Independence of hyperlogarithms over function fields via algebraic combinatorics.

M. Deneufchâtel, G. H. E. Duchamp,
Hoang Ngoc Minh *
and Allan I. Solomon †

[1]

Contents

Abstract

We obtain a necessary and sufficient condition for the linear independence of solutions of differential equations for hyperlogarithms. The key fact is that the multiplier (i.e. the factor M in the differential equation $dS = MS$) has only singularities of first order (Fuchsian-type equations) and this implies that they freely span a space which contains no primitive. We give direct applications were we extend the property of linear independence to the largest known ring of coefficients.

1 Introduction

In his 1928 study of the solutions of linear differential equations following Poincaré, Lappo-Danilevski introduced the so-called *hyperlogarithmic functions of order m*, functions of iterated integrals of the following form with logarithmic poles [12] :

$$L(a_0, \ldots, a_n|z, z_0) = \int_{z_0}^{z} \int_{z_0}^{s_n} \ldots \int_{z_0}^{s_1} \frac{ds_0}{s_0 - a_0} \ldots \frac{ds_n}{s_n - a_n}, \tag{1}$$

---

*LIPN - UMR 7030 CNRS - Université Paris 13 F-93430 Villetaneuse, France

†Open University

[1] Version compile le 22-01-2011 18:28.

where z_0 is a fixed point. It suffices that $z_0 \neq a_0$ for this iterated integral to converge. The classical polylogarithm Li_n is a particular case of these integrals [13] :

$$\mathrm{Li}_n(z) = \int_0^z \int_0^{s_n} \ldots \int_0^{s_2} \frac{ds_1}{1-s_1} \frac{ds_2}{s_2} \ldots \frac{ds_n}{s_n} = -L(1, \underbrace{0, \ldots, 0}_{n-1 \text{ times}} | z, 0). \tag{2}$$

These iterated integrals also appear in quantum electrodynamics (see [8, 15] for example). Chen [6] studied them systematically and provided a noncommutative algebraic context in which to treat them. Fliess [10, 11] encoded these iterated integrals by words over a finite alphabet and extended them to a symbolic calculus[2] for nonlinear differential equations of the following form, in the context of noncommutative formal power series :

$$\left\{ \begin{array}{rcl} y(z) & = & f(q(z)), \\ \dot{q}(z) & = & \displaystyle\sum_{i=0}^{m} \frac{A_i(q)}{z - a_i}, \\ q(z_0) & = & q_0, \end{array} \right. \tag{3}$$

where the state $q = (q_1, \ldots, q_n)$ belongs to a complex analytic manifold of dimension N, q_0 denotes the initial state, the observable f belongs to $\mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_N]\!]$, $\{A_i\}_{i=0,n}$ is the polysystem defined as follows

$$A_i(q) = \sum_{j=1}^{n} A_i^j(q) \frac{\partial}{\partial q_j}, \tag{4}$$

with, for any $j = 1, \ldots, n$, $A_i^j(q) \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_N]\!]$.
suggested addition
By introducing the encoding alphabet $X = \{x_0, \ldots, x_m\}$, the method of Fliess consists in exhibiting two formal power series over the monoid X^* :

$$F := \sum_{w \in X^*} \mathcal{A}(w) \circ f_{|q_0} \; w \quad \text{and} \quad C := \sum_{w \in X^*} \alpha_{z_0}^z(w) \; w \tag{5}$$

in order to comput the output y. These series are subjected to convergence conditions (precisely speaking, the convergence of a duality pairing), as follows

$$y(z) = \langle F || C \rangle := \sum_{w \in X^*} \mathcal{A}(w) \circ f_{|q_0} \; \alpha_{z_0}^z(w), \tag{6}$$

where

- $\mathcal{A}$ is a morphism of algebras from $\mathbb{C}\langle\langle X \rangle\rangle$ to the algebra generated by the polysystem $\{A_i\}_{i=0,n}$:

$$\begin{array}{rrcl} & \mathcal{A}(1_{X^*}) & = & \text{identity}, & (7) \\ \forall w = v x_i, x_i \in X, v \in X^*, & \mathcal{A}(w) & = & \mathcal{A}(v) A_i. & (8) \end{array}$$

---

[2] A kind of Feynman like operator calculus [9].

