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Combinatorics of Lazard Elimination and Interactions

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Combinatoire de l'élimination de Lazard et Interactions

Résumé. Ce mémoire est tout entier consacré à la réécriture des inversions dans certaines structures avec produit, leur réarrangements et les contreparties combinatoires de ces transformations pour les partitions d'alphabets c'est à dire l'élimination de Lazard de générateurs (LE) et les formules associées (en particulier dans leurs quotients). Les théorèmes du type (LE) donnent lieu à des formules uniformes pour tous les alphabets et ont des schémas similaires pour les groupes, les monoïdes, les algèbres de Lie et les algèbres associatives avec unité. Ces outils donnent lieu à de nombreux algorithmes implémentables. La forme la plus simple de (LE) se produit dans la catégorie des **k**-algèbres de Lie (**k** étant un anneau unitaire), nous nous concentrons sur les monoïdes et les algèbres de Lie et donnons des exemples sur des "smash-produits" itérés pour lesquels la réécriture des mots ("string rewriting") joue un rôle crucial non seulement dans la compréhension des formes normales, mais encore dans la façon dont on converge vers elles. La fin de cette thèse se concentre sur les utilisations supplémentaires de (LE) et de l'indexation de mots dans le contexte du half-shuffle, des algèbres de Zinbiel et de la dualité de Magnus. En outre, il aborde le sujet des hyperlogarithmes et de la théorie des caractères.

Combinatorics of Lazard Elimination and Interactions

Abstract. This memoir is all about rewriting of inversions in some product structures, reordering and their combinatorial counterparts for partition of alphabets i.e. Lazard's elimination (LE) of generators and associated formulas (in particular their quotients). (LE) theorems provide uniform formulas for every alphabet and have similar schemes for groups, monoids, Lie algebras and unital associative algebras. These tools give rise to many implementable algorithms. The most celebrated form of (LE) is on the category of Lie **k**-algebras (**k** being a unitary ring), we concentrate on monoids and Lie algebras and provide examples on iterated "smash-products", where the rewriting on words ("string rewriting") plays a crucial rôle to understand the normal forms and how one converges to them. The end of this thesis focuses on additional uses of (LE) and word indexing in the context of half-shuffle, Zinbiel algebras and Magnus duality. Furthermore, it delves into the topics of hyperlogarithms and character theory.

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Contents

Introduction

This memoir is all about rewriting of inversions in some product structures, reordering and their combinatorial counterparts for partition of alphabets i.e. (Lazard's) elimination of generators and formulas of the type

$$STRUCT\langle x_1, x_2, \ldots, x_n\rangle \cong NICE\langle x_1, x_2, \ldots, x_n\rangle \diamond STRUCT_1\langle x_1, \ldots, x_{n-1}\rangle \quad (0.1)$$

where $NICE$ et $STRUCT_1$ stand for algebraic structures generated (sometimes freely) by generators x_i. The diamond symbol being, according to the situation, a tensor product, a semi-direct product or a plain (unique) factorization. For example, with the symmetric group $\mathfrak{S}_n$ and the pure braid group $\mathcal{PB}_n$ [5] :

$$\mathfrak{S}_n \cong {}^{\mathbb{Z}}/_{n\mathbb{Z}} \diamond \mathfrak{S}_{n-1} \quad \text{and} \quad \mathcal{PB}_n \cong \mathrm{F}_{n-1} \diamond \mathcal{PB}_{n-1}.$$

Here, in the first case, $\diamond$ is only a product of permutable subgroups and the iterated decomposition helps building the infinite chain of embeddings or construct a basis of $\mathbb{Q}[\mathfrak{S}_n]$ adapted to the calculation needs of *Dynkin*'s projector [32]. In the second case we have a semi-direct product (where F_{n-1} is the Free Group with $n-1$ generators). Let us firstly see the case of two permutable subgroups[1] (where the $\diamond$ is a symbol of (unique) factorization), we have $G = G_1G_2 = G_2G_1$ (and it is required that $G = G_1.G_2$ be of unique factorization). Then, at the level of the terms, the rewriting reads

$$g_2g_1 \longrightarrow l(g_1, g_2)r(g_1, g_2) \quad (0.2)$$

and, in the case when $r(g_1, g_2) = g_2$ identically, we have a semi-direct product i.e. for every $(g_1, g_2) \in G_1 \times G_2$, $g_2g_1{g_2}^{-1} \in G_1$, so that we only need to know the factor $l(g_1, g_2)$.

[1] A common occurrence in solvability.

The categories covered here will be

1. **Set**, the category of sets.

2. **Mon**, the category of monoids.

3. **Grp**, the category of groups.

4. **k-Lie**, the category of Lie **k**-algebras (where **k** is a given commutative ring).

5. **k-AAU**, the category of unital associative **k**-algebras (where **k** is again a given commutative ring).

and (forgetful) functors between the are as follows

$$\begin{tikzcd}
 & \mathbf{Mon}(2) \arrow[dl, "F_{12}"'] & \mathbf{Grp}(3) \arrow[l, "F_{23}"'] \\
\mathbf{Set}(1) & & \mathbf{k\text{-}AAU}(5) \arrow[ul, "F_{25}"'] \arrow[dl, "F_{45}"] \\
 & \mathbf{k\text{-}Lie}(4) \arrow[ul, "F_{14}"] &
\end{tikzcd}$$

Figure 1: Similar lower diagram with algebras and **k-Mod** replacing **Set**.

This Ph. D. answers the following questions

Q1) What are the expressions of Lazard elimination (LE) in several categories where there are sort of semi-direct products?

Q2) Are these universal? (i.e. is every semi-direct product the image of some Lazard elimination? and is LE a free object?[2])

Q3) What is the Combinatorial counterpart of these investigations? (Bases, codes, formulas, etc.)

Q4) If possible, is there a deep reason for the similarity of the obtained formulas?

The results of this Ph. D. are

1. Main result: compatibility of a set of generators and relations w.r.t. a partition of the alphabet.

   - Application to the Partially Commutative Lie algebra.
   - Application to the Drinfeld-Kohno Lie algebra.

2. Every semi-direct product is the image of a Lazard elimination: Boolean gradings.

3. Generalized gradings: in the vein of the enlargement allowed by Wikipedia [58] and Bourbaki, the theory of gradings has been enlarged to additive semigroups and applied to the Hilbert series of the Drinfeld-Kohno Lie algebra (and its universal enveloping algebra) with infinite number of generators.

4. The computation of $(\mathrm{Id})_{\mathrm{gen}}$ worked out completely for any enveloping algebra[3] (it could be thought as a computation of local coordinates, see Mathoverflow [111]).

5. The infinite product, bases in duality (and Magnus setting) and their compatibility with elimination.

[2] This rather vague sounding question as well as question 3 heading subsection 2.2.3 (i.e. "About M.-P. Schützenberger's questions on the Partially Commutative Free Lie algebra") have been reinterpreted in the language of (algebraic) categories as the "search of the source of a free (forgetful) functor".

[3] Of a Lie algebra linearly free, over a $\mathbb{Q}$-algebra as announced in [94].

6. Towards a theory of domains for Polylogarithms and Harmonic sums.

Preliminary remark. *All results of this memoir (save the Section 3.4 which rests on complex analysis) are of "locally finite" nature and can be reached without topology. In particular the limiting processes of infinite sums and products boil down to the notion of summable families*[4]*.*
From time to time, concepts of general topology (like density and completion) have intentionally been kept because a purely algebraic reformulation would be lengthy, clumsy and poorly expressive.
In the same way, categories are oftentimes used as a unifying concept but many results can be rephrased without using the language of category theory.

[4]Or *pointwise finitely supported families*, see [56] Def 1.7.2.

Chapter 1

Preamble

Contents

The chapter contains general facts about "free objects" from the scheme "solutions of universal problems" for categories

$$\mathbf{Mon}, \mathbf{Grp}, \mathbf{k\text{-}AAU}, \mathbf{k\text{-}Lie} \tag{1.1}$$

$\mathbf{k}$ being a fixed unital commutative ring.

1.1 The (endo)functor Seq.

In the following, we will use the (endo)functor $\textsc{Seq} : \mathbf{Set} \to \mathbf{Set}$ (here all sets of ZFC are admissible whatever their cardinality). Then, a set X being given, $\textsc{Seq}(X)$ is the set of all sequences of elements of X i.e.

$$\textsc{Seq}(X) := \bigsqcup_{n\geq 0} X^{\{1,\cdots,n\}} \tag{1.2}$$

where, for E, F sets, F^E is the set of all maps $E \to F$, and, for $f:\ X \to Y$,

$$\textsc{Seq}[f](x_1, \cdots, x_n) := (f(x_1), \cdots, f(x_n)). \tag{1.3}$$

It is not difficult to check that $\textsc{Seq}$ is a functor.

Remark 1.1. In [50], P. Flajolet and R. Sedgewick, provide a host of functors and bifunctors with **Set** (or graded sets[1]), there called "combinatorial classes" as domain (see [50] Ch 1 §1.2 "Admissible constructions and specifications"). There can be found a host of beautiful combinatorial functors **Set** $\to$ **Set** as: $\textsc{Seq}$, $\textsc{Cyc}$, $\mathrm{M}\textsc{Set}, \ldots.$
Our functor $\textsc{Seq}$ is then similar to their $\textsc{Seq}$ functor (save a modification of its domain which is the whole sets of ZFC i.e. **Set**).

1.2 Free objects.

1.2.1 General principle.

In this subsection, we introduce the combinatorial (free) objects we will use throughout the manuscript [2]. These objects (call them $G(X)$) together with a map $j_X:\ X \to G(X)$ are all solutions of universal problems. We will recall the definition, notation and terminology about these free objects below (cf. in general Bourbaki [12] Ch IV §3 or [80] and, in particular, [10] Ch I §7.1 and Lothaire [79] Prop 1.1.1 for free monoids, Bourbaki [13] Ch II §2.2 Prop 1 and Reutenauer [94] Thm 0.4 for free Lie algebras and Bourbaki [10] for enveloping algebras i.e. towards the free associative algebras with unit), but here, we state the general principle.
The scheme is the same for all categories considered in Eq. (1.1) ($\mathbf{k}$ being a fixed ring).

[1] More rigorously, finite or denumerable sets, $\mathbb{N}$-graded by finite blocks.

[2] Freeness is here with respect to sets. To be more complete we will give in Section 1.3 an example of freeness with respect to graphs.

All objects of these categories can be considered as sets, we then have a natural "forgetful" functor F such that, $\mathcal{A}$ being an object (of one of these categories), $F(\mathcal{A})$ is the set underlying the structure $\mathcal{A}$. We are now in the position of stating the universal problem corresponding to the forgetful functor F.

Universal problem (w.r.t. F). –
For any set X ($\mathcal{C}$ being one of the categories as above) does there exist a pair $(j_X, G(X))$ ($G(X)$ being an object of $\mathcal{C}$ and $j_X : X \to F(G(X))$ a map) such that:
For any map $f : X \to F(\mathcal{A})$, there exists a unique $\widehat{f} \in Hom_{\mathcal{C}}(G(X), \mathcal{A})$ such that $f = F(\widehat{f}) \circ j_X$.

Remark 1.2. It might happen that G be not defined everywhere as shows the case with $\mathcal{C} = \mathbf{FinSet}$, F being the inclusion functor (i.e. $F(X) = X$ for every finite set and $F(f) = f$ for every set-theoretical map between finite sets).
However a solution of the universal problem (1.4), *for all* X, provides a *free functor* $G : \mathbf{Set} \to \mathcal{C}, X \mapsto G(X)$ which is left-adjoint to the forgetful functor $F : \mathcal{C} \to \mathbf{Set}$. The reader must be aware that, in general, the notion of "forgetful functor" (here constructed from algebraic structures and sets) is informal (see [112]).

$$\begin{tikzcd}
\mathbf{Set} & \mathcal{C} \arrow[l, dotted, "F"'] \\
X \arrow[r, dashed, "f"] \arrow[dr, dashed, "j_X"'] & \mathcal{A} \\
& G(X). \arrow[u, "\widehat{f}"']
\end{tikzcd} \tag{1.4}$$

1.2.2 Presented structures.

For any category of the list (1.1), the notion of a "structure defined by generators and relations" is well defined. For X a set and $\mathcal{C} \in \{\mathbf{Mon}, \mathbf{Grp}, \mathbf{k\text{-}AAU}, \mathbf{k\text{-}Lie}\}$, a relator is a set of equalities $t_i^{(1)} = t_i^{(2)}$ $(i \in I)$ where $t_i^{(j)}$ are elements of the corresponding free structure $G(X)$. We can express the universal problem of presented structures in terms of maps and co-equalization.
Let $u, v : I \to G(X)$ such that $u(i) = t_i^{(1)}$, $v(i) = t_i^{(2)}$, then the structure defined "by generators and relations" with X as set of generators and $\mathbf{r} = \{u(i) = v(i)\}_{i \in I}$ as set

of relations is noted

$$M = \langle\, X \,|\, \mathbf{r} \,\rangle_{\mathcal{C}} \;\in \mathcal{C} \tag{1.5}$$

This object comes together with an arrow $s_{\mathbf{r}} :\; G(X) \to M$. The pair $(s_{\mathbf{r}}, M)$ is a solution of the universal problem (see diagram (1.6)) such that

- $s = s_{\mathbf{r}}$ is a morphism $G(X) \to M$ of $\mathcal{C}$ such that $F(s) \circ u = F(s) \circ v$ [3] ($F :\; \mathcal{C} \to \mathbf{Set}$ being the "natural" forgetful functor) and

- For every morphism $\varphi :\; G(X) \to N$ such that $F(\varphi) \circ u = F(\varphi) \circ v$ [4] there exists a unique morphism $\widehat{\varphi} :\; M \to N$ such that $\varphi = \widehat{\varphi} \circ s$

This situation is summarized by the following diagram.

$$\begin{tikzcd}
\mathbf{Set} & \mathcal{C} \arrow[l, dotted, "F"'] & & \mathcal{C} \\
I \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & G(X) \arrow[rr, "\varphi"] \arrow[d, "s_{\mathbf{r}}"'] & & N \\
& M \arrow[urr, "\widehat{\varphi}"'] & &
\end{tikzcd} \tag{1.6}$$

Remark 1.3. When one has underlying groups (additive or multiplicative), expressions like $u_i = v_i$ can be made equivalent to $w_i = e$ (e is the neutral[5]). In these cases, the list of relations can be replaced by a list of *relators*. But, in the case of monoids (and other structures like semigroups), no such mechanism exists and we have to stick to a list of relations of type $u_i = v_i$.

1.2.3 Free monoids.

Oftentimes, we will use free monoids as monomials (i.e. in this section $\mathcal{C} = \mathbf{Mon}$). Elements of a free monoid $G(X) = X^*$ are usually called "words" which is why the generating set X is often called an alphabet (see [79]).

[3]This means that, for all $i \in I$, $s(u_i) = s(v_i)$.

[4]This means that, for all $i \in I$, $\varphi(u_i) = \varphi(v_i)$.

[5]For abelian groups (noted additively) $u_i = v_i$ is equivalent to $u_i - v_i = 0$. For groups noted multiplicatively $u_i = v_i$ is equivalent to $u_i v_i^{-1} = 1$.

Alphabets and words.

Given a set X, the free monoid on X is the set X^* of all words over the alphabet X (i.e. the empty word 1_{X^*} or non-empty words $x_1x_2\cdots x_n$, here $n \in \mathbb{N}_+$ and $x_i \in X, \forall i \in [1,n]$), made into a monoid using *concatenation* as

$$x_1\cdots x_n.y_1\cdots y_m := x_1\cdots x_ny_1\cdots y_m.$$

We notice that, for any $X \in \mathbf{Set}$, the set underlying X^* (i.e. $F(X^*)$ in the perspective of Subsection 1.2.1) is nothing but $\textsc{Seq}(X)$ with another data structure i.e. words instead of lists (i.e. under this identification, one has $F(X^*) = \textsc{Seq}(X)$).

We will denote by $|w|$ the length of a word w from X^* and a partial degree $|w|_x$ is the number of occurrences of an element x of X within w.
A word $w \in X^*$ being given, we will denote by (w) the multidegree of w, that is the family $(|w|_x)_{x\in X}$ which belongs to $\mathbb{N}^{(X)}$, the set of finitely supported families[6] of $\mathbb{N}^X$ (which is the set of all functions i.e. families $(\alpha_x)_{x\in X}$ from $X \to \mathbb{N}$) (cf. [10] Ch I §7.7).

The subset of X^* of the words of length n is denoted by

$$X^n = \{w \in X^* \,|\, |w| = n\}. \tag{1.7}$$

We also consider the subset of the words with a given multidegree $\alpha \in \mathbb{N}^{(X)}$, namely

$$X^\alpha = \{w \in X^* \,|\, (w) = \alpha\}. \tag{1.8}$$

Commentary 1. *i) When an ambient free monoid X^* is fixed, for any $Y \subset X^*$, the notation Y^* stands for the submonoid (of X^*) generated by Y, this set might not be free as a monoid. For example, in a^* (the free monoid on one letter)*

$$\{a^2, a^5\}^* = \{1, a^2, a^4\} \cup \{a^n\}_{n\geq 5} \tag{1.9}$$

is not free.
ii) Save in Example 1.3, we will only use the regular expression (see [93]) $B^*Z = \{uz\}_{u\in B^*, z\in Z}$. It can be shown that $(B^*Z)^*$ is free.

[6] Families whose support

$$\mathrm{supp}((\alpha_x)_{x\in X}) := \{x \in X \,|\, \alpha_x \neq 0\}$$

is finite.

Free monoid and its universal property.

For X a set (viewed as an alphabet of noncommutative variables), the canonical embedding of X^1 (the set of words of length one, which we identify with X) will be denoted by $j_X : \ X \hookrightarrow X^*$.
One can prove easily that the constructed pair (j_X, X^*) is a solution of the universal problem corresponding to the following diagram (see description in Subsection 1.2.1, here for $\mathcal{C} = \mathbf{Mon}$ or [94] Prop 1.1):

$$\begin{tikzcd}
\mathbf{Set} & \mathbf{Mon} \arrow[l, dotted, "F"'] \\
X \arrow[r, dashed, "f"] \arrow[dr, dashed, "j_X"'] & M \\
& X^*. \arrow[u, "\widehat{f}"']
\end{tikzcd} \tag{1.10}$$

It means that for every monoid M and set-theoretical map $f : X \to M$, there exists a unique morphism of monoids $\widehat{f} : X^* \to M$ such that $\widehat{f} \circ j_X = f$.

In Subsections 1.2.4 and 1.2.5, we will recall the construction of free objects for the category of associative **k**-algebras with unit **k-AAU** and in the category of Lie **k**-algebras **k-Lie**.

Presentation of a monoid.

Following the general scheme, the monoid presented by a set of generators X and relations $\mathbf{r} = ((u_i = v_i))_{i \in I}$ i.e.

$$M = \langle\, X \mid ((u_i = v_i))_{i \in I}\, \rangle_{\mathbf{Mon}} = X^* /_{\equiv_{\mathbf{r}}} \tag{1.11}$$

together with the natural arrow $s_{\mathbf{r}} : \ X^* \to M$ is a solution of the universal problem (see diagram (1.12)) of existence of a pair (s, M) such that

- s is a morphism of monoids $X^* \to M$ such that $F(s) \circ u = F(s) \circ v$ [7]
  ($F : \ \mathbf{Mon} \to \mathbf{Set}$ being the forgetful functor as above) and

[7] This means that, for all $i \in I$, $s(u_i) = s(v_i)$.

- For all morphisms of monoids $\varphi :\ X^* \to N$ such that $F(\varphi)\circ u = F(\varphi)\circ v$ [8] there exists a unique morphism $\widehat{\varphi} :\ M \to N$ such that $\varphi = \widehat{\varphi}\circ s$

$$\begin{tikzcd}
\mathbf{Set} & \mathbf{Mon} \arrow[l, dotted, "F"'] & & \mathbf{Mon} \\
I \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & X^* \arrow[rr, "\varphi"] \arrow[d, "s_{\mathbf{r}}"'] & & N \\
& M \arrow[urr, "\widehat{\varphi}"'] & &
\end{tikzcd} \tag{1.12}$$

Note that, due to intersection properties of congruences ([43] Ch 1 §5, see below), the arrow $s_{\mathbf{r}}$ is onto as this is the case of the analogue arrows for each category $\mathcal{C}$ of the list (1.1).

In this situation, with all set of generators X and relations $\mathbf{r} = \{(u_i = v_i)\}_{i\in I}$ ($u_i, v_i \in X^*$), the pair $(s_{\mathbf{r}}, M)$ exists. In fact, M is the quotient of X^* by the congruence generated by the family $((u_i = v_i))_{i\in I}$ (see [43] Ch 1 §5).

Example 1.1. Free group (as a presented monoid). –
Let X be an alphabet (viewed as a set of generators), we can construct an implementation version of the free group $\Gamma(X)$ as follows

- Create $\overline{X}$, a disjoint copy of X, $\widetilde{X} = X + \overline{X}$ with the involution $x \to \bar{x}$ such that $\bar{\bar{x}} = x$
- Then $\Gamma(X) = \widetilde{X}^* \big/ (x\bar{x} = 1)_{x\in\widetilde{X}}$.

Example 1.2. Dihedral group D_5 (as a presented monoid). –
The Dihedral group $D_5 = \langle\, \{s_1, s_2\} \,|\, s_1^2 = s_2^2 = (s_1 s_2)^5 = 1\,\rangle_{\mathbf{Mon}}$ is known as the symmetry group of the regular pentagon.

Definition 1.1. We now consider the following

i) In general, for any monoid M, we say that M is graded if

$$M = \biguplus_{m\in\mathbb{N}} M_m \text{ and } M_p.M_q \subseteq M_{p+q}$$

[8]This means that, for all $i \in I$, $\varphi(u_i) = \varphi(v_i)$.

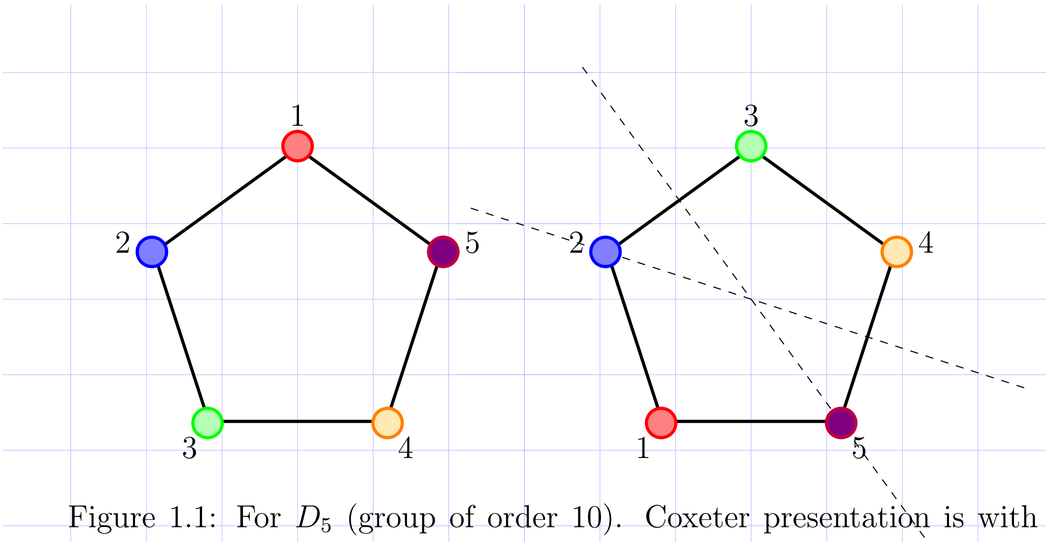


Figure 1.1: For D_5 (group of order 10). Coxeter presentation is with s_1 (symmetry w.r.t. the line passing through node 5) and s_2 (symmetry w.r.t. the line passing through node 2) and relations $[s_i^2 = 1 \ ; \ (s_1 s_2)^5 = 1]$.

with $M_0 = \{1_M\}$. It is equivalent to say that a monoid M is graded if there is an additive[9] proper[10] length function $l : M \to \mathbb{N}$ [11].

ii) If, moreover, for all $m \in \mathbb{N}$, $|M_m| < +\infty$, M is said finitely graded. It means that the additive proper length function l satisfies the property $|\{x \in M \mid l(x) = m\}| < +\infty, \forall m \in \mathbb{N}$ and then l is called a finitely additive proper length function on M.

iii) The Hilbert series of a monoid M is then the formal power series

$$Hilb(M,t) := \sum_{m \geq 0} |M_m| \, t^m \in \mathbb{N}[[t]]. \tag{1.13}$$

To have such a Hilbert series it suffices that the monoid be a finitely graded set.

Assume that X is a finite alphabet, it is an exercise to check (by the definition of finitely additive proper length function) that a monoid M presented by $M = \langle X \mid ((u_i = v_i))_{i \in I} \rangle_{\mathbf{Mon}}$ with $|u_i| = |v_i|$ for all $i \in I$ is a finitely graded monoid.

[9] $l(xy) = l(x) + l(y) \ \forall x, y \in M$.
[10] $l^{-1}(0) = \{1_M\}$.
[11] It is a morphism of monoids $M \to \mathbb{N}$, but, due to the lack of inverses, the condition $l^{-1}(0) = \{1_M\}$ does not imply that it is injective, of course.

Remark 1.4. A remark for the exercise above is that the set X is a finite alphabet and relations $\mathbf{r} = \{(u_i = v_i)\}_{i\in I}$ are homogeneous w.r.t. length meaning $|u_i|=|v_i|$ for all $i \in I$. Moreover, let $Bi = \langle\, \{a, b\} \,|\, ab = 1\, \rangle_{\mathbf{Mon}}$ be the bicyclic monoid [4], then we can prove without difficulty that Bi admits no $\mathbb{N}$-gradation because each additive proper length function l on Bi satisfies $0 = l(1) = l(ab) = l(a) + l(b) \geq 2$ which is a contradiction, this shows that we cannot relax the condition that the relator be homogeneous.
However, one can consider the Hilbert series of a set of distinguished representatives as the language $L = \{b^s a^r\}_{r,s\in\mathbb{N}}$ is a section of the presented monoid Bi.

The condition homogeneity w.r.t. length whereas it is sufficient, is not necessary by an example presented as follows: consider a homomorphism of monoids $l : M = \langle\, \{a, b\} \,|\, b^3 = ab\, \rangle_{\mathbf{Mon}} \to \mathbb{N}$ extending from a monoid homomorphism $\{a, b\}^* \to \mathbb{N}$ which sends $a \mapsto 2, b \mapsto 1$ by the coequalizer (1.12) over the relation $b^3 = ab$, then clearly l is a finitely additive proper length function on M and hence M is a finitely graded monoid.

Example 1.3. Assume that $A := \{a, b\}$ is a set of two letters with $a < b$.

1. Let us consider monoid

$$SF_0(A) := \langle A \,|\, a^2 = b^2 = 1\rangle_{\mathbf{Mon}} = \langle\{a, b\} \,|\, 1 = a^2 = b^2\rangle_{\mathbf{Mon}} \tag{1.14}$$

It is easy to show that the onto morphism $s : \{a, b\}^* \to SF_0(A)$ admits as a section the square-free language[12]

$$\begin{aligned} L &= \{a, b\}^* \setminus (\{a, b\}^* a^2 \{a, b\}^* \cup \{a, b\}^* b^2 \{a, b\}^*) && (1.15)\\ &= (ab)^*(1 + a) + (ba)^*(1 + b). && (1.16)\end{aligned}$$

(the + in 1.16 is, in this case, a dijoint union).
Alternatively, we can consider the monoid

$$SF_1(A) := \langle A \,|\, a^2 = a;\; b^2 = b\rangle_{\mathbf{Mon}} \tag{1.17}$$

It also admits the square-free language as a section, but it is by no means the same monoid (the first one is a group whereas the second is not).

[12] Description with regular expressions (extended with “set minus” for the second term, see [93]).

In fact the correct setting for $Hilb(V,t)$ is that of a free module or a vector space then and here as the monoids (1.14) and (1.17) are not graded, we give the Hilbert series of the "square free section" L where

- $L_0 = 1_{A^*}$;
- $L_1 = \{a, b\}$;
- $L_2 = \{ab, ba\}$;
- $\cdots$
- For all $m \geq 1$, one has $L_{2m} = \{(ab)^m, (ba)^m\}$ and $L_{2m+1} = \{a(ba)^m, b(ab)^m\}$.

