^{z} \Big(M_1(s)G(s) + G(s)M_2(s) \Big) ds \tag{48}$$

and $S_0 = S(z_0)$ is a constant series.
ii) Two solutions which coincide at a point do coincide everywhere.
iii) Let Δ be a closable comultiplication with constant coefficients and suppose that M_1, M_2 in (46) are primitive elements for Δ i.e.

$$\Delta(M_i) = M_i \otimes 1 + 1 \otimes M_i \ ; \ i = 1, 2 \ .$$

Then if S, a solution of (46), is group-like at a point of V, it is group-like everywhere in V.
iv) The constant term of S is contant on V (and is that of S_0), in particular, if a solution is invertible at a point, it is so everywhere (these solutions will be called regular).
v) Let S be a regular solution of an equation of type (44) with primitive multiplier. Let $\mathcal{F}$ be a filter on V (neighbourhoods of 0, of 1, of infinity etc.) and one supposes that S is asymtotically equivalent to G (w.r.t. $\mathcal{F}$ i.e. $lim_{\mathcal{F}}(G^{-1}S) = 1$). Then S is group-like.

5.0.1 Proof of theorem (5.1)

i) The integrator $H = H_{z_0}^z \in \mathrm{End}(\mathcal{H}\langle\langle \mathcal{M} \rangle\rangle)$ satisfies

$$H^n(\mathcal{H}\langle\langle \mathcal{M} \rangle\rangle) \subset \mathcal{H}_{\geq n}\langle\langle \mathcal{M} \rangle\rangle$$

for all $n \in \mathbb{N}$. This implies that $(H^n)_{n\geq 0}$ is summable for the $\mathfrak{M}$-adic topology of $\mathrm{End}^{filtr}(\mathcal{H}_{\geq 1}\langle\langle \mathcal{M} \rangle\rangle)$ given by the ideal $\mathfrak{M}$ of operators ϕ such that $\phi\Big(\mathcal{H}\langle\langle \mathcal{M} \rangle\rangle\Big) \subset \mathcal{H}_{\geq 1}\langle\langle \mathcal{M} \rangle\rangle$. Let $H^* = \sum_{n\geq 0} H^n$ be its sum. For S_0 a constant series, one has

$$\frac{d}{dz}(H^*[S_0]) = \frac{d}{dz}(I + HH^*)[S_0] =$$
$$\frac{d}{dz}(S_0) + \frac{d}{dz}(HH^*)[S_0] = M_1 H^*[S_0] + H^*[S_0]M_2 \tag{49}$$

Conversely, if S is a solution of (44), then $S_0 = (I - H)[S]$ is a constant series as

$$\frac{d}{dz}(S_0) = \frac{d}{dz}(I - H)[S] = \frac{d}{dz}S - \frac{d}{dz}(H)[S] = M_1S + SM_2 - (M_1S + SM_2) = 0 \ .$$

Moreover $S = (I - H)^{-1}[S_0] = H^*[S_0]$ and then $S_0 = S(\alpha)$.
ii) If two solutions S_1, S_2 coincide at $z_1 \in V$, one can construct the operator H with formula (48) and $\alpha = z_1$. One has then $S_1 = H^*(S_1(z_1)) = H^*(S_2(z_1)) = S_2$.
iii) A comultiplication with constant coefficients permute with the derivation operator hence, if S is a solution of (46), one has

$$\begin{aligned} \frac{d}{dz}(\Delta(S)) = \Delta(\frac{d}{dz}(S)) = \Delta(M_1S + SM_2) = \\ \Delta(M_1)\Delta(S) + \Delta(S)\Delta(M_2) \end{aligned} \tag{50}$$

this proves that $\Delta(S)$ satifies a two sided differential equation. On the other hand, $S \otimes S$ satisfies

$$\begin{aligned} \frac{d}{dz}(S \otimes S) = \frac{d}{dz}(S) \otimes S + S \otimes \frac{d}{dz}(S) = \\ (M_1S + SM_2) \otimes S + S \otimes (M_1S + SM_2) = \\ (M_1 \otimes 1 + 1 \otimes M_1)(S \otimes S) + (S \otimes S)(M_1 \otimes 1 + 1 \otimes M_1) \end{aligned} \tag{51}$$

which proves that, if M_1 and M_2 are primitive, $\Delta(S)$ and $S \otimes S$ satisfy the same differential equation. By virtue of (ii), if S is group-like at a point, it is so everywhere. In particular (and this will be used in (v)), if S is an invertible solution of an equation of type (46), with M_1 and M_2 primitive, then, for $z_0 \in V$, $S(z)S(z_0)^{-1}$ is group-like.