- $\alpha_{z_0}^z$ is a shuffle algebra morphism from $(\mathbb{C}\langle\langle X\rangle\rangle, ш)$ to $\mathcal{H}$:

$$\begin{aligned} \alpha_{z_0}^z(1_{X^*}) &= 1, && (9)\\ \forall w = vx_i, x_i \in X, v \in X^*, \quad \alpha_{z_0}^z(w) &= \int_{z_0}^z \frac{\alpha_{z_0}^s(v)}{s-a_i}. && (10) \end{aligned}$$

Formula (6) states also that the iterated integrals over the rational functions

$$u_i(z) = \frac{1}{z-a_i}, \qquad i = 0, .., n, \tag{11}$$

spans the vector space $\mathcal{H}$.
end of addition

As for the linear differential equations, the essential difficulty is to construct the fundamental system of solutions, or the Picard-Vessiot extension, to describe the space of solutions of the differential system (3) algorithmically [18]. For that, one needs to prove the linear independence of the iterated integrals in order to obtain the *universal* Picard-Vessiot extension. The $\mathbb{C}$-linear independence has already been shown by Wechsung [19]. His method consists of a recurrence based on the total degree. However this method cannot be used with variable coefficients. Another proof was given in [16] based on monodromy. In this note we describe a general theorem on differential computational algebra and show that, at the cost of using variable domains (which is the realm of germ spaces), and replacing the recurrence on total degree by a recursion on the words (with the graded lexicographic ordering), one can encompass the previous results mentioned above and obtain much larger rings of coefficients and configuration alphabets (even infinite of continuum cardinality).

2 Non commutative differential equations.

We recall here the Dirac-Schützenberger notation as in [2, 7, 17]. Let X be an alphabet and R be a commutative ring with unit. The algebra of noncommutative polynomials is the algebra $R[X^*]$ of the free monoid X^*. As an R-module, $R^{(X^*)}$ is the set of finitely supported R-valued function on X^* and, as such, it is in natural duality with the algebra of all functions on X^* (the large algebra of X^* [4]), $R^{X^*} = R\langle\langle X\rangle\rangle$, the duality being given, for $f \in R\langle\langle X\rangle\rangle$ and $g \in R[X^*]$ by

$$\langle f|g\rangle = \sum_{w\in X^*} f(w)g(w) \ . \tag{12}$$

The rôle of the ring is played here by a commutative differential k-algebra $(\mathcal{A}, d)$; that is, a k-algebra (associative and commutative with unit) $\mathcal{A}$ endowed with a distinguished derivation $d \in \mathfrak{Der}(\mathcal{A})$ (the ground field k is supposed commutative and of characteristic zero). We assume that the ring of constants $ker(d)$ is exactly k.

An alphabet X being given, one can at once extend the derivation d to a derivation of the algebra $\mathcal{A}\langle\langle X\rangle\rangle$ by

$$\mathbf{d}(S) = \sum_{w\in X^*} d(\langle S|w\rangle)w \ . \tag{13}$$

We are now in a position to state the main theorem which resolves many important questions, as we shall see some in the applications.

Theorem 2.1 *Let $(\mathcal{A}, d)$ be a k-commutative associative differential algebra with unit ($ch(k) = 0$) and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $d(\mathcal{C}) \subset \mathcal{C}$). We suppose that $S \in \mathcal{A}\langle\langle X\rangle\rangle$ is a solution of the differential equation*

$$\mathbf{d}(S) = MS \ ; \ \langle S|1\rangle = 1 \tag{14}$$

where the multiplier $M = \sum_{x\in X} u_x x \in \mathcal{C}\langle\langle X\rangle\rangle$ is a homogeneous series (a polynomial in the case of finite X) of degree 1.
The following condition are equivalent :

i) The family $(\langle S|w\rangle)_{w\in X^*}$ of coefficients of S is free over $\mathcal{C}$.

ii) The family of coefficients $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ is free over $\mathcal{C}$.

iii) The family $(u_x)_{x\in X}$ is such that, for $f \in \mathcal{C}$ and $\alpha_x \in k$

$$d(f) = \sum_{x\in X} \alpha_x u_x \Longrightarrow (\forall x \in X)(\alpha_x = 0) \ . \tag{15}$$

iv) The family $(u_x)_{x\in X}$ is free over k and

$$d(\mathcal{C}) \cap \mathrm{span}_k \Big((u_x)_{x\in X}\Big) = \{0\} \ . \tag{16}$$