However these monoids are not finitely graded for this decomposition because the grading is not compatible with the product (seen as a concatenation with rewriting) then

$$Hilb(M,t) = \sum_{m \geq 0} |L_m|\, t^m \tag{1.18}$$

$$= 1 + \sum_{m \geq 1} 2t^m = 1 + \frac{2t}{1-t} = \frac{1+t}{1-t}. \tag{1.19}$$

2. Consider another monoid with length presented as

$$M(A) := \langle A \,|\, a^2 = b^2 \rangle_{\mathbf{Mon}} = \langle \{a, b\} \,|\, a^2 = b^2 \rangle_{\mathbf{Mon}} \tag{1.20}$$

One section of it is the language of words without any b^2 factor

$$L = A^* \setminus A^* b^2 A^*. \tag{1.21}$$

In this case, $M(A)$ is a finitely graded monoid $M(A) = \biguplus_{m \in \mathbb{N}} M(A)_m$ where

- $M(A)_0 = 1_{A^*}$;
- $M(A)_1 = W_1 = \{a, b\}$;
- $M(A)_2 = W_2 = \{aa, ab, ba\}$, $|\,M(A)_2\,| = |\,M(A)_1\,| + |\,M(A)_0\,|$;
- $M(A)_3 = W_3 = \{aaa, aab, aba, baa, bab\}$, $|\,M(A)_3\,| = |\,M(A)_2\,| + |\,M(A)_1\,|$;
- $M(A)_4 = W_4 = \{aaaa, aaab, aaba, abaa, abab, baaa, baab, baba\}$, and then $|\,M(A)_4\,| = |\,M(A)_3\,| + |\,M(A)_2\,|$;

- $M(A)_5 = W_5 = \{aaaaa, aaaab, aaaba, aabaa, aabab, abaaa, abaab, ababa,$ $baaaa, baaab, baaba, babaa, babab\}$, $\mid M(A)_5 \mid = \mid M(A)_4 \mid + \mid M(A)_3 \mid$;
- $\cdots$
- For each $m \geq 2$, $M(A)_m = W_m$ the set of all words of length m without any b^2 factor and then $\mid M(A)_m \mid = \mid M(A)_{m-1} \mid + \mid M(A)_{m-2} \mid$.

We obtain that $\mid M(A)_m \mid = F(m)$ is the Fibonacci number (with initial condition $F(0) = 1,\ F(1) = 2$ [13]) and then the Hilbert series

$$\begin{aligned} Hilb(M(A), t) &= \sum_{m \geq 0} \lvert M(A)_m \rvert\, t^m = \sum_{m \geq 0} F(m) t^m \\ &= 1 + 2t + 3t^2 + 5t^3 + 8t^4 + 13t^5 + \cdots = \frac{1+t}{1-t-t^2}. \end{aligned}$$

1.2.4 Free associative algebras.

Here, we deal with free associative algebras with unit (i.e. $\mathcal{C} = \mathbf{k}$-**AAU**). See just below.

Free associative k-algebras and its universal property.

Let $\mathbf{k}$ be a commutative ring with unit. A *noncommutative polynomial* on X over $\mathbf{k}$ is a linear combination over $\mathbf{k}$ of words on X. We simply say polynomial when no confusion arises. If P is a polynomial, we write it as

$$P = \sum_{w \in X^*} \langle P \mid w \rangle w.$$

The set of all polynomials is denoted by $\mathbf{k}\langle X \rangle$. It has a $\mathbf{k}$-algebra structure, with a concatenation product

$$PQ = \sum_{w \in X^*} \langle PQ \mid w \rangle w,$$

where

$$\langle PQ \mid w \rangle = \sum_{w = uv} \langle P \mid u \rangle \langle Q \mid v \rangle.$$

The polynomial algebra $\mathbf{k}\langle X \rangle$ is the free associative $\mathbf{k}$-algebra with unit (free $\mathbf{k}$-**AAU**) generated by X. It means that, given the canonical embedding $j_X : X \to \mathbf{k}\langle X \rangle$, then

[13] Usually, the initial conditions are $0, 1, \cdots$.

for every associative $\mathbf{k}$-algebra with unit $\mathcal{A}$ and set-theoretical map $f : X \to \mathcal{A}$, there exists a unique morphism of $\mathbf{k}$-$\mathbf{AAU}$ $\widehat{f} : \mathbf{k}\langle X\rangle \to \mathcal{A}$ such that $\widehat{f} \circ j_X = f$. We get an inclusion and one can prove that the pair $(j_X, \mathbf{k}\langle X\rangle)$ is a solution of the universal problem corresponding to the following diagram (see description in Subsection 1.2.1, here for $\mathcal{C} = \mathbf{k}$-$\mathbf{AAU}$)

$$\begin{tikzcd} \mathbf{Set} & & \mathbf{k}\text{-}\mathbf{AAU} \arrow[ll, dotted, "F"'] \\ X \arrow[rr, dashed, "f"] \arrow[drr, dashed, "j_X"'] & & \mathcal{A} \\ & & \mathbf{k}\langle X\rangle . \arrow[u, "\widehat{f}"'] \end{tikzcd} \tag{1.22}$$

Graded structures of the free associative k-algebra.

We shall explain graded structures of the free associative $\mathbf{k}$-algebra with unit $\mathbf{k}\langle X\rangle$. The reader can review in Bourbaki [13] or Reutenauer [94] the concept of graded $\mathbf{k}$-algebra (in the latter, homogeneous for the multidegree is called finely homogeneous).

For $n \geq 0$, we denote by

$$\mathbf{k}_n\langle X\rangle = \mathrm{span}_{\mathbf{k}}\{X^n\} \tag{1.23}$$

the sub $\mathbf{k}$-module generated by X^n in $\mathbf{k}\langle X\rangle$ (the n-th tensor power of the free $\mathbf{k}$-module with basis X). The grading by the total degree of the free associative $\mathbf{k}$-algebra $\mathbf{k}\langle X\rangle$ can be described as a direct sum of $\mathbf{k}$-modules

$$\mathbf{k}\langle X\rangle = \bigoplus_{n\in\mathbb{N}} \mathbf{k}_n\langle X\rangle \tag{1.24}$$

with the concatenation multiplication

$$\mathbf{k}_n\langle X\rangle.\mathbf{k}_m\langle X\rangle \subseteq \mathbf{k}_{n+m}\langle X\rangle. \tag{1.25}$$

A member of one of the subspaces $\mathbf{k}_n\langle X\rangle$ (1.24) is called an *homogeneous polynomial.* On the other hand, given an $\alpha \in \mathbb{N}^{(X)}$ there is a sub $\mathbf{k}$-module of $\mathbf{k}\langle X\rangle$, namely

$$\mathbf{k}_\alpha\langle X\rangle = \mathrm{span}_{\mathbf{k}}\{X^\alpha\}. \tag{1.26}$$

The free associative $\mathbf{k}$-algebra $\mathbf{k}\langle X\rangle$ can be also graded by multidegree

$$\mathbf{k}\langle X\rangle = \bigoplus_{\alpha\in\mathbb{N}^{(X)}} \mathbf{k}_\alpha\langle X\rangle \tag{1.27}$$

with the concatenation multiplication

$$\mathbf{k}_\alpha\langle X\rangle.\mathbf{k}_\beta\langle X\rangle \subseteq \mathbf{k}_{\alpha+\beta}\langle X\rangle. \tag{1.28}$$

We also call the grading (1.27) a grading by *homogeneous submodules* or *finely homogeneous submodules.* Let $\alpha = (\alpha_x)_{x\in X} \in \mathbb{N}^{(X)}$ and denote by $\mathbf{k}_\alpha\langle X\rangle$ the space of finely homogeneous polynomials of partial degree α_x in each letter x. Notice that the set X^α is a basis of $\mathbf{k}_\alpha\langle X\rangle$ (cf. Bourbaki [13] Ch II §2.6 p.127 "multigraduation" or Reutenauer [94] p.178 Ch 8 §1.6 where this module is noted E_α).

1.2.5 Free Lie algebras.

Here, we deal with free Lie algebras (i.e. $\mathcal{C} = \mathbf{k\text{-}Lie}$). See just below.

Lie polynomials.

On the polynomial algebra $\mathbf{k}\langle X\rangle$ with the concatenation product $PQ = \sum_{w\in X^*}\langle PQ \mid w\rangle w$, a Lie bracket is as usual defined by

$$[P,Q] = PQ - QP. \tag{1.29}$$

With this bracket, $\mathbf{k}\langle X\rangle$ is a Lie algebra. A *Lie polynomial* is an element of the smallest submodule of $\mathbf{k}\langle X\rangle$ containing X and closed under the Lie bracket. The set of all Lie polynomials in $\mathbf{k}\langle X\rangle$ forms a Lie algebra and we denote it by $\mathcal{L}_\mathbf{k}(X)$. It can be shown, through Lyndon bases[14], that the Lie algebra $\mathcal{L}_\mathbf{k}(X)$ is the free Lie algebra over $\mathbf{k}$ generated by X. It means that, in view of Subsection 1.2.1, the pair $(j_X, \mathcal{L}_\mathbf{k}(X))$ is a solution of the universal problem corresponding to following diagram:

$$\begin{tikzcd}
\mathbf{Set} & \mathbf{k\text{-}Lie} \arrow[l, dotted, "F"'] \\
X \arrow[r, dashed, "f"] \arrow[dr, dashed, "j_X"'] & \mathfrak{g} \\
 & \mathcal{L}_\mathbf{k}(X). \arrow[u, "\hat{f}"']
\end{tikzcd} \tag{1.30}$$

[14]See, with Hall bases [13] Ch 2 §2.11 and [94] Ch 4 Prop 4.9.

Moreover, $\mathcal{L}_{\mathbf{k}}(X)$ is homogeneous for the two gradings, where

$$\mathcal{L}_{\mathbf{k}}(X) = \bigoplus_{n\in\mathbb{N}} \mathcal{L}_{\mathbf{k}}(X)_n \ ; \ \mathcal{L}_{\mathbf{k}}(X) = \bigoplus_{\alpha\in\mathbb{N}^{(X)}} \mathcal{L}_{\mathbf{k}}(X)_\alpha \tag{1.31}$$

with

$$\mathcal{L}_{\mathbf{k}}(X)_n := \mathcal{L}_{\mathbf{k}}(X) \bigcap \mathbf{k}_n\langle X\rangle \tag{1.32}$$

(homogeneous Lie polynomials of total degree n) and, for a finitely supported $\alpha \in \mathbb{N}^{(X)}$

$$\mathcal{L}_{\mathbf{k}}(X)_\alpha := \mathcal{L}_{\mathbf{k}}(X) \bigcap \mathbf{k}_\alpha\langle X\rangle \tag{1.33}$$

(homogeneous Lie polynomials of multidegree α).

Due to the fine grading of the free Lie algebra, for any (disjoint) partition $X = B + Z$, we can set

$$\mathcal{L}_{\mathbf{k}}(X)_B = \bigoplus_{\substack{\alpha\in\mathbb{N}^{(X)}\\ |\alpha|_Z=0}} \mathcal{L}_{\mathbf{k}}(X)_\alpha \tag{1.34}$$

and

$$\mathcal{L}_{\mathbf{k}}(X)_{BZ} = \bigoplus_{\substack{\alpha\in\mathbb{N}^{(X)}\\ |\alpha|_Z>0}} \mathcal{L}_{\mathbf{k}}(X)_\alpha \tag{1.35}$$

where, for any subset $Y \subset X$ we set $|\alpha|_Y := \sum_{x\in Y} \alpha(x)$. It is straightforward to see that

$$\mathcal{L}_{\mathbf{k}}(X) = \mathcal{L}_{\mathbf{k}}(X)_{BZ} \oplus \mathcal{L}_{\mathbf{k}}(X)_B$$

and that $\mathcal{L}_{\mathbf{k}}(X)_{BZ}$ is a Lie ideal. Then we set[15]

$$\mathcal{L}_{\mathbf{k}}(X) = \mathcal{L}_{\mathbf{k}}(X)_{BZ} \rtimes \mathcal{L}_{\mathbf{k}}(X)_B$$

or, in the language of SES (short exact sequences), the following is split

$$0 \longrightarrow \mathcal{L}_{\mathbf{k}}(X)_{BZ} \xhookrightarrow{j} \mathcal{L}_{\mathbf{k}}(X) \xrightarrow{p} \mathcal{L}_{\mathbf{k}}(X)_B \longrightarrow 0. \tag{1.36}$$

[15]For any Lie algebra $\mathfrak{b}, \mathfrak{h}$ and an action by derivations of the Lie algebra $\mathfrak{b}$ on $\mathfrak{h}$ i.e. a Lie homomorphism $\alpha : \mathfrak{b} \to \mathfrak{Der}(\mathfrak{h})$, we can construct $\mathfrak{g}$, a split Lie algebra extension of $\mathfrak{b}$ by $\mathfrak{h}$ whose underlying $\mathbf{k}$-module is the external direct sum of modules $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{b}$ and the Lie bracket is given by the following formula

$$[(h_1, b_1), (h_2, b_2)] = \big([h_1, h_2] + \alpha(b_1)(h_2) - \alpha(b_2)(h_1), [b_1, b_2]\big).$$

With our construction, we say that the Lie algebra $\mathfrak{g}$ is a semi-direct product of $\mathfrak{b}$ with $\mathfrak{h}$, denoted by $\mathfrak{g} := \mathfrak{h} \rtimes \mathfrak{b}$. See also Definition 2.1.

Remark 1.5. As a remark, we add for the reader the following nice mnemonic about the orientation of the vertical bar in the $\ltimes$ (or $\rtimes$) notation: the factor "who acts" is the one with the screwdriver i.e. is on the bar's side, this action being by automorphisms for groups and by derivations for Lie algebras.

We will see later that this split SES can serve as a model for every semi-direct product. The classical Lazard elimination theorem (Theorem 2.3 below) will give a better understanding of this split SES ((1.36)), identifying $\mathcal{L}_{\mathbf{k}}(X)_B$ and $\mathcal{L}_{\mathbf{k}}(X)_{BZ}$ as (isomorphic images of) concretely defined free Lie algebras (together with their alphabets).

Presentation of a Lie algebra.

We here follow Bourbaki [13] Ch 2 §2.3[16]. Let X be a set, $\mathfrak{g}$ a Lie $\mathbf{k}$-algebra and $\mathbf{a} = \{a_x\}_{x\in X}$ a subset of $\mathfrak{g}$. Let us consider a Lie morphism

$$f_{\mathbf{a}} : \mathcal{L}_{\mathbf{k}}(X) \to \mathfrak{g}, \qquad X \ni x \mapsto a_x. \tag{1.37}$$

The elements of the kernel of $f_{\mathbf{a}}$ are called the *relators* of the set $\mathbf{a}$. The set $\mathbf{a}$ is called *generating* (resp. *free, basis*) if $f_{\mathbf{a}}$ is surjective (resp. injective, bijective).
A presentation of $\mathfrak{g}$ is an ordered pair $(\mathbf{a}, \mathbf{r})$ consisting of a generating set $\mathbf{a} = \{a_x\}_{x\in X}$ and a set $\mathbf{r} = \{r_j\}_{j\in J}$ of relators of $\mathbf{a}$ generating[17] $\mathcal{J}_{\mathbf{r}} := \mathrm{Ker}(f_{\mathbf{a}})$ which is a Lie ideal of $\mathcal{L}_{\mathbf{k}}(X)$. We also say that $\mathfrak{g}$ is presented by the set $\mathbf{a}$ related by the relators $r_j (j \in J)$ and write

$$\mathfrak{g} = \langle\, \mathbf{a} \,|\, \mathbf{r} \,\rangle_{\mathbf{k}\text{-}\mathbf{Lie}} = \mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J}_{\mathbf{r}}. \tag{1.38}$$

Likewise, in other categories we could have defined a group by generators and relations (see [81]) and written

$$G = \langle\, \mathbf{a} \,|\, \mathbf{r} \,\rangle_{\mathbf{Grp}}. \tag{1.39}$$

[16] Adapted to our situation which requires WLOG that relators be gathered within a set (because we will perform traces, see Subsection 2.2.2) rather than a family.

[17] This time as a Lie ideal.

Adjoint representation and derivations of Lie algebras.

For convenience, the adjoint representation within $\mathfrak{g}$ will be extended to sequences of elements $(Q_1, Q_2, \cdots, Q_n) \in \textsc{Seq}(\mathfrak{g})$ as

$$\mathrm{ad}_{(Q_1,Q_2,\cdots,Q_n)} := \mathrm{ad}_{Q_1} \circ \mathrm{ad}_{Q_2} \circ \cdots \circ \mathrm{ad}_{Q_n} \in \mathrm{End}(\mathfrak{g}). \tag{1.40}$$

We recall the classical notion of a Lie algebra derivation

Definition 1.2. Let $\mathfrak{g}$ be a Lie $\mathbf{k}$-algebra, we denote $\mathrm{End}_{\mathbf{k}}(\mathfrak{g})$ (or $\mathrm{End}(\mathfrak{g})$) the set of $\mathbf{k}$-linear endomorphisms of $\mathfrak{g}$ viewed as a $\mathbf{k}$-module. Then $D \in \mathrm{End}_{\mathbf{k}}(\mathfrak{g})$ is called a derivation of $\mathfrak{g}$ if and only if for all $u, v \in \mathfrak{g}$ we have

$$D([u, v]) = [D(u), v] + [u, D(v)]. \tag{1.41}$$

The set of derivations of $\mathfrak{g}$, noted $\mathfrak{Der}(\mathfrak{g})$, is a Lie subalgebra of $\mathrm{End}(\mathfrak{g})$ (for the usual bracket $[f, g] := f \circ g - g \circ f$).

Let $\mathfrak{g} \in$ **k-Lie**, the adjoint representation is a Lie morphism

$$\mathrm{ad}^{\mathfrak{g}} := \mathrm{ad} : \mathfrak{g} \to \mathfrak{Der}(\mathfrak{g}), \qquad x \mapsto \mathrm{ad}_x\,. \tag{1.42}$$

We can now extend the adjoint representation $\mathrm{ad}^{\mathfrak{g}}$ to $\mathcal{U}(\mathfrak{g})$ by

$$\mathrm{ad}^{\mathcal{U}(\mathfrak{g})}(g_1 \cdots g_k) := \mathrm{ad}^{\mathfrak{g}}_{g_1} \circ \cdots \circ \mathrm{ad}^{\mathfrak{g}}_{g_k}\,. \tag{1.43}$$

Moreover, our aim is to generalize this definition to the f-derivations, where f is a fixed Lie algebra morphism

Definition 1.3. For any morphism of Lie algebras $f : \mathfrak{g}_1 \to \mathfrak{g}_2$, a $\mathbf{k}$-linear map $D : \mathfrak{g}_1 \to \mathfrak{g}_2$ is called a f-derivation if $D([u, v]) = [D(u), f(v)] + [f(u), D(v)]$ for any $u, v \in \mathfrak{g}_1$.

Remark 1.6. i) We beware the reader that f-derivations do not form a Lie algebra but only form a sub $\mathbf{k}$-module of $\mathrm{Hom}_{\mathbf{k}}(\mathfrak{g}_1, \mathfrak{g}_2)$. We will denote this sub $\mathbf{k}$-module by $\mathfrak{Der}^{(f)}(\mathfrak{g}_1, \mathfrak{g}_2)$.

ii) It is easily checked that, for $D \in \mathfrak{Der}^{(f)}(\mathfrak{g}_1, \mathfrak{g}_2)$ the kernel $\mathrm{Ker}(D)$ is Lie subalgebra of $\mathfrak{g}_1$.

We recall and prove some technical results as corollaries of Prop 8, [13] Ch II §2.8.

Lemma 1.1. *(Corollary of Prop 8, [13] Ch II §2.8) Let X be a set. Every mapping of X into $\mathcal{L}_{\mathbf{k}}(X)$ can be extended uniquely to a derivation $D: \mathcal{L}_{\mathbf{k}}(X) \to \mathcal{L}_{\mathbf{k}}(X)$ i.e. $D \in \mathfrak{Der}(\mathcal{L}_{\mathbf{k}}(X))$.*

Lemma 1.2. *Let X be a set, $\mathfrak{g}_2$ be a Lie algebra and let $f : \mathcal{L}_{\mathbf{k}}(X) \to \mathfrak{g}_2$ be a morphism of Lie algebras. Every mapping of X into $\mathfrak{g}_2$ can be extended uniquely to a f-derivation $D: \mathcal{L}_{\mathbf{k}}(X) \to \mathfrak{g}_2$ i.e. $D \in \mathfrak{Der}^{(f)}(\mathcal{L}_{\mathbf{k}}(X), \mathfrak{g}_2)$. We remark that Lemma 1.1 is as a consequence of Lemma 1.2 for $f = \mathrm{Id}_{\mathcal{L}_{\mathbf{k}}(X)}$.*

Proof. Let X be a set, let f be a morphism of Lie algebras $\mathcal{L}_{\mathbf{k}}(X) \to \mathfrak{g}_2$, and let d be a mapping of X into $\mathfrak{g}_2$. Notice that $\mathfrak{g}_2$ is a $\mathcal{L}_{\mathbf{k}}(X)$-module determined by the formula:

$$u.a = [f(u), a], \text{ for any } u \in \mathcal{L}_{\mathbf{k}}(X) \text{ and } a \in \mathfrak{g}_2.$$

Prop 8, [13] Ch II §2.8 say that there exists one and only one linear mapping D of $\mathcal{L}_{\mathbf{k}}(X)$ into $\mathfrak{g}_2$ extending d and satisfying the relation:

$$D([u, v]) = u.D(v) - v.D(u) \text{ for any } u, v \in \mathcal{L}_{\mathbf{k}}(X),$$

it means that

$$D([u, v]) = [f(u), D(v)] - [f(v), D(u)] = [D(u), f(v)] + [f(u), D(v)]$$

for any $u, v \in \mathcal{L}_{\mathbf{k}}(X)$. We obtain Lemma 1.2. □

The following (easy) proposition gathers properties needed for the proof of our main result in Subsection 2.2.2.

Lemma 1.3. *Let $\mathfrak{g}$ be a Lie $\mathbf{k}$-algebra, $D \in \mathrm{End}(\mathfrak{g})$ (i.e. the set of linear endomorphism of $\mathfrak{g}$ viewed as a $\mathbf{k}$-module) and $\mathcal{J}$ a Lie ideal of $\mathfrak{g}$. Then one checks easily that*

1. *In order that exists a $\mathbf{k}$-linear map $[D]$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{D} & \mathfrak{g} \\ {\scriptstyle s_{\mathcal{J}}}\downarrow & & \downarrow{\scriptstyle s_{\mathcal{J}}} \\ \mathfrak{g}/\mathcal{J} & \xrightarrow{[D]} & \mathfrak{g}/\mathcal{J} \end{array} \tag{1.44}$$

   it is necessary and sufficient that $D(\mathcal{J}) \subset \mathcal{J}$, in this case the ideal $\mathcal{J}$ is usually called D-invariant.

2. *If, moreover, D is a derivation of $\mathfrak{g}$, then one has the induced map $[D] \in \mathfrak{Der}(\mathfrak{g}/\mathcal{J})$.*

3. *The inverse image $D^{-1}(\mathcal{J})$ (D being still a derivation) is a Lie subalgebra of $\mathfrak{g}$.*

Remark 1.7. The reader is invited to remark that the proof of the above does not use the defining identities of a Lie algebra ($[x,x] = 0$ identically and Jacobi) and holds true *mutatis mutandis* to any **k**-algebra (non necessarily associative nor unital).

1.3 An example of free object with respect to graphs.

We now introduce a particular class of monoids. They are known under various names (monoid of commutations and rearrangements, monoid of traces, partially commutative monoid, Cartier-Foata monoid) and has many applications: in mathematics and combinatorics [22] as well as in computer sciences [29]. It was introduced by Cartier and Foata in 1969 and defined by a presentation consisting in commutations between certain pairs of generators.

As relations of these presentations are only commutations, they can be read within the categories **Mon**, **Grp**, **k-AAU** and **k-Lie** (cf. Duchamp and Krob [41]). In fact, for (at least) these categories and M an object of one of them, the set of pairs $(x,y) \in M^2$ such that x and y commute is a reflexive and symmetric graph $\theta_M \subset M^2$. We introduce the definition of an alphabet with commutations.

Definition 1.4. Let $X \in$ **Set** be a set viewed as a alphabet. A commutation relation on X is a reflexive and symmetric graph $\theta \subset X^2$ (i.e. $\theta = \theta^{-1}$ and $\Delta_X := \{(x,x)\}_{x \in X}$, the diagonal of X, is a subset of θ).

The pairs (X, θ) where θ is a commutation relation on X form a category **CommAlph**, the arrows of which are $f : (X, \theta_X) \to (Y, \theta_Y)$ such that

- $f : X \to Y$ is a map
- for all $(x_1, x_2) \in \theta_X$ we have $(f(x_1), f(x_2)) \in \theta_Y$.

What has been said before the definition shows us that for $\mathcal{C}$ one of the categories in the list (1.1) we can assign to each $M \in \mathcal{C}$ an object $F(M) := (M, \theta_M)$ of **CommAlph** where θ_M is the set of all pairs (x, y) of commuting elements of M (where the meaning of "commuting" depends on $\mathcal{C}$). It is not difficult to check that, in each case, $F : \mathcal{C} \to$ **CommAlph** is a functor. Then, for each category $\mathcal{C}$ of the list (1.1), one can state an universal problem in the style of Subsection 1.2.1.

A pair $(X, \theta_X) \in$ ***CommAlph*** *being given, does there exist a pair* $(j_{(X,\theta_X)}, G(X, \theta_X))$ *such that, for all* $\mathcal{A} \in \mathcal{C}$ *and arrow* $f : (X, \theta_X) \to F(\mathcal{A})$*, we have a unique* $\widehat{f} \in \mathrm{Hom}_{\mathcal{C}}(G(X, \theta_X), \mathcal{A})$ *such that* $f = F(\widehat{f}) \circ j_X$. Diagrammatically

$$\begin{tikzcd} \mathbf{CommAlph} & & \mathcal{C} \arrow[ll, dotted, "F"'] \\ (X,\theta_X) \arrow[rr, dashed, "f"] \arrow[drr, dashed, "j_{(X,\theta_X)}"'] & & \mathcal{A} \\ & & G(X,\theta_X). \arrow[u, "\widehat{f}"'] \end{tikzcd} \tag{1.45}$$

The theory of free partially commutative structures [41], which is the last one (in characteristic zero) for which Magnus theory[18] holds (see [40] Thm 3) provides, as for the case of alphabets (sets), free objects. Let us give the two structures we will need as well as their constructions

- The free partially commutative monoid $M(X, \theta)$
- The free partially commutative Lie algebra $\mathcal{L}_{\mathbf{k}}(X, \theta)$.

As all categories in the list (1.1) have a mechanism of presentation[19], then, unsurprisingly $M(X, \theta)$ and $\mathcal{L}_{\mathbf{k}}(X, \theta)$ are quotients.

Firstly, $M(X, \theta)$ is the quotient of X^* by the congruence[20] generated by the family $(xy = yx)_{(x,y)\in\theta}$. We will consider the canonical surjection $s_\theta : X^* \to M(X, \theta)$ as well as $j_\theta : M(X, \theta) \to X^*$ an arbitrary (but fixed) set-theoretical section of it. We will also need the notion of a Terminal Alphabet which is, in the model of Viennot, the set of

[18] Magnus transformation, Lower central series of the Free group, Möbius counting of dimensions of the free Lie algebra, &c.

[19] Which means definition of objects by generators and relations.

[20] An equivalence relation compatible with right and left translations of the monoid X^* (see [43] Ch 1 §5)

pieces resting on the floor (see [105] Fig 1 "Heaps of Dimers"). This alphabet $\mathrm{TAlph}(t)$ (where $t \in M(X,\theta)$) can be characterized as the set of last letters of preimages of t w.r.t. s_θ, it means that

$$\mathrm{TAlph}(t) = \{x \in X \mid t \in M(X,\theta).x\}.$$

Secondly, $\mathcal{L}_{\mathbf{k}}(X,\theta)$ is the quotient of $\mathcal{L}_{\mathbf{k}}(X)$ by the ideal generated by the relator $\mathbf{r}_\theta = \{[x,y]\}_{(x,y)\in\theta}$.

