1. Post-Doc Training I (vers. 20-07-2020 10:40)

1.1. Exercise 1 & ff

Ex1) The aim of this first problem is to show that $T = \log(1+x_0)x_1$ is not in $Dom^{loc}(\mathrm{Li})$.

a) Give, for each length, the homogeneous component $T_n = \sum_{|u|=n} \langle T \mid u \rangle u$

b) Compute Li_{T_n} and show that it is Taylor developable for $|z| < 1$, give the coefficients.

c) Is $\sum_{n \geq 0} \mathrm{Li}_{T_n}$ convergent (compactly in $D_{<1}$) ?

d) Is it absolutely convergent ?

1.2. Exercise 2 & ff

Ex2) The aim of this second problem is to understand better certain parts of your two Ph. D.

a) Give the definition of a *summable family* of (noncommutative formal power) series [1].

b) On tries to define the logarithm of a letter as $\log(x_0)$

  i) Compute formally setting $x_0 = 1 + (x_0 - 1)$, one should obtain a sum $\sum_{n \geq 1} a_n (x_0 - 1)^n$

  ii) Explain why $(a_n (x_0 - 1)^n)_{n \geq 1}$ is **not** summable.

c) Let X be an alphabet, R a $\mathbb{Q}$-algebra and $S \in R\langle\langle X \rangle\rangle$ be a proper series (i.e. such that $\langle S \mid 1_{X^*} \rangle = 0$, their set is usually denoted $R_+\langle\langle X \rangle\rangle$).

  i) Explain why, for each one-variable series $\varphi = \sum_{n \geq 0} c_n z^n \in R[[z]]$, the family $(c_n S^n)_{n \geq 0}$ is summable. One will note the result $\varphi(S)$.

  ii) Show carefully that $\exp(\log(1_{X^*} + S)) = 1_{X^*} + S$ and $\log(\exp(S)) = S$

d) (From now on $z \in \mathbb{C}$) What is the radius of convergence of $\log(1+z) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} z^n$?

e) Explain how to extend this function to $\mathbb{C} \setminus]-\infty, 0]$ (concretely)

f) (Difficult) Give the domains and images of the complex functions $\exp, \log$ and explain why they are **not** inverse one from the other, give the domain of $\exp \circ \log$ and $\log \circ \exp$.

Task (2) Blah 1

[1] In your Ph.D.(s) (C,26), in french *sommable*.

1.3. Exercise 3 & ff (Combinatorial ranking & use of asymptotic expansions).

Ex3) Let $q \geq 1$ be an integer, we form the sequence $(v_k)_{k\geq 1}$ by taking q odd numbers in increasing order (starting from 1), then one even number, then the q following odd numbers and the following even number and again and again and again ... For example, with $q = 3$, it reads

$$1, 3, 5, \mathbf{2}, 7, 9, 11, \mathbf{4}, 13, 15, 17, \mathbf{6}, 19, \ldots \quad (1)$$

in general, it reads

$$1, \ldots, 2q-1, \mathbf{2}, 2q+1, 2q+3, \ldots 4q-1, \mathbf{4}, 4q+1, \ldots 6q-1, \mathbf{6}, 6q+1, \ldots \quad (2)$$

a) For each k, compute v_k
(two cases $k = m(q+1)$ and $k = m(q+1) + r,\ 1 \leq r \leq q$).

We set $v_k = \sigma(k)$

b) Show that σ is a bijection $\mathbb{N}_+ \to \mathbb{N}_+$ (here $\mathbb{N}_+ = \mathbb{N}_{\geq 1}$)
Hint: Buid a cadidate τ for the inverse of σ, for example, with $q = 3$, $(\tau(k))_{k\geq 1}$ reads

$$1, \mathbf{4}, 2, \mathbf{8}, 3, \mathbf{12}, 5, \mathbf{16}, 6, \ldots$$

and show that $\sigma \circ \tau = \tau \circ \sigma = Id_{\mathbb{N}_+}$.

c) For $q = 2$, compute $\sum_{k\geq 1} \frac{(-1)^{\sigma(k)-1}}{\sigma(k)}$

d) Compute $\sum_{k\geq 1} \frac{(-1)^{\sigma(k)-1}}{\sigma(k)}$ for general $q \geq 1$.
Hint:

- For partial sums, have a look here
  `https://en.wikipedia.org/wiki/Series_(mathematics)`
- Evaluate N packets of $(q+1)$ terms $S_N = \sum_{k=1}^{N(q+1)} \frac{(-1)^{\sigma(k)-1}}{\sigma(k)}$.
  Then use the original Euler expansion
  `https://en.wikipedia.org/wiki/Euler%E2%80%93Mascheroni_constant`
  $$\gamma \sim H_n - \ln n - \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \cdots$$
- Not forget to mention that the general term tends to zero (think why)

e) (General, bisection method, warming) We consider a partition of $\mathbb{N}_+$ into two infinite subsets i.e. $\mathbb{N}_+ = U_0 \sqcup V_0$ (see Comments & refs Section2.1, item