Proof — (i)$\Longrightarrow$(ii) Obvious.
(ii)$\Longrightarrow$(iii)
Suppose that the family $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ (coefficients taken at letters and the empty word) of coefficients of S is free over $\mathcal{C}$ and let us consider a relation as in eq. (15)

$$d(f) = \sum_{x\in X} \alpha_x u_x \ . \tag{17}$$

We form the polynomial $P = -f1_{X^*} + \sum_{x\in X} \alpha_x x$. One has $\mathbf{d}(P) = -d(f)1_{X^*}$ and

$$d(\langle S|P\rangle) = \langle \mathbf{d}(S)|P\rangle + \langle S|\mathbf{d}(P)\rangle = \langle MS|P\rangle - d(f)\langle S|1_{X^*}\rangle = (\sum_{x\in X} \alpha_x u_x) - d(f) = 0 \tag{18}$$

whence $\langle S|P\rangle$ must be a constant, say $\lambda \in k$. For $Q = P - \lambda.1_{X^*}$, we have

$$\mathrm{supp}(Q) \subset X \cup \{1_{X^*}\} \text{ and } \langle S|Q\rangle = \langle S|P\rangle - \lambda\langle S|1_{X^*}\rangle = \langle S|P\rangle - \lambda = 0 \ .$$

This implies that $Q = 0$ and, as $Q = -(f + \lambda)1_{X^*} + \sum_{x\in X} \alpha_x x$, one has, in particular, all the $\alpha_x = 0$.
(iii)$\Longleftrightarrow$(iv)
Obvious, (iv) being a geometric reformulation of (iii).
(iii)$\Longleftrightarrow$(i)
Let $\mathcal{K}$ be the kernel of $P \mapsto \langle S|P\rangle$ (a linear form $\mathcal{C}\langle X\rangle \to \mathcal{C}$) i.e.

$$\mathcal{K} = \{P \in \mathcal{C}\langle X\rangle | \langle S|P\rangle = 0\} \ . \tag{19}$$

If $\mathcal{K} = \{0\}$, we are done. Otherwise, let us adopt the following strategy.
First, we order X by some well-ordering $<$ ([3] III.2.1) and X^* by the graded lexicographic ordering $\prec$ defined by

$$u \prec v \Longleftrightarrow |u| < |v| \text{ or } (u = pxs_1 \ , \ v = pys_2 \text{ and } x < y). \tag{20}$$

It is easy to check that $\prec$ is also a well-ordering relation. For each nonzero polynomial P, we denote by $lead(P)$ its leading monomial; i.e. the greatest element of its support $\mathrm{supp}(P)$ (for $\prec$).
Now, as $\mathcal{R} = \mathcal{K} - \{0\}$ is not empty, let w_0 be the minimal element of $lead(\mathcal{R})$ and choose a $P \in \mathcal{R}$ such that $lead(P) = w_0$. We write

$$P = fw_0 + \sum_{u \prec w_0} \langle P|u\rangle u \ ; \ f \in \mathcal{C} - \{0\} \ . \tag{21}$$

The polynomial $Q = \frac{1}{f}P$ is also in $\mathcal{R}$ with the same leading monomial, but the leading coefficient is now 1 and Q is given by

$$Q = w_0 + \sum_{u \prec w_0} \langle Q|u\rangle u \ . \tag{22}$$

Differentiating $\langle S|Q\rangle = 0$, one gets

$$\begin{aligned} 0 = \langle \mathbf{d}(S)|Q\rangle + \langle S|\mathbf{d}(Q)\rangle = \langle MS|Q\rangle + \langle S|\mathbf{d}(Q)\rangle = \\ \langle S|M^\dagger Q\rangle + \langle S|\mathbf{d}(Q)\rangle = \langle S|M^\dagger Q + \mathbf{d}(Q)\rangle \end{aligned} \tag{23}$$

with

$$M^\dagger Q + \mathbf{d}(Q) = \sum_{x \in X} u_x(x^\dagger Q) + \sum_{u \prec w_0} d(\langle Q|u\rangle)u \in \mathcal{C}\langle X\rangle \ . \tag{24}$$

It is impossible that $M^\dagger Q + \mathbf{d}(Q) \in \mathcal{R}$ because it would be of leading monomial strictly less than w_0, hence $M^\dagger Q + \mathbf{d}(Q) = 0$. This is equivalent to the recursion

$$d(\langle Q|u\rangle) = -\sum_{x \in X} u_x \langle Q|xu\rangle \ ; \text{ for } x \in X \ , \ v \in X^*. \tag{25}$$