iv) Indeed

$$\begin{aligned} \langle S|1_{X^*}\rangle = \langle H^*[S_0]|1_{X^*}\rangle = \langle (I + HH^*)[S_0]|1_{X^*}\rangle = \\ \langle S_0|1_{X^*}\rangle + \langle HH^*[S_0]|1_{X^*}\rangle = \langle S_0|1_{X^*}\rangle \ . \end{aligned} \tag{52}$$

v) For $z_0 \in V$, one defines

$$R(z, z_0) = S(z)S(z_0)^{-1}G(z_0)G(z)^{-1} = S(z)\Big(G(z_0)^{-1}S(z_0)\Big)^{-1}G(z)^{-1} \ .$$

$R = R(z, z_0)$ is the product of two group-like series ($S(z)S(z_0)^{-1}$ and $G(z_0)G(z)^{-1}$). Thus, z being fixed, one has

$$\begin{aligned} \Big(S(z) \otimes S(z)\Big)\Big(G(z)^{-1} \otimes G(z)^{-1}\Big) = (S(z)G(z)^{-1}) \otimes (S(z)G(z)^{-1}) = \\ lim_{z_0:\mathcal{F}} R(z, z_0) \otimes R(z, z_0) = lim_{z_0:\mathcal{F}} \Delta(R(z, z_0)) = \\ \Delta(lim_{z_0:\mathcal{F}}(R(z, z_0)) = \Delta(S(z)G(z)^{-1}) = \Delta(S(z))\Delta(G(z)^{-1}) = \\ \Delta(S(z))\Big(G(z)^{-1} \otimes G(z)^{-1}\Big) \end{aligned} \tag{53}$$

and, finally $\Delta(S) = S \otimes S$. □

Remark 5.2 *The proof of the theorem provides an integrator*

$$H(G) = \sum_{w \in \mathcal{M}} \Big(\sum_{uv=w} \int_{z_0}^{z} \Big(\langle M_1|u\rangle(s)\langle G|v\rangle(s) + \langle G|u\rangle(s)\langle M_2|v\rangle(s)\Big) ds\Big) w$$

but any similar operator H such that $\frac{d}{dz}(H(G)) = M_1G+GM_2$ would do. In particular, one can construct operators with varied lower integration bounds. For example, the operator

$$H(G) = \sum_{w\in\mathcal{M}} \Big(\sum_{uv=w} \Big(\int_{a(u)}^{z} \langle M_1|u\rangle(s)\langle G|v\rangle(s) + \int_{b(v)}^{z} \langle G|u\rangle(s)\langle M_2|v\rangle(s) \Big) ds \Big) w \tag{54}$$

is fairly admissible. We will see in Paragraph (6.2) an application of such a principle.

6 Coordinates of group-like elements.

6.1 Through the looking glass: passing from right to left.

Let $S \in \mathcal{H}\langle\langle X\rangle\rangle$, we call $\mathcal{F}(S)$ the $\mathbb{C}$-vector space generated by the coefficients of S, one has

$$\mathcal{F}(S) = \{\langle S|P\rangle\}_{P\in\mathbb{C}\langle X\rangle} \ . \tag{55}$$

We will use the following increasing filtrations

$$\mathcal{F}_{\leq\alpha}(S) = \{\langle S|P\rangle\}_{P\in\mathbb{C}_{\leq\alpha}\langle X\rangle} \ . \tag{56}$$

or

$$\mathcal{F}_{\leq n}(S) = \{\langle S|P\rangle\}_{P\in\mathbb{C}_{\leq n}\langle X\rangle} \ . \tag{57}$$

Proposition 6.1 *We have the following properties :*
i) If $T \in \mathbb{C}\langle\langle X\rangle\rangle$ then $\mathcal{F}(ST) \subset \mathcal{F}(S)$ and one has equality if T is invertible.
ii) If S is group-like, then $\mathcal{F}(S)$ is a unital sub-algebra of $\mathcal{H}$, which is filtered w.r.t. (56) and (57) i.e.