1) and a (boolean choice) function $h : \mathbb{N}_+ \to \{0,1\}$. We construct a map $\varphi = \varphi_h : \mathbb{N}_+ \to \mathbb{N}_+$ by

$$U_0 = U;\ V_0 = V \quad ; \qquad \text{Init. Data}$$

$$\varphi(k) = \begin{Bmatrix} \min(U_{k-1}) & \text{if} & h(k) = 0 \\ \min(V_{k-1}) & \text{otherwise} & (h(k) = 1) \end{Bmatrix} \qquad \text{Rec. Function}$$

$$U_k = U_{k-1} \setminus \{\varphi(k)\}; \quad V_k = V_{k-1} \setminus \{\varphi(k)\} \qquad \text{Rec. Data}$$

For example, for $U = \{2k-1\}_{k\geq 1}, V = \{2k\}_{k\geq 1}$ (odd and even numbers) and with h the choice function (written as a string) of period 7

$$\underbrace{0001010}_{period}000101000010100001010000101000010 10\ldots \tag{3}$$

One has φ (also written as a string)

$$\underbrace{1,3,5,2,7,4,9}_{\substack{corresponding to\\ the first period}},\underbrace{11,13,15,6,17,8,19}_{},21,23,25,10,27,12,29,\ldots \tag{4}$$

Explain the data and choice function $h : \mathbb{N}_+ \to \{0,1\}$ used to construct the bijection σ above.

f) Prove that φ is injective in all cases. Here is a training to (correct) proofs. A proof of degree 3. (see below) is required.
The string to be used as reference expresses that φ is injective (or into). It reads

$$(\forall m,n \in \mathbb{N}_+)((\varphi(m) = \varphi(n)) \implies (m = n)) \tag{5}$$

g) Give a construct h for which φ is not surjective.

h) (Difficult) We suppose that h is not ultimately constant, i.e.

$$(\forall n \in \mathbb{N}_+)(\exists m > n)(h(n) \neq h(m)) \tag{6}$$

prove that φ is surjective (onto).

i) Until the end of this training, we use the partition of $\mathbb{N}_+$ in odd and even numbers (then $U = \{2k-1\}_{k\geq 1}, V = \{2k\}_{k\geq 1}$) and, for $p,q \geq 1$, $\varphi_{p,q}(k)$ is constructed such that we take

- p odd numbers
- q even numbers
- and again ...

for example, the list $(\varphi_{3,2}(k))_{k\geq 1}$ reads

$$1,3,5,2,4,7,9,11,6,8,13,15,17,10,12,\cdots \tag{7}$$

give explicitely $h_{p,q}(k)$ and $\sigma_{p,q}$ constructed with these data.

j) For $\varphi = \varphi_{p,q}$, prove that $\sum_{k=1}^{\infty} \frac{(-1)^{\sigma(k)-1}}{\sigma(k)}$ is convergent and give its limit $l_{p,q}$ (use Euler formula).

1.4. Exercise 4 & ff (Eilenberg-Schützenberger quotients)

Ex4) For

- $S \in R\langle\langle X \rangle\rangle$ (R ring, X alphabet)
- $u \in X^*$

one defines $u^{-1}S$ (resp. Su^{-1})[2] by

$$u^{-1}S := \sum_{w \in X^*} \langle S \mid uw \rangle\, w \text{ and } Su^{-1} := \sum_{w \in X^*} \langle S \mid wu \rangle\, w \tag{8}$$

a) Show that, when $S = w$ (a single word i.e. a noncommutative monomial), one has

$$u^{-1}w = \begin{Bmatrix} v & \text{if} & w = uv \\ 0 & \text{otherwise.} & \end{Bmatrix} \quad \text{and} \quad wu^{-1} = \begin{Bmatrix} v & \text{if} & w = vu \\ 0 & \text{otherwise.} & \end{Bmatrix}$$

b) Prove that

$$u^{-1}S := \sum_{w \in X^*} \langle S \mid w \rangle\, u^{-1}w \text{ and } Su^{-1} := \sum_{w \in X^*} \langle S \mid w \rangle\, wu^{-1} \tag{9}$$

c) (From now on, we suppose that R is a $\mathbb{Q}$-algebra.)
Show that

$$\pi_X\Big(\sum_{k \geq 1} \frac{(-1)^{k-1}}{k} y_k\Big) = x_0^{-1}\big(\log(1 + x_0)x_1\big)$$

d) Compute $\big(\log(1 + x_0)x_1\big)x_0^{-1}$ and deduce that the operators x_0^{-1} ? and ? x_0^{-1} are different.

[2]Respectively known as left and right quotient by u.

