Combinatorial Physics

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Combinatorial Physics

When you're asked : "By the way, what is Combinatorial Physics ?" (and you have one minut to reply), what to say ?

Tentative definition

It is the art of solving physical problems with combinatorics and, conversely, to solve combinatorial problems using physical insights and methods.

Some topics (all relating to Physics, see also later JoCP)

- Combinatorics of data structures
  - graphs
  - words
  - discrete matchings
  - boards, diagrams, cells, polyominoes, etc ...
- Enumeration of data stuctures
  - constructible species
  - ranking
  - OGF, EGF, other denominators
  - asymptotic and exact properties

Some topics /2

- Representation theory
  - special functions and polynomials
  - symmetric polynomials
  - combinatorial modules
  - Fock spaces, normal forms
- Special functions (noncommutative)
  - noncommutative classical functions (trigonometry, etc ...)
  - noncommutative continued fractions
  - Frobenius characteristics

Some topics /3

- Combinatorial algebras, coalgebras and their actions
  - combinatorial and diagrammatic Hopf algebras
  - coherent states
  - (pro-)polynomial realizations
  - infinite dimensional Lie groups and differential equations
  - automata theory, representative functions, duality (dual laws, structure constants)
- Geometry of graphs, maps and matroids
  - constructions and operations on graphs
  - polynomials associated to graphs (Szymanczyk, Jones, Tutte polynomials)
  - braids and links
  - geometric and diagrammatic representation of tensors

Some topics /4

- Evolution groups
  - one-parameter groups of infinite matrices
  - one-parameter groups of differential operators
  - combinatorics of infinite-dimensional Lie algebras and their integration (local and global)
- Probabilities, distributions and measures
  - statistical mechanics
  - resolutions of unity for coherent states
  - moment problems, growth, unicity
  - orthogonal polynomials
  - parametric moment problems

Classical Fock space for bosons and q-ons

- Heisenberg-Weyl (two-dimensional) algebra is defined by two generators (a^+, a) which fulfill the relation

$$[a\ ,\ a^+] = aa^+ - a^+a = 1$$

- Known to have no (faithful) representation by bounded operators in a Banach space.

There are many « combinatorial » (faithful) representations by operators. The most famous one is the Bargmann-Fock representation

$$a \dashrightarrow d/dx\ ;\ a+ \dashrightarrow x$$

where a has degree -1 and a^+ has degree 1.

- These were bosons, there are also fermions. The relation for fermions is

$$aa^{+} + a^{+}a = 1$$

- This provides a framework for the q-analogue which is defined by

$$[a\ ,\ a^{+}]_q = aa^{+} - q a^{+} a = 1$$

- For which Bargmann-Fock representation reads

$$a \dashrightarrow D_q\ ;\ a+ \dashrightarrow x$$

where a has degree -1 and a^{+} has degree 1 and D_q is the (classical) q-derivative.

• For a faithful representation, one needs an infinite-dimensional space. The smallest, called Fock space, has a countable basis $(e_n)_{n\geq 0}$ (the actions are described below, each e_n is represented by a circled state « n »).

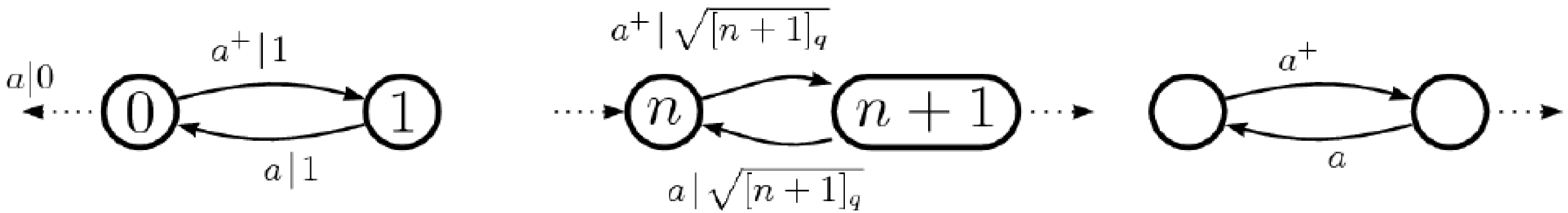


Figure 1: Classical Fock space

Revisiting the contruction of the Hopf algebra LDIAG

In a relatively recent paper Bender, Brody and Meister (*) introduce a special Field Theory described by a product formula (a kind of Hadamard product for two exponential generating functions - EGF) in the purpose of proving that any sequence of numbers could be described by a suitable set of rules applied to some type of Feynman graphs (see third Part of this talk).

These graphs label monomials.

Bender, C.M, Brody, D.C. and Meister,
Quantum field theory of partitions, J. Math. Phys. Vol 40 (1999)

How these diagrams arise and which data structures are around them

Let F, G be two EGFs.

$$F=\sum_{n\geq 0} a_n \frac{y^n}{n!};\ G=\sum_{m\geq 0} b_m \frac{y^m}{m!};\ \mathcal{H}(F,G):=\sum_{n\geq 0} a_n b_n \frac{y^n}{n!}$$

$$\mathcal{H}(F,G)=F(y\frac{d}{dx})G(x)|_{x=0}$$

Called « product formula » in the QFTP of Bender, Brody and Meister.

