# Post-Doc Training III : Trees, words and diagrams. (vers. 20-07-2020 10:49)

## 1. Introduction

To be done after interactions

## 2. Binary trees

In this first part, we deal with so-called *full binary trees*[1].

### 2.1. Definition

A binary tree $t$ is

- Either an isolated node $t = \bullet$ (the number of leaves $leaves(t)$ is $= 1$), or
- The composition $t = (t_{left}, t_{right})$ of two previously defined trees.

in the second case, we have $leaves(t) = leaves(t_{left}) + leaves(t_{right})$
The set $\mathcal{T}$ of binary trees is more formally described by the following grammar, called $G_{btrees}$

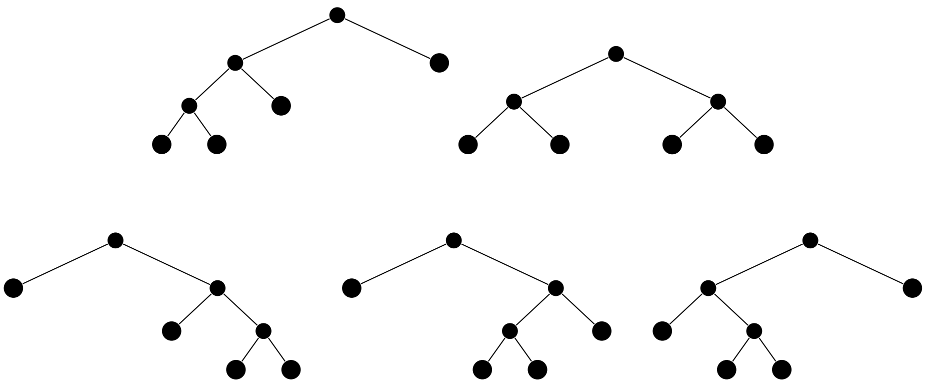
$$\mathcal{T} = \bullet + \begin{array}{ccc} & \bullet & \\ \diagup & & \diagdown \\ \mathcal{T} & & \mathcal{T} \end{array} \qquad (G_{btrees})$$

As such, it comes graded by the number of leaves, we define

$$\mathcal{T}_n = \{t \in \mathcal{T} \mid leaves(t) = n\} \tag{1}$$

We have then, by definition, $\mathcal{T} = \sqcup_{n \geq 1} \mathcal{T}_n$.
Below the set $\mathcal{T}_4$

[1]There is a lot of species of binary trees: BST, unordered, complete, almost complete, infinite complete, have a look, at [9]



1) a) Give the number $c_n = \#\mathcal{T}_n$ for $n = 1 \cdots 5$.
b) Using the grammar $(G_{btrees})$, prove that

$$c_1 = 1 \ ; \ c_n = \sum_{p+q=n} c_p \cdot c_q \tag{2}$$

c) Set $T = \sum_{n \geq 1} c_n y^n$ and, using $(G_{btrees})$, show that, in $\mathbb{C}[[y]]$, we have

$$T = y + T^2 \ .$$

Solving this equation in $\mathbb{C}[[y]]$ (please justify *as much as possible* your computations[2], each step must be legal), prove that

$$T = \frac{1 - \sqrt{1 - 4y}}{2} \tag{3}$$

d) Give the numbers $c_n = \#\mathcal{T}_n$ for $n = 1 \cdots 15$. Control the results in Sloane

`https://oeis.org/`

2) For a given set $X$, we built the set $\mathcal{T}(X)$ of binary trees with leaves in $X$, by the grammar

$$\mathcal{T} \ = \ \bullet + \begin{array}{ccc} & \bullet & \\ \diagup & & \diagdown \\ \mathcal{T}(X) & & \mathcal{T}(X) \end{array} \qquad (G_{btrees})$$

and define the grading $\mathcal{T}(X)_n$ as previously.
a) For a finite alphabet $X$, give the number $c_n(X)$ of trees with $n$ leaves as a function of $c_n$ and $\#X$.
b) ($X$ is finite and $\#X = k$. Give the generating series $T(X) = \sum_{n \geq 1} c_n(X) y^n$ as in (3).

3) Application: Construction of the free magma over $X$.