1.5. Exercise 5 & ff (Understanding the wonderful identity)

Ex5) The aim of this exercise is to establish a "Holomorphic functional calculus" [3] within the stuffle algebra, taking the proper series as arguments. Here, we set $Y = \{y_i\}_{i\geq 1}$ and assume that the identity

$$(\sum_{i\geq 1} \alpha_i\, y_i)^* \uplus (\sum_{j\geq 1} \beta_j\, y_j)^* = (\sum_{i\geq 1} \alpha_i\, y_i + \sum_{j\geq 1} \beta_j\, y_j + \sum_{i,j\geq 1} \alpha_i\beta_j\, y_{i+j})^* \qquad (10)$$

has been proved elsewhere. As the alphabet is infinite we will use homogeneous series of degree one as

$$\sum_{i\geq 1} \alpha_i\, y_i \qquad (11)$$

this sum is not necessarily finite (it is, in general, a series) but can be so. Series like (11) form a vector space (called by Pr. Schützenberger "the plane of letters"), noted in our works $\widehat{\mathbb{C}.Y}$.

a) (Warming: log-exp correspondence) Let S be a proper series[3] and $T = \sum_{n\geq 0} a_n z^n \in \mathbb{C}[[z]]$, prove that $(a_n S^{\uplus\, n})_{n\geq 0}$ is summable (see below (2.2) and use the weight).

Definition 1. *For $T \in \mathbb{C}[[z]]$ and $S \in \mathbb{C}_+\langle\langle Y\rangle\rangle$, we note*

$$T_{\uplus}(S) := \sum_{n\geq 0} \langle T \mid z^n\rangle\, S^{\uplus\, n} \qquad (12)$$

b) For $S \in \mathbb{C}_+\langle\langle Y\rangle\rangle$, show that

$$\log_{\uplus}(1_{Y^*} + S) \text{ and } \exp_{\uplus}(S) - 1_{Y^*} \text{ belong to } \mathbb{C}_+\langle\langle Y\rangle\rangle$$

and **prove carefully that**

$$\exp_{\uplus}(\log_{\uplus}(1_{Y^*} + S)) = 1_{Y^*} + S \text{ and } \log_{\uplus}(\exp_{\uplus}(S)) = S \qquad (13)$$

c) (Commutation and polynomial type coefficients) For $S, T \in \mathbb{C}_+\langle\langle Y\rangle\rangle$ and $P(z) \in \mathbb{C}[z]$, prove carefully that

$$\exp_{\uplus}(S + T) = \exp_{\uplus}(S) \uplus \exp_{\uplus}(T)\ ;\ \exp_{\uplus}(P(z).S) \in \mathbb{C}[z]\langle\langle Y\rangle\rangle \qquad (14)$$

and[4]

$$\mathbf{d}(\exp_{\uplus}(P(z).S)) = (P'(z).S) \uplus \exp_{\uplus}(P(z).S) \qquad (16)$$

[3] i.e. such that $\langle S \mid 1_{Y^*}\rangle = 0$

[4] An alphabet $\mathcal{X}$ being given, one can at once extend the derivation $\frac{d}{dz}$ to a derivation of the algebra $\mathbb{C}[z]\langle\langle \mathcal{X}\rangle\rangle$ by

$$\mathbf{d}(S) = \sum_{w\in X^*} \frac{d}{dz}(\langle S \mid w\rangle)w\ . \qquad (15)$$

d) (Coding the plane by Umbral calculus, see below section (2.3)) Let x be a auxiliary letter. Show that the map

$$\pi_Y^{Umbra} : \sum_{n\geq 1} \alpha_n\, x^n \mapsto \sum_{n\geq 1} \alpha_n\, y_n \tag{17}$$

from $\mathbb{C}_+[[x]]$ to $\widehat{\mathbb{C}.Y}$ is linear and bijective. We will call π_x^{Umbra} its inverse.

e) For $S, T \in \mathbb{C}_+[[x]]$, show that

$$(\pi_Y^{Umbra}(S))^* \uplus (\pi_Y^{Umbra}(T))^* = (\pi_Y^{Umbra}((1+S)(1+T)-1))^* \tag{18}$$

f) For $z \in \mathbb{C}$ and $T \in \mathbb{C}_+[[x]]$, one sets

$$G(z) = (\pi_Y^{Umbra}(e^{z.T}-1))^* \tag{19}$$

from (18) deduce that, for $z_1, z_2 \in \mathbb{C}$, one has

$$G(z_1+z_2) = G(z_1) \uplus G(z_2)\ ;\ G(0) = 1_{Y^*} \tag{20}$$

(this is called a "stuffle one parameter group").

g) Prove that

$$\frac{d}{dz}(G(z)) = (\pi_Y^{Umbra}(T)) \uplus G(z) \tag{21}$$

and deduce that

$$G(z) = e_{\uplus}^{z.\pi_Y^{Umbra}(T)} \tag{22}$$

h) From what precedes, show that, for each $P = \sum_{i\geq 1}\langle P \mid y_i\rangle\, y_i \in \widehat{\mathbb{C}.Y}$

$$\log_{\uplus}(P^*) = \pi_Y^{Umbra}(\log(1+\pi_x^{Umbra}(P))) \tag{23}$$

i) Using (23), show that

$$(ty_k)^* \quad = \quad \exp_{\uplus}\Big(\sum_{n\geq 1}\frac{(-1)^{n-1}t^n y_{nk}}{n}\Big) \tag{24}$$

and compute $(2ty_1 + t^2y_2)^*$ under the form of an exponential.

