On a Theorem of R. Jungen

Author(s): M. P. Schützenberger
Source: *Proceedings of the American Mathematical Society*, Vol. 13, No. 6 (Dec., 1962), pp. 885-890
Published by: American Mathematical Society
Stable URL: http://www.jstor.org/stable/2034080
Accessed: 29-01-2016 03:20 UTC

# ON A THEOREM OF R. JUNGEN

M. P. SCHÜTZENBERGER

Let us recall the following elementary result in the theory of analytic functions in one variable.

THEOREM (R. JUNGEN [7]). *If a is rational and b algebraic their Hadamard product c is algebraic; if, further, b is rational, c also is rational.*

For several variables, Jungen's proof shows that the theorem is still true for the Bochner-Martin [2] Hadamard product. It does not hold for the Cameron-Martin [3] and for the Haslam-Jones [6] Hadamard products. In this note we give a version of Jungen's theorem which is valid for a restricted interpretation of the notions involved when $a$ and $b$ are formal power series in a finite number of noncommuting variables.

1. **Notations.** Let $R$ be a fixed not necessarily commutative ring with unit 1. For any finite set $Z$, $F(Z)$ is the free monoid generated by $Z$ and $R_{\text{pol}}(Z)$ is the free module on $F(Z)$ over $R$. An element $a$ of $R_{\text{pol}}(Z)$ will usually be written in the form $a=\sum\{(a,f)\cdot f: f\in F(Z)\}$ where the coefficients $(a,f)$ are in $R$; $R_{\text{pol}}(Z)$ is graded in the usual manner and $\pi_n a=\sum\{(a,f)\cdot f: f\in F(Z),\ \deg f\leqq n\}$. We identify $R$ with $\pi_0 R_{\text{pol}}(Z)$. $R_{\text{pol}}(Z)$ is also a ring with product $aa' =\sum\{(a,f')(a',f'')\cdot f: f,f',f''\in F(Z), f=f'f''\}$.

It is well known (cf., e.g., [4; **3**]) that these notions extend to the ring $R(Z)$ of the formal power series (with coefficients in $R$) in the noncommuting variables $z\in Z$; $R(Z)$ is topologized in the same manner as a ring of commutative formal power-series and $aa' =\lim_{n,n'\to\infty}(\pi_n a)(\pi_{n'}a')$. Any $b\in R^*(Z)=\{a\in R(Z):\pi_0 a=0\}$ has a quasi-inverse $(-b)^*=\lim_{n\to\infty}\sum_{n'<n}(-b)^{n'}$. If $a$ is invertible, $a^{-1}=(1+b^*)(\pi_0 a^{-1})$ where $b=-(\pi_0 a^{-1})(a-\pi_0 a)\in R^*(Z)$. We shall say that $S^*\subset R^*(Z)$ is *rationally closed* if $r, r'\in R$, $b, b'\in S^*$ imply $rb+b'r'$, $bb'$, $b^*\in S^*$. If this is so, the set of those elements $a$ of $R(Z)$ such that $a-\pi_0 a\in S^*$ is a ring containing the inverses of its invertible elements.

DEFINITION 1. $R^*_{\text{rat}}(X)$ is the least rationally closed subset (of $R(X)$) containing $X$.

Now let $Y=\{y_j\}$ be a set of a finite number $M$ of new variables and $R^M(X\cup Y)$ (resp. $R^M_{\text{pol}}(X\cup Y)$) the cartesian product of $M$ copies

Received by the editors December 6, 1961.

of the $R$-module $R(X \cup Y)$ (resp. $R^M_{\text{pol}}(X \cup Y)$). For each $q = (q_1, \cdots, q_m) \in R^M(X \cup Y)$, $\pi_n q = (\pi_n q_1, \cdots, \pi_n q_m)$. If $q \in R^{*M}(X \cup Y)$ (i.e., if $\pi_0 q = 0$) let $\lambda_q$ be the homomorphism of the monoid $F(X \cup Y)$ into the multiplicative monoid structure of $R(X \cup Y)$ that is induced by $\lambda_q x = x$ if $x \in X$ and $\lambda_q y_j = q_j$ if $y_j \in Y$. Since $\pi_0 q = 0$, $\lambda_q$ can be extended to an endomorphism of the $R$-module $R(X \cup Y)$ by $\lambda_q a = \sum\{(a, f)\lambda_q f : f \in F(X \cup Y)\}$; also, $\lambda_q p = (\lambda_q p_1, \cdots, \lambda_q p_M)$ for any $p \in R^M(X \cup Y)$.