In case *F(0)=G(0)=1*, one can set

$$F(y) = exp(\sum_{n\geq 1} L_n \frac{y^n}{n!}) \quad G(x) = exp(\sum_{n\geq 1} V_m \frac{x^m}{m!})$$

and then,

$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0} =$$

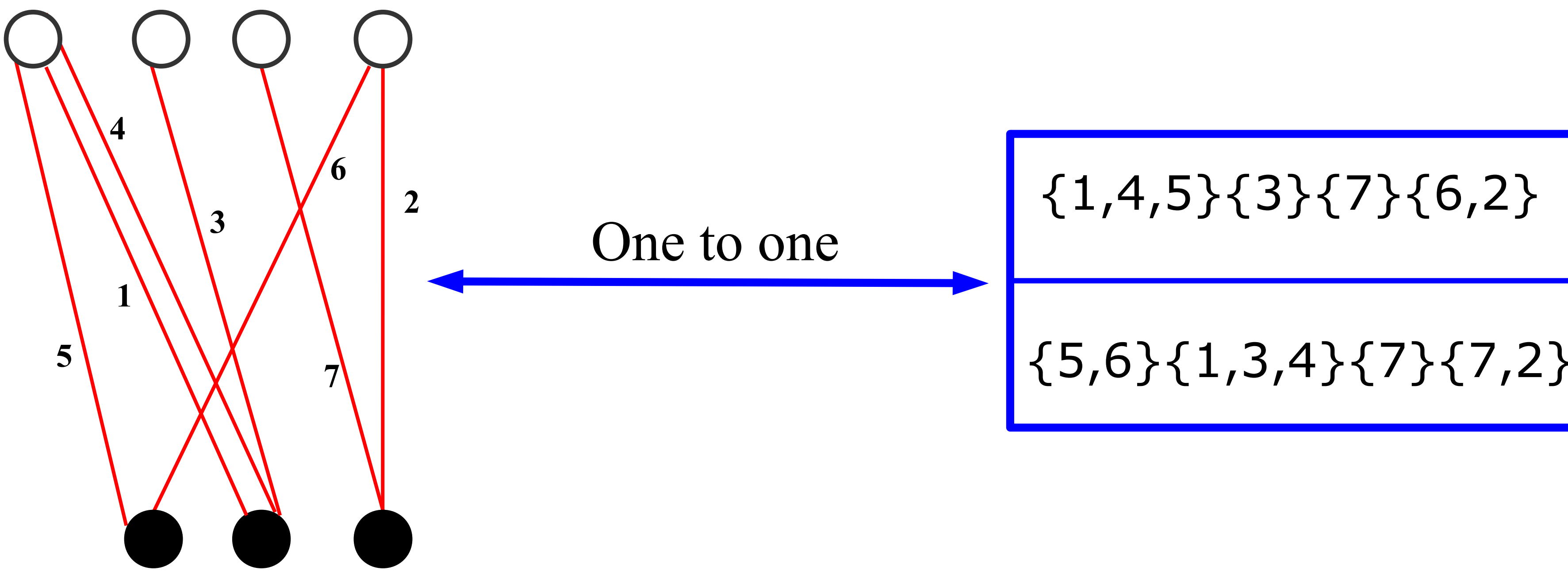
$$\sum_{n\geq 0} \frac{y^n}{n!} \sum_{|\alpha|=|\beta|=n} numpart(\alpha)numpart(\beta)\mathbb{L}^{\alpha}\mathbb{V}^{\beta}$$

with α, $\beta\in$☠$^{(☠*)}$ multiindices

$$numpart(\alpha) = \frac{|\alpha|!}{(1!)^{a_1}(2!)^{a_2}\cdots(r!)^{a_r}(a_1)!(a_2)!\cdots(a_r)!}$$

Remark that the coefficient *numpart*(α)*numpart*(β) is the number of pairs of set partitions (P1,P2) with type(P1)=α, type(P2)=β.

The original idea of Bender and al. was to introduce a special data structure suited to this enumeration.