The monoid $M(X,\theta)$ received a strikingly intuitive and powerful interpretation in terms of heaps of pieces by G. X. Viennot in [105] (see also in Krattenthaler [72]). Let $\mathcal{B}$ be a set (of pieces) that is identified with the set X, together with a symmetric and reflexive binary relation $\mathcal{R}$ defined by $x\mathcal{R}x$ for any $x \in \mathcal{B}$ and $x\mathcal{R}y$ if $(x,y) \notin \theta$ for any $x, y \in \mathcal{B}$. Let $\mathcal{H}(\mathcal{B},\mathcal{R})$ be the set of all heaps consisting of pieces from $\mathcal{B}$, including the empty heap, denoted by $\emptyset$. By introducing an (associative) binary operation $\circ$ which is a composition of heaps (cf. Krattenthaler [72] Def 2.5), the reader can verify that $(\mathcal{H}(\mathcal{B},\mathcal{R}),\circ)$ is a monoid with unit $\emptyset$. We observe that words in the monoid $M(X,\theta)$ can be encoded by heaps in the monoid $\mathcal{H}(\mathcal{B},\mathcal{R})$. Indeed, for a word $t = x_1 \cdots x_n \in M(X,\theta)$, one can define a heap $H_t = (P_t, \preceq_t, \ell_t)$ by the following

- A pair $(P_t, \preceq_t)$ is a poset, where $P_t = \{x_1, \cdots, x_n\}$ is a set of n pieces and $\preceq_t$ is a reflexive, antisymmetric and transitive binary relation defined on P_t by $x_i \preceq_t x_i$ for all $i \in \{1, \cdots, n\}$, $x_i \preceq_t x_j$ if x_i appears before x_j in the word t (from right to left) and $x_i\mathcal{R}x_j$.

- ℓ_t is a labelling of the elements of P_t by elements of $\mathcal{B}$.

Of course, an equivalence class of the word t in $M(X,\theta)$ corresponds to the same heap H_t in $\mathcal{H}(\mathcal{B},\mathcal{R})$. Under this correspondence, the composition of equivalence classes of words induced by concatenation of words corresponds exactly to the composition of heaps. We thus obtain an isomorphism of monoids

$$H_{C-F} : M(X,\theta) \to \mathcal{H}(\mathcal{B},\mathcal{R}), \qquad [t] \mapsto H_t. \tag{1.46}$$

Example 1.4. Let $X = \{b_1, b_2, \cdots, b_7\}$ be a set. Given a commutation relation θ on X as in the list below (not mentioning $\{(b_i,b_i)\}_{i=1,\cdots,7}$ the diagonal of X; if a relation

$(b_i, b_j) \in \theta$ then also $(b_j, b_i) \in \theta$)

$$(b_1, b_2),\ (b_1, b_5),\ (b_1, b_6),\ (b_2, b_3),\ (b_2, b_4),\ (b_2, b_6),\ (b_2, b_7),$$

$$(b_3, b_4),\ (b_3, b_5),\ (b_3, b_7),\ (b_4, b_5),\ (b_4, b_6),\ (b_4, b_7),\ (b_5, b_6),\ (b_5, b_7).$$

Then the pieces are $\mathcal{B} = \{b_1, b_2, \cdots, b_7\}$ and the relations are (not mentioning the relations of the form $b_i \mathcal{R} b_i$; if a relation $b_i \mathcal{R} b_j$ holds then also $b_j \mathcal{R} b_i$)

$$b_1 \mathcal{R} b_3,\ b_1 \mathcal{R} b_4,\ b_1 \mathcal{R} b_7,\ b_2 \mathcal{R} b_5,\ b_3 \mathcal{R} b_6,\ b_6 \mathcal{R} b_7. \tag{1.47}$$

For a word $t = x_1x_2x_3x_4x_5x_6x_7x_8 = b_7b_6b_1b_3b_4b_5b_1b_2$ in $M(X, \theta)$, an equivalence class of the word t will be the following sequence of interchanges

$$b_7b_6b_1b_3b_4b_5b_1b_2 \sim b_7b_6b_1b_4b_3b_5b_1b_2 \sim b_7b_6b_1b_4b_5b_3b_1b_2$$

$$\sim b_7b_6b_1b_5b_4b_3b_1b_2 \sim b_7b_6b_5b_1b_4b_3b_1b_2 \sim b_7b_5b_6b_1b_4b_3b_1b_2$$

$$\sim b_5b_7b_6b_1b_4b_3b_1b_2 \sim b_5b_7b_1b_6b_4b_3b_1b_2 \sim b_5b_7b_1b_4b_6b_3b_1b_2$$

$$\sim b_5b_7b_1b_4b_6b_3b_2b_1 \sim b_5b_7b_1b_4b_6b_2b_3b_1 \sim b_5b_7b_1b_4b_2b_6b_3b_1$$

$$\sim b_5b_7b_1b_2b_4b_6b_3b_1 \sim b_5b_7b_2b_1b_4b_6b_3b_1 \sim \cdots$$

Therefore, the corresponding heap $H_t = (P_t, \preceq_t, \ell_t)$ (where $P_t = \{x_1, x_2, \cdots, x_8\} = \{b_7, b_6, b_1, b_3, b_4, b_5, b_1, b_2\}$ is the set of 8 pieces) under the isomorphism H_{C-F} (1.46) can be illustrated in Figure 1.2 as below. We notice that, contrariwise to [72] we read our word t (from a heap of pieces H_t in Figure 1.2) from top to bottom and left to right.

Remark 1.8. Let $t = x_1 \cdots x_n \in M(X, \theta)$. One can show that,

$$\mathrm{TAlph}(t) = \{x_i \mid i \in [1, n] \text{ with } (x_j, x_i) \in \theta \text{ for all } j \in [i+1, n]\}.$$

Geometrically, under the isomorphism of monoids H_{C-F} (1.46), when successive pieces are piled from bottom to top, the set Terminal Alphabet $\mathrm{TAlph}(t)$ can be described (as already said) as pieces (in the heap H_t) resting on the floor.

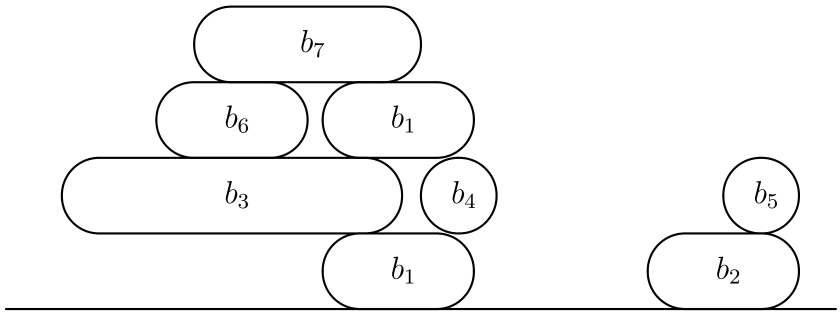


Figure 1.2: A heap of pieces H_t corresponding to the word

$$t = b_7 b_6 b_1 b_3 b_4 b_5 b_1 b_2 \in M(X, \theta)$$

here $\mathrm{TAlph}(t) = \{b_1, b_2\}$ even if b_1 and b_6 commute. From Krattenthaler [72], order of the letters has been reversed.

Remark 1.9. For (at least) the categories $\mathbf{Mon}, \mathbf{Grp}, \mathbf{k\text{-}Lie}, \mathbf{k\text{-}AAU}$, there is a forgetful functor to the category **CommAlph** of *alphabets with commutations*, that of pairs (X, θ) where $\Delta_X \subset \theta \subset X \times X$ and $\theta = \theta^{-1}$ (in other words θ is a reflexive and symmetric relation on X). An arrow $f : (X, \theta_X) \to (Y, \theta_Y)$ is a map $f : X \to Y$ such that for all $(x, y) \in \theta_X$ we have $(f(x), f(y)) \in \theta_Y$. All structures of (1.45) are free with respect to these functors.

Chapter 2

Lazard's elimination

Contents

Originated from the study of continuous groups [76], the theory of Lie algebras began mostly by (differential) geometry and then algebra. The rôle of presented Lie algebras grew in importance since the study of combinatorial group theory and its infinitesimal counterpart on the one hand and the study (for mathematics, physics and number theory) of infinite dimensional Lie algebras (see, for example the two books with the same title "Infinite Dimensional Lie algebras" [65, 106], 1968 for Kac's definition and 2001 for Wakimoto's book, showing the continuous trail of interest).

It appears that Lazard's elimination principle, expressed in the categories

$$\mathbf{Mon}, \mathbf{Grp}, \mathbf{k\text{-}Lie}, \mathbf{k\text{-}AAU}, \tag{2.1}$$

has the same form

$$\mathrm{Free}(B+Z) = \mathrm{Free}(C_B[Z]) \rtimes \mathrm{Free}(B) \tag{2.2}$$

where $X = B + Z$ is a set of generators (an alphabet) divided in two sectors (set partition) and $C_B[Z]$ is some code (i.e. a set built from the data of B and Z), Free is the free functor attached to the considered category and $\rtimes$ is a sort of semi-direct product[1].

The study of quotients of Lazard's eliminations for **k-Lie** is particularly interesting. It arises from dividing the classical elimination over a relator on the free Lie algebra $\mathcal{L}_{\mathbf{k}}(X)$ which is compatible with the alphabet partition. An emblematic example is that of infinitesimal braid relations and their decomposition. Let us first see how they arise.

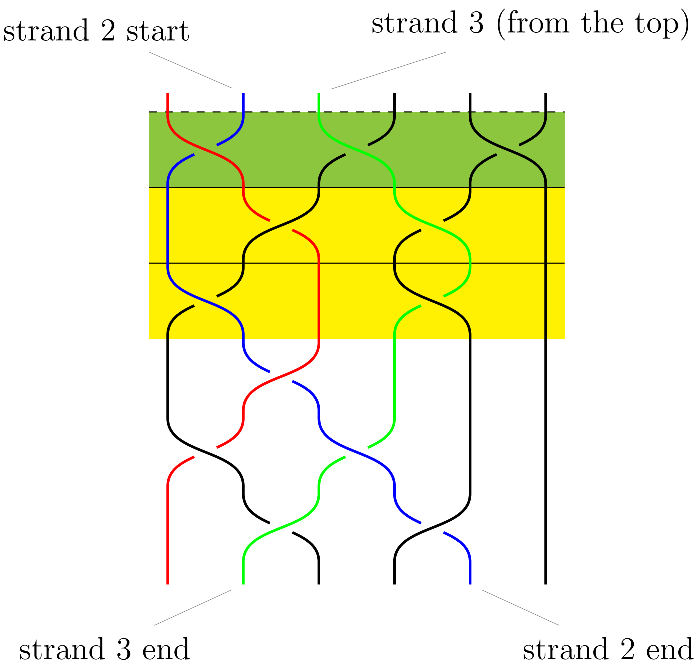


Figure 2.1: A braid on 6 strands.

[1]Classical for the categories **Grp** and **k-Lie**, smash product for **k-AAU** (see the book [87] by Susan Montgomery) and by left translations for **Mon**.

A braid with n strands is pictured geometrically as a collection of n pieces of string joining n points at the top of the diagram with n points at the bottom in Euclidean 3-space (see Figure 2.1 above).

The *braid group* $\mathcal{B}_n$ on n strands is the group of isotopy classes of n-braids with concatenation given by glueing (see Figure 2.2).

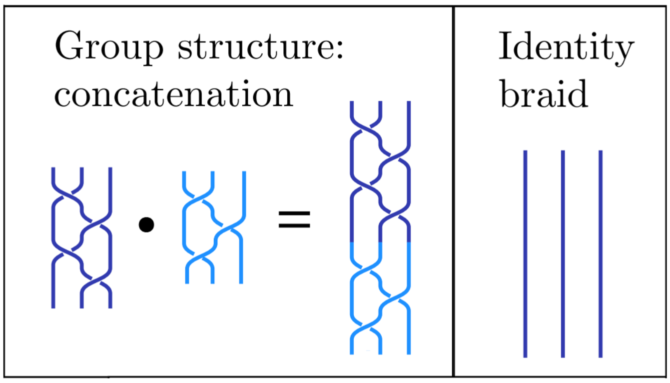


Figure 2.2: Glueing of braids and the identity of the braid group $\mathcal{B}_n$.

Due to Emil Artin, $\mathcal{B}_n$ is also the group generated by the symbols $\sigma_1, \cdots, \sigma_{n-1}$ modulo the relations:

- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $i = 1, \cdots, n-2$.
- $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $| i - j | \geq 2$.

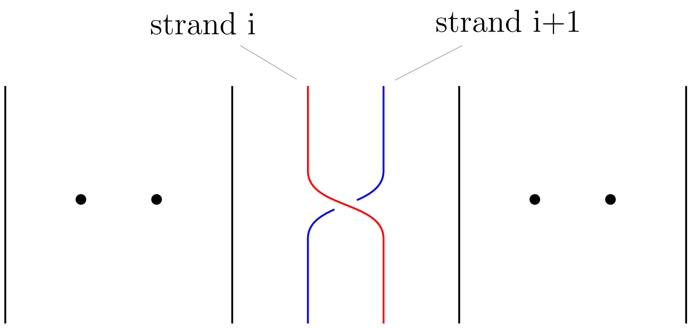


Figure 2.3: A standard generator σ_i of the braid group $\mathcal{B}_n$.

To each braid in $\mathcal{B}_n$ one can associate the permutation of the marked points, that is an element of the symmetric group $\mathfrak{S}_n$. This leads to a natural homomorphism of group $\pi : \mathcal{B}_n \to \mathfrak{S}_n$ characterized by

$$\sigma_i \mapsto (i, i+1).$$

The kernel of π is precisely the subgroup of $\mathcal{B}_n$ formed by braids inducing the trivial permutation. It is called the *pure braid group* on n strands and denoted by $\mathcal{PB}_n$. Geometrically, the group $\mathcal{PB}_n$ consists of those braids such that each strand starts and ends at the same point (see Figure 2.4).

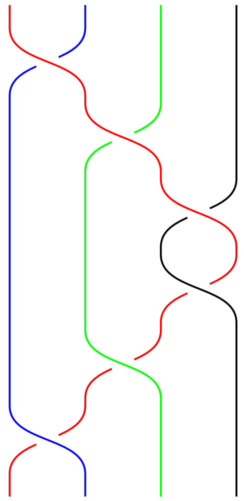


Figure 2.4: A pure braid on 4 strands.

Moreover, we can verify that the pure braid group $\mathcal{PB}_n$ coincides with the fundamental group of the complex configuration space

$$\mathbb{C}^n_* = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ if } i \neq j\}$$

i.e. $\mathcal{PB}_n = \pi_1(\mathbb{C}^n_*)$.

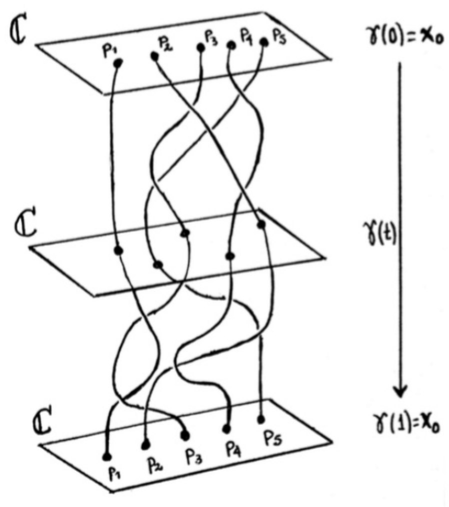


Figure 2.5: A loop $\gamma(t)$ in $\pi_1(\mathbb{C}^5_*)$.

Via Artin's presentation, $\mathcal{PB}_n$ is the group generated by the twists

$$x_{ij} := (\sigma_{j-1} \cdots \sigma_{i+1})\sigma_i^2(\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1})$$

(for $i < j$, $i, j = 1, \cdots, n$, see Figure 2.6) modulo the relations

- $(x_{ij}, x_{kl}) = 1$ for $i < j < k < l$,
- $(x_{il}, x_{jk}) = 1$ for $i < j < k < l$,
- $(x_{ij}x_{ik}, x_{jk}) = (x_{ij}, x_{ik}x_{jk}) = 1$ for $i < j < k$,
- $(x_{kl}x_{ik}x_{kl}^{-1}, x_{jl}) = 1$ for $i < j < k < l$,

where, in a group $(g, h) = g^{-1}h^{-1}gh$ is the commutator between g and h (see [13] Ch II §4.4).

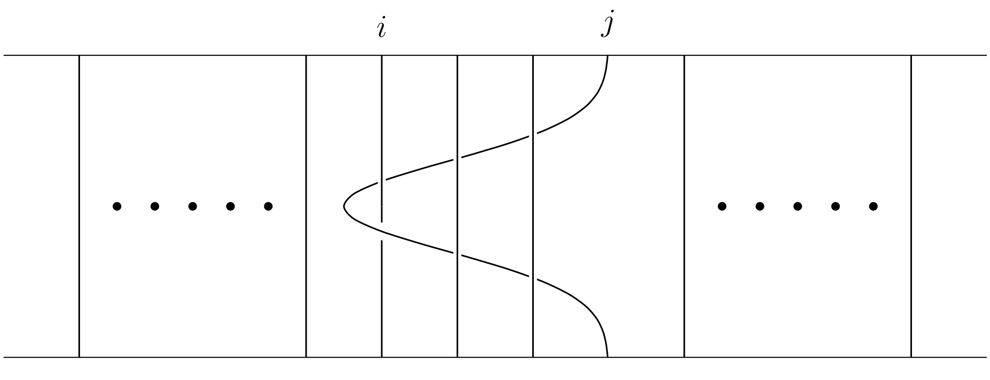


Figure 2.6: A standard twist generator of the pure braid group $\mathcal{PB}_n$.
Here, with $i = j-3$, we have $x_{ij} = \sigma_{j-1}\sigma_{j-2}\sigma_{j-3}^2\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$ (diagram: top to bottom and formula: left to right).

Given a pure braid $\beta \in \mathcal{PB}_n$, we can remove its last strand and obtain a pure braid $\rho(\beta) \in \mathcal{PB}_{n-1}$. This yields a surjective homomorphism of groups $\rho : \mathcal{PB}_n \to \mathcal{PB}_{n-1}$. Let F_{n-1} denote the free group on $n-1$ letters $x_1, \cdots, x_{n-1}$. It can be proved (cf. Kassel and Turaev [68] Ch I §1.3) that we have a short exact sequence

$$0 \longrightarrow \mathrm{F}_{n-1} \xrightarrow{\ \iota\ } \mathcal{PB}_n \underset{\mu\,=\,\text{insert straight last strand}}{\overset{\rho\,=\,\text{forget last strand}}{\rightleftarrows}} \mathcal{PB}_{n-1} \longrightarrow 0 \tag{2.3}$$

where $\iota(x_i) := x_{in} = (\sigma_{n-1}\cdots\sigma_{i+1})\sigma_i^2(\sigma_{i+1}^{-1}\cdots\sigma_{n-1}^{-1})$ for $i = 1, \cdots, n-1$. Furthermore, one sees easily that this exact sequence is split (see the move μ in (2.3)), so that $\mathcal{PB}_n \cong \mathrm{F}_{n-1} \rtimes \mathcal{PB}_{n-1}$.

For any group G, a graded $\mathbb{Z}$-module is associated to G. It is defined (cf. Bourbaki [13] Ch II §4.6) as

$$gr_{\mathbb{Z}}(G) := \bigoplus_{m=1}^{\infty} gr_m(G) \tag{2.4}$$

where the associated m-th quotient

$$gr_m(G) := {}^{G_m}\big/_{G_{m+1}}$$

is itself defined over $\{G_m\}_{m\geq 1}$, the lower central series (cf. Bourbaki [10] Ch I §6.3 Def 5 and [13] Ch II §4.6) of G is made up of normal subgroups[2]

$$G_1 := G \supseteq G_2 := (G_1, G) \supseteq \cdots \supseteq G_{m+1} := (G_m, G) \supseteq \cdots .$$

It is remarkable that this module, $gr_{\mathbb{Z}}(G)$, due to the identities of P. Hall and M. Lazard (see [74]) is, in fact, a Lie $\mathbb{Z}$-algebra (the bracket being the projection of the commutator in the group).

Then, calling $t_{i,j}$, the projection of x_{ij} into the graded Lie algebra $gr_{\mathbb{Z}}(\mathcal{PB}_n)$, it can be found in some paper by Ihara (cf. [64] Cor 3.1.6 and Prop 3.2.1 and ff) that (2.3) induces a split short exact sequence of Lie algebras

$$0 \longrightarrow gr_{\mathbb{Z}}(\mathrm{F}_{n-1}) \longrightarrow gr_{\mathbb{Z}}(\mathcal{PB}_n) \longrightarrow gr_{\mathbb{Z}}(\mathcal{PB}_{n-1}) \longrightarrow 0\ . \tag{2.5}$$

[2]For two subgroups $A, B \subset G$, (A,B) is the subgroup generated the commutators $(x,y) := x^{-1}y^{-1}xy,\ x \in A, y \in B$.

Remark 2.1. The transfer by $gr(-)$ of exact sequences is studied in Bourbaki [13] Ch II §4 Exercise 2 (b), but this exercise does not guarantee the "split" part which is, in fact, not automatic. In the case of pure braid groups, this can be recovered directly by our main result, Theorem 2.6 and the presentation below (2.6).

Due to Ihara [64] and Kohno [70], the graded Lie algebra $gr_{\mathbb{Z}}(\mathcal{PB}_n)$ is isomorphic to *Drinfeld-Kohno Lie algebra* $\mathrm{DK}_{\mathbb{Z},n}$ that is the quotient of the free Lie algebra $\mathcal{L}_{\mathbb{Z}}(\mathcal{T}_n)$ generated by the set of noncommutative variables $\mathcal{T}_n = \{t_{ij}\}_{1\leq i<j\leq n}$ modulo the Lie ideal generated by *infinitesimal pure braid relations*[3] (the corresponding set of relators will be denoted by $\mathbf{R}[\mathbf{n}]$, see formula (2.6))

$$\mathbf{R}[\mathbf{n}] = \begin{cases} \mathbf{R_1}[\mathbf{n}] \quad [t_{i,j}, t_{i,k} + t_{j,k}] & \text{for } 1 \leq i < j < k \leq n, \\ \mathbf{R_2}[\mathbf{n}] \quad [t_{i,j} + t_{i,k}, t_{j,k}] & \text{for } 1 \leq i < j < k \leq n, \\ \mathbf{R_3}[\mathbf{n}] \quad [t_{i,j}, t_{k,l}] \quad \text{for } \begin{matrix} 1 \leq i < j \leq n, \\ 1 \leq k < l \leq n, \end{matrix} & \text{and } |\{i,j,k,l\}| = 4. \end{cases} \tag{2.6}$$

Now, we extend $\mathbb{Z}$ to $\mathbf{k}$. Readers that are not keen on categories may skip the following paragraph and suppose $\mathbf{k} = \mathbb{Z}$.

For a commutative ring $\mathbf{k}$ with unit, let us consider an additive functor of abelian categories

$$- \otimes_{\mathbb{Z}} \mathbf{k} : \mathbb{Z}\text{-}\mathbf{Mod} \to \mathbf{k}\text{-}\mathbf{Mod}.$$

As a consequence of Proposition 8.3.14, Kashiwara and Schapira [66] Ch 8 §8.3, then any additive functor of abelian categories sends split exact sequences into split exact sequences. Furthermore, Drinfeld-Kohno Lie algebra over $\mathbf{k}$ coincides with the image of Drinfeld-Kohno Lie algebra over $\mathbb{Z}$ under the functor

$$- \otimes_{\mathbb{Z}} \mathbf{k} : \mathbb{Z}\text{-}\mathbf{Lie} \to \mathbf{k}\text{-}\mathbf{Lie}$$

i.e. there exists an isomorphism of Lie $\mathbf{k}$-algebras

$$\mathrm{DK}_{\mathbf{k},n} \cong \mathrm{DK}_{\mathbb{Z},n} \otimes_{\mathbb{Z}} \mathbf{k}. \tag{2.7}$$

[3]There are three ways to reach these relations

- The lower central series of $\mathcal{PB}_n$ and
- A theorem of Kohno on "small" representations of $\mathcal{PB}_n$,
- Integrability condition for the KZ_n differential equation.

Thus, an important consequence of our formulations is that we can construct a commutative diagram of $\mathbf{k}$-modules with split short exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & gr_{\mathbf{k}}(\mathrm{F}_{n-1}) & \longrightarrow & gr_{\mathbf{k}}(\mathcal{PB}_n) & \longrightarrow & gr_{\mathbf{k}}(\mathcal{PB}_{n-1}) & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} & & \\ 0 & \longrightarrow & \mathcal{L}_{\mathbf{k}}(x_1, \cdots, x_{n-1}) & \longrightarrow & \mathrm{DK}_{\mathbf{k},n} & \longrightarrow & \mathrm{DK}_{\mathbf{k},n-1} & \longrightarrow & 0 \end{array} \tag{2.8}$$

where the graded Lie algebra $gr_{\mathbf{k}}(G) = gr_{\mathbb{Z}}(G) \otimes_{\mathbb{Z}} \mathbf{k}$ for any group G. In particular, we obtain an isomorphism of $\mathbf{k}$-modules

$$\mathrm{DK}_{\mathbf{k},n} \cong \mathcal{L}_{\mathbf{k}}(x_1, \cdots, x_{n-1}) \oplus \mathrm{DK}_{\mathbf{k},n-1}. \tag{2.9}$$

A natural question is how to construct a Lie isomorphism from the Drinfeld-Kohno Lie algebra to a semi-direct product of Lie algebras

$$\mathrm{DK}_{\mathbf{k},n} \xrightarrow{\cong} \mathcal{L}_{\mathbf{k}}(x_1, \cdots, x_{n-1}) \rtimes \mathrm{DK}_{\mathbf{k},n-1}.$$

We call the phenomenon by *the decomposition of Drinfeld-Kohno Lie algebra* and it will be completely achieved in Corollary 2.17. Furthermore, we realize that this decomposition can be read directly in terms of alphabets partitioned in two blocks $\mathcal{T}_n = \mathcal{T}_{n-1} + T_n$, this is the "raison d'être" of our main result here (the reader can refer to Subsection 2.2.2). We introduce the notion of *quotients of Lazard's eliminations* which generalizes the classical elimination on the free Lie algebra $\mathcal{L}_{\mathbf{k}}(X)$ to a more general scheme. More precisely, given $X = B + Z$ a set partitioned in two blocks and a relator $\mathbf{r} \subset \mathcal{L}_{\mathbf{k}}(X)$ which is compatible with the alphabet partition, our main results are to give a Lie isomorphism [see Theorem 2.6 point (iii)]

$$\mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J} \xrightarrow{\cong} \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}_{BZ}^{Z} \rtimes \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B$$

and to determine a necessary and sufficient condition (see Proposition 2.9) so that

$$\mathcal{L}_{\mathbf{k}}(Z) \cong \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}_{BZ}^{Z}.$$

The case of the infinitesimal pure braid relator is an example of a good relator satisfying all hypotheses in our main theorem and Proposition 2.9, thus it gives rise to the existence of the decomposition of Drinfeld-Kohno Lie algebra i.e there is $\mathrm{DK}_{\mathbf{k},n} \cong \mathcal{L}_{\mathbf{k}}(x_1, \cdots, x_{n-1}) \rtimes \mathrm{DK}_{\mathbf{k},n-1}$. However, we consider the case of a relator over free partial commutations θ (see Definition 1.4), although this case does not satisfy the necessary and sufficient condition mentioned just above, we are able to use our main result and recover, by our setting complete answers to Pr. Schützenberger's questions about the Partially Commutative Free Lie algebra (see Corollary 2.16).

The structure of the chapter is the following:

The two first sections contain the main results of this chapter together with their proofs. In first Section 2.1, we will introduce the notion of equivariant diagram of Lie algebras which gives rise to a structure theorem analogous of the semi-direct product of Lie algebras. Then, in Subsection 2.2.1 we recall internal and external versions of Lazard's elimination theorem. In Subsection 2.2.2, we introduce a relator on the free Lie algebra being compatible with an elimination scheme, then we establish the main theorem (Theorem 2.6). In last subsection 2.2.3, we first construct the necessary and sufficient condition so that the left factor of the semi-direct product appeared in Theorem 2.6 point (iii) to be a free Lie algebra by Proposition 2.9. Then we will apply our main theorem to treat the special cases about Pr. Schützenberger's questions on the Partially Commutative Free Lie algebra in Corollary 2.16. Further, we first introduce the notion of the Knizhnik-Zamolodchikov equation and explain the relationship with Drinfeld-Kohno Lie algebra. Finally, we will obtain the existence of the decomposition of Drinfeld-Kohno Lie algebra in Corollary 2.17 coming from standard materials such as our main theorem together with Proposition 2.9 as already mentioned above.

