Post-Doc Training V: Algebras on words
(A significant part of Schützenberger intellectual and operational inheritage)

1. Algebras on words

Exercice 1 : Exchangeable series and polynoms.

Let k be a ring and $\mathcal{X}$ an alphabet, as usual we have $k\langle\mathcal{X}\rangle$, the algebra of noncommutative polynomials (over the noncommutative variables $\mathcal{X}$ and with coefficients in k) and $k\langle\langle\mathcal{X}\rangle\rangle$, the algebra of noncommutative series (idem). A series $S \in A\langle\langle\mathcal{X}\rangle\rangle$ is called *syntactically exchangeable* if and only if it is constant on multi-homogeneous classes, *i.e.*

$$(\forall u, v \in \mathcal{X}^*)([(\forall x \in \mathcal{X})(|u|_x = |v|_x)] \quad \Rightarrow \quad \langle S \mid u\rangle = \langle S \mid v\rangle). \tag{1}$$

The set of these series, a shuffle subalgebra of $A\langle\langle\mathcal{X}\rangle\rangle$, will be denoted $A_{\mathrm{exc}}^{\mathrm{synt}}\langle\langle\mathcal{X}\rangle\rangle$.
a) Prove that a series $S \in A\langle\langle\mathcal{X}\rangle\rangle$ is syntactically exchangeable iff it is of the form

$$S = \sum_{\alpha\in\mathbb{N}^{(\mathcal{X})},\mathrm{supp}(\alpha)=\{x_1,\ldots,x_k\}} s_\alpha\, x_1^{\alpha(x_1)} ⧢ \ldots ⧢ x_k^{\alpha(x_k)}. \tag{2}$$

b) Prove that all series of the form

$$S = \sum_{\alpha\in\mathbb{N}^{(\mathcal{X})}} c_\alpha\, \mathcal{X}^{⧢\, \alpha}. \tag{3}$$

is syntactically exchangeable. What is the correspondence between the coefficients s_α and c_β ?
c) Deduce from (b) that, if A is a $\mathbb{Q}$-algebra, every syntacticallly exchangeable series is of the form (3).
d) Give, in $\mathbb{Z}\langle\mathcal{X}\rangle$, polynomials which are syntactically exchangeable but cannot be put in the form (3).

Exercice 2 : Exchangeable polynoms.

a) Prove that if A is a $\mathbb{Q}$-algebra, the shuffle subalgebra $A_{\mathrm{exc}}^{\mathrm{synt}}\langle\langle\mathcal{X}\rangle\rangle \cap A\langle\mathcal{X}\rangle$ is exactly the shuffle subalgebra generated by $\mathcal{X}$ (**Hint**. — Call $\mathcal{A}$ shuffle subalgebra generated by $\mathcal{X}$ and proceed by double inclusion).
b) What happens when A is not a $\mathbb{Q}$-algebra ?

2. Cartier-Quillen-Milnor-Moore theorem.

2.1. Preliminaries on words

Exercice 3 : Dualizable and moderate series.

a) Reprove Th2 in [4] (as much as possible in a CBT[1] mode).
Let $\mathcal{X}$ be a set[2], we define q-infiltration (see [4] p8) co-product on the letters by

$$\Delta_{\uparrow_q}(x) = x \otimes 1 + 1 \otimes x + q\, x \otimes x \tag{4}$$

and extend by bi-concatenation.
b) Show that, for $q = 0$, $\Delta_{\uparrow_0} = \Delta_{\sqcup\!\sqcup}$ and, for $q = 1$, $\Delta_{\uparrow_1}$ is dual of the ordinary infiltration product (see [5, 9]).
c) Prove the beautiful combinatorial formula (5) in [8]

$$\Delta(w) = \sum_{I \cup J = [1..|w|]} q^{|I \cap J|}\, w[I] \otimes w[J] \quad (\mathbf{5}) \text{ in MO}$$

By the way, some references here

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[7] `https://en.wikipedia.org/wiki/Holomorphic_functional_calculus`

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[1] Closed Book Training
[2] An alphabet for Theoretical Computer Scientists

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