$$\mathcal{F}_{\leq\alpha}(S)\mathcal{F}_{\leq\beta}(S) \subset \mathcal{F}_{\leq\alpha+\beta}(S) \tag{58}$$

Proof — (i) The space $\mathcal{F}(ST)$ is spanned by the

$$\langle ST|w\rangle = \sum_{uv=w} \langle S|u\rangle\langle T|v\rangle \in \mathcal{F}(S)$$

if T is invertible one has $\mathcal{F}(S) = \mathcal{F}(STT^{-1}) \subset \mathcal{F}(ST)$ which proves the equality.
ii) If S is group-like, one has

$$\langle S|u\rangle\langle S|v\rangle = \langle S\otimes S|u\otimes v\rangle = \langle \Delta(S)|u\otimes v\rangle = \langle S|u ⧢ v\rangle \tag{59}$$

In the case when all functions $\langle S|w\rangle$ are $\mathbb{C}$-linearly independant, one has a correspondence between the Differential Galois group of a differential equation of type (44) (acting on the right) and the group of automorphisms of $\mathcal{F}(S)$ compatible with the preceding filtration (they turn out to be unipotent).

Proposition 6.2 *Let S be a group-like series. The following conditions are equivalent:*
i) For every $x \in X$, $ker_{\mathbb{C}}(S) \subset ker_{\mathbb{C}}(Sx)$.
ii) For every $x \in X$, there is a derivation $\delta_x \in \mathfrak{Der}(\mathcal{F}(S))$ such that

$$\delta_x(S) = Sx \tag{60}$$

iii) For every $x \in X$, there is a one-paramater group of automorphisms $\phi_x^t \in Aut(\mathcal{F}(S))$; $t \in \mathbb{R}$ such that

$$\phi_x^t(S) = Se^{tx} \tag{61}$$

iv) For every $C \in \mathrm{Lie}_C\langle\langle X\rangle\rangle$, there is $\delta \in \mathfrak{Der}(\mathcal{F}(S))$ such that

$$\delta(S) = SC \tag{62}$$

v) For every $C \in \mathrm{Lie}_C\langle\langle X\rangle\rangle$, there is $\phi \in Aut(\mathcal{F}(S))$ such that

$$\phi(S) = Se^C \tag{63}$$

vi) The functions $(\langle S|w\rangle)_{w\in X^*}$ are $\mathbb{C}$-linearly independant.

Proof — *i)* $\Longrightarrow$ *ii)* From the inclusion, we derive that, for all $x \in X$ there exists a $\mathbb{C}$-linear mapping $\phi \in \mathrm{End}(\mathcal{F}(S))$ such that for all $w \in \mathcal{M}$, $\phi(\langle S|w\rangle) = \langle Sx|w\rangle$. It must be a derivation of $\mathcal{F}(S)$ as

$$\begin{aligned}\phi(\langle S|u\rangle\langle S|v\rangle) = \phi(\langle S|u\sqcup\!\sqcup v\rangle) = \langle Sx|u\sqcup\!\sqcup v\rangle = \langle S|(u\sqcup\!\sqcup v)x^{-1}\rangle = \\ \langle S|(ux^{-1}\sqcup\!\sqcup v) + (u\sqcup\!\sqcup vx^{-1})\rangle = \langle S|(ux^{-1}\sqcup\!\sqcup v)\rangle\langle S|(u\sqcup\!\sqcup vx^{-1})\rangle = \\ \langle Sx|u\rangle\langle S|v\rangle + \langle S|u\rangle\langle Sx|v\rangle = \phi(\langle S|u\rangle)\langle S|v\rangle + \langle S|u\rangle\phi(\langle S|v\rangle)\end{aligned} \tag{64}$$

from the fact that $(\langle S|w\rangle)_{w\in X^*}$ spans $\mathcal{F}(S)$.
ii) $\Longrightarrow$ *iv)* As $(\langle S|w\rangle)_{w\in X^*}$ spans $\mathcal{F}(S)$, the derivation ϕ is uniquely defined. Let us note it δ_x and notice that, doing so, we have constructed a mapping $\Phi : X \to \mathfrak{Der}(\mathcal{F}(S))$ (which is Lie algebra. Therefore, there is a unique extension of this mapping as a morphism $\mathrm{Lie}_{\mathbb{C}}\langle X\rangle \to \mathfrak{Der}(\mathcal{F}(S))$. This correspondence, which we will note $P \to \delta(P)$ is (uniquely) recursively defined by

$$\delta(x) = \delta_x \ ; \ \delta([P,Q]) = [\delta(P), \delta(Q)] \ . \tag{65}$$

