Post-Doc Training IV: Universal problems (Linear and algebraic dependences, Equivalence relations, Localization)

1. Introduction

https://en.wikipedia.org/wiki/Universal_property

If someone wants to know "where all this goes", one can read [2] IV §3 which is less categorical but is a good preparation to categories and, of course, our papers. Today, we will deal with some kinds of universal problems i.e. factorization of arrows, linked with our current research i.e. Linear and algebraic independences, Equivalence relations, Localization.

1.1. Ex1. — Linear independence.

We will consider k, a field which will serve as "rings of coefficients" (and, in fact, all what will be said and seen there holds true for k being a ring and, therefore, within the category of modules).
Let $\mathcal{X}$ be a set[1], the vector space $k^{\mathcal{X}}$ is the set of all functions (i.e. families $(\alpha_x)_{x\in\mathcal{X}}$) from $\mathcal{X} \to k$. The structure of k-vector space is given as usual, but we recall it here

1. Addition: For all $(alpha_x)_{x\in\mathcal{X}},\ (\beta_x)_{x\in\mathcal{X}} \in k^{\mathcal{X}}$
$$(alpha_x)_{x\in\mathcal{X}} + (\beta_x)_{x\in\mathcal{X}} := (\alpha_x + \beta_x)_{x\in\mathcal{X}} \tag{1}$$
2. Multiplication (external): For all $\lambda \in k$ and $(alpha_x)_{x\in\mathcal{X}} \in k^{\mathcal{X}}$
$$\lambda \cdot (alpha_x)_{x\in\mathcal{X}} := (\lambda.\alpha_x)_{x\in\mathcal{X}} \tag{2}$$

The support of a family $(alpha_x)_{x\in\mathcal{X}} \in k^{\mathcal{X}}$ is given by
$$\mathrm{supp}((alpha_x)_{x\in\mathcal{X}}) := \{x \in \mathcal{X} \,|\, \alpha_x \neq 0\} \tag{3}$$
a) We define $k^{(\mathcal{X})}$ as the set of finitely supported families of $k^{\mathcal{X}}$. Prove that $k^{(\mathcal{X})}$ is a vector subspace of $k^{\mathcal{X}}$ and make precise the embedding $j_1 :\ \mathcal{X} \hookrightarrow k^{(\mathcal{X})}$
b) Let V be a k-vector space and $f :\ \mathcal{X} \to V$ be any set-theoretical[2] map. Show that there exists a unique linear mapping $\hat{f} :\ k^{(\mathcal{X})} \to V$ such that the following triangle Fig. 1 commutes.
c) In case $\mathcal{X} \subset V$ and f is the canonical embedding, show that

1. $\mathcal{X}$ is k-free iff $\hat{f}$ is into
2. $\mathcal{X}$ is k-generating iff $\hat{f}$ is onto
3. $\mathcal{X}$ is a basis of V iff $\hat{f}$ is one-to-one

[1] Think of an alphabet, finite (like X) or infinite (like Y) or, indeed, any set.
[2] Do not forget that $\mathcal{X}$ is only a set with no particular structure.

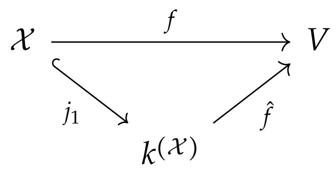

Figure 1: Universal property from Sets to k-Vect

1.2. Ex2. — Algebraic independence

This time, instead of k-vector spaces, we will deal with commutative k-AAU[3].

Let $\mathcal{A}$ be a commutative k-AAU. For fixed $\mathcal{X}$, we consider the algebra of commutative polynomials $k[\mathcal{X}]$.

a) For $\alpha \in \mathbb{N}^{(\mathcal{X})}$ explain what is the monomial $\mathcal{X}^\alpha$ (multiindex notation) and detail the multiplicative law in $k[\mathcal{X}]$ (i.e., what is the value of $\mathcal{X}^\alpha.\mathcal{X}^\beta$?) and make precise the embedding $j_2 : \mathcal{X} \hookrightarrow k[\mathcal{X}]$.

b) Let $\mathcal{A}$ be a commutative k-AAU and $f : \mathcal{X} \to \mathcal{A}$ be any set-theoretical map.

Show that there exists a unique morphism of commutative k-AAU $\hat{f} : k[\mathcal{X}] \to \mathcal{A}$ such that the following triangle Fig. 2 commutes.

Hint: Use the fact that $(\mathcal{X}^\alpha)_{\alpha\in\mathbb{N}^{(\mathcal{X})}}$ is a basis of $k[\mathcal{X}]$.

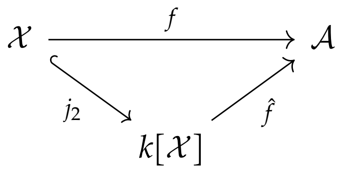

Figure 2: Universal property from Sets to k-CAAU

c) (Definitions here, nothing to prove) In a similar way that in case of linear independence (see section 1.1 c.), when $\mathcal{X} \subset\subset \mathcal{A}$ and f is the canonical embedding, we say that

1. $\mathcal{X}$ is algebraically free iff $\hat{f}$ is into

2. $\mathcal{X}$ is algebraically generating subset iff $\hat{f}$ is onto

3. $\mathcal{X}$ is a transcendence basis (formerly called algebraic basis) of $\mathcal{A}$ iff $\hat{f}$ is one-to-one

Application 1. —

d1) Let us suppose k to be of characteristic zero (hence $\mathbb{Q} \hookrightarrow k$) and take $\mathcal{A} = (k\langle\mathcal{X}\rangle, ш, 1_{\mathcal{X}^*})$.