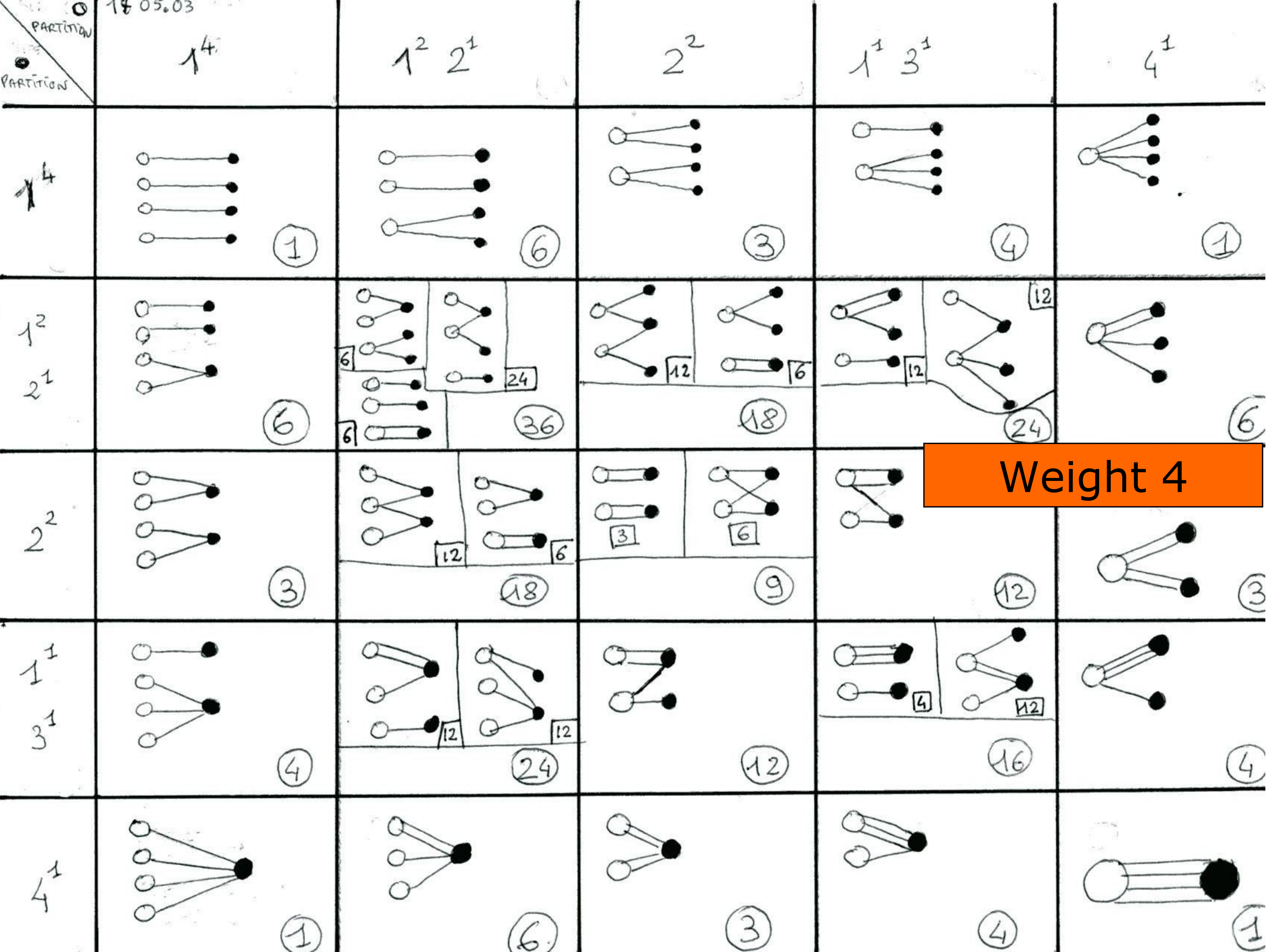


Now the product formula for EGFs reads

$$\mathcal{H}(F,G) = \sum_{d\ FB-diagram} \frac{y^{|d|}}{|d|!} \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)}$$

$$\mathcal{H}(F,G) = \sum_{d \in \mathbf{diag}} \frac{y^{|d|}}{|d|!} mult(d) \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)}$$

The main interest of these new forms is that we can impose rules on the counted graphs and we can call these (and their relatives) graphs : Feynman Diagrams of this theory (i.e. QFTP).



18.05.03
PARTITION
PARTITION
1^4
$1^2\ 2^1$
2^2
$1^1\ 3^1$
4^1
1
6
3
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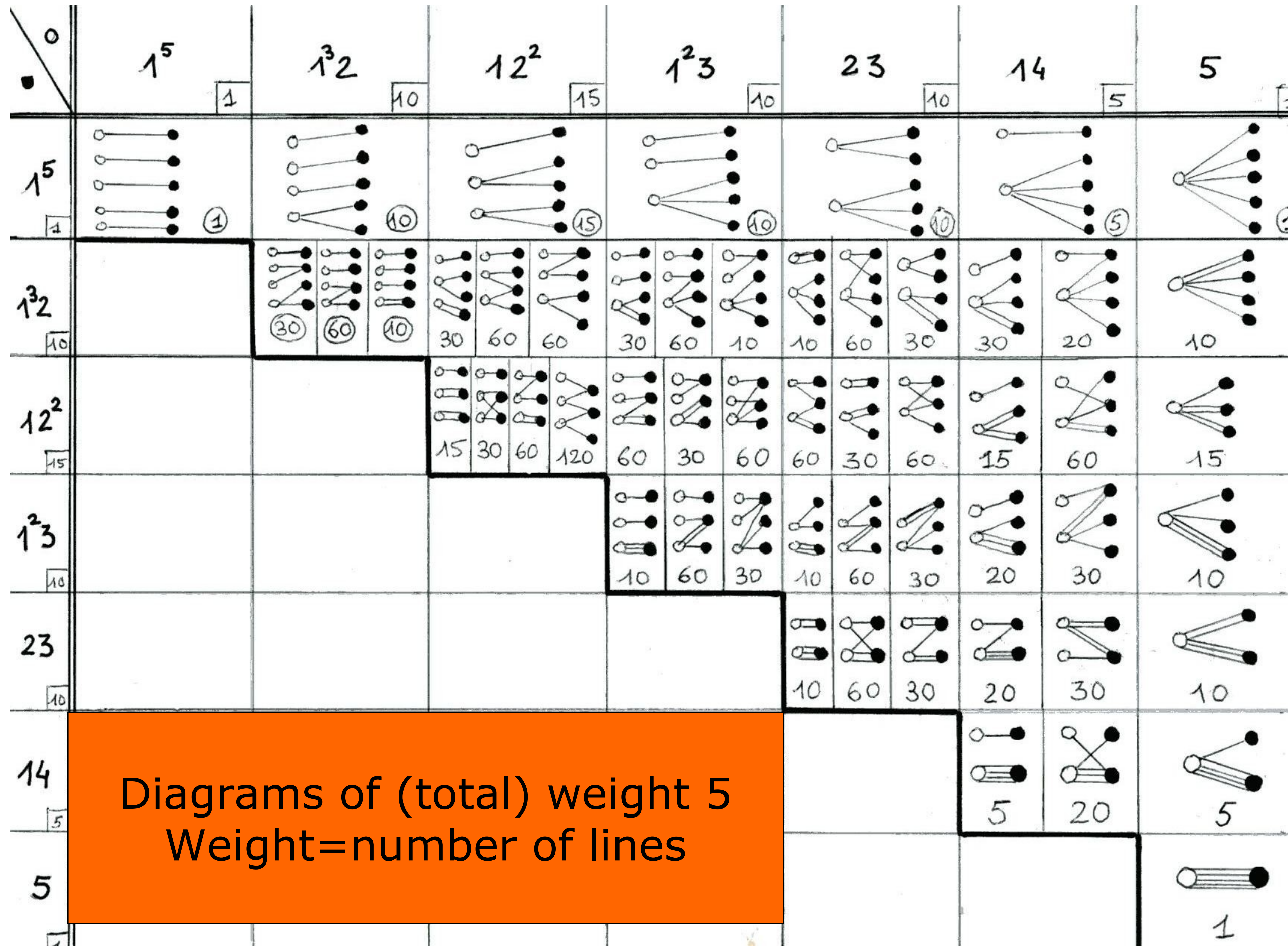

