Post-Doc Training II : Computation with infinite sums and products. (vers. 20-07-2020 10:40)

1. Preamble

An infinite[1] sum (resp. product) is a *symbolic expression* [2] of the form

$$\sum_{i\in I} a_i;\ \prod_{\in I} b_i; \sum_{n\geq 1}\frac{(-1)^n}{n};\ \overleftarrow{\prod_{\in I}} a_i;\ \overrightarrow{\prod_{\in I}} a_i;\ \overset{\searrow}{\prod_{\in I}} a_i; \tag{1}$$

save for the two first, the evaluation/computation is directed (and I has to be ordered).

2. Monoids, polynomials and series

2.1. Binary laws

2.1.1.

A *monoid* $(M,*,1_M)$ is a set endowed with a binary law $*$, associative with neutral.
1) Prove that, in $\mathcal{M}(n,\mathbb{C})$ (the set of n by n complex matrices), the laws given by

$$S\oplus T = S+T+I_{n\times n}\ ;\ S*T := S+T+ST$$

are associative.
2) Do they have a neutral ? if yes, which one ?
3) Prove that $*$ is distributive over $\oplus$.
4) Could you explain why ?
5) Read the following question (and the answer which got a bounty)

`https://math.stackexchange.com/questions/1055259/`
`binary-operation-commutative-associative-and-distributive-over-multiplication`

6) Have look at the questions tagged [semigroups-and-monoids] there.
7) For the free commutative monoid, have a look here

`https://mathoverflow.net/questions/316732/`
`generating-totally-ordered-free-commutative-monoids/354834#354834`

[1]In fact, I can be finite as well but the wording of this domain is to stress the fact that I *can be* infinite and the limiting process of these expressions should coincide with the finite version when I, the indexing set, is finite).
[2]Symbolic expressions are semistructured data in human-readable text form.

2.1.2. Expansion of question 5 above

The aim of this exercise is to classify completely the binary laws over $\mathbb{Q}$ which are commutative and distributive over multiplication. Please follow the questions step-by-step. In the following $*$ stands for any binary law $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ which is distributive over multiplication. All harvested cases will be frameboxed.
1) Prove that $1 * 1 \in \{0, 1\}$.
1.a) In case $1 * 1 = 0$ prove that the law is the null law.
1.b) Prove that the null law satifies distributivity. $\boxed{x * y \equiv 0}$

Hint [Why 1.b ?]: For a classification, you must prove (even if it is obvious) that all properties **implied** lead to satisfaction of the conditions because sometimes the "tree of discussion" leads to dead ends.

From now on, we assume that $1 * 1 = 1$.
2) Prove that $1 * (-1) \in \{-1, 1\}$.
2.1) Prove that $1 * (-1) = -1$ is impossible.
From now on, we assume that $\boxed{1 * (-1) = (-1) * 1 = 1}$.
2.2) Prove that $(-1) * (-1) \in \{-1, 1\}$.
From now on, we set $u = (-1) * (-1)$ and $\mathbb{Q}_+^* := \{r \in \mathbb{Q} \mid z > 0\}$.
3)