Using the duality between $(P_w)_{w\in\mathcal{X}^*}$ and $(S_w)_{w\in\mathcal{X}^*}$, show that $(S_l)_{l\in\mathcal{L}yn(\mathcal{X})}$ is an algebraic basis of $\mathcal{A}$. Can we deduce that $(l)_{l\in\mathcal{L}yn(\mathcal{X})}$ is an algebraic basis of $\mathcal{A}$? How ?

[3] Associative Algebra with Unit.

(Radford's theorem).
d2) Show that, in case k is of characteristic p (a prime number), the result does not hold (**Hint**: For $a \in \mathcal{X}$, a^p is not in the shuffle algebra generated by the Lyndon words.). Prove that the family $(l)_{l\in\mathcal{L}yn(\mathcal{X})}$ remains algebraically free (hard).
Application 2. — Let S be the sphere in $\mathbb{R}^3$ and $x(M)$, $y(M)$, $z(M)$ be the three coordinates functions $S \to \mathbb{R}$ (to a point $M \in S$ associates its coordinates). The ambiant algebra is that of functions $S \to \mathbb{R}$ with the ordinary pointwise product (i.e. $\mathcal{A} = \mathbb{R}^S$). Show that $X_1 = \{x(M), y(M)\} \subset \mathcal{A}$ is algebraically free, but not
$X_2 = \{x(M), y(M), z(M)\} \subset \mathcal{A}$.
Application 3 (Ex 1-2). — **What is a free singleton ?**
A3-1) In the context of Ex 1 and with $x \in V$, a vector space, prove

$$\{x\} \text{ is linearly } k\text{-free} \iff x \neq 0_V$$

A3-2) In the context of Ex 2 and with $x \in \mathcal{A}$, a k-CAAU, prove

$$\{x\} \text{ is algebraically } k\text{-free} \iff (\forall P \in k[t])(P(x) = 0 \Longrightarrow P = 0_{k[t]})$$

In this case, we say that x is *transcendent* "w.r.t. k and within $\mathcal{A}$".
A3-3) Let X be a set and $\mathcal{A}$ be the algebra k^X (all functions $X \to k$) prove that for $f \in \mathcal{A}$

$$f \text{ is transcendent within } \mathcal{A} \text{ wrt } k \iff f(X) \text{ is infinite} \tag{4}$$

Rq. — An element is not transcendent *per se* (i.e. in itself) but with respect to some coefficients and within a precise algebra (this can be, and is often, understated when the context is clear). `https://en.wikipedia.org/wiki/Transcendence_degree`

1.3. Ex3. — Noncommutative polynomials

This time, instead of k-CAAU, we will deal with (commutative or not) k-AAU.

Let $\mathcal{A}$ be a k-AAU. For fixed $\mathcal{X}$, we consider the free monoid $\mathcal{X}^*$ and its algebra[4] $k[\mathcal{X}^*] = k\langle\mathcal{X}\rangle$.
a) What is the multiplication law in $(k\langle X\rangle, conc, 1_{\mathcal{X}^*})$ (recall it in detail). Make precise the embedding $j_3 : \mathcal{X} \hookrightarrow k\langle\mathcal{X}\rangle$.
b) Let $\mathcal{A}$ be a k-AAU and $f : \mathcal{X} \to \mathcal{A}$ be any set-theoretical map.
Show that there exists a unique morphism of k-AAU $\hat{f} : k\langle\mathcal{X}\rangle \to \mathcal{A}$ such that the following triangle Fig. 3 commutes.
Hint: Use the fact that $(\mathcal{X}^*$ is a basis of $k\langle X\rangle$.

[4]Mind: in algebra the notation $k[?]$ is a polymorphism. If ? is a set it stands for "the algebra of (commutative) polynomials with variables in ?, If ? is a monoid (resp. a semigroup), $k[?]$ is the algebra of the monoid (resp. the semigroup).

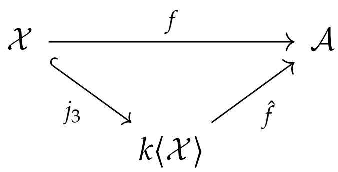

Figure 3: Universal property from Sets to k-AAU

1.4. Ex4. — Free monoids

a) What are words ? what is the free monoid ? concatenation ?
b) Make precise the embedding $j_4 : \mathcal{X} \hookrightarrow \mathcal{X}^*$.
b) Let M be a monoid and $f : \mathcal{X} \to M$ be any set-theoretical map.
Show that there exists a unique morphism of monoids $\hat{f} : \mathcal{X}^* \to \mathcal{A}$ such that the following triangle Fig. 4 commutes.

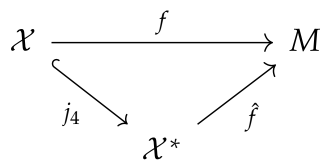

Figure 4: Universal property from Sets to Monoids

1.5. Ex4. — Localization, generalities

à faire

1.6. Ex5. — Localization of differential algebras

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By the way, some references here

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