Diagrams of (total) weight 5

Weight=number of lines

One has now 3 types of diagrams :

- the diagrams with labelled edges (from 1 to |d|). Their set is denoted (see above) FB-diagrams.

- the unlabelled diagrams (where permutation of black and white spots). Their set is denoted (see above) **diag.**

- the diagrams, as drawn, with black (resp. white) spots ordered i.e. labelled. Their set is denoted **ldiag.**

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Hopf algebra structure

$$(H,\mu,\Delta,1_H,\varepsilon,\alpha)$$

Satisfying the following axioms

$(H,\mu,1_H)$ is an associative k-algebra with unit (here k will be a – commutative - field)

(H,Δ,ε) is a coassociative k-coalgebra with counit

$\Delta : H \to H\otimes H$ is a morphism of algebras

$\alpha : H \to H$ is an anti-automorphism (the antipode) which is the inverse of Id for convolution.

Convolution is defined on End(H) by

$$\varphi\bullet\psi = \mu\,(\varphi \otimes \psi)\,\Delta$$

with this law End(H) is endowed with a structure of associative algebra with unit $1_H\varepsilon$.

First step: Defining the spaces

$Diag = \oplus_{d \in \text{diagrams}} \mathbf{C}\, d \qquad LDiag = \oplus_{d \in \text{labelled diagrams}} \mathbf{C}\, d$

(functions with finite supports on the set of diagrams). At this stage, we have a natural arrow *LDiag* → *Diag*.

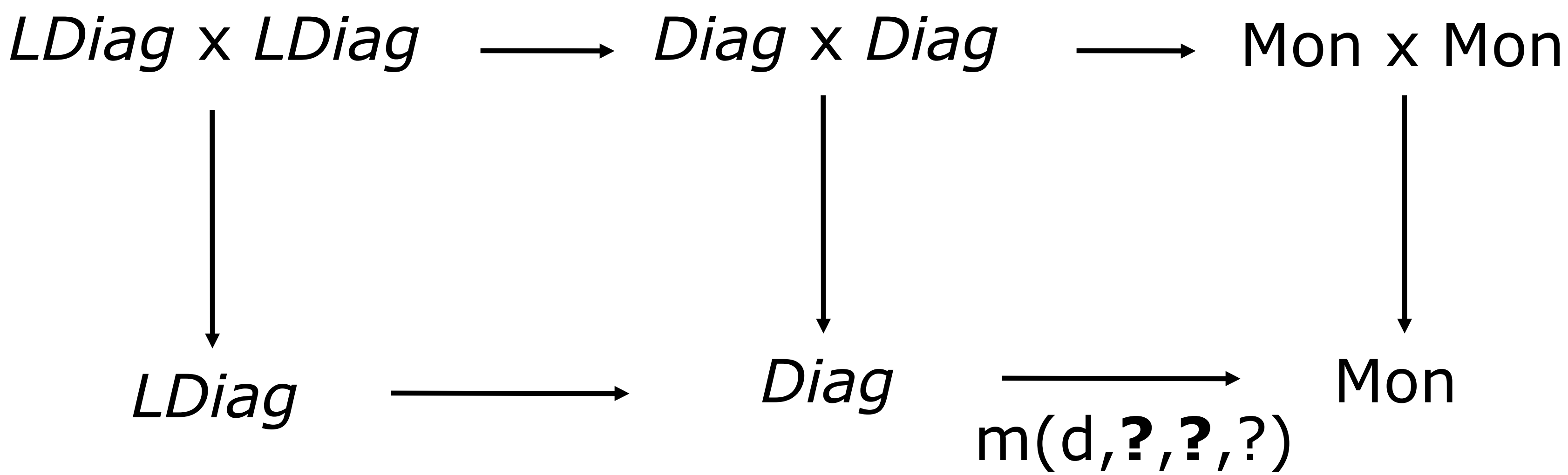
Second step: The product on *Ldiag* is just the concatenation of diagrams

d_1 儔 $d_2 = d_1\, d_2$

And, setting $m(d,\mathbf{L},\mathbf{V},z)=\mathbf{L}^{\alpha(d)}\mathbf{V}^{\beta(d)}\, z^{|d|}$
one gets

$$m(d_1 * d_2,\mathbf{L},\mathbf{V},z) = m(d_1,\mathbf{L},\mathbf{V},z)\, m(d_2,\mathbf{L},\mathbf{V},z)$$

This product is associative with unit (the empty diagram). It is compatible with the arrow *LDiag* → *Diag* and so defines the product on *Diag* which, in turn, is compatible with the product of monomials.