In Section 2.3, we will discuss more categorical frameworks for Lazard's elimination in **k-Lie**. In Subsection 2.3.1, we describe the category of Short Exact Sequences with Section (SESS) in **k-Lie**. In the last subsection 2.3.2, we first introduce S-graded objects in each category (2.1), where S is an additive commutative semigroup. We are in particular interested in $\mathbb{B}$-graded objects in **k-Lie** (where $\mathbb{B} = (\{0,1\}, \vee)$ is the Boolean semigroup). Then we can formalize these objects by the category of $\mathbb{B}$-graded Lie algebras, that is equivalent to the category of SESS in **k-Lie** by considering Proposition 2.18. Furthermore, we can perform Lazard's elimination as a free functor from the category of double sets to the category of $\mathbb{B}$-graded Lie algebras. Finally, in Subsection 2.3.3, we will concentrate on the Drinfeld-Kohno Lie algebra with an infinite number of generators: from strange to generalized gradings.

Section 2.4 is devoted to investigate more formal aspects of Lazard's elimination in **k-AAU** and their interactions with Sections 2.1 and 2.2. We will first study crossed and smash product of algebras by Theorem 2.25 and Corollary 2.26 respectively. Furthermore, Example 2.7 gives us a natural and important example of smash product algebras, this leads to Proposition 2.27 as an extension of Proposition 2.2 in Sections 2.1 under the universal enveloping functor. At the end of the day our real interest are Lazard's elimination and the quotient of Lazard's elimination in **k-AAU** discussed in the last two examples of this section, building only on the fundamental knowledge of smash product of algebras and the main results appeared in Section 2.2.

In the last Section 2.5, we introduce a useful table to summarize our results of this chapter for Lazard's elimination principle (2.2) in all of categories **Mon**, **Grp**, **k-Lie** and **k-AAU**.

2.1 Groups and Lie algebras.

We here construct and study the equivariant diagram property in the category of groups and then Lie algebras which gives rise to a structure theorem analogous of the semi-direct product of Lie algebras. These give at last main tools to establish the proof of quotients of Lazard's eliminations and the decomposition of Drinfeld-Kohno Lie algebra appearing in Subsection 2.2.2 and Subsection 2.2.3.

We shall now study a Theorem of N. Bourbaki [14] which allows to consider semi-direct products as categorical colimits. In the language of MO [108] [4], it reads

Theorem 2.1. *([14] §2.10 Prop 27) Let H and N be two groups and $\phi : H \to Aut(N)$ be a homomorphism of groups written as a indexed automorphism $\phi_h(n)$. Assume that*

[4] See the discussion after this question, especially Andreas Thom's answer.

there exist $f\colon N \to G$, $g\colon H \to G$ two homomorphisms into a group G such that

$$f(\phi_h(n)) = g(h)f(n)g(h^{-1}) \tag{2.10}$$

for all $n \in N$, $h \in H$. Then there is a unique homomorphism $k\colon N \rtimes H \to G$ extending f and g in the usual sense (i.e. $k(n,h) = f(n).g(h)$ the product in G between $f(n)$ and $g(h)$), where the group operation in $N \rtimes H$ is given by $(n_1,h_1).(n_2,h_2) = (n_1\phi_{h_1}(n_2), h_1h_2)$.

This is for groups, but we will tailor a similar property for Lie algebras remarking that (2.1) can be, in bivariate notations, reformutated as

$$f(\phi(h,n)) = \mathrm{Ad}^G(g(h), f(n)), \tag{2.11}$$

this is equivalent to saying that the following diagram commutes

$$\begin{tikzcd} H \times N \arrow[r, "g\times f"] \arrow[d, "\phi"] & G \times G \arrow[d, "\mathrm{Ad}^G"] \\ N \arrow[r, "f"] & G. \end{tikzcd}$$

As we want to highlight the similarity between semi-direct products of groups and Lie algebras, we recall below the definition of it.

Definition 2.1. For any Lie algebras $\mathfrak{b}, \mathfrak{h}$ and an action by derivations of the Lie algebra $\mathfrak{b}$ on $\mathfrak{h}$ i.e. a Lie homomorphism $\alpha : \mathfrak{b} \to \mathfrak{Der}(\mathfrak{h})$, we can construct $\mathfrak{g}$, a split Lie algebra extension of $\mathfrak{b}$ by $\mathfrak{h}$ whose underlying $\mathbf{k}$-module is the external direct sum of modules $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{b}$ and the Lie bracket is given by the following formula

$$[(h_1,b_1),(h_2,b_2)] = \big([h_1,h_2] + \alpha(b_1)(h_2) - \alpha(b_2)(h_1), [b_1,b_2]\big).$$

We denote this Lie algebra by $\mathfrak{g} := \mathfrak{h} \rtimes \mathfrak{b}$ and call it "the *semi-direct product* of $\mathfrak{b}$ with $\mathfrak{h}$".

Remark 2.2. It is not difficult to see that, likewise, semi-direct products of Lie algebras are appropriate colimits[5], see Commentary 2 below. As groups act on themselves

[5]For colimits, see Appendix 5.2. Moreover, we can survey the categorical framework of semi-direct products in the context of **Grp** by discussions in MO [108].

by automorphisms (inner adjoint representation Ad), Lie algebras, similarly, act on themselves by derivations (resp. inner adjoint representation ad). We will have the choice between the indexed notation $\mathrm{Ad}_h(g) := hgh^{-1}$ (resp. $\mathrm{ad}_h(g) = [h, g]$) and a bivariate one $\mathrm{Ad}(h, g) = hgh^{-1}$ (resp. $\mathrm{ad}(h, g) = [h, g]$). In order to ease the writing of diagrams and equivariance, we will adopt below the bivariate notation.

Adaptation of Bourbaki's Proposition (i.e. Theorem 2.1) to our situation is then the following:

Proposition 2.2. *Let* $\mathbf{k}$ *be a commutative ring with unit and* $\mathfrak{g}_i,\ i = 1, 2$ *be two Lie* $\mathbf{k}$*-algebras. We suppose given also a Lie* $\mathbf{k}$*-algebra morphism* $\alpha :\ \mathfrak{g}_2 \to \mathfrak{Der}(\mathfrak{g}_1)$*. Then let* $f_1 \colon \mathfrak{g}_1 \to \mathfrak{g}$*,* $f_2 \colon \mathfrak{g}_2 \to \mathfrak{g}$ *be two homomorphisms into a Lie* $\mathbf{k}$*-algebra* $\mathfrak{g}$*, such that*

$$f_1(\alpha(b, a)) = \mathrm{ad}^{\mathfrak{g}}(f_2(b), f_1(a)) \tag{2.12}$$

for all $b \in \mathfrak{g}_2$*,* $a \in \mathfrak{g}_1$ *(*α *and* $\mathrm{ad}^{\mathfrak{g}}$ *are here written in bivariate notation in the obvious way). Then there is a unique homomorphism* $f \colon \mathfrak{g}_1 \rtimes \mathfrak{g}_2 \to \mathfrak{g}$ *extending* f_1 *and* f_2 *in the usual sense.*

Again, this is equivalent to saying that the diagram (2.13) below commutes.

We are led to the following definition of SDT ("Semi-Direct Twist") for the categories of groups and Lie algebras

Definition 2.2. i) A semi-direct twist in the category of groups is a triplet (G_2, G_1, α) where $\alpha :\ G_2 \times G_1 \to G_1$ is such that $b \mapsto \alpha(b, -)$ is a group morphism $G_2 \to Aut(G_1)$.
ii) A semi-direct twist in the category of Lie $\mathbf{k}$-algebras is a triplet $(\mathfrak{g}_2, \mathfrak{g}_1, \alpha)$ where $\alpha :\ \mathfrak{g}_2 \times \mathfrak{g}_1 \to \mathfrak{g}_1$ is such that $b \mapsto \alpha(b, -)$ is a Lie algebra morphism $\mathfrak{g}_2 \to \mathfrak{Der}(\mathfrak{g}_1)$.

In **Grp** (resp. **k-Lie**), a morphism between (G_2, G_1, α) and (H_2, H_1, β) (resp. $(\mathfrak{g}_2, \mathfrak{g}_1, \alpha)$ and $(\mathfrak{h}_2, \mathfrak{h}_1, \beta)$) is an equivariant pairs of group morphisms $f_i :\ G_i \to H_i$ (resp. Lie algebra morphisms $f_i :\ \mathfrak{g}_i \to \mathfrak{h}_i$) i.e. such that (in both cases) the following diagram

$$\begin{tikzcd} G_2 \times G_1 \arrow[r, "f_2 \times f_1"] \arrow[d, "\alpha"] & H_2 \times H_1 \arrow[d, "\beta"] \\ G_1 \arrow[r, "f"] & H_1 \end{tikzcd}$$

and

$$\begin{tikzcd}
\mathfrak{g}_2 \times \mathfrak{g}_1 \arrow[r, "f_2 \times f_1"] \arrow[d, "\alpha"] & \mathfrak{h}_2 \times \mathfrak{h}_1 \arrow[d, "\beta"] \\
\mathfrak{g}_1 \arrow[r, "f"] & \mathfrak{h}_1
\end{tikzcd}$$

commute. We are now led to define $\mathbf{SDT}^{\mathbf{Grp}}$ the category of semi-direct twists in **Grp** and $\mathbf{SDT}^{\mathbf{k}\text{-}\mathbf{Lie}}$ the category of semi-direct twists in **k-Lie**.

We have the following useful comments

Commentary 2. *i) The functor F from **Grp** to $\boldsymbol{SDT}^{\boldsymbol{Grp}}$ defined by $G \mapsto \big(G, G, \mathrm{Ad}^G\big)$ is right-adjoint to the functor $\boldsymbol{SDT}^{\boldsymbol{Grp}} \to$ **Grp** defined by $(G_2, G_1, \alpha) \mapsto G_1 \rtimes G_2$ and the same holds for the analogous functors in Lie algebras.*
ii) More precisely, one has the following universal diagrams

$$\begin{tikzcd}
\boldsymbol{SDT}^{\boldsymbol{Grp}} & & \boldsymbol{Grp} \arrow[ll, dotted, "F"'] \\
(G_2, G_1, \alpha) \arrow[rr, dashed, "f_2 \times f_1"] \arrow[drr, dashed, "j"'] & & G \\
& & G_1 \rtimes G_2 \arrow[u, "\widehat{f}"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\boldsymbol{SDT}^{\mathbf{k}\text{-}\boldsymbol{Lie}} & & \mathbf{k}\text{-}\boldsymbol{Lie} \arrow[ll, dotted, "F"'] \\
(\mathfrak{g}_2, \mathfrak{g}_1, \alpha) \arrow[rr, dashed, "f_2 \times f_1"] \arrow[drr, dashed, "j"'] & & \mathfrak{g} \\
& & \mathfrak{g}_1 \rtimes \mathfrak{g}_2. \arrow[u, "\widehat{f}"']
\end{tikzcd}$$

Remark 2.3. We would like here to formulate two remarks about equation (2.12) Firstly, it can be proved only on generators.
More precisely, let $G_2 = \{b_j\}_{j\in J}$ and $G_1 = \{a_i\}_{i\in I}$ be set of generators of the Lie algebras $\mathfrak{g}_2$ and $\mathfrak{g}_1$ respectively and assume that the diagram (2.13) commutes for $(b, a) \in G_2 \times G_1$.
Now, given a fixed generator b_j and using Jacobi identity, the two $\mathbf{k}$-linear maps $D_1 : \mathfrak{g}_1 \to \mathfrak{g}, a \mapsto f_1 \circ \alpha_{\otimes}(b_j \otimes a)$ and $D_2 : \mathfrak{g}_1 \to \mathfrak{g}, a \mapsto \mathrm{ad}^{\mathfrak{g}}_{\otimes} \circ (f_2 \otimes f_1)(b_j \otimes a)$ are f_1-derivations i.e. $D_1, D_2 \in \mathfrak{Der}^{(f_1)}(\mathfrak{g}_1, \mathfrak{g})$. By assumption, we have that $D_1 - D_2$ is zero on generators $\{a_i\}_{i\in I}$ of $\mathfrak{g}_1$ thus, by Remark 1.6 (ii), $D_1 - D_2 = 0$. This means that

for all $a \in \mathfrak{g}_1$ one has

$$f_1 \circ \alpha_\otimes(b_j \otimes a) = \mathrm{ad}^{\mathfrak{g}}_\otimes \circ (f_2 \otimes f_1)(b_j \otimes a)$$

and this is true for all fixed $b_j \in G_2$. To summarize, the above equation is true for $(a, b) \in \mathfrak{g}_1 \times G_2$. We then define the following submodule

$$\mathfrak{m} = \{b \in \mathfrak{g}_2 \mid (\forall a \in \mathfrak{g}_1)\ f_1 \circ \alpha_\otimes(b \otimes a) = \mathrm{ad}^{\mathfrak{g}}_\otimes \circ (f_2 \otimes f_1)(b \otimes a)\}.$$

and leave to the reader to check (by Jacobi identity) that $\mathfrak{m}$ is a Lie subalgebra of $\mathfrak{g}_2$. Then, as inclusion $G_2 \subseteq \mathfrak{m}$ follows from the summary, one derives $\mathfrak{m} = \mathfrak{g}_2$ and then Eq. (2.12) is established.

Secondly Eq. (2.12) is equivalent to saying that the following commutes

$$\begin{tikzcd} \mathfrak{g}_2 \otimes \mathfrak{g}_1 \arrow[r, "f_2 \otimes f_1"] \arrow[d, "\alpha_\otimes"] & \mathfrak{g} \otimes \mathfrak{g} \arrow[d, "\mathrm{ad}^{\mathfrak{g}}_\otimes"] \\ \mathfrak{g}_1 \arrow[r, "f_1"] & \mathfrak{g} \end{tikzcd} \tag{2.13}$$

(here $\alpha_\otimes$ and $\mathrm{ad}^{\mathfrak{g}}_\otimes$ are linear maps respectively induced from the bilinear maps $\alpha : \mathfrak{g}_2 \times \mathfrak{g}_1 \to \mathfrak{g}_1$ and $\mathrm{ad}^{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$).

2.2 A generalization of Lazard's elimination theorem.

We here introduce the classical Lazard's elimination theorem and extend it to a more general scheme, namely theory of quotients of Lazard's eliminations. The main applications are to derive answers to Pr. Schützenberger's questions about the Partially Commutative Free Lie algebra (cf. Duchamp and Krob [41]) and the decomposition of Drinfeld-Kohno Lie algebra described in Subsection 2.2.3.

2.2.1 Classical Lazard's elimination.

Let us recall briefly Lazard's elimination theorem in our setting.

Theorem 2.3 (Lazard's elimination theorem, see also in [13] Ch II §2.9 Props 9 and 10). *Let $X = B + Z$ be a set partitioned in two blocks. We have an isomorphism of split short exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}_{\mathbf{k}}(B^*Z) \arrow[r, hook, "j_{B|Z}\,(=rn)"] \arrow[d, "rn"] & \mathcal{L}_{\mathbf{k}}(X) \arrow[r, "p_{B|Z}"] \arrow[d, "\mathrm{Id}"] & \mathcal{L}_{\mathbf{k}}(B) \arrow[r] \arrow[d, "j_B"] & 0 \\ 0 \arrow[r] & \mathcal{L}_{\mathbf{k}}(X)_{BZ} \arrow[r, hook, "j"] & \mathcal{L}_{\mathbf{k}}(X) \arrow[r, "p"] & \mathcal{L}_{\mathbf{k}}(X)_B \arrow[r] & 0 \end{tikzcd} \tag{2.14}$$

where B^*Z is the set of words $B^*Z = \{uz\}_{u\in B^*, z\in Z}$ (all letters are in B save the last one which is in Z) and where the maps rn and j_B are as follows

- *the mapping rn is, by universal property, the unique Lie morphism $\mathcal{L}_{\mathbf{k}}(B^*Z) \to \mathcal{L}_{\mathbf{k}}(X)_{BZ}$ such that, for $u = b_1 \cdots b_k \in B^*$ and $z \in Z$ we get*

$$rn(uz) = \left(\mathrm{ad}_{b_1}^{\mathcal{L}_{\mathbf{k}}(X)} \circ \cdots \circ \mathrm{ad}_{b_k}^{\mathcal{L}_{\mathbf{k}}(X)}\right)(z) =: \mathrm{ad}_{(u)}^{\mathcal{L}_{\mathbf{k}}(X)}(z) \tag{2.15}$$

  bracketing, see [94] Ch 1 §3 p.20 (indeed, we here use the same symbol rn for the Lie morphism $\mathcal{L}_{\mathbf{k}}(B^*Z) \to \mathcal{L}_{\mathbf{k}}(X)$ and for the restriction to its image as in the diagram (2.14)*).*

- *for convenience to describe a fixed double symbol formed (j, p) of any short exact sequence, the mapping $j_{B|Z}$ in the first arrow of the diagram* (2.14) *is in fact the morphism $rn : \mathcal{L}_{\mathbf{k}}(B^*Z) \to \mathcal{L}_{\mathbf{k}}(X)$ and $p_{B|Z}$ is the Lie algebra homomorphism $\mathcal{L}_{\mathbf{k}}(X) \to \mathcal{L}_{\mathbf{k}}(B)$ that sends each $b \in B$ to b and sends each $z \in Z$ to* 0*.*

- *if $j_B : \ \mathcal{L}_{\mathbf{k}}(B) \to \mathcal{L}_{\mathbf{k}}(X)$ is the subalphabet embedding, (so that the restriction to its image is the Lie isomorphism $j_B : \ \mathcal{L}_{\mathbf{k}}(B) \to \mathcal{L}_{\mathbf{k}}(X)_B$) then $j_B \circ p_{B|Z}$ is the projector on*

$$\mathcal{L}_{\mathbf{k}}(X)_B = \bigoplus_{\substack{\alpha\in\mathbb{N}^{(X)} \\ |\alpha|_Z = 0}} \mathcal{L}_{\mathbf{k}}(X)_\alpha .$$

  The kernel of $p_{B|Z}$ is

$$\mathcal{L}_{\mathbf{k}}(X)_{BZ} = \bigoplus_{\substack{\alpha\in\mathbb{N}^{(X)} \\ |\alpha|_Z > 0}} \mathcal{L}_{\mathbf{k}}(X)_\alpha .$$

- *diagram* (2.14) *is a split SES, one of its section is given by j_B.*

Commentary 3. *i) The map rn acts as a substitution, for example*

$$rn\left([b_1b_2z_1,\ b_3z_2]\right) = \left[[b_1,[b_2,z_1]]\,,\ [b_3,z_2]\right]$$

for $b_i \in B$ and $z_i \in Z$.
ii) What says Theorem 2.3 above is that

$$rn:\ \mathcal{L}_{\mathbf{k}}(B^*Z) \to \mathcal{L}_{\mathbf{k}}(X)_{BZ}$$

is an isomorphism i.e. given $Q \in \mathcal{L}_{\mathbf{k}}(X)_{BZ}$ (each Lie monomial has at least one $z \in Z$ in its multidegree), there exists

a) words $(u_1z_1, \cdots, u_nz_n)$

b) a Lie polynomial $P = P\left(u_1z_1, \cdots, u_nz_n\right) \in \mathcal{L}_{\mathbf{k}}(B^*Z)$

such that $rn(P) = P\left(rn(u_1z_1), \cdots, rn(u_nz_n)\right) = Q$ and that P is unique.
For example with three letters $\{a, b, z\}$ and $Q = [[a,b],z]$ one has $P = abz - baz$.

Proof of Theorem 2.3 (Sketch of) :

- For all $b \in B$, the left translation $t_b^{(0)}:\ B^*Z \to B^*Z$ defined by $t_b^{(0)}(uz) := buz$ can be extended as a derivation $t_b^{(1)}$ of $\mathcal{L}_{\mathbf{k}}(B^*Z)$ (see Lemma 1.1 i.e. Corollary of Prop 8, [13] Ch II §2.8).
  For example, we have here,

  $$t_b^{(1)}\left([b_1b_2z_1,\ b_3z_2]\right) = [bb_1b_2z_1,\ b_3z_2] + [b_1b_2z_1,\ bb_3z_2]\,.$$

- By universal property, the map $b \mapsto t_b^{(1)}:\ B \to \mathfrak{Der}(\mathcal{L}_{\mathbf{k}}(B^*Z))$ is extended as a morphism of Lie algebras $Q \mapsto t_Q^{(2)}:\ \mathcal{L}_{\mathbf{k}}(B) \to \mathfrak{Der}(\mathcal{L}_{\mathbf{k}}(B^*Z))$ rewritten at once bivariately as $t_Q^{(2)}(P) =: \alpha(Q,P) \in \mathcal{L}_{\mathbf{k}}(B^*Z)$ (i.e. we define a map $\alpha : \mathcal{L}_{\mathbf{k}}\left(B\right) \times \mathcal{L}_{\mathbf{k}}\left(B^*Z\right) \to \mathcal{L}_{\mathbf{k}}\left(B^*Z\right)$ by $\alpha\left(Q,P\right) = t_Q^{(2)}\left(P\right)$).

- We get a map $\alpha_{\otimes}$ as in (2.13) (with $\mathfrak{g}_2 = \mathcal{L}_{\mathbf{k}}(B)$ and $\mathfrak{g}_1 = \mathcal{L}_{\mathbf{k}}(B^*Z)$). This action α (or $\alpha_{\otimes}$) allows us to construct $\mathcal{L}_{\mathbf{k}}\left(B^*Z\right) \rtimes \mathcal{L}_{\mathbf{k}}\left(B\right)$ (supported by $\mathcal{L}_{\mathbf{k}}(B^*Z) \oplus \mathcal{L}_{\mathbf{k}}(B)$) according to Definition 2.1 (see also [13] Ch 1 §1.8 and our Proposition 2.2).

- The pair of Lie homomorphisms $f_1 : \mathcal{L}_{\mathbf{k}}(B^*Z) \to \mathcal{L}_{\mathbf{k}}(X)$ and $f_2 : \mathcal{L}_{\mathbf{k}}(B) \to \mathcal{L}_{\mathbf{k}}(X)$ defined by $f_1(uz) := rn(uz)$ (where rn is the right-normed bracketing, see above) and $f_2(b) := b$ satisfy the equivariance condition[6] of (2.12) w.r.t. α, hence, by Prop. Proposition 2.2, there exists a unique Lie $\mathbf{k}$-algebra morphism $f : \mathcal{L}_{\mathbf{k}}(B^*Z) \rtimes \mathcal{L}_{\mathbf{k}}(B) \to \mathcal{L}_{\mathbf{k}}(X)$ extending $f_i,\ i = 1, 2$.

- In the inverse direction, a Lie algebra morphism $\mathcal{L}_{\mathbf{k}}(X) \to \mathcal{L}_{\mathbf{k}}(B^*Z) \rtimes \mathcal{L}_{\mathbf{k}}(B)$ can be constructed by the universal property of $\mathcal{L}_{\mathbf{k}}(X)$, sending each generator $b \in B$ to $(0, b)$ and each generator $z \in Z$ to $(z, 0)$.

- The two obtained arrows are proven mutually inverse by a direct computation on the generators.

- So far, we have proved that $\mathcal{L}_{\mathbf{k}}(X) \cong \mathcal{L}_{\mathbf{k}}(B^*Z) \rtimes \mathcal{L}_{\mathbf{k}}(B)$. In order to complete the proof of Theorem 2.3, we remark that what precedes establishes the semidirect product i.e. the top row of diagram 2.14; the bottom row follows by identifying $\mathcal{L}_{\mathbf{k}}(B^*Z)$ and $\mathcal{L}_{\mathbf{k}}(B)$ with their images in $\mathcal{L}_{\mathbf{k}}(X)$ and the fact that any nested Lie bracket with a factor in X can be expressed as a linear combination of right-normed B^*Z-brackets.

$\square$

Remark 2.4. (Dynkin combs and their evaluations.)
i) In an algebra $(\mathcal{A}, *)$ (not necessarily associative), we define an operator $ev_{\mathcal{A}}(\mathscr{T}, (x_1, \cdots, x_n))$ (where $\mathscr{T}$ is a binary tree with n leaves (noted $|\mathscr{T}|_l = n$) and $(x_1, \cdots, x_n) \in \mathcal{A}^n$) by the recursion

$$ev_{\mathcal{A}}(\mathscr{T}, (x_1, \cdots, x_n)) = \begin{cases} x_1 \text{ if } |\mathscr{T}|_l = 1, \\ ev_{\mathcal{A}}(\mathscr{T}_1, (x_1, \cdots, x_p)) * ev_{\mathcal{A}}(\mathscr{T}_2, (x_{p+1}, \cdots, x_n)) \\ \text{ if } \mathscr{T} = (\mathscr{T}_1, \mathscr{T}_2) \text{ and } |\mathscr{T}_1|_l = p. \end{cases} \tag{2.16}$$

As a particular case, we can define an operator $ev_{\mathfrak{g}}(\mathscr{T}, (x_1, \cdots, x_n))$ on a Lie $\mathbf{k}$-algebra

[6] It is sufficient to test the equivariance on generators as stressed in the last part of remark (2.3).

$\mathfrak{g}$ by replacing the algebra operation $*$ by the Lie bracket $[\cdot,\cdot]$ i.e. for $(x_1,\cdots,x_n)\in\mathfrak{g}^n$

$$ev_{\mathfrak{g}}(\mathscr{T},(x_1,\cdots,x_n))=\begin{cases} x_1 \text{ if } |\mathscr{T}|_l=1, \\ [ev_{\mathfrak{g}}(\mathscr{T}_1,(x_1,\cdots,x_p)),ev_{\mathfrak{g}}(\mathscr{T}_2,(x_{p+1},\cdots,x_n))] \\ \text{ if } \mathscr{T}=(\mathscr{T}_1,\mathscr{T}_2) \text{ and } |\mathscr{T}_1|_l=p. \end{cases} \tag{2.17}$$

Now, we define a special sequence of trees (right-normed tree or Dynkin combs) with n leaves, noted $\mathscr{D}_n$, defined recursively by

$$\mathscr{D}_1=\bullet\ ;\ \mathscr{D}_{n+1}=(\bullet,\mathscr{D}_n). \tag{2.18}$$

Then, for a Lie $\mathbf{k}$-algebra $\mathfrak{g}$ and any sequence $(x_1,\cdots,x_n)\in\mathfrak{g}^n$ (or written as a word $x_1\cdots x_n\in\mathfrak{g}^*$ when there is no ambiguity), we can set

$$rn((x_1,\cdots,x_n))=rn(x_1\cdots x_n):=ev_{\mathfrak{g}}(\mathscr{D}_n,(x_1,\cdots,x_n)). \tag{2.19}$$

The action of *rn as a Lie morphism* is therefore that of a substitution: for example, its action on Lie monomials of $\mathcal{L}_{\mathbf{k}}(B^*Z)$ is as follows. For every tree with n leaves $\mathscr{T}$ and list of words $u_iz_i\in B^*Z$ (could be called blocks, because B^*Z is a code in the sense of [79], Prop 1.2.1)

$$rn\Big(ev_{\mathcal{L}_{\mathbf{k}}(B^*Z)}(\mathscr{T},(u_1z_1,\cdots,u_nz_n))\Big)=ev_{\mathcal{L}_{\mathbf{k}}(X)_{BZ}}\Big(\mathscr{T},(rn(u_1z_1),\cdots,rn(u_nz_n))\Big). \tag{2.20}$$

For example, let $X=B+Z$ be a set partitioned in two blocks $B=\{b_1,b_2\}$ and $Z=\{z_1,z_2,z_3\}$. Then the image of $\Big[\big[b_1z_3,b_1b_2z_1\big],b_3b_3z_2\Big]$ under the Lie algebra morphism rn is $rn\left(\Big[\big[b_1z_3,b_1b_2z_1\big],b_3b_3z_2\Big]\right)=\Big[\big[rn(b_1z_3),rn(b_1b_2z_1)\big],rn(b_3b_3z_2)\Big]=\Big[\big[[b_1,z_3],[b_1,[b_2,z_1]]\big],[b_3,[b_3,z_2]]\Big]$.

ii) We have already called "monomial" (see Proposition 3.1) the bases of the free Lie algebra (like Hall, Lyndon, Viennot, Schützenberger) coming from the bracketing of a family of binary trees $(\mathscr{T}_i)_{i\in I}$ through evaluation (see (2.20)). Lazard elimination provides an algorithmic way to create new families of monomial bases in the following way

- Partition the alphabet X as $X=B+Z$
- Totally order the new alphabet B^*Z

- Choose any process (Hall, Lyndon, Viennot, Schützenberger) and get the associated family of trees $(T_i)_{i\in I}$ such that $(rn(T_i))_{i\in I}$ is a (linear) basis of $\mathcal{L}_{\mathbf{k}}(B^*Z)$ and remark that each $rn(T_i)$ is the evaluation of the tree obtained from T_i by appending Dynkin combs (corresponding to each uz) to the leaves of T_i.