For $C = \sum_{n\geq 0} C_n \in \mathrm{Lie}_{\mathbb{C}}\langle\langle X\rangle\rangle$ with $C_n \in \mathrm{Lie}_{\mathbb{C}}\langle X\rangle_n$, we remark that the sequence $\langle S\sum_{0\leq n\leq N} C_n|w\rangle$ is stable (for large N). Set $\delta_{\leq N} := \delta(\sum_{0\leq n\leq N} C_n)$. We see that $\delta_{\leq N}$ is stable (for large N) on every $\mathcal{F}_\alpha$ and we note $\delta(C)$ its limit. It is clear that this limit is a derivation and that it corresponds to C.
iv) $\Longrightarrow$ *v)* For evey $C = \sum_{n\geq 0} C_n \in \mathrm{Lie}_{\mathbb{C}}\langle\langle X\rangle\rangle$, the exponential e^C defines a mapping $\phi \in \mathrm{End}(\mathcal{F}(S))$ as indeed $e^{\delta_{\leq N}}$ is stationnary. It is easily checked that this mapping is an automorphism of algebra of $\mathcal{F}(S)$.
v) $\Longrightarrow$ *iii)* For $C_i \in \mathrm{Lie}_{\mathbb{C}}\langle\langle X\rangle\rangle; \imath = 1, 2$ which commute we have

$$Se^{C_1}e^{C_2} = \phi_{C_1}(S)e^{C_2} = \phi_{C_1}(Se^{C_2}) = \phi_{C_1}\phi_{C_2}(S) \tag{66}$$

this proves the existence, for a $C \in \mathrm{Lie}_{\mathbb{C}}\langle\langle X\rangle\rangle$ of a one-parameter (rational) group ϕ_C^t in $\mathrm{Aut}(\mathcal{F}(S))$ such that $Se^{tC} = \phi_C^t(S)$. This one-parameter (rational) group can be extended to $\mathbb{R}$ as continuity is easily checked by taking the scalar products $\langle\phi_C^t(S)|w\rangle = \langle Se^{tC}|w\rangle$ and it suffices to specialize the result to $C = x$.
iii) $\Longrightarrow$ *ii)* By stationary limits one has

$$\langle Sx|w\rangle = \lim_{t\to 0}\frac{1}{t}(\langle Se^{tx}|w\rangle - \langle S|w\rangle) = \lim_{t\to 0}\frac{1}{t}(\langle\phi_x^t(S)|w\rangle - \langle S|w\rangle) \tag{67}$$

$v) \Longrightarrow i)$ Let $x \in X, t \in \mathbb{R}$, we take $C = tx$ and $\phi_t \in \mathrm{Aut}(\mathcal{F}(S))$ s.t. $\phi_t(S) = Se^{tx}$. It there is $P \in \mathbb{C}\langle X\rangle$ such that $\langle S|P\rangle = 0$ one has

$$0 = \langle S|P\rangle = \phi_t(\langle S|P\rangle) = \langle \phi_t(S)|P\rangle = \langle Se^{tx}|P\rangle = \sum_{n=0}^{\deg(P)} \frac{t^n}{n!}\langle Sx^n|P\rangle \tag{68}$$

and then, for all $z \in V$, the polynomial

$$\sum_{n=0}^{\deg(P)} \frac{t^n}{n!}\langle S(z)x^n|P\rangle \tag{69}$$

is identically zero over $\mathbb{R}$ hence so are all of its coeefficients in particular $\langle S(z)x|P\rangle$ for all $z \in V$. This proves the claim.
$i) \Longrightarrow vi)$ Let $P \in ker_{\mathbb{C}}(S)$ if $P \neq 0$ take it of minimal degree with this property. For all $x \in X$, one has $P \in ker_{\mathbb{C}}(Sx)$ which means $\langle Sx|P\rangle = 0$ and then $Px^\dagger = 0$ as $\deg(Px^\dagger) = \deg(P) - 1$. The reconstruction lemma implies that

$$P = \langle P|1\rangle + \sum_{x\in X}(Px^\dagger)x = \langle P|1\rangle \tag{70}$$

Then, one has $0 = \langle S|P\rangle = \langle S|1\rangle\langle P|1\rangle = \langle P|1\rangle$ which shows that $ker_{\mathbb{C}}(S) = \{0\}$. This is equivalent to the statement (vi).
$vi) \Longrightarrow i)$ Is obvious as $ker_{\mathbb{C}}(S) = \{0\}$.
□

It is possible to enlarge somehow the range of proposition (6.1) to coefficients that are analytic functions $f:\ dom(f) \to \mathbb{C}$.