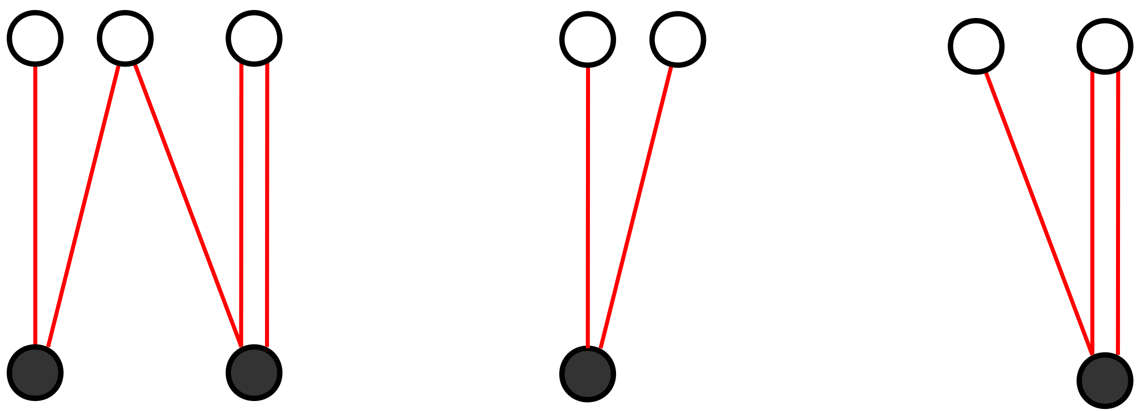


The coproduct needs to be compatible with m(d,**?**,**?**,?). One has two symmetric possibilities. The « white spots coproduct » reads

$$\Delta_{BS}(d) = \sum d_I \otimes d_J$$

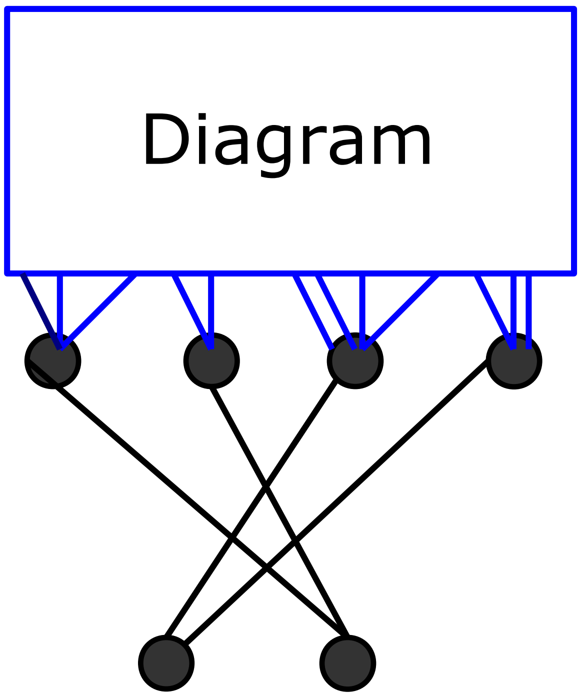
the sum being taken over all the decompositions, (I,J) of the Black Spots of d into two subsets.

For example, with the following diagrams d, d_1 and d_2



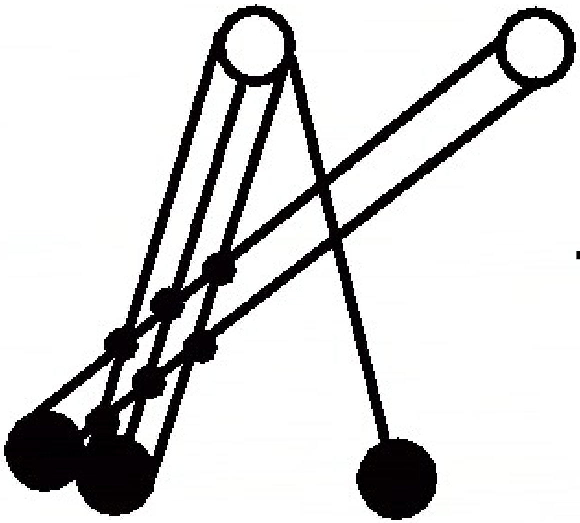
one has $\Delta_{BS}(d) = d \otimes \emptyset + \emptyset \otimes d + d_1 \otimes d_2 + d_2 \otimes d_1$

In order to connect these Hopf algebras to others of interest for physicists, we have to deform the product. The most popular technic is to use a monoidal action with many parameters (as braiding etc.). Here, it is the symmetric semigroup which acts on the black spots

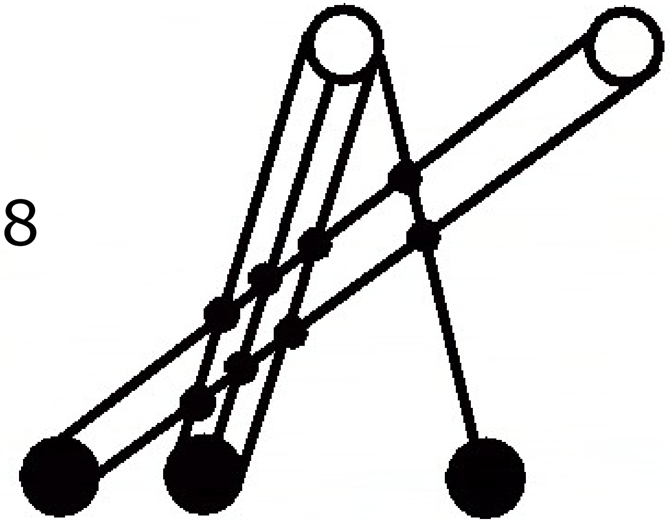


We tried to weight the shuffle with superpositions (stuffle). The weights being given by the intersection numbers.