Let us now consider a situation where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b}$, $\mathfrak{h}$ being an ideal and $\mathfrak{b}$ a Lie subalgebra (hence we have the - internal - semidirect product $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b}$). Implementation of the associated arrows is the following

Theorem 2.4 (Ladder LET (internal version)). *Let $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b}$ be a semi-direct product of Lie algebras, $\mathfrak{h}$ (resp. $\mathfrak{b}$) being an ideal (resp. a Lie subalgebra) of $\mathfrak{g}$* [7]*.*
Let $X = B \sqcup Z$ be a set partitioned in two blocks and $\varphi : X \to \mathfrak{g}$ a (set-theoretical) map such that[8]

1. *$\varphi(B)$ is a generating set of $\mathfrak{b}$ as a Lie algebra.*
2. *$\varphi(Z)$ is a generating set of $\mathfrak{h}$ as a Lie ideal of $\mathfrak{g}$.*

Then

1. *$\varphi(X)$ is a generating set of $\mathfrak{g}$ as a Lie algebra.*
2. *One has a commutative diagram of Lie algebras with split short exact rows (and commuting sections)*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}_{\mathbf{k}}(B^*Z) \arrow[r, "j_{B|Z}\,(=rn)"] \arrow[d, "\varphi_1"] & \mathcal{L}_{\mathbf{k}}(X) \arrow[r, "p_{B|Z}"] \arrow[d, "\varphi_3"] & \mathcal{L}_{\mathbf{k}}(B) \arrow[r] \arrow[d, "\varphi_2"] & 0 \\
0 \arrow[r] & \mathfrak{h} \arrow[r, "j"] & \mathfrak{g} \arrow[r, "p"] & \mathfrak{b} \arrow[r] & 0.
\end{tikzcd} \tag{2.21}$$

The arrows $j_{B|Z}$ and $p_{B|Z}$ here are the ones constructed as in Theorem 2.3, j, p being the canonical maps for the (internal) decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b}$ and

- *φ_3, φ_2 are extensions of φ by universal property of Diagram* (1.30)

[7] This means that $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b}$ is an internal semi-direct product.
[8] The maps $pr_i,\ i = 1, 2$ standing respectively for the first and second projections of the cartesian product.

- *for* $u = b_1 \cdots b_k \in B^*$, $z \in Z$, *we have*

$$\varphi_1(uz) := \left(\mathrm{ad}^{\mathfrak{g}}_{\varphi(b_1)} \circ \cdots \circ \mathrm{ad}^{\mathfrak{g}}_{\varphi(b_k)}\right)(\varphi(z)). \tag{2.22}$$

In order to make a smooth transition with the subsequent point (i.e. 2.2.2), let us give an external version of Theorem 2.4.

Theorem 2.5 (Ladder LET (external version))**.** *Let* $\mathfrak{g}_3 = \mathfrak{g}_1 \rtimes \mathfrak{g}_2$ *be a semi-direct product of Lie algebras, constructed after a morphism of Lie* $\mathbf{k}$*-algebras* $\alpha : \ \mathfrak{g}_2 \to \mathfrak{Der}(\mathfrak{g}_1)$*.*
Let $X = B + Z$ *be a set partitioned in two blocks and* $\varphi : \ X \to \mathfrak{g}_3$ *a (set-theoretical) map such that*

1. $\varphi(B) \subset \{0\} \times \mathfrak{g}_2$ *and* $pr_2 \circ \varphi(B)$ *is a generating set of* $\mathfrak{g}_2$ *as a Lie algebra.*

2. $\varphi(Z) \subset \mathfrak{g}_1 \times \{0\}$ *and* $pr_1 \circ \varphi(Z)$ *is a generating set of* $\mathfrak{g}_1$ *as a Lie algebras with operators (here, operators are provided by* $\alpha : \ \mathfrak{g}_2 \to \mathfrak{Der}(\mathfrak{g}_1)$*).*

Then

1. $\varphi(X)$ *is a generating set of* $\mathfrak{g}_3$ *as a Lie algebra.*

2. *One has a commutative diagram of Lie algebras with split short exact rows*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}_{\mathbf{k}}(B^*Z) \arrow[r, "j_{B|Z}\,(=rn)"] \arrow[d, "\varphi_1"] & \mathcal{L}_{\mathbf{k}}(X) \arrow[r, "p_{B|Z}"] \arrow[d, "\varphi_3"] & \mathcal{L}_{\mathbf{k}}(B) \arrow[r] \arrow[d, "\varphi_2"] & 0 \\ 0 \arrow[r] & \mathfrak{g}_1 \arrow[r, "j_1"] & \mathfrak{g}_3 \arrow[r, "pr_2"] & \mathfrak{g}_2 \arrow[r] & 0. \end{tikzcd} \tag{2.23}$$

The arrows $j_{B|Z}$ *and* $p_{B|Z}$ *here are the ones constructed as in Theorem 2.3,* j_1, pr_2 *being the canonical maps for the (external) decomposition* $\mathfrak{g}_3 = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ *and*

- φ_3 *(resp.* φ_2*) is the extension of* φ *(resp.* $pr_2 \circ \varphi$*) by universal property of Diagram* (1.30)

- *for* $u = b_1 \cdots b_k \in B^*$, $z \in Z$, *we have*

$$\varphi_1(uz) := \alpha[\varphi(b_1)] \circ \cdots \circ \alpha[\varphi(b_k)](\varphi(z)). \tag{2.24}$$

2.2.2 Quotients of Lazard's eliminations.

In this subsection, we deal with a special kind of relators i.e. relators being compatible with an elimination scheme. This situation encompasses presented Lie algebras like Drinfeld-Kohno or partially commutative ones (cf. Duchamp and Krob [41]). The situation will be described in the subsequent section.

Definition of the compatibility and an example.

Let $\mathbf{k}$ be a ring. Let $X = B + Z$ be a set partitioned in two blocks. We suppose given a relator $\mathbf{r} = \{r_j\}_{j \in J} \subset \mathcal{L}_{\mathbf{k}}(X)$ (cf. [13] Ch II §2.3[9]) which is compatible with the alphabet partition i.e. there exists a partition of the set of indices $J = J_Z \sqcup J_B$ such that

- $\mathbf{r}_B = \{r_j\}_{j \in J_B} = \mathbf{r} \cap \mathcal{L}_{\mathbf{k}}(X)_B$ and $\mathbf{r}_Z = \{r_j\}_{j \in J_Z} = \mathbf{r} \cap \mathcal{L}_{\mathbf{k}}(X)_{BZ}$.

The notations being as above, we construct the ideals

- $\mathcal{J}_B$ is the Lie ideal of $\mathcal{L}_{\mathbf{k}}(X)_B$ generated by $\{r_j\}_{j \in J_B}$
- $\mathcal{J}, \mathcal{J}_Z$ and $\mathcal{J}_{BZ}$ are the Lie ideals of $\mathcal{L}_{\mathbf{k}}(X)$ generated respectively by $\mathbf{r}, \mathbf{r}_Z$ and $\mathbf{r}_{BZ} := \{\mathrm{ad}_Q\, z\}_{Q \in \mathcal{J}_B, z \in Z}$.

Example 2.1. i) Let us recall that $\mathcal{T}_{n+1} = \{t_{ij}\}_{1 \le i < j \le n+1}$ is the set of variables and the infinitesimal pure braid relator $\mathbf{R}[\mathbf{n+1}]$ (2.6) in the free Lie algebra $\mathcal{L}_{\mathbf{k}}(\mathcal{T}_{n+1})$. A typical example is for the graded set $\mathcal{T}_{n+1} = \mathcal{T}_n \sqcup T_{n+1}$ (i.e. $T_{n+1} = \{t_{i,n+1}\}_{1 \le i \le n}$) and the infinitesimal pure braid relator $\mathbf{r} := \mathbf{R}[\mathbf{n+1}] \subset \mathcal{L}_{\mathbf{k}}(\mathcal{T}_{n+1})$. In this case, we observe that the relator $\mathbf{r}_{\mathcal{T}_n}$ is equal to $\mathbf{R}[\mathbf{n+1}] \cap \mathcal{L}_{\mathbf{k}}(\mathcal{T}_{n+1})_{\mathcal{T}_n} = \mathbf{R}[\mathbf{n}]$ and the relator $\mathbf{r}_{T_{n+1}} = \mathbf{R}[\mathbf{n+1}] \cap \mathcal{L}_{\mathbf{k}}(\mathcal{T}_{n+1})_{\mathcal{T}_n T_{n+1}}$ is equal to the following formulas

$$\begin{cases} \mathbf{R}^\dagger_{\mathbf{1}}[\mathbf{n+1}] & [t_{i,j}, t_{i,n+1} + t_{j,n+1}] & \text{for } 1 \le i < j \le n, \\ \mathbf{R}^\dagger_{\mathbf{2}}[\mathbf{n+1}] & [t_{i,j} + t_{i,n+1}, t_{j,n+1}] & \text{for } 1 \le i < j \le n, \\ \mathbf{R}^\dagger_{\mathbf{3}}[\mathbf{n+1}] & \pm[t_{i,j}, t_{k,n+1}] & \text{for } \begin{matrix} 1 \le i < j \le n, \\ 1 \le k \le n, \end{matrix} \text{ and } |\{i,j,k\}| = 3. \end{cases} \tag{2.25}$$

Then we can construct the following Lie ideals

[9] With $I = X$.

- $\mathcal{J}_{\mathcal{T}_n} = \mathcal{J}_{\mathbf{R[n]}}$ is the Lie ideal of $\mathcal{L}_{\mathbf{k}}(\mathcal{T}_n)$ generated by the infinitesimal pure braid relator $\mathbf{r}_{\mathcal{T}_n} = \mathbf{R}[\mathbf{n}]$.

- $\mathcal{J}_{T_{n+1}}$ (resp. $\mathcal{J}_{\mathcal{T}_n T_{n+1}}$) is the Lie ideal of $\mathcal{L}_{\mathbf{k}}(\mathcal{T}_{n+1})$ generated by the relator $\mathbf{r}_{T_{n+1}}$ (resp. $\mathbf{r}_{\mathcal{T}_n T_{n+1}} = \{\mathrm{ad}_Q\, z\}_{Q\in\mathcal{J}_{\mathbf{R[n]}}, z\in T_{n+1}}$).

- $\mathcal{J} = \mathcal{J}_{\mathbf{R[n+1]}}$ is the Lie ideal of $\mathcal{L}_{\mathbf{k}}(\mathcal{T}_{n+1})$ generated by the infinitesimal pure braid relator $\mathbf{R}[\mathbf{n+1}]$.

ii) Of course, for instance, if we considered the relator $\mathbf{R}[4] \cup \{[t_{1,2}, t_{2,3}] + [t_{1,4}, t_{3,4}]\}$, then compatibility would no longer be fulfilled, for the added relator would neither belong to $\mathcal{L}_{\mathbf{k}}\left(X\right)_B$ nor to $\mathcal{L}_{\mathbf{k}}\left(X\right)_{BZ}$, and thus J would not be $J_Z \sqcup J_B$ any more.

Main Result: Quotients of Lazard's eliminations.

When we have such a type of relator, we can state the following theorem.

Theorem 2.6. *(Main Result) Let* $\mathbf{k}$ *be a ring and* $X = B + Z$ *be a set partitioned in two blocks. We suppose given a relator* $\mathbf{r} = \{r_j\}_{j\in J} \subset \mathcal{L}_{\mathbf{k}}(X)$ *(cf. [13] Ch II §2.3*[10]*) which is compatible with the alphabet partition i.e. there exists a partition of the set of indices* $J = J_Z \sqcup J_B$ *such that*

- $\mathbf{r}_B = \{r_j\}_{j\in J_B} = \mathbf{r} \cap \mathcal{L}_{\mathbf{k}}(X)_B$ *and* $\mathbf{r}_Z = \{r_j\}_{j\in J_Z} = \mathbf{r} \cap \mathcal{L}_{\mathbf{k}}(X)_{BZ}$.

With these data, we construct the ideals

- $\mathcal{J}_B$ *is the Lie ideal of* $\mathcal{L}_{\mathbf{k}}(X)_B$ *generated by* $\{r_j\}_{j\in J_B}$

- $\mathcal{J}, \mathcal{J}_Z$ *and* $\mathcal{J}_{BZ}$ *are the Lie ideals of* $\mathcal{L}_{\mathbf{k}}(X)$ *generated respectively by* $\mathbf{r}, \mathbf{r}_Z$ *and* $\mathbf{r}_{BZ} := \{\mathrm{ad}_Q\, z\}_{Q\in\mathcal{J}_B, z\in Z} = \{[Q, z]\}_{Q\in\mathcal{J}_B, z\in Z}$.

With these notations, we get the following properties:

i) we have $\mathcal{J}_{BZ}^Z := \mathcal{J}_Z + \mathcal{J}_{BZ} \subset \mathcal{L}_{\mathbf{k}}(X)_{BZ}$. *Moreover,* $\mathcal{J}_{BZ}^Z$ *is a Lie ideal of* $\mathcal{L}_{\mathbf{k}}(X)_{BZ}$ *(and even, by definition, of* $\mathcal{L}_{\mathbf{k}}(X)$*)*

[10] With $I = X$.

ii) the action of $\mathcal{L}_{\mathbf{k}}(X)_B$ *on* $\mathfrak{Der}(\mathcal{L}_{\mathbf{k}}(X)_{BZ})$ *(by internal* ad*) passes to quotients as an action*

$$\alpha:\ \mathcal{L}_{\mathbf{k}}(X)_B \to \mathfrak{Der}\left(\mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ}\right) \tag{2.26}$$

such that $\mathbf{r}_B \subset \mathrm{Ker}(\alpha)$ *and then, we get an action*

$$[\alpha]:\ \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B \to \mathfrak{Der}\left(\mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ}\right) \tag{2.27}$$

iii) we can construct an isomorphism (and its inverse) from presented Lie algebra $\mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J}$ *by the set* $\mathbf{r} = \{r_j\}_{j\in J}$ *of relators onto the semi-direct product of Lie algebras* $\mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ} \rtimes \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B$ *which will be denoted by*

$$\beta_{25}: \mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J} \xrightarrow{\cong} \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ} \rtimes \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B \tag{2.28}$$

iv) in fact, one has a commutative diagram of Lie algebras with split short exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X)_{BZ} & \overset{j}{\hookrightarrow} & \mathcal{L}_{\mathbf{k}}(X) & \overset{p}{\longrightarrow} & \mathcal{L}_{\mathbf{k}}(X)_B & \longrightarrow & 0 \\ & & \big\downarrow s_{\mathcal{J}^Z_{BZ}} & & \big\downarrow s_{\mathcal{J}} & & \big\downarrow s_{\mathcal{J}_B} & & \\ 0 & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ} & \xrightarrow{[j]} & \mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J} & \xrightarrow{[p]} & \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B & \longrightarrow & 0 \end{array} \tag{2.29}$$

where, for any ideal $\mathcal{I}$*,* $s_{\mathcal{I}}$ *stands for the natural quotient map.*

Proof. i) The formula (1.35) implies that $\mathcal{L}_{\mathbf{k}}(X)_{BZ} = \bigoplus_{\substack{\alpha\in\mathbb{N}^{(X)} \\ |\alpha|_Z>0}} \mathcal{L}_{\mathbf{k}}(X)_\alpha$ is a Lie ideal of $\mathcal{L}_{\mathbf{k}}(X)$ which contains $\mathbf{r}_Z = \mathbf{r} \cap \mathcal{L}_{\mathbf{k}}(X)_{BZ}$. Hence, $\mathcal{J}_Z$ (the Lie ideal of $\mathcal{L}_{\mathbf{k}}(X)$ generated by $\mathbf{r}_Z$) is a subset of $\mathcal{L}_{\mathbf{k}}(X)_{BZ}$. Similarly for the case $\mathbf{r}_{BZ} = \{\mathrm{ad}_Q\, z\}_{Q\in\mathcal{J}_B, z\in Z} \subset \mathcal{L}_{\mathbf{k}}(X)_{BZ}$, we easily show that $\mathcal{J}_{BZ} \subset \mathcal{L}_{\mathbf{k}}(X)_{BZ}$. Therefore $\mathcal{J}^Z_{BZ} = \mathcal{J}_Z + \mathcal{J}_{BZ} \subset \mathcal{L}_{\mathbf{k}}(X)_{BZ}$. Then by taking the intersection, it is seen that $\mathcal{J}^Z_{BZ} \cap \mathcal{J}_B \subset \mathcal{L}_{\mathbf{k}}(X)_{BZ} \cap \mathcal{L}_{\mathbf{k}}(X)_B = \{0\}$ (see (2.14)) and then $\mathcal{J}^Z_{BZ} \cap \mathcal{J}_B = \{0\}$. The second assertion of (i) can be obtained from the fact that $\mathcal{J}^Z_{BZ}$ is a Lie ideal of $\mathcal{L}_{\mathbf{k}}(X)$, thus $\mathcal{J}^Z_{BZ} = \mathcal{J}^Z_{BZ} \cap \mathcal{L}_{\mathbf{k}}(X)_{BZ}$ is a Lie ideal of $\mathcal{L}_{\mathbf{k}}(X)_{BZ}$.

ii) Let us recall briefly the adjoint representation

$$\mathrm{ad}:\ \mathcal{L}_{\mathbf{k}}(X)_B \to \mathfrak{Der}(\mathcal{L}_{\mathbf{k}}(X)_{BZ}) \tag{2.30}$$

which is defined by $\mathrm{ad}_Q(P) = [Q, P]$ for any $Q \in \mathcal{L}_{\mathbf{k}}(X)_B$ and $P \in \mathcal{L}_{\mathbf{k}}(X)_{BZ}$. It is well-known that ad is a Lie algebra morphism. Let $Q \in \mathcal{L}_{\mathbf{k}}(X)_B$ then, due to the fact that $\mathcal{J}_{BZ}^{Z} = \mathcal{J}_Z + \mathcal{J}_{BZ}$ is ad_Q-invariant and by Lemma 1.3, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_{\mathbf{k}}(X)_{BZ} & \xrightarrow{\quad \mathrm{ad}_Q \quad} & \mathcal{L}_{\mathbf{k}}(X)_{BZ} \\ {\scriptstyle s_{\mathcal{J}_{BZ}^{Z}}}\downarrow & & \downarrow{\scriptstyle s_{\mathcal{J}_{BZ}^{Z}}} \\ \mathcal{L}_{\mathbf{k}}(X)_{BZ} \big/ \mathcal{J}_{BZ}^{Z} & \xrightarrow{\quad \alpha(Q) \quad} & \mathcal{L}_{\mathbf{k}}(X)_{BZ} \big/ \mathcal{J}_{BZ}^{Z}. \end{array} \tag{2.31}$$

This construction is sufficient in order to get a well-defined morphism of Lie algebras

$$\alpha : \ \mathcal{L}_{\mathbf{k}}(X)_B \to \mathfrak{Der}\left(\mathcal{L}_{\mathbf{k}}(X)_{BZ} \big/ \mathcal{J}_{BZ}^{Z}\right) \tag{2.32}$$

that is induced from the adjoint representation ad. Let us show that

$$\mathbf{r}_B \subset \mathrm{Ker}(\alpha). \tag{2.33}$$

As $\mathrm{Ker}(\alpha)$ is a Lie ideal, showing (2.33) is equivalent to showing

Lemma 2.7. *With the notations and conditions* (2.32) *and* (2.33), *we have*

(a) $\mathcal{J}_B \subset \mathrm{Ker}(\alpha)$.

(b) There is a Lie algebra morphism

$$[\alpha] : \ \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B \to \mathfrak{Der}(\mathcal{L}_{\mathbf{k}}(X)_{BZ} \big/ \mathcal{J}_{BZ}^{Z}) \tag{2.34}$$

which factorizes α *as* $\alpha = [\alpha] \circ s_{\mathcal{J}_B}$.

Proof. We recall that the adjoint representation within $\mathcal{L}_{\mathbf{k}}(X)$ can be extended to sequences of Lie polynomials $(Q_1, Q_2, \cdots, Q_n) \in \mathrm{S{\small EQ}}(\mathcal{L}_{\mathbf{k}}(X))$ by equation (1.40)

$$\mathrm{ad}_{(Q_1,Q_2,\cdots,Q_n)} = \mathrm{ad}_{Q_1} \circ \mathrm{ad}_{Q_2} \circ \cdots \circ \mathrm{ad}_{Q_n} \in \mathrm{End}(\mathcal{L}_{\mathbf{k}}(X)).$$

Subsequences of letters will be noted as words as follows

$$(\underbrace{b_1, \cdots, b_p}_{u}, Q, \underbrace{b_{p+1}, \cdots, b_{p+q}}_{v}) = (u, Q, v). \tag{2.35}$$

With these notations, we have the following

Lemma 2.8. *Let $Q \in \mathcal{J}_B$ and $u = b_1 \cdots b_k \in B^*$, then there exists two finite sequences $(u_1, \cdots, u_N) \in (B^*)^N$ and $(Q_1, \cdots, Q_N) \in \mathcal{J}_B^N$ such that*

$$\mathrm{ad}_{(Q,u)} = \mathrm{ad}_{(Q,b_1,\cdots,b_k)} = \sum_{i=1}^{N} \mathrm{ad}_{(u_i,Q_i)} \quad . \tag{2.36}$$

Proof. We prove the fact by induction on k.

If $k = 0$, we are tautologically done.

If $k > 0$, we write $u = b_1 v$ with $b_1 \in B$, then

$$\begin{aligned}
&\mathrm{ad}_{(Q,u)} = \mathrm{ad}_{(Q,b_1 v)} = \mathrm{ad}_Q \circ \mathrm{ad}_{b_1} \circ \mathrm{ad}_{(v)} = \\
&[\mathrm{ad}_Q, \mathrm{ad}_{b_1}] \circ \mathrm{ad}_{(v)} + \mathrm{ad}_{b_1} \circ \mathrm{ad}_Q \circ \mathrm{ad}_{(v)} = \\
&\mathrm{ad}_{[Q,b_1]} \circ \mathrm{ad}_{(v)} + \mathrm{ad}_{b_1} \circ \mathrm{ad}_Q \circ \mathrm{ad}_{(v)} = \mathrm{ad}_{[Q,b_1]} \circ \mathrm{ad}_{(v)} + \mathrm{ad}_{b_1} \circ \mathrm{ad}_{(Q,v)} \, .
\end{aligned}$$

Now we observe that $[Q, b_1] \in \mathcal{J}_B$ and $|v| = |u| - 1$, then applying the induction hypothesis for $([Q, b_1], v) \in \mathcal{J}_B \times B^{k-1}$ and $(Q, v) \in \mathcal{J}_B \times B^{k-1}$, we get the result (of Lemma 2.8). □

End of the proof of Lemma 2.7. –

Let $Q \in \mathcal{L}_{\mathbf{k}}(X)_B$, due to the fact that $\mathcal{L}_{\mathbf{k}}(X)_{BZ}$ and $\mathcal{J}_{BZ}^Z$ are ad_Q-invariant (they are ideals)[11], we have a commutative diagram (where, for any ideal $\mathcal{J}$, $s_{\mathcal{J}}$ stands for the natural quotient map)

$$\begin{tikzcd}
\mathcal{L}_{\mathbf{k}}(X)_{BZ} \arrow[r, "\mathrm{ad}_Q"] \arrow[d, "s_{\mathcal{J}_{BZ}^Z}"'] & \mathcal{L}_{\mathbf{k}}(X)_{BZ} \arrow[d, "s_{\mathcal{J}_{BZ}^Z}"] \\
\mathcal{L}_{\mathbf{k}}(X)_{BZ} \big/ \mathcal{J}_{BZ}^Z \arrow[r, "\alpha(Q)"] & \mathcal{L}_{\mathbf{k}}(X)_{BZ} \big/ \mathcal{J}_{BZ}^Z .
\end{tikzcd} \tag{2.37}$$

Now, for $Q \in \mathcal{J}_B$ and $(u, z) \in B^* \times Z$, let us show that $\mathrm{ad}_Q(\mathrm{ad}_{(u)}(z)) \in \mathcal{J}_{BZ}$. From equation (2.36) of Lemma 2.8, we can write

$$\mathrm{ad}_Q(\mathrm{ad}_{(u)}\, z) = \mathrm{ad}_{(Q,u)}(z) = \sum_{i \in F} \mathrm{ad}_{(u_i,Q_i)}(z) = \sum_{i \in F} \mathrm{ad}_{(u_i)} \circ \mathrm{ad}_{Q_i}(z) \tag{2.38}$$

with $Q_i \in \mathcal{J}_B$ and $u_i \in B^*$, then clearly the sum belongs to $\mathcal{J}_{BZ}$ by the definition of $\mathbf{r}_{BZ} = \{\mathrm{ad}_Q\, z\}_{Q \in \mathcal{J}_B, z \in Z}$. In other words, for all $(u, z) \in B^* \times Z$, $\mathrm{ad}_{(u)}(z) \in \mathrm{ad}_Q^{-1}(\mathcal{J}_{BZ})$. But from the point 3 of Lemma 1.3 we know that $\mathrm{ad}_Q^{-1}(\mathcal{J}_{BZ})$ is a Lie

[11] For D-invariance, see Lemma 1.3.

subalgebra of $\mathcal{L}_{\mathbf{k}}(X)_{BZ}$ but we also know that $\{\mathrm{ad}_{(u)}(z)\}_{(u,z)\in B^*\times Z}$ is a generating set of $\mathcal{L}_{\mathbf{k}}(X)_{BZ}$, it follows that $\mathrm{ad}_Q^{-1}(\mathcal{J}_{BZ}) = \mathcal{L}_{\mathbf{k}}(X)_{BZ}$ i.e. $\mathrm{ad}_Q(\mathcal{L}_{\mathbf{k}}(X)_{BZ}) = \mathcal{J}_{BZ}$ and then

$$\alpha(Q) \circ s_{\mathcal{J}^Z_{BZ}} = s_{\mathcal{J}^Z_{BZ}} \circ \mathrm{ad}_Q = 0. \tag{2.39}$$

From the fact that $s_{\mathcal{J}^Z_{BZ}}$ is onto, we see that $\alpha(Q) = 0$.
As a conclusion ($Q \in \mathcal{J}_B$) implies ($\alpha(Q) = 0$) which is the claim. $\square$

As a consequence of the construction of the Lie algebra morphism $[\alpha]$ the $\mathbf{k}$-module

$$\mathfrak{g}_X := \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ} \oplus \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B$$

is endowed with the structure of a Lie algebra given by the semi-direct product

$$\mathfrak{g}_X := \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ} \rtimes \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B .$$

Here, the Lie bracket is given by the following formula

$$[(P_0, Q_0), (P_1, Q_1)] = ([P_0, P_1] + [\alpha](Q_0)(P_1) - [\alpha](Q_1)(P_0), [Q_0, Q_1]). \tag{2.40}$$

With our construction, we thus have the split short exact sequence of Lie algebras

$$0 \to \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ} \to \mathfrak{g}_X \to \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B \to 0. \tag{2.41}$$

iii) In order to prove that $\mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J} \cong \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ} \rtimes \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B$ in **k-Lie**, we follow the steps below:

a. A mapping $\beta_{21} : X \to \mathfrak{g}_X$ is given by the formula

$$\beta_{21}(x) := \begin{cases} (0, [x]) & \text{if} \quad x \in B \\ ([x], 0) & \text{if} \quad x \in Z, \end{cases} \tag{2.42}$$

here $[x] = x + \mathcal{J}_B$ if $x \in B$ and $[x] = x + \mathcal{J}^Z_{BZ}$ if $x \in Z$. The universal property of the free Lie algebra $\mathcal{L}_{\mathbf{k}}(X)$ shows that there is a Lie algebra morphism $\beta_{23} :\ \mathcal{L}_{\mathbf{k}}(X) \to \mathfrak{g}_X$ such that the following triangle

$$\begin{tikzcd} X \arrow[rr, hook, "\beta_{22}"] \arrow[dr, "\beta_{21}"'] & & \mathcal{L}_{\mathbf{k}}(X) \arrow[dl, "\beta_{23}"'] \\ & \mathfrak{g}_X & \end{tikzcd} \tag{2.43}$$

is commutative, where β_{22} is the embedding map. We claim that

$$\beta_{23}(\mathrm{ad}_{(u)}(z)) = (\mathrm{ad}_{(u)}(z) + \mathcal{J}_{BZ}^{Z}, 0), \tag{2.44}$$

for any $u \in B^*$ and $z \in Z$. Indeed, this follows by induction on the length of word u,

- If $|u| = 0$: for any $z \in Z$, by equation (2.42) and diagram (2.43) we have $\beta_{23}(z) = \beta_{21}(z) = (z + \mathcal{J}_{BZ}^{Z}, 0)$.
- If $|u| = 1$: for any $u = b \in B$ and $z \in Z$, by formulas (2.40), (2.42) and diagram (2.43) we have

$$\begin{aligned}\beta_{23}(\mathrm{ad}_{(u)}(z)) &= \beta_{23}([b,z]) = [\beta_{23}(b), \beta_{23}(z)] \\ &= [(0,[b]),([z],0)] = ([\alpha]([b])([z]),0) \\ &= ([b,z] + \mathcal{J}_{BZ}^{Z}, 0) = (\mathrm{ad}_{(u)}(z) + \mathcal{J}_{BZ}^{Z}, 0).\end{aligned}$$

- Assume that $\beta_{23}(\mathrm{ad}_{(u)}(z)) = (\mathrm{ad}_{(u)}(z) + \mathcal{J}_{BZ}^{Z}, 0)$ for any $u \in B^*$ and $|u| = k-1$. For this assumption, for any $u = b_1 b_2 \cdots b_k \in B^*$ and $z \in Z$, by formulas (2.40), (2.42) and diagram (2.43) we also have

$$\begin{aligned}\beta_{23}(\mathrm{ad}_{(u)}(z)) &= \beta_{23}\big([b_1, \mathrm{ad}_{(b_2\cdots b_k)}(z)]\big) = \big[\beta_{23}(b_1), \beta_{23}(\mathrm{ad}_{(b_2\cdots b_k)}(z))\big] \\ &= \big[(0,[b_1]),([\mathrm{ad}_{(b_2\cdots b_k)}(z)],0)\big] \text{ (by ind. on } k) \\ &= \big([\alpha]([b_1])([\mathrm{ad}_{(b_2\cdots b_k)}(z)]),0\big) = \big([b_1, \mathrm{ad}_{(b_2\cdots b_k)}(z)] + \mathcal{J}_{BZ}^{Z}, 0\big) \\ &= \big(\mathrm{ad}_{(b_1\cdots b_k)}(z) + \mathcal{J}_{BZ}^{Z}, 0\big) = \big(\mathrm{ad}_{(u)}(z) + \mathcal{J}_{BZ}^{Z}, 0\big).\end{aligned}$$

b. If $j \in J = J_Z \sqcup J_B$, then $\beta_{23}(r_j)$ is, according to the properties of (2.42) and (2.43), equal to

$$\beta_{23}(r_j) = \begin{cases} (0,[r_j]) = (0,0) & \text{if} \quad j \in J_B \\ ([r_j],0) = (0,0) & \text{if} \quad j \in J_Z. \end{cases} \tag{2.45}$$

beacause, from (2.42), $\beta_{23}(r_j)$ arrives in the sectors of

$$\mathfrak{g}_X := \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}_{BZ}^{Z} \oplus \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B$$

as precised in 2.45 and we know that, each time, the result is zero.