We shall say that $p \in R^{*M}(X \cup Y)$ is a *proper system* if $(p_j, y_{j'}) = 0$ for all $j, j' \leqq M$. Then, if $q \in R^{*M}(X)$, $\lambda_q p \in R^{*M}(X)$ and $\pi_{n+1}\lambda_q p = \pi_{n+1}\lambda_{\pi_n q} p$ for all $n$. Consider now the infinite sequence $p(0) = 0$, $p(1) = \lambda_{p(0)} p, \cdots, p(m+1) = \lambda_{p(m)} p, \cdots$. Trivially, $\pi_{m'} p(m') = \pi_{m'} p(m'+m'') \in R^{*M}(X)$ for $m' = 0$ and all $m''$. If these relations hold for $m' \leqq m$, they still hold for $m+1$ because

$$\pi_{m+1} p(m+1) = \pi_{m+1}\lambda_{p(m)} p = \pi_{m+1}\lambda_{\pi_m p(m)} p = \pi_{m+1}\lambda_{\pi_m p(m+m'')} p$$
$$= \pi_{m+1}\lambda_{p(m+m'')} p = \pi_{m+1} p(m+1+m'').$$

Hence, $p(\infty) = \lim_{m\to\infty} p(m)$ exists and it satisfies $p(\infty) \in R^{*M}(X)$, $\pi_0 p(\infty) = 0$, $p(\infty) = \lambda_{p(\infty)} p$. In fact, $p(\infty)$ is the only element to satisfy these equations because if $\pi_0 p' = 0$ and $p' = \lambda_{p'} p$, any relation $\pi_m p(\infty) = \pi_m p'$ implies $\pi_{m+1} p' = \pi_{m+1}\lambda_{\pi_m p'} p = \pi_{m+1}\lambda_{\pi_m p(\infty)} p = \pi_{m+1} p(\infty)$. For this reason we call $p(\infty)$ *the solution* of $p$.

DEFINITION 2. $R^*_{\text{alg}}(X)$ is the least subset (of $R^*(X)$) that contains every coordinate of the solution of any proper system having its coordinates in $R^*_{\text{pol}}(X \cup Y)$.

(REMARK. It can easily be shown that $R^*_{\text{alg}}(X)$ is rationally closed and that it contains every coordinate of the solution of any proper system having its coordinates in $R^*_{\text{alg}}(X \cup Y)$.)

DEFINITION 3. For any

$$a, b \in R(X), \quad a \odot b = \sum\{(a, f)(b, f) \cdot f : f \in F(X)\}.$$

## 2. Main result.

*Property* 2.1. The element $a$ of $R^*(X)$ belongs to $R^*_{\text{rat}}(X)$ if and only if there exists a finite integer $N \geqq 2$ and a homomorphism $\mu$ of $F(X)$ into the multiplicative monoid of $R^{N\times N}$ (the ring of the $N \times N$ matrices with entries in $R$) such that $a = \sum\{\mu f_{1,N} \cdot f : f \in F(X)\}$ (abbreviated as $\sum \mu f_{1,N} \cdot f$).

PROOF. (1) *The condition is necessary.* This is trivial if $a = \pi_1 a$. Hence it suffices to show that for any $r, r' \in R$, $a = \sum \mu f_{1,N} \cdot f$ and $a' = \sum \mu' f_{1,N'} \cdot f$ one can construct suitable homomorphisms giving $ra + a'r'$, $aa'$ and $a^*$. This is done below, defining the homomorphisms by their restriction to $X$.

*Addition.* Let $N''=N+N'+2$ and $\mu''x\in R^{N''\times N''}$ defined for each $x\in X$ by

$$\mu''x_{i,1} = \mu''x_{N'',i} = 0 \qquad \text{for } 1 \leqq i \leqq N'';$$

$$\mu''x_{1,i+1} = r\mu x_{1,i} \quad \text{and} \quad \mu''x_{i+1,N''} = \mu x_{i,N} \qquad \text{for } 1 \leqq i \leqq N;$$

$$\mu''x_{1,i+N+1} = \mu' x_{1,i} \quad \text{and} \quad \mu''x_{i+N+1,N''} = \mu' x_{i,N'}\cdot r' \qquad \text{for } 1 \leqq i \leqq N';$$

$$\mu''x_{i,i'} = \text{the direct sum of } \mu x \text{ and } \mu' x \qquad \text{for } 2 \leqq i, i' \leqq N'' - 1;$$

$$\mu''x_{1,N''} = r\mu x_{1,N} + \mu' x_{1,N'}r'.$$

The verification is trivial.