$q_c^2 q_s^6$



$+q_c^8$



Maple 11 - [sh_sth.mws - [Server 1]]

```
> sth([1,2],[3,4]):
> [1, 2, 3, 4];[1, [2, 3], 4];[1, 3, 2, 4];[1, 3, [2, 4]];[1, 3, 4, 2];[[1, 3], 2, 4];[[1, 3], [2, 4]];[[1, 3], 4, 2];[3, 1, 2, 4];[3, 1, [2, 4]];[3, 1, 4, 2];[3, [1, 4], 2];[3, 4, 1, 2];
```

[1, 2, 3, 4]

[1, [2, 3], 4]

[1, 3, 2, 4]

[1, 3, [2, 4]]

[1, 3, 4, 2]

[[1, 3], 2, 4]

[[1, 3], [2, 4]]

[[1, 3], 4, 2]

[3, 1, 2, 4]

[3, 1, [2, 4]]

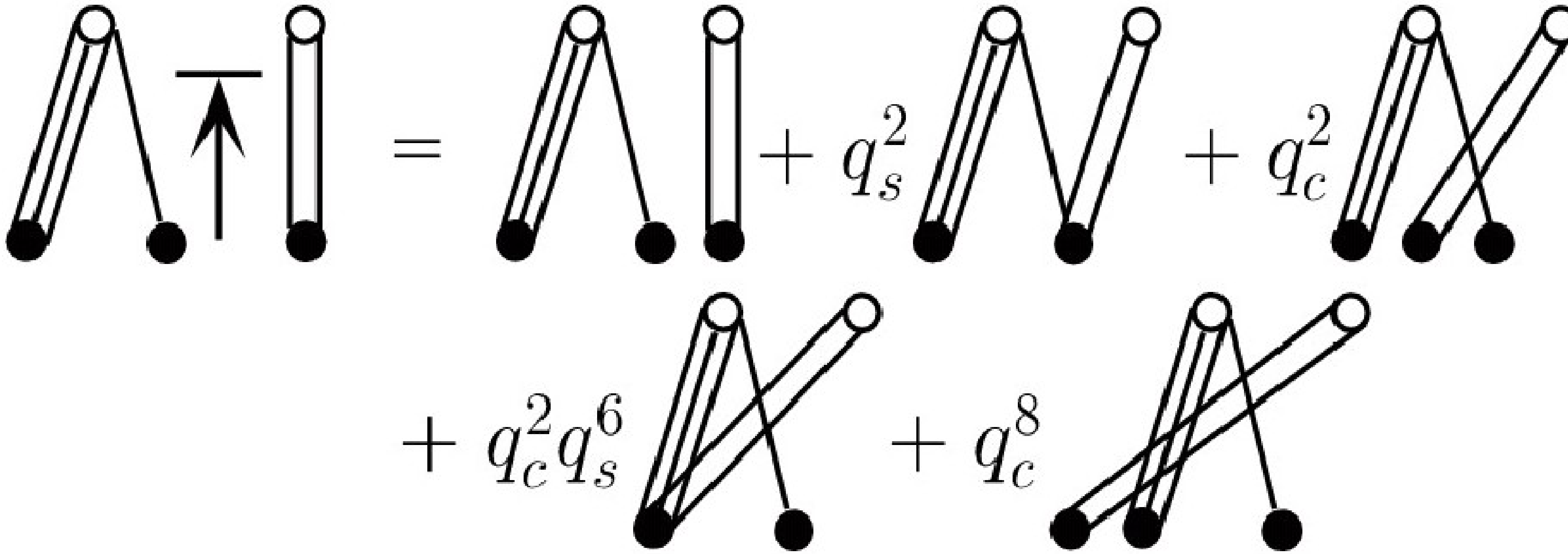
[3, 1, 4, 2]

[3, [1, 4], 2]

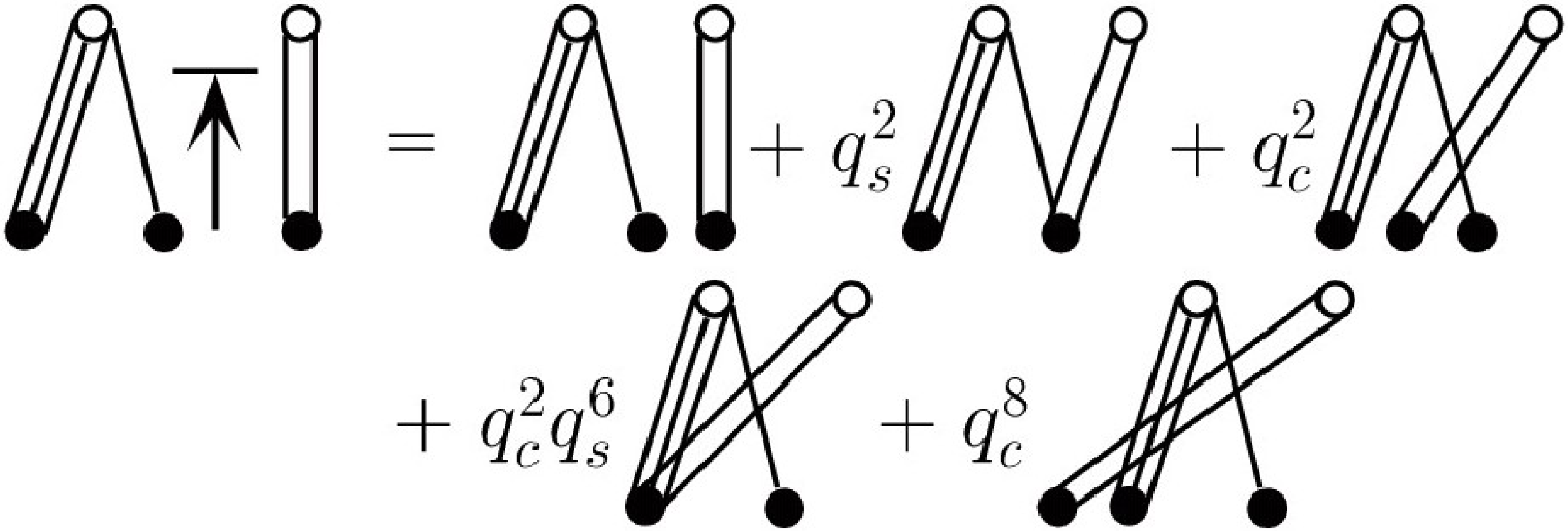
[3, 4, 1, 2]

The 13 terms of the stuffle of [1,2] with [3,4]

Time: 0.3s Bytes: 384K Available: 321M



What is striking is that this law is associative.