For a given set $X$ (finite or infinite), we define $j : X \hookrightarrow \mathcal{T}(X)$ as the canonical embedding (because $\mathcal{T}(X)_1 = X$).

[2]Have a look at section "Notes".

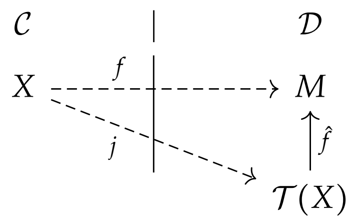



Figure 1: Universal property from Sets to Magma

a) Let $\mathcal{C}$ be the class of sets and $\mathcal{D}$ be the class of magmas. The definition can be found there (just a set equipped with a binary operation, no condition required). A morphism between two magmas is a map compatible with the operations (definition as well in the following link)

`https://en.wikipedia.org/wiki/Magma_(algebra)`

Show that $T(X)$ equipped with the operation "append" (i.e. the composition sending the pair $(t_1, t_2)$ to the new tree $(root, t_1, t_2)$) is a magma.
b) Show that for all (set-theoretical) mapping $f : X \to M$, where $M$ is a magma, it exists a unique morphism of magmas $\hat{f} \in \hom_{\mathcal{D}}(\mathcal{T}(X), M)$ such that "set-theoretically" $f = \hat{f} \circ j$. c) Compare what has been done with construction [3], Ch 1 §7.1. Proposition 1.
d) Read the following

`https://mathoverflow.net/questions/39862/free-commutative-magma-over-a-set\\`

# 3. Words

## 3.1. Free monoid: Warming

Let $X$ be an alphabet (a set), $(X^*, conc, 1_{X^*})$ be the monoid of words built over $X$ and set $j \hookrightarrow X^*$ be the canonical embedding.
1) a) Show that we have an analogue of Figure 1 for $\mathcal{C}$ (resp. $\mathcal{D}$) the category of sets (resp. of monoids), i.e.
For all set-theoretical map $f : X \to M$, where $(M, *, 1_M)$ is a monoid, it exists a unique morphism of monoids $\hat{f} \in \hom_{\mathcal{D}}(\mathcal{T}(X), M)$ such that "set-theoretically" $f = \hat{f} \circ j$ (see Figure 2).
b) Compare what has been done with construction [3], Ch 1 §7.2 (where $X^*$ is called $Mo(X)$) Proposition 3 and [6] Prop 1.1.1.
c) Read the following

`https://en.wikipedia.org/wiki/Free_monoid`

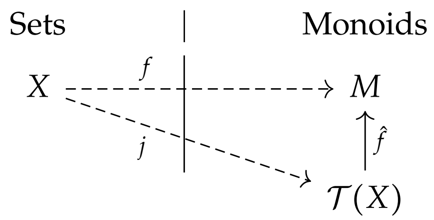



Figure 2: Universal property from Sets to Monoid

## 3.2. Lexicographic ordering and Lyndon words

### 3.2.1. Preamble on relations and graphs

In general, a *small* graph is a set $G \subset E \times F$ ($E, F$ being sets). When $E = F$, we speak of an endograph (or simply a graph when the context is clear as below). The set of these graphs is $2^{E\times E}$.
On $2^{E\times E}$, there is a binary law called the compostion of graphs, given by

$$R_1 \circ R_2 = \Big\{(x,z)|(\exists y \in E)\Big((x,y) \in R_1 \text{ and } (y,z) \in R_2\Big)\Big\} \tag{4}$$

we also need the diagonal $\Delta = \Delta_E = \{(x,x)\}_{x\in E}$.
1) Show that $(2^{E\times E}, \circ, \Delta)$ is a monoid.
2) As an application show that

$$\mathcal{M}_1 = \{R \in 2^{E\times E} \mid \Delta \subset R\}$$

is a submonoid of $2^{E\times E}$.
Write the multiplication table of $\mathcal{M}_1$ for $E = \{1,2\}$.
A relation $R \subset E \times E$ is said

1. *reflexive* if $\Delta \subset R$
2. *irreflexive* if $\Delta \cap R = \varnothing$
3. *transitive* if $R \circ R \subset R$
4. *symmetric* if $s(R) \subset R$ (where $s : E \times E \to E \times E$ is the canonicaal symmetry $s(x,y) = (y,x)$