Definition 6.3 *We call here differential field of germs w.r.t. a filter basis $\mathbb{B}$ of open connected subsets of V, a map $\mathcal{C}$ defined on $\mathbb{B}$ such that for every $U \in \mathbb{B}$, $\mathcal{C}[U]$ is a subring of $C^\omega(U,\mathbb{C})$ and*

1. *$\mathcal{C}$ is compatible with restrictions i.e. if $U, V \in \mathbb{B}$ and $V \subset U$, one has*
$$res_{VU}(\mathcal{C}[U]) \subset \mathcal{C}[V]$$
2. *if $f \in \mathcal{C}[U] \setminus \{0\}$ then there exists $V \in \mathbb{B}$ s.t. $V \subset U - \mathcal{O}_f$ and f^{-1} (defined on V) is in $\mathcal{C}[V]$.*

For any $U \in \mathbb{B}$, we note $\mathcal{C}[U]$ the ring of functions in $\mathcal{C}$ defined on U and restricted to this set.

There are important cases when the conditions (6.1) are satified as shows the following theorem.

Theorem 6.4 *Let V be a simply connected non-void open subset of $\mathbb{C} - \{a_0, \cdots a_n\}$ ($\{a_0, \cdots a_n\}$ are distinct points), $M = \sum_{i=0}^n \frac{\lambda_i x_i}{z-a_i}$ be a multiplier on $X = \{x_0, \cdots x_n\}$ with all $\lambda_i \neq 0$ and S be any regular solution of*

$$\frac{d}{dz}S = MS\ . \tag{71}$$

Then, let $\mathcal{C}$ be a differential field of functions defined on V which do not contain linear combinations of logarithms on any domain but which contains z and the constants (as, for example the rational functions).
If U is a non-void domain of $\mathcal{C}$ and $P \in \mathcal{C}[U]\langle X\rangle$, one has

$$\langle S|P\rangle = 0 \Longrightarrow P = 0 \tag{72}$$

Proof — Let $U \in \mathbb{B}$. For every non-zero $Q \in \mathcal{C}[U]\langle X\rangle$, we note $lead(Q)$ the greatest word in the support of Q for the graded lexicographic ordering $\prec$ (we have endowed X with any linear ordering) and call Q monic if the leading coefficient $\langle Q|lead(Q)\rangle$ is 1. A monic polynomial then reads

$$Q = w + \sum_{u \prec w} \langle Q|u\rangle u \ . \tag{73}$$

Suppose now that it is possible to find U and $P \in \mathcal{C}[U]\langle X\rangle$ (not necessarily monic) such that $\langle S|P\rangle = 0$, we choose P with $lead(P)$ minimal for $\prec$.

Then

$$P = f(z)w + \sum_{u \prec w} \langle P|u\rangle u \tag{74}$$

with $f \not\equiv 0$. Thus $U_1 = U \setminus \mathcal{O}_f \in \mathbb{B}$ and $Q = \frac{1}{f(z)} P \in \mathcal{C}[U_1]\langle X\rangle$ is monic and satisfies

$$\langle S|Q\rangle = 0 \ . \tag{75}$$

Differentiating eq. (75), we get

$$0 = \langle S'|Q\rangle + \langle S|Q'\rangle = \langle MS|Q\rangle + \langle S|Q'\rangle = \langle S|Q' + M^\dagger Q\rangle \ . \tag{76}$$

Remark that one has

$$Q' + M^\dagger Q \in \mathcal{C}[U_1]\langle X\rangle \tag{77}$$

If $Q' + M^\dagger Q \neq 0$, one has $lead(Q' + M^\dagger Q) \prec lead(Q)$ and this is not possible because of the minimality hypothesis of $lead(Q) = lead(P)$. Hence, one must have $R = Q' + M^\dagger Q = 0$. With $|w| = n$, write now

$$Q = Q_n + \sum_{|u|<n} \langle Q|u\rangle u \ . \tag{78}$$

where $Q_n = \sum_{|u|=n} \langle Q|u\rangle u$ is the dominant homogeneous component of Q. For every $|u| = n$ we have

$$(\langle Q|u\rangle)' = -\langle M^\dagger Q|u\rangle = -\langle Q|Mu\rangle = 0 \tag{79}$$

thus all the coefficients of Q_n are constant.