The labelled diagrams are in one to one correspondence with the packed matrices of MQSym and we can see easily that the product of the latter is obtained for

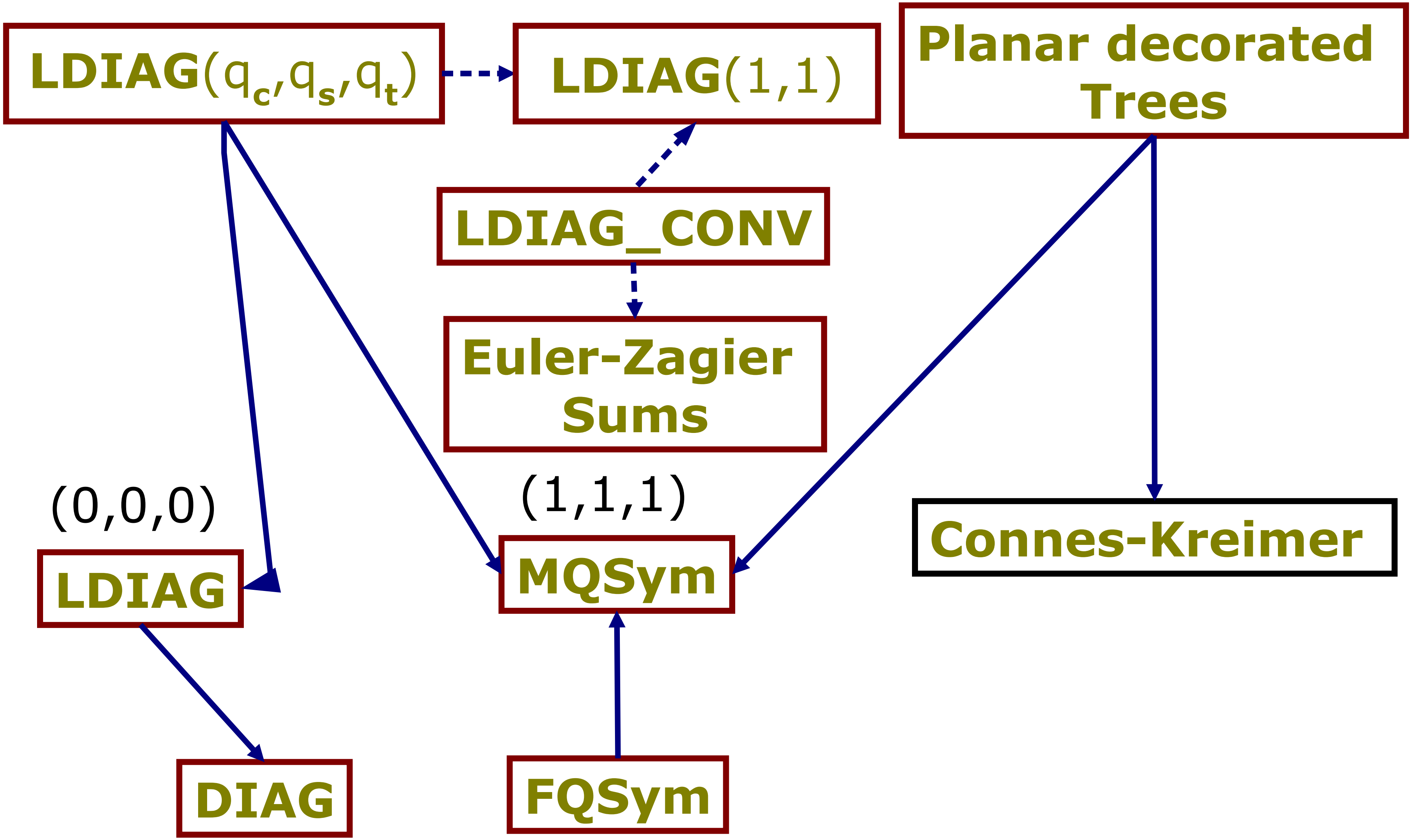
$$q_c = 1 = q_s$$

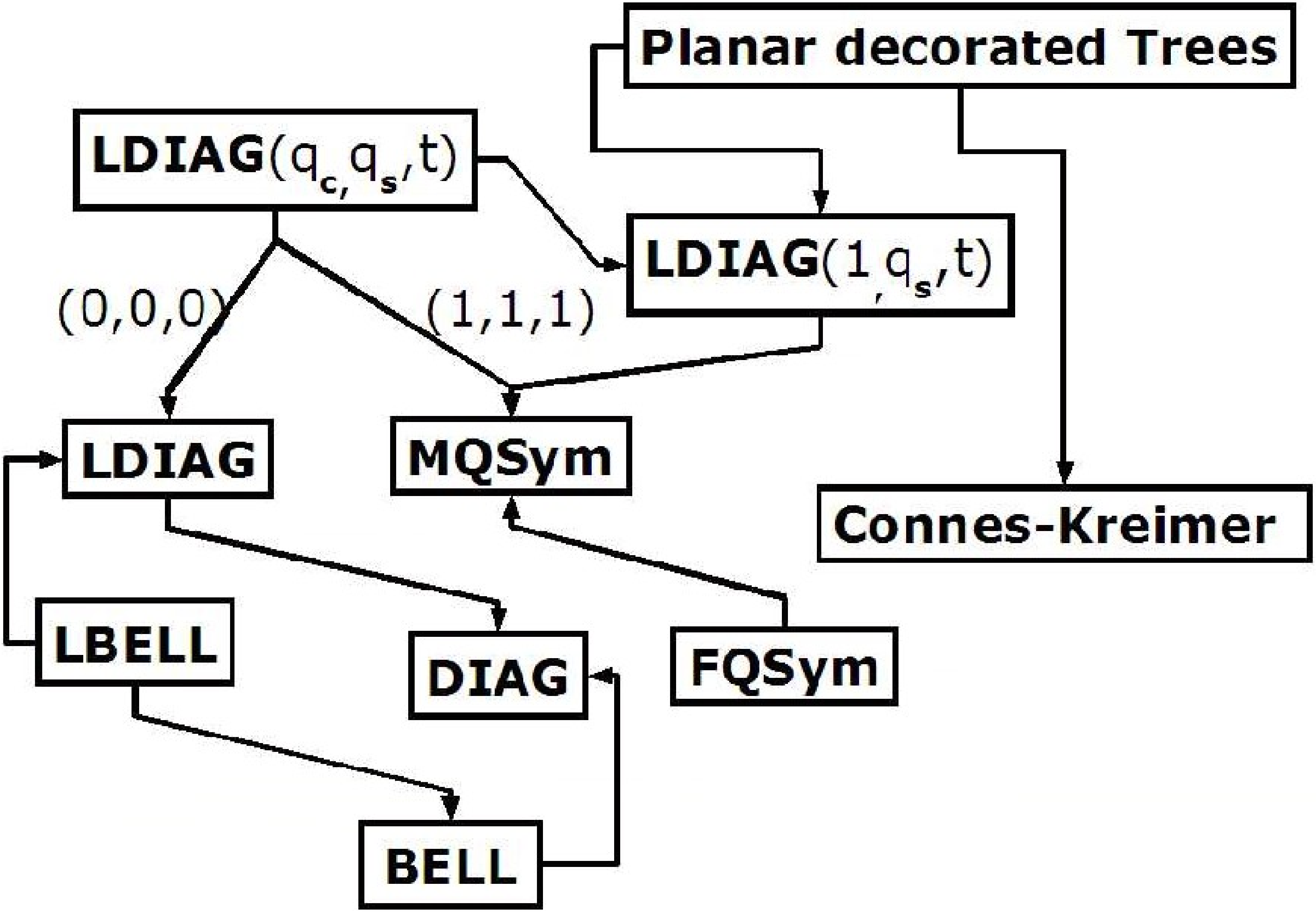
Hopf interpolation : One can see that the more intertwined the diagrams are the less connected components they have. This is the main argument to prove that LDIAG($q_{c,}$ q_s) is free. Therefore one can define a coproduct on the generators by

$$\Delta_t=(1-t)\Delta_{BS}+t\ \Delta_{MQSym}$$

this is LDIAG($q_{c,}$ q_s,t). (Rq =t is boolean).

Images and Specializations





The arrow *Planar Dec. Trees* → *LDIAG(1,q_s,t) is due to L. Foissy*

The Independance of the polylogarithms rests on the following general theorem.

Theorem 1. *Let* $(\mathcal{A}, d)$ *be a* k*-commutative associative differential algebra with unit* ($ch(k) = 0$) *and* $\mathcal{C}$ *be a differential subfield of* $\mathcal{A}$ *(i.e.* $d(\mathcal{C}) \subset \mathcal{C}$*). We suppose that* $S \in \mathcal{A}\langle\langle X \rangle\rangle$ *is a solution of the differential equation*

$$\mathbf{d}(S) = MS \ ; \ \langle S|1\rangle = 1 \tag{15}$$

where the multiplier M *is a homogeneous series (a polynomial in the case of finite* X*) of degree* 1*, i.e.*