From this last relation, we deduce that $\langle Q|w\rangle \in k$ for every w of length $deg(Q)$ and, because $\langle S|1\rangle = 1$, one must have $deg(Q) > 0$. Then, we write $w_0 = x_0 v$ and compute the coefficient at v

$$d(\langle Q|v\rangle) = -\sum_{x \in X} u_x \langle Q|xv\rangle = \sum_{x \in X} \alpha_x u_x \tag{26}$$

with coefficients $\alpha_x = -\langle Q|xv\rangle \in k$ as $|xv| = \deg(Q)$ for all $x \in X$. Condition **PI** implies that all coefficients $\langle Q|xu\rangle$ are zero; in particular, as $\langle Q|x_0u\rangle = 1$, we get a contradiction. This proves that $\mathcal{K} = \{0\}$.
$\square$

3 Applications

Let V be a connected and simply connected analytic variety (for example, the doubly cut plane $\mathbb{C} - (]-\infty, 0[\cup]1, +\infty[)$, or the universal covering of $\mathbb{C} - \{0, 1\}$), and let $\mathcal{H}$ be the space of analytic functions on V.

It is possible to enlarge the range of scalars to coefficients that are analytic functions with variable domains $f: \ dom(f) \to \mathbb{C}$.

Definition 3.1 *We define a differential field of germs as the data of a filter basis $\mathcal{B}$ of open connected subsets of V, and a map $\mathcal{C}$ defined on $\mathcal{B}$ such that for every $U \in \mathcal{B}$, $\mathcal{C}[U]$ is a subring of $C^\omega(U, \mathbb{C})$ and*

1. *$\mathcal{C}$ is compatible with restrictions i.e. if $U, V \in \mathcal{B}$ and $V \subset U$, one has*

$$res_{VU}(\mathcal{C}[U]) \subset \mathcal{C}[V]$$

2. *if $f \in \mathcal{C}[U] \setminus \{0\}$ then there exists $V \in \mathcal{B}$ s.t. $V \subset U - \mathcal{O}_f$ and f^{-1} (defined on V) is in $\mathcal{C}[V]$.*

There are important cases where the conditions (3.2) are satisfied as shown by the following theorem.

Theorem 3.2 *Let V be a simply connected non-void open subset of $\mathbb{C} - \{a_1, \cdots a_n\}$ ($\{a_1, \cdots a_n\}$ are distinct points), $M = \sum_{i=1}^n \frac{\lambda_i x_i}{z - a_i}$ be a multiplier on $X = \{x_1, \cdots x_n\}$ with all $\lambda_i \neq 0$ and S be any regular solution of*

$$\frac{d}{dz} S = MS \ . \tag{27}$$

Then, let $\mathcal{C}$ be a differential field of functions defined on V which does not contain linear combinations of logarithms on any domain but which contains z and the constants (as, for example the rational functions).
If U is a non-void domain of $\mathcal{C}$ and $P \in \mathcal{C}[U]\langle X \rangle$, one has

$$\langle S | P \rangle = 0 \Longrightarrow P = 0 \tag{28}$$

Proof — Let $U \in \mathcal{B}$. For every non-zero $Q \in \mathcal{C}[U]\langle X \rangle$, we denote by $lead(Q)$ the greatest word in the support of Q for the graded lexicographic ordering $\prec$. We endow X with an arbitrary linear ordering, and call Q monic if the leading coefficient $\langle Q | lead(Q) \rangle$ is 1. A monic polynomial is then given by

$$Q = w + \sum_{u \prec w} \langle Q | u \rangle u \ . \tag{29}$$

Now suppose that it is possible to find U and $P \in \mathcal{C}[U]\langle X \rangle$ (not necessarily monic) such that $\langle S | P \rangle = 0$; we choose P with $lead(P)$ minimal for $\prec$.