2. Q & A

2.1. Comments & refs

1. Bisections were introduced by Berstel and Perrin for the theory of codes [1] and by Viennot for the theory of flips (*bascules* in french) [4] (to construct new bases of the free Lie algebra by successive eliminations i.e. Lazard codes).

Some references here

[1] Jean Berstel and Dominique Perrin, *Theory of codes*, Pure and Applied Mathematics, **117**, Academic Press, Inc., New York, 1985

[2] N. Bourbaki, *Theory of sets*, Springer-Verlag Berlin Heidelberg 2004

[3] `https://en.wikipedia.org/wiki/Holomorphic_functional_calculus`

[4] Xavier Viennot, *Factorisations des monoïdes libres, bascules et algèbres de Lie libres*, Séminaire Dubreil : Algèbre, 25e année, 1971/72, Fasc. 2 : Journées sur les anneaux et les demi-groupes [1972. Paris], **J5** | Numdam | MR 419649 | Zbl 0355.20059

[5] Jean Dieudonné, *Infinitesimal calculus*, Houghton Mifflin (1971)

[6] Christophe Reutenauer, *Free Lie Algebras*, Université du Québec a Montréal, Clarendon Press, Oxford (1993)

2.2. Summable families

In [6] is given a very simple definition of *summable families* in the context of formal ppower series. Let R be a ring and $\mathcal{X}$ an alphabet. We say that a family $(S_j)_{j\in J}$ is *summable* if, for each word $w \in \mathcal{X}^*$, the function $j \mapsto \langle S_j \mid w\rangle$ from J to R is finitely supported. The sum $\sum_{j\in J} S_j$ is by definition the series

$$S = \sum_{w\in\mathcal{X}^*} \big(\sum_{j\in J}\langle S_j \mid w\rangle\big)\, w \tag{25}$$

Let $u_{mn} = 1/(m^2 - n^2)$ if $m \neq n$ and $u_{mn} = 0$ if $m = n$.

Note that

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \sum_{m=1, m\neq n}^{\infty} \frac{1}{m^2 - n^2} \neq \sum_{m=1}^{\infty} \sum_{n=1, n\neq m}^{\infty} \frac{1}{m^2 - n^2} = -\frac{\pi^2}{12}.$$

By anti-symmetry, the double sum changes sign with an interchange of indices.

To find the sum, use

$$\sum_{m=1,m\neq n}^{\infty} \frac{1}{m^2-n^2} = \lim_{M\to\infty} \frac{1}{2n} \sum_{m=1,m\neq n}^{M} \left(\frac{1}{m-n} - \frac{1}{m+n}\right) \tag{26}$$

$$= \lim_{M\to\infty} \frac{1}{2n} \left(\sum_{k=1}^{M-n} \frac{1}{k} - \sum_{k=1}^{n-1} \frac{1}{k} - \sum_{k=n+1}^{M+n} \frac{1}{k}\right) \tag{27}$$

$$= \lim_{M\to\infty} \frac{1}{2n} \left(\frac{1}{n} - \frac{1}{M-n+1} - \ldots - \frac{1}{M+n}\right) \tag{28}$$

$$= \frac{1}{2n^2} \tag{29}$$

2.3. Umbral calculus

Technique used in combinatorics consisting in the substitution $x^n \mapsto B_n$ (lowering exponents). Have a look there

`https://en.wikipedia.org/wiki/Umbral_calculus`

Here we use the coding $x^n \mapsto y_n$.

2.4. Proofs by degree of security

1. (the lowest and NOT a proof) Computer program (can detect errors however and acts rather as a check), the highest order the best (that's why, in Chiên's PhD, we mentioned that the computations were done "up to order twelve").

2. Proofs in natural language (arguments: "then | thus | however | we have" &c.) acts on the base of convincing, but not always sound (people can be swayed by strong words or statements).

3. Formalized proofs of proofs based on formalized strings. High degree of security. Formalized strings look like

$$(\forall w \in X^*)(i \mapsto \langle S_i \mid w \rangle \text{ is finitely supported})$$

   or better (for logic)

$$(\forall K \subset_{compact} \Omega))(\forall \epsilon > 0)(\exists \alpha_0 \in A)(\forall \alpha \geq \alpha_0)(||f - f_\alpha||_K < \epsilon)$$

4. Certified proofs (our future :), have a look at Thomas Fernique's talk here

   `https://www-lipn.univ-paris13.fr/~duchamp/Conferences/CAP6_2019.html`

   and the story of Thomas Hales here

   `https://en.wikipedia.org/wiki/Kepler_conjecture`

5. Proofs certified by two or more independant programs.

2.5. Training to proofs

I begin in citing the introduction of [2] which gives a good glimpse of proof-checking.