$$M = \sum_{x \in X} u_x x \in \mathcal{C}\langle\langle X \rangle\rangle \ . \tag{16}$$

The following conditions are equivalent :

i) The family $(\langle S|w\rangle)_{w \in X^*}$ *of coefficients of* S *is free over* $\mathcal{C}$.

ii) The family of coefficients $(\langle S|y\rangle)_{y \in X \cup \{1_{X^*}\}}$ *is free over* $\mathcal{C}$.

iii) The family $(u_x)_{x \in X}$ *is such that, for* $f \in \mathcal{C}$ *and* $\alpha_x \in k$

$$d(f) = \sum_{x \in X} \alpha_x u_x \Longrightarrow (\forall x \in X)(\alpha_x = 0) \ . \tag{17}$$

iv) The family $(u_x)_{x \in X}$ *is free over* k *and*

$$d(\mathcal{C}) \cap \mathrm{span}_k\left((u_x)_{x \in X}\right) = \{0\} \ . \tag{18}$$

Concluding remarks and future

i) $LDIAG(q_c,q_s,t)$ is neither commutative nor cocommutative.

ii) The deformation above is likely to be decomposed in two deformation processes ; twisting (already investigated in NCSFIII) and shifting. Also, it could have a connection with other well known associators.

iii) The identity on the symmetric semigroup can be lifted to a more general monoid which takes into account the operations of concatenation and stacking. This, associated with new crossing and superposing rules could embrace the case of coloured polyzetas

Thank you