If $n = 0$, $Q \neq 0$ is constant which is impossible by eq. (75) and because S is regular. If $n > 0$, for any word $|v| = n - 1$, we have

$$(\langle Q|v\rangle)' = -\langle M^\dagger Q|v\rangle = -\langle Q|Mv\rangle = -\sum_{i=0}^{n} \frac{\lambda_i}{z - a_i} \langle Q|x_i v\rangle = -\sum_{i=0}^{n} \frac{\lambda_i}{z - a_i} \langle Q_n|x_i v\rangle \tag{80}$$

Because all $x_i v$ are of length n.
Then

$$\langle Q|v\rangle = -\sum_{i=0}^{n} \langle Q_n|x_i v\rangle \int_\alpha^z \frac{\lambda_i}{s - a_i} ds + const \tag{81}$$

But all the functions $\int_\alpha^z \frac{\lambda_i}{s-a_i} ds$ are linearly independant over $\mathbb{C}$ and not all the scalars $\langle Q_n | x_i v\rangle$ are zero (write $w = x_k v$ and choose v such). This contradicts the fact that $Q \in \mathcal{C}[U_1]\langle X\rangle$ as $\mathcal{C}$ does contain no linear combination of logarithms. $\square$

Remark 6.5 *i) If a series satifies the equivalent conditions of the theorem (6.2), then every Se^C so does.*
ii) Series as the one of polylogarithms and all the exponential solutions of equation

$$\frac{d}{dz}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z})S \tag{82}$$

satisfy conditions of the theorem (6.2) as shows theorem (6.4).
iii) One could ask oneself what happens when these conditions are not satisfied. In fact the set of Lie series $C \in \mathrm{Lie}_{\mathbb{C}}\langle\langle \mathbb{X}\rangle\rangle$ such that it exists a $\phi \in \mathrm{End}(\mathcal{F}(S))$ (then a derivation) s.t. $SC = \phi(S)$ is a closed Lie subalgebra of $\mathrm{Lie}_{\mathbb{C}}\langle\langle \mathbb{X}\rangle\rangle$ which we will note Lie_S. For example

- *for $X = \{x_0, x_1\}$ and $S = e^{zx_0}$ one has $x_0 \in Lie_S$; $x_1 \notin Lie_S$*
- *for $X = \{x_0, x_1\}$ and $S = e^{z(x_0+x_1)}$, one has $x_0, x_1 \notin Lie_S$ but $(x_0 + x_1) \in Lie_S$.*

6.2 Polylogarithms and related functions

Here X is still the finite alphabet $\{x_0, x_1\}$ equipped with the order $x_0 < x_1$ and let $\mathcal{C}$ be the ring $\mathbb{C}[z, z^{-1}, (1-z)^{-1}]$.

The iterated integral over ω_0, ω_1 associated to $w = x_{i_1} \cdots x_{i_k}$ over X and along the integration path $z_0 \rightsquigarrow z$ is the following multiple integral defined by

$$\int_{z_0 \rightsquigarrow z} \omega_{i_1} \cdots \omega_{i_k} = \int_{z_0}^{z} \omega_{i_1}(t_1) \int_{z_0}^{t_1} \omega_{i_2}(t_2) \ldots \int_{z_0}^{t_{r-2}} \omega_{i_r}(t_{r-1}) \int_{z_0}^{t_{r-1}} \omega_{i_r}(t_r), \tag{83}$$

where $t_1 \cdots t_{r-1}$ is a subdivision of the path $z_0 \rightsquigarrow z$. In a shortened notation, we denote this integral by $\alpha_{z_0}^z(w)$ and[1] $\alpha_{z_0}^z(1_{X^*}) = 1$. One can check that the polylogarithm $\mathrm{Li}_{s_1,\ldots,s_r}$ is also the value of the iterated integral over ω_0, ω_1 and along the integration path $0 \rightsquigarrow z$ [?, ?] :

$$\mathrm{Li}_w(z) = \alpha_0^z(x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1). \tag{84}$$