If, as happens again and again, doubts arise as to the correctness of the text under consideration, they concern ultimately the possibility of translating it unambiguously into such a formalized language : either because the same word has been used in different senses according to the context, or because the rules of syntax have been violated by the unconscious use of modes of argument which they do not specifically authorize, or again because a material error has been committed. Apart from this last possibility, the process of rectification, sooner or later, invariably consists in the construction of texts which come closer and closer to a formalized text until, in the general opinion of mathematicians, it would be superfluous to go any further in this direction. In other words, the correctness of a mathematical text is verified by comparing it, more or less explicitly, with the rules of a formalized language.

2.6. Hamel basis

Every k-vector space W admits a basis. Even more, in ZFC (and many systems) every k-free family $(v_i)_{i\in I}$ can be completed to a (larger) family $\mathcal{B} = (v_i)_{i\in J}$ $(I \subset J)$ such that $\mathcal{B}$ is a basis of W. For example $(1, \sqrt{2})$ is $\mathbb{Q}$-free in $\mathbb{R}$ (prove this by Euclid's argument or using continued fractions, it then exists a basis of $\mathbb{R}$ (as a $\mathbb{Q}$-vector space) $\mathcal{B} = (v_i)_{i\in J}$ and $j_1, j_2 \in J$ such that $v_{j_1} = 1, v_{j_2} = \sqrt{2}$. So one can define $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$(\forall x, y \in \mathbb{R})(\varphi(x+y) = \varphi(x) + \varphi(y) \text{ and } \varphi(1) = 0,\ \varphi(\sqrt{2}) = 1) \tag{30}$$

this φ cannot be of the form $f(x) = a \cdot x$. It is *additive* but not *linear*.

2.7. Statements Jig

Proposition 1. *i) Let $(S_i)_{i\in I}$ (resp. $(T_j)_{j\in J}$ be summable families*[5]*, then $(S_i \sqcup\!\!\sqcup T_j)_{(i,j)\in I\times J}$ is summable and*

$$\sum_{(i,j)\in I\times J} S_i \sqcup\!\!\sqcup T_j = (\sum_{i\in I} S_i) \sqcup\!\!\sqcup (\sum_{j\in J} T_j) \tag{31}$$

ii) We suppose now that $I = J = \mathbb{N}$[6]*, then $(\sum_{p+q=n} S_p \sqcup\!\!\sqcup T_q)_{n\geq 0}$ is summable and*

$$(\sum_{p\in\mathbb{N}} S_p) \sqcup\!\!\sqcup (\sum_{q\in\mathbb{N}} T_q) = \sum_{n\geq 0}(\sum_{p+q=n} S_p \sqcup\!\!\sqcup T_q) \tag{32}$$

Proposition 2. *For $S \in \mathbb{C}\langle\langle Y\rangle\rangle$, we define $\varpi(S)$ to be the least weight of the words in the support.*

$$\varpi(S) := \min\{(w) | w \in supp(S)\} \in \mathbb{N} \cup \{+\infty\}$$

i) For $S, T \in \mathbb{C}\langle\langle Y\rangle\rangle$

$$\varpi(S \sqcup\!\!\sqcup T) \geq \varpi(S) + \varpi(T)$$

where usual sum is extended as follows

$+$	p	$+\infty$
q	$p+q$	$+\infty$
$+\infty$	$+\infty$	$+\infty$

ii) Let $S \in \mathbb{C}_+\langle\langle Y\rangle\rangle$ (a proper series, then), $(a_n S^{\sqcup\!\!\sqcup n})_{n\geq 0}$ is summable.

[5] Series not necessarily proper.
[6] and still that $(S_i)_{i\in\mathbb{N}}$ (resp. $(T_j)_{j\in\mathbb{N}}$ are summable.

2.8. Corrections

Of course, as you are (like everyone of us four) on an "improvement track", the title of this section means:

Corrections, examples and discussion

so never hesitate if you see something to improve or clarify.

Ex3 e) f) g) and h). —
e) The data are

$$\begin{cases} U = \{2k-1\}_{k\geq 1};\ V = \{2k\}_{k\geq 1} & & \text{the odd and even numbers} \\ h(k) = 0 & \text{if} & k \neq 0 \mod q+1 \\ h(k) = 1 & \text{otherwise.} & \end{cases}$$

(U, V) is a partition as $U \cap V = \emptyset$ and $U \cup V = \mathbb{N}_+$.
f) Prove that φ is injective in all cases.
We will prove injectivity by the equivalent property (contraposition of the given string)

$$(\forall m, n \in \mathbb{N}_+)((m \neq n) \Longrightarrow (\varphi(m) \neq \varphi(n))) \tag{33}$$