A *strict order relation* (on $E$) is a relation which is *irreflexive* and *transitive*, an *order relation* is a relation which is *reflexive*, *antisymmetric* and *transitive*. The set of order relations (rep. strict order relations) will be called $OR(E)$ (resp. $SOR(E)$).
3) Show that $R \to \widetilde{R} = R \cup \Delta$ is a bijection $SOR(E) \to OR(E)$.
4) Show that $\Delta_E \in OR(E)$ and that, if $R \in OR(E)$ (resp. $R \in SOR(E)$) then $s(R) \in OR(E)$ (resp. $s(R) \in SOR(E)$); $s(R)$ is then called the oposite order (resp. strict order).
5) Use (4) to enumerate all strict order relations within $E = \{a,b\}$ (3 solutions) and within $E = \{a,b,c\}$.

### 3.2.2. Lexicographic ordering

Let $X$ be an alphabet (i.e. a set, finite or infinite)[3] and $<$ a total ordering[4] on $X$.
We define the (strict) lexicographic ordering between words by

$$(LO) \qquad u <_{lex} v \iff \begin{cases} v = us \text{ with } s \neq 1_{X^*} & (LO_1) \\ u = pxs_1 \; v = pys_2 \text{ with } x,y \in X \text{ and } x < y & (LO_2) \end{cases} \tag{5}$$

$LO_1$) expresses that $u$ is a (strict) prefix of $v$ and $(LO_2)$ expresses that, at the first position where they differ the comparison is made at this position. As a convenient, we will note these relations, respectively, $<_1$, $<_2$.

### 3.2.3. Preparation one: The Finite case

1) We suppose that the alphabet $X$ is finite and numbered $X = \{a_1, a_2, \cdots, a_N\}$, let $B = N+1$ and send every word $w = a_{i_1} a_{i_2} \cdots a_{i_k}$ to $j_B(w) = \sum_{1\leq s\leq k} i_s \cdot B^{-s}$. In numeral notation this number is

$$j_B(w) = \overline{0.i_1 i_2 \cdots i_k}^B$$

a) Show that $j_B$ is into, what is its image ?
b) Show that

$$u < v \iff j_B(u) < j_B(v) \tag{6}$$

c) Deduce that $<_{lex}$ is a (strict) total ordering on the words (i.e. $X^*$).

### 3.2.4. Two: The infinite (arbitrary) case

d) Prove the following lemma

**Lemma 1.** *Let $(S, R)$ be a set endowed with a binary relation $R$.
We suppose that for every subset set $P = \{a, b, c\} \subset S$, it exists an injective map $j$ to a totally ordered set $(T, <)$ such that for all $x, y \in P$*

$$R(x,y) \iff j(x) < j(y) \tag{7}$$

*Then $R$ is a total (strict) order on $S$.*

e) Apply lemma (7) and section (3.2.3) to show that $<$ is a strict total order on $X^*$.
**Hint:** *$X$ being still totally ordered by $<$, for any subset $P = \{u, v, w\} \subset X^*$ consider the finite alphabet $A = \mathrm{alph}(u) \cup \mathrm{alph}(v) \cup \mathrm{alph}(w)$ (note that $A$ can have any cardinality, but is finite nevertheless) $A = \{a_1 < a_2 < \cdots < a_N\} \subset X$ is totally ordered by the order inherited from X. Then, with $B = \#A + 1$ construct $j_B : A^* \to [0, 1[$ and apply lemma (7) and section*

[3] Its elements of $X$ will be called letters.

[4] The possibility of constructing a total order on a set is linked to the system of axioms. In particular, have a look at [2] or `https://en.wikipedia.org/wiki/Well-order`. In combinatorics, we are in ZFC, so this is not a problem here.