Thus, considering $\mathbf{r} = \{r_j\}_{j \in J}$ as a set, we have $\beta_{23}(\mathbf{r}) = 0$ and then $\beta_{23}(\mathcal{J}) = 0$.

This shows that β_{23} induces a morphism of Lie algebras

$$\beta_{25} := [\beta_{23}] : \mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J} \to \mathfrak{g}_X. \tag{2.46}$$

Given the natural surjection here named $\beta_{24} : \mathcal{L}_{\mathbf{k}}(X) \twoheadrightarrow \mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J}$, we can construct a commutative diagram, namely

$$\begin{tikzcd} X \arrow[r, "\beta_{21}"] \arrow[d, "\beta_{22}"'] & \mathfrak{g}_X \\ \mathcal{L}_{\mathbf{k}}(X) \arrow[ur, "\beta_{23}"] \arrow[r, "\beta_{24}"] & \mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J}. \arrow[u, "\beta_{25}"'] \end{tikzcd} \tag{2.47}$$

c. In this part, using the equivariant property of Lie algebras of Section 2.1, we describe the inverse isomorphism (called here β_{33}) as follows

$$\beta_{33} : \mathfrak{g}_X = \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ} \rtimes \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B \to \mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J}$$

In fact,

- There is the obvious embedding map $j : \mathcal{L}_{\mathbf{k}}(X)_{BZ} \hookrightarrow \mathcal{L}_{\mathbf{k}}(X)$. We can easily verify that $j(\mathcal{J}^Z_{BZ}) = \mathcal{J}^Z_{BZ} \subseteq \mathcal{J}$, then j gives rise to a Lie morphism

$$g_1 := [j] : \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ} \to \mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J}.$$

  By (2.44) and (2.46), one obtains $\beta_{25} \circ g_1(\mathrm{ad}_{(u)}(z) + \mathcal{J}^Z_{BZ}) = [\beta_{23}](\mathrm{ad}_{(u)}(z) + \mathcal{J}) = \beta_{23}(\mathrm{ad}_{(u)}(z)) = (\mathrm{ad}_{(u)}(z) + \mathcal{J}^Z_{BZ}, 0)$, for any $u \in B^*$ and $z \in Z$. From the above calculation over generators $\{\mathrm{ad}_{(u)}(z)\}_{(u,z)\in B^*\times Z} + \mathcal{J}^Z_{BZ}$, it follows that

$$\beta_{25} \circ g_1(P_0) = (P_0, 0), \forall P_0 \in \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ}. \tag{2.48}$$

- Observe that there is the embedding map $s : \mathcal{L}_{\mathbf{k}}(X)_B \hookrightarrow \mathcal{L}_{\mathbf{k}}(X)$ which is a section of $p : \mathcal{L}_{\mathbf{k}}(X) \to \mathcal{L}_{\mathbf{k}}(X)_B$ (as in 2.14). Since $\mathcal{J}_B \subseteq \mathcal{J}$ and then $s(\mathcal{J}_B) \subseteq \mathcal{J}$, so there is a Lie algebra morphism

$$g_2 := [s] : \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B \to \mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J}.$$

  By (2.42), (2.43) and (2.46), one observes that

$$\beta_{25} \circ g_2(b + \mathcal{J}_B) = [\beta_{23}](b + \mathcal{J}) = \beta_{21}(b) = (0, b + \mathcal{J}_B) \ ,$$

for any $b \in B$. We calculated this over its generators, one derives the relation

$$\beta_{25} \circ g_2(Q_0) = (0, Q_0), \forall Q_0 \in \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B. \tag{2.49}$$

- Moreover, we show directly that morphisms $g_1 : \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ} \to \mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J}$ and $g_2 : \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B \to \mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J}$ in the category **k-Lie** satisfy the equivariant property (2.12) (or diagrammatically (2.13)) i.e. for any $Q_0 = Q + \mathcal{J}_B \in \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B$ and $P_0 = P + \mathcal{J}^Z_{BZ} \in \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ}$ one has

$$g_1\big([\alpha](Q_0)(P_0)\big) = [g_2(Q_0), g_1(P_0)]. \tag{2.50}$$

Indeed, we give the following our proof without difficulty

$$\begin{aligned}
&g_1\big([\alpha](Q_0)(P_0)\big) = g_1\big([\alpha](Q + \mathcal{J}_B)(P + \mathcal{J}^Z_{BZ})\big) = \\
&g_1\big([Q, P] + \mathcal{J}^Z_{BZ}\big) = [Q, P] + \mathcal{J} = [Q + \mathcal{J}, P + \mathcal{J}] \\
&= [g_2(Q_0), g_1(P_0)].
\end{aligned}$$

d. In terms of equivariant property (2.50) and by Proposition 2.2, they are sufficient to conclude that there is a unique morphism of Lie algebras

$$\beta_{33} : \ \mathfrak{g}_X = \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ} \rtimes \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B \to \mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J} \tag{2.51}$$

which extends g_1 and g_2 in the usual sense i.e. $\beta_{33}(P_0, 0) = g_1(P_0)$ and $\beta_{33}(0, Q_0) = g_2(Q_0)$, for any $P_0 \in \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ}$ and $Q_0 \in \mathcal{L}_{\mathbf{k}}(X)_B \big/ \mathcal{J}_B$. We obtain two Lie algebra morphisms $\beta_{33} \circ \beta_{25} : \ \mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J} \to \mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J}$ and $\beta_{25} \circ \beta_{33} : \ \mathfrak{g}_X \to \mathfrak{g}_X$. We now show that β_{25} is the inverse of β_{33}

- By formula (2.42) and morphisms (2.43), (2.46) and (2.51), it give us to compute in detail the following behavior: for any $x \in X$, then $\beta_{33} \circ \beta_{25}([x]) = \beta_{33} \circ [\beta_{23}]([x]) = \beta_{33}(\beta_{21}(x)) =$

$$\begin{cases} \beta_{33}((0, [x])) = g_2([x]) = [x] & \text{if } x \in B \\ \beta_{33}(([x], 0)) = g_1([x]) = [x] & \text{if } x \in Z. \end{cases}$$

Hence, $\beta_{33} \circ \beta_{25}([x]) = [x] = \mathrm{Id}_{\mathcal{L}_{\mathbf{k}}(X) / \mathcal{J}}([x])$. As a consequence we clearly derive that $\beta_{33} \circ \beta_{25} = \mathrm{Id}_{\mathcal{L}_{\mathbf{k}}(X) / \mathcal{J}}$.

- Moreover, by formulas (2.48), (2.49) and the morphism (2.51), for any $(P_0, Q_0) \in \mathcal{L}_{\mathbf{k}}(X)_{BZ} \big/ \mathcal{J}^Z_{BZ} \oplus \mathcal{L}_{\mathbf{k}}(X)_B / \mathcal{J}_B$, one has

$$\begin{aligned}
&\beta_{25} \circ \beta_{33}(P_0, Q_0) = \beta_{25} \circ \beta_{33}((P_0, 0) + (0, Q_0)) = \\
&\beta_{25} \circ (\beta_{33}(P_0, 0) + \beta_{33}(0, Q_0)) = \beta_{25}(g_1(P_0) + g_2(Q_0)) = \\
&\beta_{25}(g_1(P_0)) + \beta_{25}(g_2(Q_0)) = (P_0, 0) + (0, Q_0) = (P_0, Q_0).
\end{aligned}$$

The above calculation amounts to assert that $\beta_{25} \circ \beta_{33} = \mathrm{Id}_{\mathfrak{g}_X}$.

e. As a consequence, β_{25} is an isomorphism of Lie algebras. Hence, in the category **k-Lie**, we constructed explicitly the isomorphism and its inverse

$$\mathcal{L}_{\mathbf{k}}(X) / \mathcal{J} \underset{\beta_{33}}{\overset{\beta_{25}}{\rightleftarrows}} \mathcal{L}_{\mathbf{k}}(X)_{BZ} \big/ \mathcal{J}^Z_{BZ} \rtimes \mathcal{L}_{\mathbf{k}}(X)_B / \mathcal{J}_B . \tag{2.52}$$

iv) From the above results, one thus derives a commutative diagram of Lie algebras with split short exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}_{\mathbf{k}}(X)_{BZ} \arrow[r, hook, "j"] \arrow[d, "s_{\mathcal{J}^Z_{BZ}}"] & \mathcal{L}_{\mathbf{k}}(X) \arrow[r, "p"] \arrow[d, "s_{\mathcal{J}}"] & \mathcal{L}_{\mathbf{k}}(X)_B \arrow[r] \arrow[d, "s_{\mathcal{J}_B}"] & 0 \\
0 \arrow[r] & \mathcal{L}_{\mathbf{k}}(X)_{BZ} \big/ \mathcal{J}^Z_{BZ} \arrow[r, "{[j]}"] & \mathcal{L}_{\mathbf{k}}(X) / \mathcal{J} \arrow[r, "{[p]}"] & \mathcal{L}_{\mathbf{k}}(X)_B / \mathcal{J}_B \arrow[r] & 0.
\end{tikzcd} \tag{2.53}$$

QED

□

Notation. – An alternative proof of the Theorem above 2.6, using $\mathbb{B}$-gradings can be found in Commentary 5.

2.2.3 Applications.

Elimination of the subalphabet Z.

In certain cases (which is that of the Lie algebras $\mathrm{DK}_{\mathbf{k},n}$), it can happen that the left factor of the semi-direct product (2.28) be isomorphic to $\mathcal{L}_{\mathbf{k}}(Z)$. We start from the commutative diagram (2.29) with an additional arrow

$$\begin{tikzcd} & \mathcal{L}_{\mathbf{k}}(Z) \arrow[d, hook, "j_Z"] & & & \\ 0 \arrow[r] & \mathcal{L}_{\mathbf{k}}(X)_{BZ} \arrow[r, hook, "j"] \arrow[d, "s_{\mathcal{J}^Z_{BZ}}"] & \mathcal{L}_{\mathbf{k}}(X) \arrow[r, "p"] \arrow[d, "s_{\mathcal{J}}"] & \mathcal{L}_{\mathbf{k}}(X)_B \arrow[r] \arrow[d, "s_{\mathcal{J}_B}"] & 0 \\ 0 \arrow[r] & \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ} \arrow[r, "{[j]}"] & \mathcal{L}_{\mathbf{k}}(X) / \mathcal{J} \arrow[r, "{[p]}"] & \mathcal{L}_{\mathbf{k}}(X)_B / \mathcal{J}_B \arrow[r] & 0 \end{tikzcd} \tag{2.54}$$

where j_Z is the subalphabet embedding such that

$$\mathrm{Im}(j_Z) = \mathcal{L}_{\mathbf{k}}(X)_Z = \bigoplus_{\substack{\alpha\in\mathbb{N}^{(X)} \\ |\alpha|_B=0}} \mathcal{L}_{\mathbf{k}}(X)_\alpha. \tag{2.55}$$

We are now in the position to state the following

Proposition 2.9. *With the notations as in Theorem 2.6, let us consider the composite map* $\beta = s_{\mathcal{J}^Z_{BZ}} \circ j_Z$*, then*

a. In order that β *be injective, it is necessary and sufficient that* $\mathcal{J}^Z_{BZ} \cap \mathcal{L}_{\mathbf{k}}(X)_Z = \{0\}$.

b. In order that β *be surjective, it is necessary and sufficient that, for all* $(b,z) \in B\times Z$*, we had*

$$s_{\mathcal{J}^Z_{BZ}}([b,z]) \in s_{\mathcal{J}^Z_{BZ}}(\mathcal{L}_{\mathbf{k}}(X)_Z). \tag{2.56}$$

Proof. a. Firstly, it is clear that the composite $s_{\mathcal{J}^Z_{BZ}} \circ j_Z$ is injective $\iff$ $\mathrm{Ker}(s_{\mathcal{J}^Z_{BZ}} \circ j_Z) = \{0\} \iff \mathcal{J}^Z_{BZ} \cap \mathcal{L}_{\mathbf{k}}(X)_Z = \mathrm{Ker}(s_{\mathcal{J}^Z_{BZ}} \circ j_Z) = \{0\}$.

b. Secondly, we now have to prove that the composite map $s_{\mathcal{J}^Z_{BZ}} \circ j_Z$ is surjective if and only if for all $(b,z) \in B\times Z$, we have

$$s_{\mathcal{J}^Z_{BZ}}([b,z]) \in s_{\mathcal{J}^Z_{BZ}}(\mathcal{L}_{\mathbf{k}}(X)_Z). \tag{2.57}$$

Let us call β the composite map $s_{\mathcal{J}^Z_{BZ}} \circ j_Z$. The proof goes as follows

" $\Rightarrow$ ": Assume that β is surjective. This assumption shows that for any $(b,z) \in B\times Z$, there exits $Q \in \mathcal{L}_{\mathbf{k}}(Z)$ such that $s_{\mathcal{J}^Z_{BZ}}([b,z]) = \beta(Q) = s_{\mathcal{J}^Z_{BZ}}(j_Z(Q))$ and the fact that $j_Z(Q) \in \mathcal{L}_{\mathbf{k}}(X)_Z$ proves the claim.

" $\Leftarrow$ ": We first prove that, for all $(b,Q) \in B\times\mathcal{L}_{\mathbf{k}}(Z)$ we have

$$s_{\mathcal{J}^Z_{BZ}}([b,j_Z(Q)]) \in s_{\mathcal{J}^Z_{BZ}}(\mathcal{L}_{\mathbf{k}}(X)_Z). \tag{2.58}$$

Due to the fact that the homogeneous polynomials linearly generate $\mathcal{L}_{\mathbf{k}}(Z)$ by (1.31) and (1.32), it is sufficient to prove this for homogeneous (Lie) polynomials of degree n for all $n \in \mathbb{N}_{\geq 1}$, we will do it by induction. Let then $Q \in \mathcal{L}_{\mathbf{k}}(Z)_n$.
If $n = 1$ this is the hypothesis.
If $n \geq 2$ (and Q homogeneous), then Q is a finite sum $Q = \sum_{i \in F}[Q_{i1}, Q_{i2}]$ with $Q_{ij} \in \mathcal{L}_{\mathbf{k}}(Z)_{n_{ij}}$ and $n = n_{i1} + n_{i2}$. Now, we have

$$[b, Q] = \sum_{i \in F}[[b, Q_{i1}], Q_{i2}] + [Q_{i1}, [b, Q_{i2}]] \tag{2.59}$$

and then the claim is a consequence of the induction hypothesis[12] and the fact that $s_{\mathcal{J}^Z_{BZ}}(\mathcal{L}_{\mathbf{k}}(X)_Z)$ is a Lie subalgebra.
Now we prove, by induction on $|u|$ ($u \in B^*$) that, for all $Q \in \mathcal{L}_{\mathbf{k}}(Z)$

$$s_{\mathcal{J}^Z_{BZ}}\big(\mathrm{ad}_{(u)}(j_Z(Q))\big) \in s_{\mathcal{J}^Z_{BZ}}(\mathcal{L}_{\mathbf{k}}(X)_Z). \tag{2.60}$$

If $|u| = 0$ this is trivial, otherwise $u = bv$ for $(b, v) \in B \times B^*$. From the induction hypothesis (understated)

$$s_{\mathcal{J}^Z_{BZ}}\big(\mathrm{ad}_{(v)}(j_Z(Q))\big) = s_{\mathcal{J}^Z_{BZ}}(R)$$

for some $R \in \mathcal{L}_{\mathbf{k}}(X)_Z$. But we have

$$\begin{aligned}
&s_{\mathcal{J}^Z_{BZ}}\big(\mathrm{ad}_{(u)}(j_Z(Q))\big) = s_{\mathcal{J}^Z_{BZ}}\big(\mathrm{ad}_{(bv)}(j_Z(Q))\big)\\
&= s_{\mathcal{J}^Z_{BZ}}\big([b, \mathrm{ad}_{(v)}(j_Z(Q))]\big) = s_{\mathcal{J}^Z_{BZ}}\big([b, \mathrm{ad}_{(v)}(j_Z(Q)) - R]\big) + s_{\mathcal{J}^Z_{BZ}}([b, R])\\
&= s_{\mathcal{J}^Z_{BZ}}([b, R]) \in s_{\mathcal{J}^Z_{BZ}}(\mathcal{L}_{\mathbf{k}}(X)_Z) \text{ by (2.58).}
\end{aligned}$$

We just proved (2.60).
Thus, it permits us to verify that formula $\{s_{\mathcal{J}^Z_{BZ}}(\mathrm{ad}_{(u)}(z)) \mid (u, z) \in B^* \times Z\} \subset \mathrm{Im}(s_{\mathcal{J}^Z_{BZ}} \circ j_Z)$ which is obviously the Lie subalgebra of $\mathcal{L}_{\mathbf{k}}(X)_{BZ} \big/ \mathcal{J}^Z_{BZ}$. We remind that $\{s_{\mathcal{J}^Z_{BZ}}(\mathrm{ad}_{(u)}(z)) \mid (u, z) \in B^* \times Z\}$ is a generating set of the Lie algebra $\mathcal{L}_{\mathbf{k}}(X)_{BZ} \big/ \mathcal{J}^Z_{BZ}$, this yields

$$\mathrm{Im}(s_{\mathcal{J}^Z_{BZ}} \circ j_Z) = \mathcal{L}_{\mathbf{k}}(X)_{BZ} \big/ \mathcal{J}^Z_{BZ}. \tag{2.61}$$

The above formula i.e. (2.61) gives a consequence that $s_{\mathcal{J}^Z_{BZ}} \circ j_Z$ is surjective.

□

[12] Understated, but which can be unfolded on request.

About M.-P. Schützenberger's questions on the Partially Commutative Free Lie algebra.

About the Free (partially commutative) Lie algebras, Pr. Schützenberger asked the following questions [99]

1. Is the free partially commutative Lie algebra torsion free (over $\mathbb{Z}$)?
2. If yes (in which case it is linearly free over $\mathbb{Z}$), is it possible to construct combinatorial bases of it?
3. To which extent can it be considered as "free"? (more than "as a module").

Question 3 has been answered in Section 1.3. The two remaining ones can be answered by the following adaptation of §2.2.3.

Theorem 2.10. *Let (X, θ) be an alphabet with commutations and $M(X, \theta)$ be the free partially commutative monoid. We consider a partition of X, $X = B + Z$ such that Z is totally non-commutative i.e. no two letters of Z commute between themselves ($\theta \cap Z^2 = \Delta_Z$). As defined in Section 1.3, s_θ is the canonical surjection $X^* \to M(X, \theta)$. We also consider $j_\theta : M(X, \theta) \to X^*$, an arbitrary set-theoretical section of s_θ. For $t \in M(X, \theta)$, we define the terminal alphabet of it*

$$\mathrm{TAlph}(t) = \{x \in X \mid t \in M(X, \theta).x\} \tag{2.62}$$

as the set of last letters of preimages of t w.r.t. s_θ and the code

$$C_B(Z) = \{s_\theta(uz) | u \in B^*, z \in Z, \mathrm{TAlph}(s_\theta(uz)) = \{z\}\} \subset M(X, \theta) \tag{2.63}$$

Let $C = j_\theta(C_B(Z)) \subset B^*Z$ and j_C be the composite map $\mathcal{L}_{\mathbf{k}}(C) \hookrightarrow \mathcal{L}_{\mathbf{k}}(B^*Z) \xrightarrow{rn} \mathcal{L}_{\mathbf{k}}(X)_{BZ}$ (where rn is the Lie isomorphism as in Diagram (2.14)), we have the diagram

$$\begin{array}{ccccccccc}
 & & \mathcal{L}_{\mathbf{k}}(C) & & & & & & \\
 & & \big\downarrow{\scriptstyle j_C} & & & & & & \\
0 & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X)_{BZ} & \overset{j}{\hookrightarrow} & \mathcal{L}_{\mathbf{k}}(X) & \xrightarrow{p} & \mathcal{L}_{\mathbf{k}}(X)_B & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle s_{\mathcal{J}^Z_{BZ}}} & & \big\downarrow{\scriptstyle s_{\mathcal{J}}} & & \big\downarrow{\scriptstyle s_{\mathcal{J}_B}} & & \\
0 & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X)_{BZ} \big/ \mathcal{J}^Z_{BZ} & \xrightarrow{[j]} & \mathcal{L}_{\mathbf{k}}(X) / \mathcal{J} & \xrightarrow{[p]} & \mathcal{L}_{\mathbf{k}}(X)_B / \mathcal{J}_B & \longrightarrow & 0.
\end{array} \tag{2.64}$$

Then, with the above hypotheses (Z totally non-commutative and $C = j_\theta(C_B(Z))$), $s_{\mathcal{J}_{BZ}^Z} \circ j_C$ is an isomorphism.
In particular, the left factor of the semi-direct product (2.28), here $\mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}_{BZ}^Z$ is a free Lie algebra.

Proof. For the proof that $s_{\mathcal{J}_{BZ}^Z} \circ j_C$ is one-to-one, we will need the three following lemmas.

Lemma 2.11. *Let $\mathfrak{g}$ be a Lie $\mathbf{k}$-algebra, G a set of generators of $\mathfrak{g}$ as a Lie $\mathbf{k}$-algebra and $R \subset \mathfrak{g}$. Then, the ideal $\mathcal{J}(R)$ generated as an ideal by R is linearly (i.e. as a module) generated by the elements*

$$\{\mathrm{ad}_t(h)\}_{\substack{t\in\mathrm{SEQ}(G)\\ h\in R}}. \tag{2.65}$$

Proof of Lemma 2.11 : For M a $\mathbf{k}$-submodule of $\mathfrak{g}$, the set

$$S(M) = \{g \in \mathfrak{g} \mid \mathrm{ad}_g(M) \subset M\} \tag{2.66}$$

is a Lie subalgebra of $\mathfrak{g}$ from the Jacobi relation. Then, let N be the submodule of $\mathfrak{g}$ generated by the elements of (2.65). By the fact that $\{\mathrm{ad}_t(h)\}_{\substack{t\in\mathrm{SEQ}(G)\\ h\in R}} \subset \mathcal{J}(R)$, one has $N \subset \mathcal{J}(R)$. Moreover, we have, by construction, $G \subset S(N)$ hence $\mathfrak{g} \subset S(N)$ which proves that N is an ideal containing R and then $\mathcal{J}(R)$. We obtain the claim. □

Consequence 2.12. *Corresponding to $\mathbf{r}_\theta$, and applying Lemma 2.11 to $\mathfrak{g} = \mathcal{L}_{\mathbf{k}}(B)$, $G = B$, we see that the ideal $\mathcal{J}_B$ is generated (as a submodule) by the elements*

$$\{\mathrm{ad}_{(u)}([b_1, b_2])\}_{\substack{u\in B^*\\ (b_1,b_2)\in\theta_B}} \tag{2.67}$$

then applying Lemma 2.11 to $\mathfrak{g} = \mathcal{L}_{\mathbf{k}}(X)$, $G = X$, we see that the ideal $\mathcal{J}_Z$ is generated (as a submodule) by the elements

$$\{\mathrm{ad}_{(u)}([b, z])\}_{\substack{u\in X^*\\ (b,z)\in\theta\cap(B\times Z)}}. \tag{2.68}$$

Lemma 2.13. *Let $\mathfrak{g}$ be a Lie algebra and $(p_1, \cdots, p_n, q) \in \mathrm{SEQ}(\mathfrak{g})$, then there exists $t = \sum_\sigma c(\sigma)\, \sigma \in \mathbb{Z}[\mathfrak{S}_n]$ such that*

$$[\mathrm{ad}_{(p_1,\cdots,p_{n-1})}(p_n), q] = \sum_{\sigma\in\mathfrak{S}_n} c(\sigma)\ \mathrm{ad}_{(p_{\sigma(1)},\cdots,p_{\sigma(n)})}(q). \tag{2.69}$$

Proof of Lemma 2.13 : Let $Y = \{y_1, \cdots, y_n, z\}$ be an auxiliary alphabet of $n+1$ letters and φ be the morphism $\mathcal{L}_{\mathbf{k}}(Y) \to \mathfrak{g}$ defined by $\varphi(y_i) = p_i$ and $\varphi(z) = q$. Then, using the partition $Y = \{y_1, \cdots, y_n\} + \{z\} =: B + Z$, we see that $Q = [\mathrm{ad}_{(y_1,\cdots,y_{n-1})}(y_n), z] \in \mathcal{L}_{\mathbf{k}}(Y)_{BZ}$. Then

$$[\mathrm{ad}_{(y_1,\cdots,y_{n-1})}(y_n), z] = \sum_{u\in\{y_1,\cdots,y_n\}^*} c(u)\ \mathrm{ad}_{(u)}(z). \tag{2.70}$$

But Q is of multidegree $(1, \cdots, 1, 1) \in \mathbb{N}^{(Y)}$ and then, each u in the support of the decomposition (2.70) can be written $u = y_{\sigma(1)} \cdots y_{\sigma(n)}$ for some permutation $\sigma \in \mathfrak{S}_n$. Hence

$$[\mathrm{ad}_{(y_1,\cdots,y_{n-1})}(y_n), z] = \sum_{\sigma\in\mathfrak{S}_n} c(\sigma)\ \mathrm{ad}_{(y_{\sigma(1)},\cdots,y_{\sigma(n)})}(z). \tag{2.71}$$

Now, with the notations of Lemma 2.13,

$$\begin{aligned}
&[\mathrm{ad}_{(p_1,\cdots,p_{n-1})}(p_n), q] = \varphi([\mathrm{ad}_{(y_1,\cdots,y_{n-1})}(y_n), z]) \\
&= \sum_{\sigma\in\mathfrak{S}_n} c(\sigma)\, \varphi(\mathrm{ad}_{(y_{\sigma(1)},\cdots,y_{\sigma(n)})}(z)) \\
&= \sum_{\sigma\in\mathfrak{S}_n} c(\sigma)\ \mathrm{ad}_{(p_{\sigma(1)},\cdots,p_{\sigma(n)})}(q).
\end{aligned} \tag{2.72}$$