We choose $m, n \in \mathbb{N}_+$ with $m \neq n$ and have to prove that $\varphi(m) \neq \varphi(n)$.
Suppose that $m < n$ (the other case $n < m$ being similar) then set
$p = \#\{m, n\} \cap \varphi^{-1}(U)$ we have three cases

- $p = 0$ then $\{m, n\} \subset \varphi^{-1}(V)$ and, by construction φ is strictly increasing on $\varphi^{-1}(V)$ [7] (i.e. m is choosen strictly before n) then $\varphi(m) < \varphi(n)$, this proves the result $(\varphi(m) \neq \varphi(n))$ in this case.
- $p = 1$ then m, n are in separate blocks of the partition $[\varphi^{-1}(U), \varphi^{-1}(V)]$ their images are in separate blocks of $[U, V]$.
- $p = 2$ similar to the case $p = 0$ replacing $\varphi^{-1}(V)$ by $\varphi^{-1}(U)$.

g) Give a construct h for which φ is not surjective.
It is the case if h is ultimately constant i.e. that it exists $N > 0$ from which $h(n) = h(N)$, formally

$$(\exists N > 0)(\forall n \geq N)(h(n) = h(N)) \tag{34}$$

as a string

$$h(1)\ldots h(N)h(N)h(N)h(N)h(N)h(N)\ldots \tag{35}$$

set $U = \{u_k\}_{k\geq 1}$ (resp. $V = \{v_k\}_{k\geq 1}$) in strictly increasing order (this is possible because U, V are both infinite). Suppose $h(N) = 0$ (the other case is similar), it exists $k_0 > 0$ such that $\varphi(N) = u_{k_0}$ and, by construction $\varphi(N + r) = u_{k_0+r}$ (from rank N,

[7] and (separately) on $\varphi^{-1}(V)$

all elements are choosen in U), then only a finite number of elements are choosen in V, setting $F = \varphi(\mathbb{N}_+) \cap V$ (a finite set), the image of φ is $\varphi(\mathbb{N}_+) = U \cup F \neq \mathbb{N}_+$.

h) In the case h is not ultimately constant, prove that φ is surjective (onto).

This is the case when h always changes i.e., written as a string, h is not of the form $w.0^\infty$ or $w.1^\infty$ (w being an initial string). We again write $U = \{u_k\}_{k\geq 1}$ (resp. $V = \{v_k\}_{k\geq 1}$) in strictly increasing order and prove that every element of U (resp. V) is choosen. Let $U_{choose} = \varphi(\mathbb{N}_+) \cap U$, either $U_{choose} = U$ and we are done or $U_{choose} \subsetneq U$, in this last case, let

$$k_0 = \inf\{k \,|\, u_k \notin U_{choose}\} \tag{36}$$

(i.e. the first index for which u_k would not be choosen). Then $k_0 = 1$ is impossible because if j_0 is the first number $j \in \mathbb{N}_+$ for which $h(j) = 0$ (there is one because h is not ultimately constant), we have, by construction, $\varphi(j_0) = u_1$, then $k_0 \geq 2$, but, from the definition of k_0 we have $u_{k_0-1} \in U_{choose}$ and then $u_{k_0-1} = \varphi(m)$ for some m (hence $h(m) = 0$) now, as h is not ultimately constant, there exist indices $m' > m$ such that $h(m') = 0$ again (otherwise h would finish by 1^∞), let m'_0 be the least of these m' ($m' > m$ such that $h(m') = 0$ again), then as $(\forall k \in]m, m'[)(h(k) = 1)$ we must have $\varphi(k_0) = u_{k_0}$, a contradiction. The reasoning is similar for V and then $\varphi(\mathbb{N}_+) \supset U \cup V = \mathbb{N}_+$.

i) **Left to the reader, ranking exercise**

j) For $\sigma = \sigma_{p,q}$, prove that $\sum_{k=1}^{\infty} \frac{(-1)^{\sigma(k)-1}}{\sigma(k)}$ is convergent and give its limit $l_{p,q}$.

We use the following lemma (summation by intervals)

Lemma 1. *Let $\sum_{n\geq 1} a_n$ some series of complex numbers and $N > 0$ (pack length), then TFAE*

1. $\sum_{n\geq 1} a_n$ *is convergent*
2. $\sum_{m\geq 0} \left(\sum_{r=1}^{N} a_{mN+r} \right)$ *is convergent and* $\lim_{n\to\infty} a_n = 0$.