(3.2.3).
The associated order (with eq.) will be noted $\leq_{lex}$ or simply $\leq$ when the context is clear.
*The maps to numbers was just a probing tool and will no longer be used in the sequel.*
f) (Compatibility with left translations and cancellations) Prove that, for $u, v, w \in X^*$

$$v < w \Longleftrightarrow uv < uw$$

a principle which can be abstracted as (with all words in $X^*$)

$$\boxed{2} < \boxed{3} \Longleftrightarrow \boxed{1}\boxed{2} < \boxed{1}\boxed{3} \tag{8}$$

g) Show that, if $u < v$ and if $u$ is not a prefix of $v$, then for all $s_1, s_2 \in X$, we have $us_1 < vs_2$. A principle which can be abstracted as (with all words in $X^*$, but remark that $u <_2 v$ implies $u, v \in X^+$)

$$\boxed{1} <_2 \boxed{2} \Longrightarrow \boxed{1}\boxed{3} <_2 \boxed{2}\boxed{4} \tag{9}$$

h) Show that the result of (c) is not true in general (give a counterexample), even if $s_1 = s_2$, if we suppose only $u < v$ (no compatibility on the right, in case $LO_1$).
i) Show that

$$\boxed{1} < \boxed{2} < \boxed{1}\boxed{3} \Longrightarrow \boxed{1} <_1 \boxed{2} \text{ and } \boxed{2} = \boxed{1}\boxed{4} \text{ with } \boxed{4} < \boxed{3} \tag{10}$$

### 3.2.5. Preamble: Conjugacy classes and Lyndon words

For a word $w$ the conjugacy class (with multiplicities) is the multiset[5] $\{\{vu\}\}_{w=uv}$. For example, the conjugacy classes of $aabab$ and $abab$ are

$$CClass(aabab) = \{\{aabab, baaba, abaab, babaa, ababa\}\} \; ; \; CClass(abab) = \{\{abab, baba, abab, baba\}\}$$

A Lyndon word is a word which is the strict minimum (for $\leq_{lex}$, hence the alphabet must have been totally ordered) of its conjugacy class. In an equivalent way we set the definition

**Definition 1.** *A word $w \in X^+$ is said Lyndon iff*

$$w = ps \text{ with } p, s \in X^+ \Longrightarrow w < sp \tag{11}$$

*Their set will be noted $\mathcal{L}yn(X)$.*

**Remark 1.** *i) A factorization $w = ps$ with $p, s \in X^+$ is called a proper factorization and, in this case $p$ (resp. $s$) is a proper prefix (resp. proper suffix) of $w$. As letters have no proper factorization, we have $X \subset \mathcal{L}yn(X)$.*
*ii) For example $CClass(abab)$ has no word in $\mathcal{L}yn(X)$ and $CClass(aabab) \cap \mathcal{L}yn(X) = \{aabab\}$.*

[5]Read `https://en.wikipedia.org/wiki/Multiset`. Here in order not to confuse with sets, we will note the multiset with double curly brackets, as in the french page `https://fr.wikipedia.org/wiki/Multiensemble`.

In order to give an equivalent criterium for a word to be Lyndon, we will need the following lemma

**Lemma 2** ([6] Prop. 1.3.4. p8). *Let* $u, v, c \in X^+$ *be related by the equation*

$$uc = cv \tag{12}$$

*then it exists* $x, y \in X^+$, $t \in \mathbb{N}$ *s.t.* $u = xy, v = yx, c = (xy)^t x$.

We now have (physicists would say "We now have an equivalent definition")

**Proposition 1.** *A word* $w$ *is Lyndon iff it is less than its proper suffixes.*

In other words

$$w \in \mathcal{L}yn(X) \Longleftrightarrow [w = ps \text{ with } p, s \in X^+ \Longrightarrow w < s] \tag{13}$$

For the convenience of the reader, I reproduce here the proof here where I call here *criterium* the property

$$w = ps \text{ with } p, s \in X^+ \Longrightarrow w < s\ .$$

*Proof.*
*criterium* $\Longrightarrow w \in \mathcal{L}yn(X)$] If $w = ps$ is a proper factorization of $w$, then, by the criterium $w < s$ and, as $w$ cannot be a prefix of $s$ (because $|w| > |s|$) and then by question (c) above [principle (9)] $w < sp$.
$w \in \mathcal{L}yn(X) \Longrightarrow$ *criterium*] Let us consider $s$, a proper suffix of $w$. We have then $w = ps$ with $p, s \in X^+$, as $w \in \mathcal{L}yn(X)$ we get $w < sp$.
Let us first establish that $s$ cannot be a prefix of $w$. If it were so, we had $ps = w = st$ and then, by Lemma (2), it exists $x, y \in X^+$, $t \in \mathbb{N}$ such that $p = xy$, $t = yx$, $s = (xy)^t x$, then $w = ps < sp$ reads