Then

$$P = f(z) w + \sum_{u \prec w} \langle P | u \rangle u \tag{30}$$

with $f \not\equiv 0$. Thus $U_1 = U \setminus \mathcal{O}_f \in \mathcal{B}$ and $Q = \frac{1}{f(z)} P \in \mathcal{C}[U_1]\langle X \rangle$ is monic and satisfies

$$\langle S|Q \rangle = 0 \ . \tag{31}$$

Differentiating eq. (31), we get

$$0 = \langle S'|Q \rangle + \langle S|Q' \rangle = \langle MS|Q \rangle + \langle S|Q' \rangle = \langle S|Q' + M^\dagger Q \rangle \ . \tag{32}$$

Remark that one has

$$Q' + M^\dagger Q \in \mathcal{C}[U_1]\langle X \rangle \tag{33}$$

If $Q' + M^\dagger Q \neq 0$, one has $lead(Q' + M^\dagger Q) \prec lead(Q)$ and this is not possible because of the minimality hypothesis of $lead(Q) = lead(P)$. Hence, one must have $R = Q' + M^\dagger Q = 0$. With $|w| = n$, we now write

$$Q = Q_n + \sum_{|u|<n} \langle Q|u \rangle u \tag{34}$$

where $Q_n = \sum_{|u|=n} \langle Q|u \rangle u$ is the dominant homogeneous component of Q. For every $|u| = n$ we have

$$(\langle Q|u \rangle)' = -\langle M^\dagger Q|u \rangle = -\langle Q|Mu \rangle = 0 \tag{35}$$

thus all the coefficients of Q_n are constant.

If $n = 0$, $Q \neq 0$ is constant which is impossible by eq. (31) and because S is regular. If $n > 0$, for any word $|v| = n - 1$, we have

$$(\langle Q|v \rangle)' = -\langle M^\dagger Q|v \rangle = -\langle Q|Mv \rangle = -\sum_{i=0}^{n} \frac{\lambda_i}{z - a_i} \langle Q|x_i v \rangle = -\sum_{i=0}^{n} \frac{\lambda_i}{z - a_i} \langle Q_n|x_i v \rangle \tag{36}$$

bcause all $x_i v$ are of length n.
Then

$$\langle Q|v \rangle = -\sum_{i=0}^{n} \langle Q_n|x_i v \rangle \int_\alpha^z \frac{\lambda_i}{s - a_i} ds + const \tag{37}$$

But all the functions $\int_\alpha^z \frac{\lambda_i}{s-a_i} ds$ are linearly independent over $\mathbb{C}$ and not all the scalars $\langle Q_n|x_i v \rangle$ are zero (write $w = x_k v$ and choose v accordingly). This contradicts the fact that $Q \in \mathcal{C}[U_1]\langle X \rangle$ as $\mathcal{C}$ contains no linear combination of logarithms. $\square$

Corollary 3.3 *Let V be as above and R be the ring of functions which can be defined on some $V \cup U_{a_1} \cup U_{a_2} \cup \cdots U_{a_n}$ where U_{a_i} are open neighborhoods of $a_i, i = 1 \cdots n$ and have non-essential singularities at these points. Then, the set of hyperlogarithms $(\langle S|w \rangle)_{w \in X^*}$ are linearly independent over R.*

Remark 3.4 *i) If a series $S = \sum_{w \in X^*} \langle S|w \rangle w$ is a regular solution of (27) and satisfies the equivalent conditions of the theorem (2.1), then every Se^C (with $C \in \mathrm{Lie}_{\mathbb{C}}\langle\langle X \rangle\rangle$) does.*

ii) Series such as that of polylogarithms and all the exponential solutions of equation

$$\frac{d}{dz}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z})S \tag{38}$$

satisfy the conditions of the theorem (2.1) as shown by theorem (3.2).

iii) Call $\mathcal{F}(S)$ the vector space generated by the coefficients of the series S. One may ask what happens when the conditions for independence are not satisfied.
In fact, the set of Lie series $C \in \mathrm{Lie}_{\mathbb{C}}\langle\langle\mathbb{X}\rangle\rangle$ such that there exists a $\phi \in \mathrm{End}(\mathcal{F}(S))$ (then a derivation) such that $SC = \phi(S)$, is a closed Lie subalgebra of $\mathrm{Lie}_{\mathbb{C}}\langle\langle\mathbb{X}\rangle\rangle$ which we will denote by Lie_S. For example

- *for $X = \{x_0, x_1\}$ and $S = e^{zx_0}$ one has $x_0 \in Lie_S$; $x_1 \notin Lie_S$*
- *for $X = \{x_0, x_1\}$ and $S = e^{z(x_0+x_1)}$, one has $x_0, x_1 \notin Lie_S$ but $(x_0 + x_1) \in Lie_S$.*

iv) Theorem (3.2) holds mutatis mutandis *when the multiplier is infinite i.e.*

$$M = \sum_{i \in I} \frac{\lambda_i x_i}{z - a_i}$$

even if I is continuum infinite (say $I = \mathbb{R}$, singularities being all the reals).