Here we take $N = p + q$ and $a_n = \frac{(-1)^{n-1}}{n}$, until the m-th pack $\sum_{r=1}^{N} a_{(m-1)N+r}$, we have the odd numbers from rank 1 to mp i.e. $1, 3, \dots 2mp - 1$ and thev even numbers from rank 1 to mq i.e. $2, 4, \dots 2mq$. Then

$$\sum_{m=0}^{M-1} \Big(\sum_{r=1}^{N} a_{mN+r} \Big) = (\sum_{r=1}^{Mp} \frac{1}{2r-1}) - (\sum_{r=1}^{Mq} \frac{1}{2r}) = (\sum_{r=1}^{2Mp} \frac{1}{r} - \sum_{r=1}^{Mp} \frac{1}{2r}) - (\sum_{r=1}^{Mq} \frac{1}{2r}) =$$
$$(\sum_{r=1}^{2Mp} \frac{1}{r}) - \frac{1}{2}(\sum_{r=1}^{Mp} \frac{1}{r}) - \frac{1}{2}(\sum_{r=1}^{Mq} \frac{1}{r}) =$$
$$\log(2Mp) + \gamma + \epsilon_1(n) - \frac{1}{2}(\log(Mp) + \gamma + \epsilon_2(n)) - \frac{1}{2}(\log(Mq) + \gamma + \epsilon_3(n)) =$$
$$\log(2) + \frac{1}{2}(\log(p) - \log(q)) + \epsilon_4(n). \tag{37}$$

which proves (together with the fact that the general term tends to zero) that

$$\sum_{k=1}^{\infty} \frac{(-1)^{\sigma(k)-1}}{\sigma(k)}$$

is convergent and tends to $\log(2) + \frac{1}{2}(\log(p) - \log(q))$.
Do not forget to mention that the general term tends to zero otherwise the lemma is false (think of $\sum_{n\geq 1}(-1)^n$ with packs of length 2).

Ex5 c). —
c) We first prove (14) left part. Let us take $S, T \in \mathbb{C}_+\langle\langle Y \rangle\rangle$ and $P(z) \in \mathbb{C}[z]$. Now, we have

$$\exp_{\uplus}(S) \uplus \exp_{\uplus}(T) = \sum_{p\geq 0} \frac{1}{p!} S^{\uplus p} \uplus \sum_{q\geq 0} \frac{1}{q!} T^{\uplus q} = \sum_{p\geq 0}\sum_{q\geq 0} \frac{1}{p!q!} S^{\uplus p} \uplus T^{\uplus q} =$$
$$\sum_{n\geq 0} \frac{1}{n!} (\sum_{p+q=n} \frac{n!}{p!q!} S^{\uplus p} \uplus T^{\uplus q}) = \sum_{n\geq 0} \frac{1}{n!} (\sum_{p+q=n} \binom{n}{p} S^{\uplus p} \uplus T^{\uplus q}) =$$
$$\sum_{n\geq 0} \frac{1}{n!} (S+T)^{\uplus n} = \exp_{\uplus}(S+T) \tag{38}$$

For the right part, we have to prove that every coefficient of $\exp_{\uplus}(P(z).S)$ is a polynomial i.e. for all $w \in Y^*$, $\langle \exp_{\uplus}(P(z).S) \mid w \rangle \in \mathbb{C}[z]$. Then

$$\langle \exp_{\uplus}(P(z).S) \mid w \rangle = \sum_{n\geq 0} \frac{1}{n!} \langle (P(z).S)^{\uplus n} \mid w \rangle = \sum_{0\leq n\leq (w)} \frac{1}{n!} P(z)^n \cdot \langle S^{\uplus n} \mid w \rangle \tag{39}$$

because for $n > (w)$, $\langle (S)^{\uplus n} \mid w \rangle = 0$. As the sum $\sum_{0\leq n\leq (w)} \frac{1}{n!} P(z)^n \cdot \langle S^{\uplus n} \mid w \rangle$ is finite, this proves the polynomiality of the coefficient $\langle \exp_{\uplus}(P(z).S) \mid w \rangle$.
Now, we prove (16). We have

$$\mathbf{d}(\exp_{\uplus}(P(z).S)) = \mathbf{d}(\sum_{w\in Y^*} \sum_{0\leq n\leq (w)} \frac{1}{n!} P(z)^n \cdot \langle S^{\uplus n} \mid w \rangle\, w) =_{(1)}$$
$$\sum_{w\in Y^*} (\frac{d}{dz}(\sum_{0\leq n\leq (w)} \frac{1}{n!} P(z)^n \cdot \langle S^{\uplus n} \mid w \rangle))\, w =_{(2)} \sum_{w\in Y^*} (\sum_{1\leq n\leq (w)} \frac{1}{n!} \frac{d}{dz}(P(z)^n) \cdot \langle S^{\uplus n} \mid w \rangle)\, w =$$
$$\sum_{w\in Y^*} (\sum_{1\leq n\leq (w)} \frac{1}{n!} n P(z)^{n-1} P'(z) \cdot \langle S^{\uplus n} \mid w \rangle)\, w =$$
$$P'(z) \sum_{w\in Y^*} (\sum_{1\leq n\leq (w)} \langle S \uplus \frac{1}{(n-1)!} (P(z)^{n-1} S^{\uplus n-1} \mid w \rangle)\, w =$$
$$P'(z) \sum_{w\in Y^*} (\langle S \uplus \sum_{0\leq m\leq (w)-1} \frac{1}{m!} (P(z)^m S^{\uplus m} \mid w \rangle)\, w =_{(3)}$$
$$P'(z) \sum_{w\in Y^*} (\langle S \uplus \sum_{0\leq m} \frac{1}{m!} (P(z)^m S^{\uplus m} \mid w \rangle)\, w = (P'(z).S) \uplus \exp_{\uplus}(P(z).S)$$