$$(xy)^t xyx = xy(xy)^t x < s = (xy)^t xxy \tag{14}$$

by question (b) above [principle (8)], we can simplify and get $yx < xy$ which have the same length $|x| + |y|$. Therefore, we cannot be in the case $(L0_1)$ ($yx$ cannot be a prefix of $xy$) then we can multiply on the right by arbitrary factors obtaining

$$(yx)^{t+1} x < (xy)^{t+1} x = w$$

but the first word $(yx)^{t+1} x$ is a conjugate of $w$, a contradiction. □

### 3.2.6. Questions

a) Prove that, if $u, v \in \mathcal{L}yn(X)$ and $u < v$, then $uv \in \mathcal{L}yn(X)$ and $u < uv < v$.

For any word $|w| \geq 2$ we define $\sigma(w) = (u, v)$ such that $w = uv$ and $v$ is the smallest proper suffix of $w$.
b) With $\sigma(w) = (u, v)$, show that $v \in \mathcal{L}yn(X)$ and

$$w \in \mathcal{L}yn(X) \Longleftrightarrow (u \in \mathcal{L}yn(X) \text{ and } u < v) \tag{15}$$

**Hint:** *For the hardest part ($w \in \mathcal{L}yn(X) \Longrightarrow u \in \mathcal{L}yn(X)$), take a proper suffix of $u$, say $s$, if we had $s < u$ (to disprove), show that $s < u < uv < v < sv$ (give the reason of each inequality). Apply principle* (10) *to $s < v < sv$ concluding that $v = st$ with $t < v$ and remember that $v \in \mathcal{L}yn(X)$.*
c) In the case when $w \in \mathcal{L}yn(X)^{\geq 2}$, $\sigma(w) = (u, v)$ and $|u| \geq 2$, set $\sigma(u) = (u_1, u_2)$, show that $u_2 \geq v$

# 4. Trials

The quick brown fox jumps over the lazy dog.

# 5. Notes

N1) About equations of the second degree in a ring (here the ring is $\mathbb{C}[[y]]$).
(that's why I asked to justify because, in a ring, $\Delta$ may have no root (look at $y - 1$ in the ring is $\mathbb{R}[[y]]$) or an infinity of such. For example in a $\mathbb{C}$-algebra, you can have two non-null and distinct elements $e, f$ with

$$f^2 \neq 0 \; ; \; ef = fe = 0 \; ; \; e^2 = 0 \tag{16}$$

then $f^2$ admits as square roots, at least, the family of elements $(f + \lambda \cdot e)_{\lambda \in \mathbb{C}}$.

### Xtra-exercises (notes)

Ex-N1) Give two $3 \times 3$ complex matrices which fulfill the relations(16).
Ex-N2) Let $R$ be a commutative ring where 2 has an inverse (which will be noted, as usual, 1/2). We consider the equation

$$X^2 + bX + c = 0 \tag{17}$$

We set $\Delta = b^2 - 4c$.
a) Firstly, let us suppose that $\Delta$ has a square root which will be noted $\delta$ as in

`https://en.wikipedia.org/wiki/Quadratic_equation`

Show that equation (17) has a root $r$ (**Hint**: Take $r = \frac{1}{2}(-b + \delta)$).
b) Conversely, we suppose that (17) has a root, say $r$, show that

$$X^2 + bX + c = (X - r)(X + r + b) \tag{18}$$

deduce that $c = -r(r + b)$ and find a square root for $\Delta$.
c) Show that, if $R$ has no zero divisors equation (17) has

- two roots if $\Delta \neq 0$ and is a square (i.e. has a square root $\delta$)

- one root if $\Delta = 0$
- no root if $\Delta$ and is not a square

d) Application.
Let $R = \mathbb{Z}/3\mathbb{Z}$ (a field).
d1) What are the squares in $R$ ? (two solutions)
d2) What are the quadratic equations ((17)) which have roots ? (9 possibilities out of which only a few have roots).
d3) Solve them and give their roots. e) Returning to the matrices of (Ex-N1), show that the equation $X^2 - f = 0$ has infinitely many solutions.

**By the way, some references here**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