□

Consequence 2.14. *Still with* $\mathbf{r} = \mathbf{r}_\theta$ *and in view of §2.2.2 we have the generator* $\mathbf{r}_{BZ} := \{\mathrm{ad}_Q\, z\}_{Q\in\mathcal{J}_B, z\in Z}$. *We know, from Consequence, 2.12 that* $\mathcal{J}_B$ *is generated (as a module) by the elements* $\{\mathrm{ad}_{(u)}([b_1, b_2])\}_{\substack{u\in B^* \\ (b_1,b_2)\in\theta_B}}$. *Then, here* $\mathcal{J}_{BZ}$ *is generated by the elements*

$$\{[\mathrm{ad}_{(u)}([b_1, b_2]), z]\}_{\substack{u\in B^* \\ (b_1,b_2)\in\theta_B,\ z\in Z}} \tag{2.73}$$

and then, from Lemma 2.13 and tracking the position of $[b_1, b_2]$, $\mathcal{J}_{BZ}$ *is generated by elements*

$$\{\mathrm{ad}_{(u),[b_1,b_2],(v)}(z)\}_{\substack{u,v\in B^* \\ (b_1,b_2)\in\theta_B,\ z\in Z}} \tag{2.74}$$

(where, recalling that in the part "End of the proof of Lemma 2.7", for any $Q \in \mathcal{J}_B$ *and* $(v, z) \in B^* \times Z$, *from equation* (2.36) *of Lemma 2.8, we have shown that* $\mathrm{ad}_{(Q),(v)}(z) \in \mathcal{J}_{BZ}$. *Thus, for* $Q = [b_1, b_2]$, *we can write* $\{\mathrm{ad}_{(u),[b_1,b_2],(v)}(z)\}_{\substack{u,v\in B^* \\ (b_1,b_2)\in\theta_B,\ z\in Z}} \subset \mathcal{J}_{BZ}$*).*

The last lemma characterizes the kernel of a morphism which performs identifications and annihilation of letters

Lemma 2.15. *Let Y_1, Y_2 be alphabets and $\varphi_0 : Y_1 \to Y_2 \cup \{0\} \subset \mathcal{L}_{\mathbf{k}}(Y_2)$ with $Y_2 \subset \varphi_0(Y_1)$. Let φ_1, be the morphism $\mathcal{L}_{\mathbf{k}}(Y_1) \to \mathcal{L}_{\mathbf{k}}(Y_2)$ constructed from φ_0 by the mechanism* (1.30)*. Then the kernel of φ_1 is the ideal generated by the elements*

$$\mathbf{r}_{eq} = \{x - x'\}_{\varphi_0(x)=\varphi_0(x')\in Y_2}\ ;\ \mathbf{r}_{nil} = \{x\}_{\varphi_0(x)=0}.$$

Proof of Lemma 2.15 : Still with $\mathbf{r}_{eq}$ and $\mathbf{r}_{nil}$ defined as above, call R the relator[13] i.e. $R = \mathbf{r}_{eq} \cup \mathbf{r}_{nil}$. It is easily checked that $R \subset \mathrm{Ker}(\varphi_1)$ and then $\mathcal{J}(R) \subset \mathrm{Ker}(\varphi_1)$ ($\mathcal{J}(R)$ being the ideal generaled by R). We then have a morphism

$$\varphi_3 : \ \mathcal{L}_{\mathbf{k}}(Y_1) \Big/ \mathcal{J}(R) \to \mathcal{L}_{\mathbf{k}}(Y_2). \tag{2.75}$$

It is surjective due to the condition $Y_2 \subset \varphi_0(Y_1)$. Now let us consider the following candidate to be an inverse of φ_3. Remarking that, for $y_2 \in Y_2$ and y_1 a preimage of y_2 ($\varphi_1(y_1) = y_2$), the class $y_1 + \mathcal{J}(R)$ is *independent from the choice of* y_1. We define $\varphi_4 :\ \mathcal{L}_{\mathbf{k}}(Y_2) \to \mathcal{L}_{\mathbf{k}}(Y_1) \Big/ \mathcal{J}(R)$ by $\varphi_4(y_2) := y_1 + \mathcal{J}(R)$. A routine check shows that $\varphi_3 \circ \varphi_4 = \mathrm{Id}_{\mathrm{Dom}(\varphi_4)}$ and $\varphi_4 \circ \varphi_3 = \mathrm{Id}_{\mathrm{Dom}(\varphi_3)}$. □

End of the proof of Theorem 2.10. –

We now consider the following composition

$$\mathcal{L}_{\mathbf{k}}(C) \xhookrightarrow{j_C} \mathcal{L}_{\mathbf{k}}(X)_{BZ} \xrightarrow{s_{\mathcal{J}^Z_{BZ}}} \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}^Z_{BZ} \tag{2.76}$$

and, as $\alpha_{31} := rn :\ \mathcal{L}_{\mathbf{k}}(B^*Z) \to \mathcal{L}_{\mathbf{k}}(X)$ (see (2.14)) is injective with image $\mathcal{L}_{\mathbf{k}}(X)_{BZ}$, we also note $\alpha_{31} = rn :\ \mathcal{L}_{\mathbf{k}}(B^*Z) \to \mathcal{L}_{\mathbf{k}}(X)_{BZ}$ the corresponding isomorphism and $\beta_{13} :\ \mathcal{L}_{\mathbf{k}}(X)_{BZ} \to \mathcal{L}_{\mathbf{k}}(B^*Z)$ its inverse.

We define $\varphi :\ \mathcal{L}_{\mathbf{k}}(B^*Z) \to \mathcal{L}_{\mathbf{k}}(C)$ of the type considered in Lemma 2.15 by $\varphi :\ B^*Z \to C \cup \{0\}$ as follows

$$\varphi(uz) = j_\theta \circ s_\theta(uz) \text{ if } \mathrm{TAlph}(s_\theta(uz)) = \{z\} \text{ and } \varphi(uz) = 0 \text{ otherwise.} \tag{2.77}$$

[13]As a type, R is a mere subset of $\mathcal{L}_{\mathbf{k}}(Y_1)$ but "relator" means that it is intended to be the generating set of an ideal.

Let us prove that $\mathrm{Ker}(\varphi\circ\beta_{13}) = \alpha_{31}(\mathrm{Ker}(\varphi)) \supset \mathcal{J}_Z + \mathcal{J}_{BZ} = \mathcal{J}_{BZ}^Z$. In fact, from (2.74), we know that $\mathcal{J}_{BZ}$ is generated by elements

$$\{\mathrm{ad}_{(u),[b_1,b_2],(v)}(z)\}_{\substack{u,v\in B^*\\ (b_1,b_2)\in\theta_B,\ z\in Z}}.$$

Set $Q = \mathrm{ad}_{(u),[b_1,b_2],(v)}(z)$ and remark that from the Jacobi identity one has

$$\begin{aligned} Q &= \mathrm{ad}_{(u),b_1,b_2,(v)}(z) - \mathrm{ad}_{(u),b_2,b_1,(v)}(z) = \alpha_{31}(ub_1b_2vz) - \alpha_{31}(ub_2b_1vz)\\ &= \alpha_{31}(ub_1b_2vz - ub_2b_1vz). \end{aligned} \tag{2.78}$$

But $ub_1b_2vz - ub_2b_1vz \in \mathrm{Ker}(\varphi)$ because b_1 and b_2 commute in $M(X,\theta)$, hence one has $\alpha_{31}(\mathrm{Ker}(\varphi)) \supset \mathcal{J}_{BZ}$. Now, from (2.68), we know that $\mathcal{J}_Z$ is generated by the elements

$$\{\mathrm{ad}_{(u)}([b,z])\}_{\substack{u\in X^*\\ (b,z)\in\theta\cap(B\times Z)}}.$$

Set $Q = \mathrm{ad}_{(u)}([b,z])$ and remark that

$$Q = \mathrm{ad}_{(u)}([b,z]) = \mathrm{ad}_{(ub)}(z) - \mathrm{ad}_{(uz)}(b) = \alpha_{31}(ubz - uzb) \tag{2.79}$$

but, again, $\varphi(ubz) = \varphi(uzb)$ because b and z commute in $M(X,\theta)$. Then we arrive at $\alpha_{31}(\mathrm{Ker}(\varphi))) \supset \mathcal{J}_Z$.

And now, we can construct the factorization to quotient of $\varphi\circ\beta_{13}$ as follows

$$\begin{tikzcd}
\mathcal{L}_{\mathbf{k}}(C) \arrow[d, "j_C"] & & \mathcal{L}_{\mathbf{k}}(X)_{BZ} \big/ \mathcal{J}_{BZ}^Z \arrow[ll, "\alpha"'] \\
\mathcal{L}_{\mathbf{k}}(X)_{BZ} \arrow[r, "\beta_{13}"'] \arrow[d, "s_{\mathcal{J}_{BZ}^Z}"] & \mathcal{L}_{\mathbf{k}}(B^*Z) \arrow[ul, "\varphi"] \arrow[r, shift right, "\alpha_{31}"'] & \mathcal{L}_{\mathbf{k}}(X)_{BZ} \arrow[l, shift right, "\beta_{13}"'] \arrow[u, "s_{\mathcal{J}_{BZ}^Z}"'] \\
\mathcal{L}_{\mathbf{k}}(X)_{BZ} \big/ \mathcal{J}_{BZ}^Z . & &
\end{tikzcd}$$

Figure 2.7: Diagram of the arrows involved in the proof of Theorem 2.10 (beware this diagram is not commutative in general).

Let us show that $s_{\mathcal{J}_{BZ}^Z}\circ j_C$ and α are mutually inverse and, firstly remark that, for $uz \in B^*Z$, $j_\theta s_\theta(uz)$ is the unique representative of uz within C. Then, for $uz \in C$, we have $\beta_{13}j_C(uz) = uz$. This shows that, for $uz \in C$, we have $\varphi\beta_{13}j_C(uz) = \varphi(uz) = j_\theta s_\theta(uz) = uz$ and then $\varphi\circ\beta_{13}\circ j_C = \mathrm{Id}_{\mathcal{L}_{\mathbf{k}}(C)}$. From this, we get

$$\alpha\circ s_{\mathcal{J}_{BZ}^Z}\circ j_C = \varphi\circ\beta_{13}\circ j_C = \mathrm{Id}_{\mathcal{L}_{\mathbf{k}}(C)}\,. \tag{2.80}$$

In particular α is onto and $s_{\mathcal{J}_{BZ}^Z} \circ j_C$ is into. Let us prove (which is sufficient) that $s_{\mathcal{J}_{BZ}^Z} \circ j_C$ is onto. Our strategy is to show that $\mathrm{Im}(s_{\mathcal{J}_{BZ}^Z} \circ j_C)$ contains $s_{\mathcal{J}_{BZ}^Z}\alpha_{31}(B^*Z)$.

Let us show that, if $uz \in B^*Z$ and $\mathrm{TAlph}(s_\theta(uz)) \neq \{z\}$, we have $s_{\mathcal{J}_{BZ}^Z}\alpha_{31}(uz) = 0$. Indeed, we claim that, for any $uz = b_1 \cdots b_k z \in B^*Z$ such that $(b_1, z) \in \theta$ and $(b_1, b_j) \in \theta$ for all $j \in [2, k]$, then $\alpha_{31}(uz) \in \mathcal{J}_{BZ}^Z$. Indeed, if $|u| = 1$ then $\alpha_{31}(uz) = [b_1, z] \in \mathcal{J}_Z \subseteq \mathcal{J}_{BZ}^Z$ since $(b_1, z) \in \theta \cap (B \times Z)$. If $|u| = 2$ i.e. $u = b_1 b_2 \in B^*$, $(b_1, z) \in \theta$ and $(b_1, b_2) \in \theta$, then $\alpha_{31}(uz) = [b_1, [b_2, z]] = [b_2, [b_1, z]] + [z, [b_2, b_1]] = \mathrm{ad}_{b_2}([b_1, z]) - \mathrm{ad}_{[b_2,b_1]}(z) \in \mathcal{J}_{BZ}^Z$ since $\mathrm{ad}_{b_2}([b_1, z]) \in \mathcal{J}_Z$ and $\mathrm{ad}_{[b_2,b_1]}(z) \in \mathcal{J}_{BZ}$. If $|u| = k \geq 3$ i.e. $u = b_1 b_2 \cdots b_k \in B^*$, it is obtained from the induction hypothesis and by the formula

$$\begin{aligned} \alpha_{31}(uz) &= \mathrm{ad}_{(b_1 b_2 \cdots b_k)}(z) = [b_1, [b_2, \mathrm{ad}_{(b_3 \cdots b_k)}(z)]] \\ &= \mathrm{ad}_{b_2}([b_1, \mathrm{ad}_{(b_3 \cdots b_k)}(z)]) - \mathrm{ad}_{[b_2,b_1]}(\mathrm{ad}_{(b_3 \cdots b_k)}(z)) \\ &= \mathrm{ad}_{b_2}(\alpha_{31}(b_1 b_3 \cdots b_k z)) - \mathrm{ad}_{[b_2,b_1]}(\mathrm{ad}_{(b_3 \cdots b_k)}(z)). \end{aligned}$$

Now, if $uz = b_1 \cdots b_k z \in B^*Z$ and $\mathrm{TAlph}(s_\theta(uz)) \neq \{z\}$, by Remark 1.8 there exists $i \in [1, k]$ so that $(b_i, z) \in \theta$ and $(b_i, b_j) \in \theta$ for all $j \in [i+1, k]$. Thus, $s_{\mathcal{J}_{BZ}^Z}\alpha_{31}(uz) = 0$ because $\alpha_{31}(uz) = \mathrm{ad}_{(b_1 \cdots b_{i-1})}(\alpha_{31}(b_i \cdots b_k z)) \in \mathcal{J}_{BZ}^Z$ (here $\alpha_{31}(b_i \cdots b_k z) \in \mathcal{J}_{BZ}^Z$ by the claim above).

Otherwise, if $\mathrm{TAlph}(s_\theta(uz)) = \{z\}$, we have $j_\theta s_\theta(uz) \in C$ and

$$\alpha_{31}(uz - j_\theta s_\theta(uz)) \in \mathcal{J}_{BZ}$$

because $\mathcal{J}_{BZ}$ contains all the commutations of θ_B, then

$$\begin{aligned} & s_{\mathcal{J}_{BZ}^Z}\alpha_{31}(uz) = s_{\mathcal{J}_{BZ}^Z}\alpha_{31}(uz - j_\theta s_\theta(uz) + j_\theta s_\theta(uz)) \\ & = s_{\mathcal{J}_{BZ}^Z}\alpha_{31}(j_\theta s_\theta(uz)) \in s_{\mathcal{J}_{BZ}^Z}\alpha_{31}(C) = s_{\mathcal{J}_{BZ}^Z} j_C(C). \end{aligned} \tag{2.81}$$

All in all $s_{\mathcal{J}_{BZ}^Z}\alpha_{31}(B^*Z) \subset s_{\mathcal{J}_{BZ}^Z} \circ j_C(\mathcal{L}_{\mathbf{k}}(C))$. Thus, $s_{\mathcal{J}_{BZ}^Z} \circ j_C$ and α are mutually inverse. We proved our theorem.

QED

$\square$

It would be interesting to have alternative proofs for answers to Schützenberger's questions about the Partially Commutative Free Lie algebra (cf. Duchamp and Krob [41] Thm III.3) as a consequence of our main theorem (Theorem 2.6).

Corollary 2.16. *(Lazard's Partially Commutative Elimination) Let X be a set equipped with a commutation relation θ and B be a subset of X such that $Z = X - B$ is totally non-commutative. Then there is an isomorphism from the free partially commutative Lie algebra $\mathcal{L}_{\mathbf{k}}(X, \theta)$ to the semi-direct of product of Lie algebras, namely*

$$\mathcal{L}_{\mathbf{k}}(X, \theta) \cong \mathcal{L}_{\mathbf{k}}(C) \rtimes \mathcal{L}_{\mathbf{k}}(B, \theta_B) \text{ in } \mathbf{k}\text{-}\boldsymbol{Lie}. \tag{2.82}$$

Proof. A partial result of our main theorem [Theorem 2.6 (iii)] deals the following decomposition

$$\mathcal{L}_{\mathbf{k}}(X, \theta) \cong_{\mathbf{k}\text{-}\mathbf{Lie}} \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}_{BZ}^{Z} \rtimes \mathcal{L}_{\mathbf{k}}(B, \theta_B).$$

Therefore, in the category **k-Lie**, the formula $\mathcal{L}_{\mathbf{k}}(C) \cong \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ \mathcal{J}_{BZ}^{Z}$ is immediately seen from the fact that the composite $s_{\mathcal{J}_{BZ}^{Z}} \circ j_C$ is isomorphism of Lie algebras by our result in Theorem 2.10. We thus obtain the corollary. □

Knizhnik-Zamolodchikov equation, Drinfeld-Kohno Lie algebra and its decomposition.

In this part, we give a sample of the application of our main theorem (Theorem 2.6) and by Proposition 2.9 to the decomposition of Drinfeld-Kohno Lie algebras. We remind the reader that the decomposition has deep relations with the study of special solutions of Knizhnik-Zamolodchikov equations (KZ_n, for $n = 3, 4$) by polylogarithms and hyperlogarithms (cf. Drinfeld [30, 31], Cartier [20], Brown [16], Oi and Ueno [90]). For $n \geq 2$, assume that $\mathcal{T}_n = \{t_{i,j}\}_{1 \leq i < j \leq n}$ is a set of $\binom{n}{2}$ endomorphisms $t_{i,j}$ of W, where W is a finite-dimensional vector space over $\mathbb{C}$. We consider the *Knizhnik-Zamolodchikov (KZ) equation* (cf. Knizhnik and Zamolodchikov [69], Drinfeld [30, 31], Cartier [19], Kassel [67])

$$(\mathrm{KZ}_n) \quad \mathbf{d}\mathrm{F}(z) = \Omega_n(z)\mathrm{F}(z) \tag{2.83}$$

defined over the complex configuration space

$$\mathbb{C}_*^n = \{z = (z_1, \cdots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ for } i \neq j\},$$

where the system (so-called the *KZ connection* of 1-forms)

$$\Omega_n(z) = \sum_{1\le i<j\le n} \frac{t_{i,j}}{2\mathrm{i}\pi} d\log(z_i - z_j) \tag{2.84}$$

and $\mathrm{F} = \mathrm{F}(z)$ is a function defined on an open subset of $\mathbb{C}_*^n$ with values in the complex space $\mathrm{End}_{\mathbb{C}}(W)$.

As a consequence of a classical integrability criterium (cf. Drinfeld [30], Cartier [19], Kohno [71]), the system (2.84) is completely integrable if and only if $\mathbf{d}\Omega_n - \Omega_n \wedge \Omega_n = 0$, and this imposes relations between the endomorphisms $t_{i,j}$. These relations are precisely *the infinitesimal pure braid relations.* Considering now $t_{i,j}$ as abstract variables (or generators) and no longer endomorphisms, we repeat these relations here i.e. $\mathcal{T}_n = \{t_{i,j}\}_{1\le i<j\le n}$ satisfy the following infinitesimal pure braid relations

$$\mathbf{R[n]} = \begin{cases} \mathbf{R_1[n]} & [t_{i,j}, t_{i,k} + t_{j,k}] \qquad \text{for } 1 \le i < j < k \le n, \\ \mathbf{R_2[n]} & [t_{i,j} + t_{i,k}, t_{j,k}] \qquad \text{for } 1 \le i < j < k \le n, \\ \mathbf{R_3[n]} & [t_{i,j}, t_{k,l}] \quad \text{for } \begin{matrix} 1 \le i < j \le n, \\ 1 \le k < l \le n, \end{matrix} \text{ and } |\{i,j,k,l\}| = 4. \end{cases} \tag{2.85}$$

Remark 2.5. It is remarkable that these relations be the same as those obtained by the functor $gr_{\mathbb{Z}}(-)$ (see (2.6)).

Example 2.2. For $n = 3$, by Drinfeld transformation (see [30, 31, 67]), the KZ equation (2.83) has a solution of the form

$$\mathrm{F}(z) = (z_3 - z_1)^{\frac{1}{2\mathrm{i}\pi}(t_{1,2}+t_{1,3}+t_{2,3})} S\left(\frac{z_2 - z_1}{z_3 - z_1}\right) \tag{2.86}$$

where $S(z)$ satisfies the first order differential equation with three regular singular points at $0, 1$ and ∞

$$\frac{d}{dz} S(z) = \frac{1}{2\mathrm{i}\pi} \left(\frac{t_{1,2}}{z} + \frac{t_{2,3}}{z-1} \right) S(z).$$

By setting $x_0 := \frac{t_{1,2}}{2\mathrm{i}\pi}$ and $x_1 := -\frac{t_{2,3}}{2\mathrm{i}\pi}$, we then transform $X := \{x_0, x_1\}$ as an object in **Set**, this arises to consider the first order noncommutative differential equation (see (3.38))

$$\begin{cases} \mathbf{d}(S) = (\omega_0(z)x_0 + \omega_1(z)x_1)S, & (NCDE) \\ \lim\limits_{z\in\Omega, z\to 0} S(z)e^{-x_0\log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle}, & \text{asymptotic initial condition,} \end{cases}$$

where, $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$ are two differential forms on the complement of the union of the real half-lines $]-\infty, 0]$ and $[1, +\infty[$ in the complex plane i.e. the simply-connected domain $\Omega = \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$ and for any series $S \in \mathcal{H}(\Omega)\langle\langle X \rangle\rangle$ over $\mathcal{H}(\Omega)$ the algebra (for the pointwise product) of complex-valued functions which are holomorphic on Ω and $\mathbf{d}$ stands for the term by term derivation $\mathbf{d}(S) = \sum_{w \in X^*} \frac{d}{dz}(\langle S \mid w \rangle)w$.

Assume that $\mathbf{k}$ is a commutative ring with unit. We also transform the set of endomorphisms $\mathcal{T}_n$ as an object in **Set**. The *Drinfeld-Kohno Lie algebra*[14] $\mathrm{DK}_{\mathbf{k},n}$ is then presented as

$$\mathrm{DK}_{\mathbf{k},n} = \langle\, \mathcal{T}_n \mid \mathbf{R}[\mathbf{n}]\,\rangle_{\mathbf{k}\text{-}\mathbf{Lie}} = \mathcal{L}_{\mathbf{k}}(\mathcal{T}_n) \Big/ \mathcal{J}_{\mathbf{R}[\mathbf{n}]} \tag{2.87}$$

where $\mathcal{J}_{\mathbf{R}[\mathbf{n}]}$ is the Lie ideal of the free Lie algebra $\mathcal{L}_{\mathbf{k}}(\mathcal{T}_n)$ generated by $\mathbf{R}[\mathbf{n}]$ (2.6).

By using the Knizhnik-Zamolodchikov equations, Kohno proved in [70] that $\mathrm{DK}_{\mathbb{Z},n}$ can be identified with $gr_{\mathbb{Z}}(\mathcal{PB}_n)$ the graded Lie algebra of the pure braid group $\mathcal{PB}_n$. The Drinfeld-Kohno Lie algebra $\mathrm{DK}_{\mathbf{k},n} \cong \mathrm{DK}_{\mathbb{Z},n} \otimes_{\mathbb{Z}} \mathbf{k}$ is also called the *Lie algebra of infinitesimal braids.*

Corollary 2.17. *(Decomposition of Drinfeld-Kohno Lie algebra, cf. Etingof et al. [44]) Given $\mathbf{k}$ a commutative ring with unit and $n \geq 0$, there is an isomorphism of Lie algebras from Drinfeld-Kohno Lie algebra to the semi-direct product of Lie algebras*

$$\mathrm{DK}_{\mathbf{k},n+1} \cong \mathcal{L}_{\mathbf{k}}(X_n) \rtimes \mathrm{DK}_{\mathbf{k},n}, \tag{2.88}$$

where X_n is any alphabet of cardinality n.

Proof. Recall in Example 2.1, we decompose $\mathcal{T}_{n+1} = \mathcal{T}_n + T_{n+1}$ and consider the infinitesimal pure braid relator $\mathbf{r} = \mathbf{R}[\mathbf{n+1}] \subset \mathcal{L}_{\mathbf{k}}(\mathcal{T}_{n+1})$. We then can easily check through direct calculation that $\mathbf{r}$ is a good relator satisfying all hypotheses in Theorem 2.6 and Proposition 2.9. This provides us with effective tools to derive the existence of the decomposition of Drinfeld-Kohno Lie algebra, namely, there is a Lie isomorphism $\mathrm{DK}_{\mathbf{k},n+1} \cong \mathcal{L}_{\mathbf{k}}(X_n) \rtimes \mathrm{DK}_{\mathbf{k},n}$, where we identified T_{n+1} with the alphabet X_n. □

[14] It was originally invented and named such in Kohno [70] and Etingof et al. [44] respectively.

Remark 2.6. The above method gives, as a consequence, that the Drinfeld-Kohno Lie algebra is an iterated semi-direct product of free Lie algebras, more precisely

$$\mathrm{DK}_{\mathbf{k},n+1} \cong \mathcal{L}_{\mathbf{k}}(X_n) \rtimes (\mathcal{L}_{\mathbf{k}}(X_{n-1}) \rtimes (\cdots \rtimes \mathcal{L}_{\mathbf{k}}(X_1))\cdots)$$

in the category **k-Lie** (see Etingof et al. [44] §3.10). We also remark that, in [25] Cor 4.4, the authors used Gröbner-Shirshov bases for the Drinfeld-Kohno Lie algebra to also show that $\mathrm{DK}_{\mathbf{k},n+1}$ is an iterated semi-direct product of free Lie algebras (the reader can study an exposition[15] of the theory of Gröbner-Shirshov bases for associative algebras, Lie algebras, groups, semigroups, Ω-algebras, operads, etc. in the survey of L. A. Bokut and Y. Chen [7]).

We will return to this point with strange and generalized gradings in Subsection 2.3.3.

2.3 Lazard elimination as a free object.

In this section, we investigate more categorical frameworks for Lazard's elimination in **k-Lie**. In the first subsection, we introduce the category of Short Exact Sequences with Section (SESS) in **k-Lie**. In the last subsection, we will study the category of $\mathbb{B}$-graded Lie algebras, where $\mathbb{B} = (\{0,1\}, \vee)$ is the Boolean semigroup, which can be proved to be equivalent to the previous category in Proposition 2.18. Finally, we will perform Lazard's elimination as a free functor from the category of double sets to the category of $\mathbb{B}$-graded Lie algebras.

2.3.1 Category of SESS in Lie algebras.

Assume that we have a SES of Lie algebras

$$0 \longrightarrow \mathfrak{g}^{\mathfrak{l}} \xrightarrow{j} \mathfrak{g} \xrightarrow{p} \mathfrak{g}^{\mathfrak{r}} \longrightarrow 0, \tag{2.89}$$

(we also say that $\mathfrak{g}$ is an extension of $\mathfrak{g}^{\mathfrak{r}}$ by $\mathfrak{g}^{\mathfrak{l}}$). The extension of Lie algebras is said to split if SES (2.89) is split i.e. there is a Lie algebra homomorphism $\sigma : \mathfrak{g}^{\mathfrak{r}} \to \mathfrak{g}$ such that $p \circ \sigma = \mathrm{Id}_{\mathfrak{g}^{\mathfrak{r}}}$ (σ is called a section of p. In this case (2.89) can be pictured as below

[15] Where the semi-direct decomposition of $\mathrm{DK}_{\mathbf{k},n+1}$ is also established.

(see (2.92) and look there for a formal definition of the category of SSES) will be called "Short Exact Sequence with Section (SESS)"). Thus, if the extension of Lie algebras is split, then there is a Lie ideal $\mathfrak{h} := \mathrm{Ker}(p)$ and a Lie subalgebra $\mathfrak{b} := \mathrm{Im}(\sigma)$ such that $\mathfrak{g}$ can be uniquely decomposed as an internal direct sum of submodules $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b}$. Then p (resp. σ) induces a Lie isomorphism $\mathfrak{h} \cong \mathfrak{g}^{\mathfrak{l}}$ (resp. $\mathfrak{b} \cong \mathfrak{g}^{\mathfrak{r}}$) and then a module isomorphism $\mathfrak{g} \cong \mathfrak{g}^{\mathfrak{l}} \oplus \mathfrak{g}^{\mathfrak{r}}$ (external direct sum of modules). Moreover, one clearly defines an action of the Lie algebra $\mathfrak{b}$ (resp.$\mathfrak{g}^{\mathfrak{r}}$) on $\mathfrak{h}$ (resp. $\mathfrak{g}^{\mathfrak{l}}$) by derivations. In other words, at the level of elements $\mathfrak{b}$ (resp.$\mathfrak{g}^{\mathfrak{r}}$) acts on $\mathfrak{h}$ by $D_b(h) = [\sigma(b), h]$ (resp.internal brackets i.e. the adjoint representation).