*Product.* Let $N''=N+N'$ and define $\nu f\in R^{N''\times N''}$ for each $f\in F(X)$ by $\nu f_{i,i'}=\mu f_{i,N}$ if $f\neq 1$, $1\leqq i\leqq N$, $i'=N+1$; $\nu f_{i,i'}=0$, otherwise. Then, if $\mu''x=\bar{\mu}x+\nu x$ where $\bar{\mu}x$ is the direct sum of $\mu x$ and $\mu' x$, one has for each $f = x^{(1)}x^{(2)} \cdots x^{(n)}$, $\mu'' f = \bar{\mu} f + \sum\{\bar{\mu}f'\nu x^{(j)}\bar{\mu}f'' : f'x^{(j)}f'' = f\}$. Since $\nu f x^{(j)} = \bar{\mu} f \nu x^{(j)}$ and $(\nu f'''\bar{\mu}f'')_{1,N''}=0$ when $f''=1$, one has $\mu''f_{1,N''}=\sum\{(\mu f'_{1,N})(\mu' f''_{1,N'}) : f'f''=f\}$. Hence, $\sum \mu'' f_{1,N''}\cdot f = aa'$.

*Quasi-inverse.* Let $N''=N$ and define $\nu f\in R^{N\times N}$ for each $f\in F(X)$ by $\nu f_{i,i'}=\mu f_{i,N}$ if $f\neq 1$, $1\leqq i\leqq N$, $i'=1$; $\nu f_{i,i'}=0$, otherwise. Then $\mu''x = \mu x + \nu x$ and since $\mu f \nu x = \nu f x$ identically one has $\mu'' f = \sum \nu f^{(1)} \nu f^{(2)} \cdots \nu f^{(k)} \mu f^{(k+1)}$ where the summation is over all the factorisations $f=f^{(1)}f^{(2)} \cdots f^{(k+1)}$ of $f$ in an arbitrary number of factors. The $(1, N)$ entry of any of these products is zero unless all its factors are different from 1 and under this condition, it is equal to $\mu f^{(1)}_{1,N}\, \mu f^{(2)}_{1,N} \cdots \mu f^{(k+1)}_{1,N}$. Hence, $\sum \mu'' f_{1,N}\cdot f = \sum_{n>0} a^n = a^*$ and the first part of the proof is completed.

(2) *The condition is sufficient.* We say that the proper system $p$ is *linear* if for each $j\leqq M$, $p_j = q_{j,0} + \sum_{j'} q_{j,j'} y_{j'}$ where all the $q$'s belong to $R^*_{\text{rat}}(X)$ and we verify that all coordinates of the solution of such a system belong to $R^*_{\text{rat}}(X)$.

This is trivial if $M=1$ because $p(\infty) = (1-q_{1,1})^{-1} q_{1,0} (= (1+q^*_{1,1}) q_{1,0})$. If it is true for $M'<M$ it is still true for $M$. Indeed, because $p(\infty)_M = (1-q_{M,M})^{-1}(q_{M,0} + \sum_{j<M} q_{M,j'} p(\infty)_{j'})$, the proper linear system $p'$ defined by $p'_j = p_j - q_{j,M} y_M + q_{j,M} p_M$ for $j < M$ and $p'_M = (1-q_{M,M})^{-1}(p_M - q_{M,M} y_M)$ is such that $p(\infty) = p'(\infty)$. Since its first $M-1$ coordinates do not involve $y_M$ the result follows from the induction hypothesis.

Now, given a homomorphism $\mu$ of $F(X)$ into $R^{M\times M}$, the $M$ elements $a_j = \sum\{\mu f_{j,M}\cdot f : f\in F(X),\ f\neq 1\}$ are such that $(a_j, xf) = \sum_{j'} \mu x_{j,j'}(a_{j'}, f)$. Hence $(a_1, \cdots, a_M)$ is the solution of the linear proper system such that $q_{j,0} = \sum\{\mu x_{j,M}\cdot x : x \in X\}$, $q_{j,j'} = \sum\{\mu x_{j',j}\cdot x : x\in X\}$ for each $j, j'$ and 2.1 is proved.

We now consider two subrings $R'$ and $R''$ of $R$ that commute element-wise.

*Property* 2.2. If $a=\sum \mu' f_{1,N}\cdot f \in R'^*_{\text{rat}}(X)$ where $\mu'$ is a homomorphism into $R'^{N\times N}$ and if $b=p(\infty)_1 \in R''^*_{\text{alg}}(X)$ where the proper system $p$ has its coordinates in $R''^*_{\text{pol}}(X\cup Y)$, then $a \odot b \in R^*_{\text{alg}}(X)$. If, further, $b\in R''^*_{\text{rat}}(X)$ then $a \odot b \in R^*_{\text{rat}}(X)$.