$=_{(1)}$ is from the definition of $\mathbf{d}$, $=_{(2)}$ because $\frac{d}{dz}(P(z)^0)\cdot\langle S^{⊔⊔ 0}\mid w\rangle) = 0$, $=_{(3)}$ because, due to the presence of $(S ⊔⊔\ ?)$ all terms are zero for $m > (w) - 1$.

d) **Hint**. — For linearity just prove that the map is additive and homogeneous of degree 1. For one-to-one just construct explicitely its inverse π_x^{Umbra}.

e) Let $S, T \in \mathbb{C}_+[[x]]$, one can write $S = \sum_{p\geq 1} a_p\, x^p, T = \sum_{q\geq 1} b_q\, x^q$. Then

$$(\pi_Y^{Umbra}(S))^* ⊔⊔ (\pi_Y^{Umbra}(T))^* = (\sum_{p\geq 1} a_p\, y_p)^* ⊔⊔ (\sum_{q\geq 1} b_q\, y_q)^* =_{(1)}$$
$$(\sum_{p\geq 1} a_p\, y_p + \sum_{q\geq 1} b_q\, y_q + \sum_{p,q\geq 1} a_p b_q\, y_{p+q})^* = (\pi_Y^{Umbra}(S+T+ST))^*$$
$$(\pi_Y^{Umbra}((1+S)(1+T)-1))^*$$

$=_{(1)}$ is from (10).

f) We have $G(0) = (0_{\mathbb{C}\langle\langle Y\rangle\rangle})^* = 1_{Y^*}$, now

$$G(z_1) ⊔⊔ G(z_2) = (\pi_Y^{Umbra}(e^{z_1.T}-1))^* ⊔⊔ (\pi_Y^{Umbra}(e^{z_2.T}-1))^* =_{(1)}$$
$$(\pi_Y^{Umbra}[(e^{z_1.T}-1+1)(e^{z_2.T}-1+1)-1])^* = (\pi_Y^{Umbra}(e^{(z_1+z_2).T}-1))^* = G(z_1+z_2)$$

$=_{(1)}$ is from (18).

g) One can show that, for all $z \in \mathbb{C}$,

$$\frac{d}{dz}(G(z)) = \lim_{h\to 0}\frac{G(z+h)-G(z)}{h} \text{ (pointwise convergence)}$$

then

$$\frac{d}{dz}(G(z)) = \lim_{h\to 0}\frac{G(z) ⊔⊔ (G(h)-1)}{h} =_{(1)} G(z) ⊔⊔ \lim_{h\to 0}\frac{(G(h)-1)}{h} =$$
$$G(z) ⊔⊔ \lim_{h\to 0}\frac{((\pi_Y^{Umbra}(e^{h.T}-1))^*-1)}{h} = G(z) ⊔⊔ \lim_{h\to 0}\frac{\pi_Y^{Umbra}(\sum_{n\geq 1}\frac{1}{n!}h^n T^n))^+}{h} =$$
$$G(z) ⊔⊔ (\pi_Y^{Umbra}(T)) = (\pi_Y^{Umbra}(T)) ⊔⊔ G(z)$$

Now $G(z)$ and $R(z) = e_{⊔⊔}^{z.\pi_Y^{Umbra}(T)}$ satisfy the same differential equation with the same initial condition which is

$$\frac{d}{dz}(f(z)) = (\pi_Y^{Umbra}(T)) ⊔⊔ f(z) \text{ with } f(0) = 1_{Y^*}$$

then we classically[8] form $F(z) = G(z) ⊔⊔ R(-z)$, then[9]

$$\frac{d}{dz}(F(z)) = \frac{d}{dz}(G(z) ⊔⊔ R(-z)) = \frac{d}{dz}(G(z)) ⊔⊔ R(-z) + G(z) ⊔⊔ \frac{d}{dz}(R(-z)) =$$
$$(\pi_Y^{Umbra}(T)) ⊔⊔ G(z) ⊔⊔ R(-z) + G(z) ⊔⊔ (\pi_Y^{Umbra}(-T)) ⊔⊔ R(-z) = 0$$

[8]When one has linear differential equations, the product of one solution with the inverse of the other is usually considered.

[9]One has to prove first that we are within a diifferential algebra (refs in your Ph. D.s).

From this, we deduce that $G(z)) \uplus R(-z)$ is constant and equal to its value at $z = 0$ [10] then $G(z)) \uplus R(-z) = 1_{Y^*}$ and then, for all z, $G(z) = R(z) = e_{\uplus}^{z.\pi_Y^{Umbra}(T)}$.

[10]One has to prove first that the constants are $\mathbb{C} \cdot 1_{Y^*}$ (todo).