The definition of polylogarithms is extended over the words $w \in X^*$ by putting $\mathrm{Li}_{x_0}(z) := \log z$. The $\{\mathrm{Li}_w\}_{w\in X^*}$ are $\mathcal{C}$-linearly independent [?, ?]. In order to, define $L = \sum_{w \in X^*} \mathrm{Li}_w w$, one also can use an integrator with variable lower integration bounds as one described by (54) with $M_2 = 0$, $a(u) = 1$ for $u \in x_0^*$ and $a(u) = 0$ for $u \in X^*x_1X^*$.
Indeed, L is group-like but, to show this one cannot use Thm 5.1 (iii) because the lower bounds of the integrals are different. So one first shows that

$$lim_{z\to 0} \exp(-x_0 \log z)\mathrm{L}(z) = lim_{z\to 0}\mathrm{L}(z)\exp(-x_0 \log z) = 1 \tag{85}$$

[1] Here, 1_{X^*} stands for the empty word over X.

which can be done as follows. One first remarks that, in case w contains at least ons x_1 (i.e. $|w|_{x_1} \geq 1$) and for every k

$$lim_{z \to 0} \log(z)^k \langle \mathrm{L(z)}|w\rangle = 0 \tag{86}$$

then, setting $\mathrm{L}^+(\mathrm{z}) = \sum_{|w|_{x_1} \geq 1} \langle \mathrm{L(z)}|w\rangle w$, one has

$$lim_{z \to 0} \exp(-x_0 \log z)\mathrm{L}^+(\mathrm{z}) = \mathrm{L}^+(\mathrm{z}) \exp(-x_0 \log z) = 0 \tag{87}$$

and as

$$L(z) = \mathrm{L}^+(\mathrm{z}) + \sum_{w \in (x_0)^*} \langle \mathrm{L(z)}|w\rangle w = \mathrm{L}^+(\mathrm{z}) + \exp(x_0 \log z) \tag{88}$$

the result follows.

The following functions

$$\forall w \in X^*, \qquad \mathrm{P}_w(z) = (1-z)^{-1}\mathrm{Li}_w(z), \tag{89}$$

are also $\mathcal{C}$-linearly independent, as $\mathcal{C}$ is an integral domain, by the following lemma easy to check

Lemma 6.6 *Let $\mathcal{A}$ be an integral domain and M an $\mathcal{A}$-module. If $(x_i)_{i \in I}$ is a linearly independant family and $b \neq 0$ in $\mathcal{A}$, then $(bx_i)_{i \in I}$ is linearly independant.*

Since, for any $w \in Y^*, \mathrm{P}_w$ is the ordinary generating function of the sequence $\{\mathcal{H}_w(N)\}_{N \geq 0}$:

$$\mathrm{P}_w(z) = \sum_{N \geq 0} \mathcal{H}_w(N)\ z^N \tag{90}$$

then, as a consequence of the classical isomorphism between convergent Taylor series and their associated sums, the harmonic sums $\{\mathcal{H}_w\}_{w \in Y^*}$ are $\mathbb{C}$-linearly independent. Firstly, $\ker \mathrm{P} = \{0\}$ and $\ker \mathcal{H} = \{0\}$, and secondly, P is a morphism transporting the stuffle to the Hadamard product :

$$\mathrm{P}_u(z) \odot \mathrm{P}_v(z) = \sum_{N \geq 0} \mathcal{H}_u(N)\mathcal{H}_v(N) z^N = \sum_{N \geq 0} \mathcal{H}_{u \uplus v}(N) z^N = \mathrm{P}_{u \uplus v}(z). \tag{91}$$

7 Conclusion

To sum up what has been done in this paper? we can state that the deformed algebra $\mathbf{LDIAG}(q_c, q_s)$, which originates from a special quantum field theory [**?**], is free and its law can be constructed from very general procedures: it is a shifted twisted law. Before shifting, one can observe that the law is, in fact, dual to a comultiplication on a free algebra. This comultiplication is a perturbation, with q_s (the superposition parameter) of the shuffle comultiplication on this free algebra. The parameter q_s is obtained by addition of a perturbating factor which is just dual to a (diagonally) deformed law of a semigroup whereas the crossing parameter q_c is obtained by extending to the tensor structure (i.e. to words) a colour factor of an algebra.

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