v) Theorem (3.2) does no longer hold with singularities of higher order (i.e. not fuchsian). For example, with

$$M = \frac{x_0}{z^2} + \frac{x_1}{(1-z)^2} \ . \tag{39}$$

Firstly, the differential field $\mathcal{C}$ generated by

$$u_0 = \frac{1}{z^2}, \ u_1 = \frac{1}{(1-z)^2} \tag{40}$$

contains

$$\frac{1}{2}(\frac{1}{u_0} - \frac{1}{u_1} + \frac{1}{2}\frac{d^2}{dz^2}(\frac{1}{u_0})) = z \tag{41}$$

and hence $\mathcal{C} = \mathbb{C}(z)$ the field of rational functions over $\mathbb{C}$. Condition (ii) of Theorem (2.1) is not fulfilled (as $z^2 u_0 - (1-z)^2 u_1 = 0$). Indeed, one has the dependance relation

$$\langle S|x_1 x_0\rangle = -\langle S|x_0^2\rangle - \langle S|x_1^2\rangle - \langle S|x_1\rangle \ . \tag{42}$$

4 Conclusion

In this paper we showed that by using fields of germs, some difficult results can be considerably simplified and extended. We believe that this procedure is not only of theoretical importance, but can be taken into account at the very computational level because every formula (especially analytic) carries with it its domain of validity. As a matter of fact, having at hand the linear independance of coordinate functions over large rings allows to express uniquely solutions of systems like (3) in the basis of hyperlogarithms.

Suggested addition

A nice perspective would be to determine the asymptotic expansion at infinity of the Taylor coefficients of the $y(z)$ as given in (6) for the general case which has already been done for only singularities at $\{0, 1\}$ and for different purposes (see arXiv:1011.0523v2 and http://fr.arxiv.org/abs/0910.1932).

end addition

References

[1] E. Abe, *Hopf algebras.* Cambridge Univ. Press, 1980.

[2] J. Berstel, C. Reutenauer, *Rational series and their languages.* EATCS Monographs on Theoretical Computer Science, Springer, 1988.

[3] N. Bourbaki, *Theory of sets*, Springer (2004).

[4] N. Bourbaki, *Algebra, chapter III*, Springer (1970)

[5] N. Bourbaki, *Lie algebras and Lie groups, chapter II and III*, Springer (1970)

[6] K.T. Chen.– *Iterated path integrals*, Bull. Amer. Math.Soc., vol 83, 1977, pp. 831-879.

[7] G. Duchamp, C. Reutenauer, *Un critère de rationalité provenant de la géométrie noncommutative* Invent. Math. **128** 613-622. (1997).

[8] F.J. Dyson.– *The radiation theories of Tomonaga, Schwinger and Feynman*, Physical Rev, vol 75, 1949, pp. 486-502.

[9] R. Feynman.– *An Operator Calculus Having Applications in Quantum Electrodynamics*, Phys. Rev. 84, 108128 (1951).

[10] M. Fliess.– *Fonctionnelles causales non linéaires et indéterminées non commutatives*, Bull. Soc. Math. France, N°109, 1981, pp. 3-40.

[11] M. Fliess.– *Réalisation locale des systèmes non linéaires, algèbres de Lie filtrées transitives et séries génératrices*, Invent. Math., t 71, 1983, pp. 521-537.

[12] J.A. Lappo-Danilevsky.– *Théorie des systèmes des équations différentielles linéaires*, Chelsea, New York, 1953.

[13] L. Lewin.– *Polylogarithms and associated functions*, North Holland, New York and Oxford, 1981.

[14] L. Lewin.– *Structural properties of polylogarithms*, Mathematical survey and monographs, Amer. Math. Soc., vol 37, 1992.

[15] W. Magnus.– On the Exponential Solution of Differential Equation for a Linear Operator, Comm. Pure Appl. Math. 7, 1954, pp. 649-673.

[16] Hoang Ngoc Minh, M. Petitot and J. Van der Hoeven.– Polylogarithms and Shuffle Algebra, *Proceedings of FPSAC'98*, 1998.

[17] C. Reutenauer, *Free Lie algebras*, Oxford University Press, 1993.

[18] M. van der Put and M.F. Singer.– Galois Theory of Linear Differential Equation, Comprehensive Studies in Mathematics, vol. 328, Springer-Verlag, Berlin, (2003).

[19] G. Wechsung.– Functional Equations of Hyperlogarithms, in [14]