PROOF. We verify first the case of $b\in R''^*_{\text{rat}}(X)$, i.e., of $b=\sum \mu'' f_{1,N''}\cdot f$ for some $N''$ and $\mu''$. Then $a \odot b = \sum (\mu'\otimes\mu'') f_{1,NN''}\cdot f$ where the kroneckerian product $\mu'\otimes\mu''$ is a homomorphism of $F(X)$ into $R^{NN''\times NN''}$ because $R'$ and $R''$ commute and the result is proved.

For the general case we denote by $K(Z)$ for any set $Z$ the ring of the $N\times N$ matrices with entries in $R(Z)$. We shall have to consider several homomorphisms of module $\sigma: R^M(Z')\to K^M(Z'')$ where $Z'$ and $Z''$ are two finite sets. In each case $\sigma$ is defined by a mapping $Z'\to K(Z'')$ which is extended in a natural fashion to a homomorphism of the monoid $F(Z')$ into the multiplicative structure of $K(Z'')$. Then for each

$$a = (a_1, \cdots, a_M) \in R^M(Z'), \quad \sigma a_j = \sum\{(a_j, g)\cdot \sigma g\colon g \in F(Z')\}$$

and $\sigma a = (\sigma a_1, \cdots, \sigma a_M)$.

More specifically, $\mu: R^M(X)\to K^M(X)$ is induced by a mapping $\mu: X\to K(X)$ such that the entries of each $\mu x$ belong to $R'^*(X)$.

For each $q\in R''^{*M}(X)$, $\lambda_{\mu q}: R(X\cup Y)\to K^M(X)$ is induced by $\lambda_{\mu q} f = \mu f$ if $f\in F(X)$ and $\lambda_{\mu q} y_j = \mu q_j$ if $y_j\in Y$. Hence, since $R'$ and $R''$ commute element-wise, $\mu\lambda_q g = \lambda_{\mu q} g$ for each $g\in F(X\cup Y)$ (with $\lambda_q$ as previously defined). Consequently, $\mu\lambda_q p = \lambda_{\mu q} p$ for any $p\in R''^M(X\cup Y)$.

Let now $Z=\{z_{j,i,i'}\}$ $(1\leqq j\leqq M;\ 1\leqq i,\ i'\leqq N)$, a set of $M\times N\times N$ new variables and $\nu: R^M(X\cup Y)\to K^M(X\cup Z)$ induced by $\nu f=\mu f$ if $f\in F(X)$, $\nu y_j=$ the $N\times N$ matrix with entries $z_{j,i,i'}$ if $y_j\in Y$. Also $\lambda_{\nu q}: R(X\cup Z)\to R(X)$ is induced by $\lambda_{\nu q} f = f$ if $f\in F(X)$ and $\lambda_{\nu q} z_{j,i,i'} = (\nu q_j)_{i,i'}$ if $z_{j,i,i'}\in Z$. We extend $\lambda_{\nu q}$ to a homomorphism $K^M(X\cup Z)\to K^M(X)$ by defining $\lambda_{\nu q}\boldsymbol{m}$ for any $\boldsymbol{m}\in K(X\cup Z)$ as the $N\times N$ matrix with entries $\lambda_{\nu q}(\boldsymbol{m}_{i,i'})$.

Because $R'$ and $R''$ commute, $\lambda_{\mu q} g = \lambda_{\nu q}\nu g$ for each $g\in F(X\cup Y)$ and, consequently, $\lambda_{\mu q} p = \lambda_{\nu q}\nu p$ for each $p\in R''^{*M}(X\cup Y)$. Hence, if $p$ is a proper $M$-dimensional system with coordinates in $R''^*(X\cup Y)$ we have $\mu p(\infty) = \mu\lambda_{p(\infty)} p = \lambda_{\mu p(\infty)} p$. Since $\mu$ and $\nu$ coincide on $R''^{*M}(X)$, we have also $\mu p(\infty) = \nu p(\infty) = \lambda_{\mu p(\infty)} p = \lambda_{\nu p(\infty)}\nu p$.

However, the $M\times N\times N$ elements $p'_{j,i,i'} = (\nu p_j)_{i,i'}$ all belong to $R^*(X\cup Z)$ and they constitute a proper system $p'$ of dimension $MN^2$. Thus, by construction, $(\mu p(\infty)_j)_{i,i'} = p'(\infty)_{j,i,i'}$ identically. If, further, $p \in R''^{*M}_{\text{pol}}(X \cup Y)$ all the entries appearing in $\nu p$ belong to $R^*_{\text{pol}}(X \cup Z)$ and then finally $(\mu p(\infty)_j)_{i,i'} \in R^*_{\text{alg}}(X)$.

This completes the proof because

$$a \odot b = \sum \{(b, f)\mu' f_{1,N} \cdot f : f \in F(X)\}$$
$$= \sum \{(b, f)\mu f_{1,N} : f \in F(X)\} = \mu b_{1,N}$$

where for each $x \in X$, $\mu$ is defined by $\mu x_{i,i'} = \mu' x_{i,i'} \cdot x$.

REMARK 1. Definitions 1, 2, and 3 and the computations of this section used only the structure of monoid of the additive groups considered. Hence, the results are still valid when an arbitrary *semiring* $S$ is taken in place of $R$. For $S$ consisting of two Boolean elements, Jungen's theorem and its special case for $b$ rational have been obtained in a different form by Y. Bar-Hillel, M. Perles and E. Shamir [1] (also by S. Ginsburg and G. F. Rose [5]) and by S. Kleene [8] respectively as by-products of more sophisticated theories.

REMARK 2. Let $R = C$, the field of complex numbers; and $p$ a proper system of dimension $M$. Introducing $4M$ new symbols $z_j$ and replacing each $y_j$ by $z_{4j} + iz_{4j+1} - z_{4j+2} - iz_{4j+3}$ in the $p_j$s we can deduce from $p$ a new system of dimension $4M$ in which all the coefficients are non-negative real numbers and whose solution is simply related to $p(\infty)$.

Assume now that $p \in C^{*M}_{\text{pol}}(X \cup Y)$ has only real non-negative coefficients and denote by $\alpha$ a homomorphism of $C_{\text{pol}}(X \cup Y)$ into $C$. Because of the assumption that $(p_j, y_{j'}) = (p_j, 1) = 0$, identically, we can find an $\epsilon > 0$ such that $|\alpha p_j| < \epsilon$ for all $j$ when $|\alpha x| \leqq \epsilon$ and $|\alpha y| \leqq 2\epsilon$ for all $x \in X$ and $y \in Y$. Since the sequence $\alpha p(0)$, $\alpha p(1)$, $\cdots$, $\alpha p(n)$, $\cdots$ is monotonically increasing it converges to a finite solution (cf., e.g., [10]).

Hence, the canonical epimorphism of $C_{\text{pol}}(X \cup Y)$ onto the ring of the ordinary (commutative) polynomials can be extended to an epimorphism of $C_{\text{alg}}(X)$ onto the ring of the Taylor series of the algebraic functions.

**Acknowledgment.** Acknowledgment is made to the Commonwealth Fund for the grant in support of the visiting professorship of biomathematics in the Department of Preventive Medicine at Harvard Medical School.

## REFERENCES

1. Y. Bar-Hillel, M. Perles and E. Shamir, *On formal properties of simple phrase structure grammars*, Technical Report No. 4. Information System Branch, Office of Naval Research, 1960.

**2.** S. Bochner and W. T. Martin, *Singularities of composite functions in several variables*, Ann. of Math. **38** (1938), 293–302.

**3.** R. H. Cameron and W. T. Martin, *Analytic continuation of diagonals*, Trans. Amer. Math. Soc. **44** (1938), 1–7.

**4.** K. T. Chen, R. H. Fox and R. C. Lyndon, *Free differential calculus*. IV, Ann. of Math. (2) **68** (1958), 81–95.

**5.** S. Ginsburg and G. F. Rose, *Operations which preserve definability*, System Development Corporation, Santa Monica, Calif., SP-511, October, 1961.

**6.** U. S. Haslam-Jones, *An extension of Hadamard multiplication theorem*, Proc. London Math. Soc. II. Ser. **27** (1928), 223–232.

**7.** R. Jungen, *Sur les series de Taylor n'ayant que des singularités algébrico-logarithmiques sur leur cercle de convergence*, Comment. Math. Helv. **3** (1931), 226–306.

**8.** S. Kleene, *Representation of events in nerve nets and finite automata*, Automata Studies, Princeton Univ. Press, Princeton, N. J., 1956.

**9.** M. Lazard, *Lois de groupes et analyseurs*, Ann. Sci. Ecole Norm. Sup. (4) **72** (1955), 299–400.

**10.** A. M. Ostrowski, *Solutions of equations and systems of equations*, Academic Press, New York, 1960.

HARVARD MEDICAL SCHOOL