In this vein, we have the following

Remark 2.7. i) In general, any SES (2.89) is not necessarily split. For example, the Heisenberg Lie algebra (see Blasiak et al. [6]), denoted by $\mathcal{L}_{\mathcal{H}}$, is presented as

$$\mathcal{L}_{\mathcal{H}} = \langle\, a^\dagger, a, e \,|\, [a, a^\dagger] = e, [a^\dagger, e] = [a, e] = 0 \,\rangle_{\textbf{k-Lie}}. \tag{2.90}$$

If we consider the one-dimensional Lie algebra

$$\mathcal{L}_1 = \langle\, z \,|\, \emptyset \,\rangle_{\textbf{k-Lie}}$$

and the two-dimensional abelian Lie algebra

$$\mathcal{L}_2 = \langle\, x, y \,|\, [x, y] = 0 \,\rangle_{\textbf{k-Lie}},$$

the reader can verify that $0 \longrightarrow \mathcal{L}_1 \xrightarrow{j} \mathcal{L}_{\mathcal{H}} \xrightarrow{p} \mathcal{L}_2 \longrightarrow 0$ is a SES of Lie algebras, but not split, where $j : \mathcal{L}_1 \to \mathcal{L}_{\mathcal{H}}, z \mapsto e$ is an injective Lie homomorphism and $p : \mathcal{L}_{\mathcal{H}} \to \mathcal{L}_2, a^\dagger \mapsto x, a \mapsto y, e \mapsto 0$ is a surjective Lie homomorphism (but there is no existence of a section s of p as, otherwise, this central extension would be commutative, see Lemma 5.1).

ii) With the semi-direct construction (see Definition 2.1), we give a SES of Lie algebras

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b} \longrightarrow \mathfrak{b} \longrightarrow 0 \tag{2.91}$$

but it is better than a simple extension because this SES is split and the Lie algebra $\mathfrak{g}$ is a semi-direct product of $\mathfrak{b}$ with $\mathfrak{h}$, denoted by $\mathfrak{g} := \mathfrak{h} \rtimes \mathfrak{b}$.

As a consequence of the remark above, if SES (2.89) is split then $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b}$ and $\mathfrak{g} \simeq \mathfrak{g}^{\mathfrak{r}} \ltimes \mathfrak{g}^{\mathfrak{l}}$ in **k-Lie**.

Let us define the category of SESS in **k-Lie**, denoted by **k-SSLie**, as follows

- Objects: an object is a SESS of Lie algebras

$$
\begin{tikzcd} 0 \arrow[r] & \mathfrak{g}^{\mathfrak{l}} \arrow[r, "j"] & \mathfrak{g} \arrow[r, shift left, "p"] & \mathfrak{g}^{\mathfrak{r}} \arrow[l, shift left, "\sigma"] \arrow[r] & 0 \end{tikzcd} \tag{2.92}
$$

- Morphisms: a morphism between two objects

$$
\begin{tikzcd} 0 \arrow[r] & \mathfrak{g}_1^l \arrow[r, "j_1"] & \mathfrak{g}_1 \arrow[r, shift left, "p_1"] & \mathfrak{g}_1^r \arrow[l, shift left, "\sigma_1"] \arrow[r] & 0 \end{tikzcd}
$$

and

$$
\begin{tikzcd} 0 \arrow[r] & \mathfrak{g}_2^l \arrow[r, "j_2"] & \mathfrak{g}_2 \arrow[r, shift left, "p_2"] & \mathfrak{g}_2^r \arrow[l, shift left, "\sigma_2"] \arrow[r] & 0 \end{tikzcd}
$$

is a commutative diagram in **k-Lie** with SESS arrows

$$
\begin{tikzcd} 0 \arrow[r] & \mathfrak{g}_1^l \arrow[r, "j_1"] \arrow[d, "\varphi^l"] & \mathfrak{g}_1 \arrow[r, shift left, dotted, "p_1"] \arrow[d, "\varphi"] & \mathfrak{g}_1^r \arrow[l, shift left, "\sigma_1"] \arrow[r] \arrow[d, "\varphi^r"] & 0 \\ 0 \arrow[r] & \mathfrak{g}_2^l \arrow[r, "j_2"] & \mathfrak{g}_2 \arrow[r, shift left, dotted, "p_2"] & \mathfrak{g}_2^r \arrow[l, shift left, "\sigma_2"] \arrow[r] & 0. \end{tikzcd} \tag{2.93}
$$

2.3.2 An equivalence of categories and a Lazard elimination functor.

Given a set S, an object X in **Set** is said to be S-graded if it can be written as $X = \bigsqcup_{s\in S} X_s$ a disjoint union structure (i.e. the coproduct) of a family of subsets $\{X_s\}_{s\in S}$. Furthermore, if S is an additive commutative semigroup, an object M (resp. $\mathfrak{g}, \mathcal{A}$) in one of categories

$$\textbf{Mon} \text{ or } \textbf{Grp} \text{ (resp. } \textbf{k-Lie}, \textbf{k-AAU})$$

is said to be S-graded if it can be written as $M = \biguplus_{s\in S} M_s$ a disjoint union of a family of subsets $\{M_s\}_{s\in S}$ (resp. $\mathfrak{g} = \bigoplus_{s\in S} \mathfrak{g}_s$ an internal direct sum of a family of **k**-submodules $\{\mathfrak{g}_s\}_{s\in S}$, $\mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s$ an internal direct sum of a family of **k**-submodules $\{\mathcal{A}_s\}_{s\in S}$) such that the binary operation respects this gradation $M_s.M_t \subseteq M_{s+t}$ (resp.

the Lie bracket respects this gradation $[\mathfrak{g}_s, \mathfrak{g}_t] \subseteq \mathfrak{g}_{s+t}$, the multiplication respects this gradation $\mathcal{A}_s.\mathcal{A}_t \subseteq \mathcal{A}_{s+t}$).

Commentary 4. *We summarize these definitions by the following table*

Structure	*Grading support*	*Formula*	*Internal structure*	*Global structure*
Set	*Set*	$X = \bigsqcup_{s\in S} X_s$	*subsets*	*No*
Mon *or* ***Grp***	*Monoid*	$M = \biguplus_{s\in S} M_s$	*subsets*	$M_s.M_t \subseteq M_{s+t}$
k-*Lie*	*Semigroup*	$\mathfrak{g} = \bigoplus_{s\in S} \mathfrak{g}_s$	*submodules*	$[\mathfrak{g}_s, \mathfrak{g}_t] \subseteq \mathfrak{g}_{s+t}$
k-*AA*	*Semigroup*	$\mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s$	*submodules*	$\mathcal{A}_s.\mathcal{A}_t \subseteq \mathcal{A}_{s+t}$
k-*AAU*	*Monoid*	$\mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s$	*submodules*	$\mathcal{A}_s.\mathcal{A}_t \subseteq \mathcal{A}_{s+t}$

Table 1: S-graded structures for the list of categories, where "Internal structure" means "Algebraic internal structures of the components".

Remark 2.8. 1. In case $M \in$ **Mon** and $(\mathbb{N}, +, 0)$ the commutative monoid of non-negative integers, if we assume that $M_0 = \{1_M\}$ then $M = \biguplus_{m\in\mathbb{N}} M_m$ a $\mathbb{N}$-graded monoid is called succinctly "graded monoid" in Definition 1.1. We recall the Hilbert series of a finitely graded moinoid M is the formal power series

$$Hilb(M, t) = \sum_{m\geq 0} |\, M_m \,|\; t^m \in \mathbb{N}[[t]] \; (\subset \mathbb{Q}[[t]]).$$

More generally, this definition is still valid when internal structures of the components are subsets (i.e. for **Set**, **Mon** or **Grp**).

2. From now on and until the end of this remark $\mathbf{k}$ is assumed to be a field. Then, in case of **k-AAU**, an $\mathbb{N}$-graded (or graded for short) associative algebra with unit $\mathcal{A} = \bigoplus_{m\in\mathbb{N}} \mathcal{A}_m$ is said finitely graded if each $\mathbf{k}$-module $\mathcal{A}_m$ is finite dimensional. The Hilbert series of a finitely graded associative algebra $\mathcal{A}$ is the formal power series

$$Hilb(\mathcal{A}, t) := \sum_{m\geq 0} \dim_{\mathbf{k}} \mathcal{A}_m.t^m \in \mathbb{Q}[[t]]. \tag{2.94}$$

More generally, this definition is still valid when internal structures of the components are free submodules (i.e. for **k-Lie** or **k-AAU**).

3. The Hilbert series of direct sum and standard tensor product of two finitely graded modules $\mathcal{A}$ and $\mathcal{B}$ is $Hilb(\mathcal{A}\oplus\mathcal{B},t) = Hilb(\mathcal{A},t)+Hilb(\mathcal{B},t)$ and $Hilb(\mathcal{A}\otimes\mathcal{B},t) = Hilb(\mathcal{A},t).Hilb(\mathcal{B},t)$, respectively.

Now we attempt to find a bivariate Hilbert series of $\mathcal{U}(\mathrm{DK}_{\mathbf{k},n+1})$ the universal enveloping algebra of the Drinfeld-Kohno Lie algebra that has a nice description in the next subsection 2.3.3 (see Definition 2.3 and Example 2.5). Firstly, we say that the monovariate Hilbert series are known.

Example 2.3. (cf. Kohno [70], Etingof et al. [44]) The Hilbert series of the universal enveloping algebra of the Drinfeld-Kohno Lie algebra is

$$Hilb(\mathcal{U}(\mathrm{DK}_{\mathbf{k},n+1}),t) = \prod_{i=2}^{n+1} \frac{1}{1-(i-1)t} \in \mathbb{Q}[[t]].$$

Remark 2.9. Bourbaki's book [10], Algebra Ch II §11.1 deals with graded $\mathbf{k}$-algebras over a commutative monoid. However, for other structures (like Lie algebras), semigroups can replace monoids with the same crucial properties (homogeneous components, generators &c.). This will be sufficient for our purposes (indeed for the infinite Drinfeld-Kohno Lie algebra $\mathrm{DK}_{\mathbf{k},\infty}$, we will need the additive commutative semigroup $(\mathbb{N}_{\geq 2},\vee)\times(\mathbb{N}_{\geq 1},+)$, see in Subsection 2.3.3).

Here, we will use semigroups for Lie algebras as in the above table.

The category of $\mathbb{B}$-graded Lie algebras.

For a Boolean semigroup $\mathbb{B} = (\{0,1\},\vee)$ [16], a $\mathbb{B}$-graded Lie algebra $\mathfrak{g}$ can be described as follows: $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0$ is the direct sum of two submodules $\mathfrak{g}_1$ and $\mathfrak{g}_0$ such that

$$\begin{aligned}
[\mathfrak{g}_0,\mathfrak{g}_0] &\subseteq \mathfrak{g}_{0\vee 0} = \mathfrak{g}_0,\\
[\mathfrak{g}_0,\mathfrak{g}_1] &\subseteq \mathfrak{g}_{0\vee 1} = \mathfrak{g}_1,\\
[\mathfrak{g}_1,\mathfrak{g}_0] &\subseteq \mathfrak{g}_{1\vee 0} = \mathfrak{g}_1,\\
[\mathfrak{g}_1,\mathfrak{g}_1] &\subseteq \mathfrak{g}_{1\vee 1} = \mathfrak{g}_1.
\end{aligned} \tag{2.95}$$

Thus, the $\mathbb{B}$-graded Lie algebra $\mathfrak{g}$ can be written as $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{b}$, where $\mathfrak{h} := \mathfrak{g}_1$ being a Lie ideal and $\mathfrak{b} := \mathfrak{g}_0$ a Lie subalgebra (hence we have the semi-direct product $\mathfrak{g} = \mathfrak{h}\rtimes\mathfrak{b}$).

[16] In fact, $\mathbb{B}$ is a monoid.

We now define the category of $\mathbb{B}$-graded Lie algebras over $\mathbf{k}$, denoted by $\mathbb{B}$-**GrLie**, as follows

- Objects: an object is a Lie $\mathbf{k}$-algebra $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b}$, $\mathfrak{h}$ is a Lie ideal and $\mathfrak{b}$ a Lie subalgebra of $\mathfrak{g}$.

- Morphisms: a morphism of two objects $\mathfrak{g}_1 = \mathfrak{h}_1 \rtimes \mathfrak{b}_1$ and $\mathfrak{g}_2 = \mathfrak{h}_2 \rtimes \mathfrak{b}_2$ is a Lie algebra homomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\varphi(\mathfrak{h}_1) \subseteq \mathfrak{h}_2$ and $\varphi(\mathfrak{b}_1) \subseteq \mathfrak{b}_2$.

We then can define a functor F : **k-SSLie** $\to$ $\mathbb{B}$-**GrLie** from the category of SESS in **k-Lie** to the category of $\mathbb{B}$-graded Lie algebras by the following

- $F((2.92)) \overset{def}{=} (\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b})$, where $\mathfrak{h} = \mathrm{Ker}(p)$ the Lie ideal and $\mathfrak{b} = \mathrm{Im}(\sigma)$ the Lie subalgebra of $\mathfrak{g}$, that is an object in $\mathbb{B}$-**GrLie**;

- $F((2.93)) \overset{def}{=} (\varphi : \mathfrak{g}_1 = \mathfrak{h}_1 \rtimes \mathfrak{b}_1 \to \mathfrak{g}_2 = \mathfrak{h}_2 \rtimes \mathfrak{b}_2)$, here $F((2.93))$ is a morphism in $\mathbb{B}$-**GrLie** because $\varphi(\mathfrak{h}_1) = \varphi(\mathrm{Ker}(p_1)) = \varphi(\mathrm{Im}(j_1)) = \varphi \circ j_1(\mathfrak{g}_1^l) = j_2 \circ \varphi^l(\mathfrak{g}_1^l) \subseteq j_2(\mathfrak{g}_2^l) = \mathrm{Im}(j_2) = \mathrm{Ker}(p_2) = \mathfrak{h}_2$ and $\varphi(\mathfrak{b}_1) = \varphi(\mathrm{Im}(\sigma_1)) = \varphi \circ \sigma_1(\mathfrak{g}_1^r) = \sigma_2 \circ \varphi^r(\mathfrak{g}_1^r) \subseteq \sigma_2(\mathfrak{g}_2^r) = \mathrm{Im}(\sigma_2) = \mathfrak{b}_2$.

Moreover, one has the following

Proposition 2.18. *The functor F :* **k-*SSLie*** $\to$ $\mathbb{B}$-***GrLie*** *is*

- *an essentially surjective functor (i.e. every object $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b} \in \mathbb{B}$-**GrLie**, there exist an object* (2.92) $\in$ **k-*SSLie*** *and an isomorphism $F((2.92))) \cong \mathfrak{g}$ in $\mathbb{B}$-**GrLie**, see [66] Def 1.2.11) and*

- *a fully faithful functor.*

Proof. We first prove that F is essentially surjective. Indeed, for every object $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b} \in \mathbb{B}$-**GrLie**, we take the following natural SESS in **k-Lie**

$$0 \longrightarrow \mathfrak{h} \xrightarrow{j} \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b} \underset{\sigma}{\overset{p}{\rightleftarrows}} \mathfrak{b} \longrightarrow 0 \tag{2.96}$$

which is an object in **k-SSLie**. We then easily see that $F((2.96)) = \mathfrak{g}$ in $\mathbb{B}$-**GrLie** which corresponds to $\mathrm{Id}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$ because $F((2.96)) = (\mathfrak{g} = \mathrm{Ker}(p) \rtimes \mathrm{Im}(\sigma) = \mathfrak{h} \rtimes \mathfrak{b})$. We verified that F is an essentially surjective functor. Further, for each pair of objects

$$0 \longrightarrow \mathfrak{g}_1^l \xrightarrow{j_1} \mathfrak{g}_1 \underset{\sigma_1}{\overset{p_1}{\rightleftarrows}} \mathfrak{g}_1^r \longrightarrow 0 \tag{2.97}$$

and

$$0 \longrightarrow \mathfrak{g}_2^l \xrightarrow{j_2} \mathfrak{g}_2 \underset{\sigma_2}{\overset{p_2}{\rightleftarrows}} \mathfrak{g}_2^r \longrightarrow 0 \tag{2.98}$$

in **k-SSLie**, for short we will designate SESS (2.97) by x_1 and SESS (2.98) by x_2, then the function

$$\begin{aligned} F : \mathrm{Hom}_{\textbf{k-SSLie}}(x_1, x_2) &\rightarrow \mathrm{Hom}_{\mathbb{B}\textbf{-GrLie}}(F(x_1), F(x_2)) \\ (2.93) &\mapsto \varphi : \mathfrak{g}_1 = \mathfrak{h}_1 \rtimes \mathfrak{b}_1 \rightarrow \mathfrak{g}_2 = \mathfrak{h}_2 \rtimes \mathfrak{b}_2 \end{aligned}$$

is a bijection in **Set** by the following properties

- Surjectivity: in fact, for any morphism $\varphi : \mathfrak{g}_1 = \mathfrak{h}_1 \rtimes \mathfrak{b}_1 \rightarrow \mathfrak{g}_2 = \mathfrak{h}_2 \rtimes \mathfrak{b}_2$ in $\mathrm{Hom}_{\mathbb{B}\textbf{-GrLie}}(F(x_1), F(x_2))$, we construct two Lie homomorphisms $\varphi^l := r_2 \circ \varphi \circ j_1 : \mathfrak{g}_1^l \rightarrow \mathfrak{g}_2^l$ and $\varphi^r := p_2 \circ \varphi \circ \sigma_1 : \mathfrak{g}_1^r \rightarrow \mathfrak{g}_2^r$, where $r_2 : \mathfrak{g}_2 \rightarrow \mathfrak{g}_2^l$ is a retract of j_2 i.e. r_2 is a Lie algebra homomorphism such that $r_2 \circ j_2 = \mathrm{Id}_{\mathfrak{g}_2^l}$. One notes that since $\varphi(\mathfrak{h}_1) \subseteq \mathfrak{h}_2$ by assumption, we get that $\varphi \circ j_1(\mathfrak{g}_1^l) = \varphi(\mathfrak{h}_1) \subseteq \mathfrak{h}_2 = j_2(\mathfrak{g}_2^l)$ and then for any $a_1^l \in \mathfrak{g}_1^l$ there exists $a_2^l \in \mathfrak{g}_2^l$ such that $\varphi \circ j_1(a_1^l) = j_2(a_2^l)$ (a_2^l is unique since j_2 is injective). We can now obtain that $\varphi^l(a_1^l) = r_2 \circ \varphi \circ j_1(a_1^l) = r_2 \circ j_2(a_2^l) = a_2^l$, and then $j_2 \circ \varphi^l(a_1^l) = j_2(a_2^l) = \varphi \circ j_1(a_1^l)$. Thus, $j_2 \circ \varphi^l = \varphi \circ j_1$. Moreover, we also remark that $\varphi(\mathfrak{b}_1) \subseteq \mathfrak{b}_2$ by assumption, then $\varphi \circ \sigma_1(\mathfrak{g}_1^r) = \varphi(\mathfrak{b}_1) \subseteq \mathfrak{b}_2 = \sigma_2(\mathfrak{g}_2^r)$, thus for any $a_1^r \in \mathfrak{g}_1^r$ there exists $a_2^r \in \mathfrak{g}_2^r$ such that $\varphi \circ \sigma_1(a_1^r) = \sigma_2(a_2^r)$ (a_2^r is unique since σ_2 is injective). It is sufficient to obtain that $\varphi^r(a_1^r) = p_2 \circ \varphi \circ \sigma_1(a_2^r) = p_2 \circ \sigma_2(a_2^r) = a_2^r$, we thus have $\sigma_2 \circ \varphi^r(a_1^r) = \sigma_2(a_2^r) = \varphi \circ \sigma_1(a_1^r)$. We arrive at $\sigma_2 \circ \varphi^r = \varphi \circ \sigma_1$. As a consequence, we can obtain the following commutative diagram in **k-Lie** with SESS arrows

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{g}_1^l \arrow[r, "j_1"] \arrow[d, "\varphi^l"] & \mathfrak{g}_1 \arrow[r, dotted, shift left, "p_1"] \arrow[d, "\varphi"] & \mathfrak{g}_1^r \arrow[l, shift left, "\sigma_1"] \arrow[r] \arrow[d, "\varphi^r"] & 0 \\ 0 \arrow[r] & \mathfrak{g}_2^l \arrow[r, "j_2"] & \mathfrak{g}_2 \arrow[r, dotted, shift left, "p_2"] & \mathfrak{g}_2^r \arrow[l, shift left, "\sigma_2"] \arrow[r] & 0. \end{tikzcd} \tag{2.99}$$

We therefore conclude that the function F is a surjection because $F((2.99)) = (\varphi : \mathfrak{g}_1 = \mathfrak{h}_1 \rtimes \mathfrak{b}_1 \rightarrow \mathfrak{g}_2 = \mathfrak{h}_2 \rtimes \mathfrak{b}_2)$, where (2.99) is a morphism in $\mathrm{Hom}_{\textbf{k-SSLie}}(x_1, x_2)$.

- Injectivity: we start with a morphism $\varphi : \mathfrak{g}_1 = \mathfrak{h}_1 \rtimes \mathfrak{b}_1 \rightarrow \mathfrak{g}_2 = \mathfrak{h}_2 \rtimes \mathfrak{b}_2$ in $\mathrm{Hom}_{\mathbb{B}\textbf{-GrLie}}(F(x_1), F(x_2))$ as above, we assume that there is an another morphism

$$
\begin{tikzcd}
0 \arrow[r] & \mathfrak{g}_1^l \arrow[r, "j_1"] \arrow[d, "\psi^l"] & \mathfrak{g}_1 \arrow[r, dotted, shift left, "p_1"] \arrow[d, "\varphi"] & \mathfrak{g}_1^r \arrow[l, shift left, "\sigma_1"] \arrow[r] \arrow[d, "\psi^r"] & 0 \\
0 \arrow[r] & \mathfrak{g}_2^l \arrow[r, "j_2"] & \mathfrak{g}_2 \arrow[r, dotted, shift left, "p_2"] & \mathfrak{g}_2^r \arrow[l, shift left, "\sigma_2"] \arrow[r] & 0
\end{tikzcd}
\tag{2.100}
$$

in $\mathrm{Hom}_{\textbf{k-SSLie}}(x_1, x_2)$ such that $F((2.100)) = (\varphi : \mathfrak{g}_1 = \mathfrak{h}_1 \rtimes \mathfrak{b}_1 \to \mathfrak{g}_2 = \mathfrak{h}_2 \rtimes \mathfrak{b}_2)$. By (2.99) and (2.100), observe that for any $a_1^l \in \mathfrak{g}_1^l$ then $j_2 \circ \psi^l(a_1^l) = \varphi \circ j_1(a_1^l) = j_2 \circ \varphi^l(a_1^l)$, one has $\psi^l(a_1^l) = \varphi^l(a_1^l)$ because j_2 is injective. We obtain that $\psi^l = \varphi^l$. Similarly, it is not hard to show that $\psi^r = \varphi^r$. Consequently two morphisms (2.99) and (2.100) are equal in $\mathrm{Hom}_{\textbf{k-SSLie}}(x_1, x_2)$. We proved the injectivity of the function F.

Thus, F is a fully faithful functor. We proved our proposition. $\square$

It follows from the above proposition that F is an equivalence of these categories. Thus, two categories **k-SSLie** and $\mathbb{B}$**-GrLie** are equivalent[17]. In a more explicit way, we can construct an inverse functor $G : \mathbb{B}\textbf{-GrLie} \to \textbf{k-SSLie}$ as follows

- For an object $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b} \in \mathbb{B}\textbf{-GrLie}$, by (2.96), we set

$$
G(\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b}) \overset{def}{=} \left(0 \longrightarrow \mathfrak{h} \xrightarrow{\;j\;} \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b} \underset{\sigma}{\overset{p}{\rightleftarrows}} \mathfrak{b} \longrightarrow 0\right)
$$

  which is obviously an object in **k-SSLie**;

- For a morphism $\varphi : \mathfrak{g}_1 = \mathfrak{h}_1 \rtimes \mathfrak{b}_1 \to \mathfrak{g}_2 = \mathfrak{h}_2 \rtimes \mathfrak{b}_2$ in $\mathbb{B}$**-GrLie**, we set $G(\varphi) \overset{def}{=}$

$$
\begin{tikzcd}
0 \arrow[r] & \mathfrak{h}_1 \arrow[r, "j_1"] \arrow[d, "\varphi_{|\mathfrak{h}_1}"] & \mathfrak{g}_1 \arrow[r, dotted, shift left, "p_1"] \arrow[d, "\varphi"] & \mathfrak{b}_1 \arrow[l, shift left, "\sigma_1"] \arrow[r] \arrow[d, "\varphi_{|\mathfrak{b}_1}"] & 0 \\
0 \arrow[r] & \mathfrak{h}_2 \arrow[r, "j_2"] & \mathfrak{g}_2 \arrow[r, dotted, shift left, "p_2"] & \mathfrak{b}_2 \arrow[l, shift left, "\sigma_2"] \arrow[r] & 0
\end{tikzcd}
$$

  which is a commutative diagram in **k-Lie** with SESS arrows i.e. a morphism in **k-SSLie**.

[17] Two categories $\mathcal{C}$ and $\mathcal{D}$ are called *equivalent* if there exists an equivalence between them i.e. there are functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\alpha : 1_{\mathcal{C}} \to G \circ F$, $\beta : 1_{\mathcal{D}} \to F \circ G$. Moreover, Proposition 2.18 is a criterion for testing whether any functor $F : \mathcal{C} \to \mathcal{D}$ is a part of an equivalence of categories in this sense, see [66] Prop 1.3.13.

Commentary 5. *i) Let us provide an alternative proof of Theorem 2.6 using the tools above.*

1. *The fact that the alphabet X is bisected as $X = B + Z$ induces a $\mathbb{B}$-grading in $\mathcal{L}_{\mathbf{k}}(X)$ with $B \to 0$, $Z \to 1$ (in fact, by Lemma 5.3, a regrading of the fine grading of $\mathcal{L}_{\mathbf{k}}(X)$).*

2. *Likewise the partitioning $\mathbf{r} = \mathbf{r}_B \sqcup \mathbf{r}_Z$ means that the relators are homogeneous w.r.t. this $\mathbb{B}$-grading and, hence the Lie ideal $\mathcal{J}$ generated by $\mathbf{r}$ is itself $\mathbb{B}$-graded, let us denote $\mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_0$ be its homogeneous decomposition.*

3. *The only thing we have to prove is that $\mathcal{J}_0 = \mathcal{J}_B$ and $\mathcal{J}_1 = \mathcal{J}_Z + \mathcal{J}_{BZ}$ which can be done as follows*

   (a) *The inclusion $\mathcal{J}_B + \mathcal{J}_Z + \mathcal{J}_{BZ} \subset \mathcal{J}$ is straightforward considering the definitions of the summands.*

   (b) *We observe that $\mathcal{J}_B + \mathcal{J}_Z + \mathcal{J}_{BZ}$ is a Lie ideal of $\mathcal{L}_{\mathbf{k}}(X)$. This is due to the fact that, for all $b \in B$ and $z \in Z$, we have $[b, \mathcal{J}_B] \subset \mathcal{J}_B$ and $[z, \mathcal{J}_B] \subset \mathcal{J}_{BZ}$ by definition, the other summands ($\mathcal{J}_Z$ and $\mathcal{J}_{BZ}$) being Lie ideals.*

   (c) *Then, as $\mathbf{r} \subset \mathcal{J}_B + \mathcal{J}_Z \subset \mathcal{J}_B + \mathcal{J}_Z + \mathcal{J}_{BZ}$, we get the reverse inclusion $\mathcal{J}_B + \mathcal{J}_Z + \mathcal{J}_{BZ} \supset \mathcal{J}$.*

   (d) *Finally we have $\mathcal{J}_0 = \mathcal{J} \cap \mathcal{L}_{\mathbf{k}}(X)_B = \mathcal{J}_B$ and $\mathcal{J}_1 = \mathcal{J} \cap \mathcal{L}_{\mathbf{k}}(X)_{BZ} = \mathcal{J}_Z + \mathcal{J}_{BZ}$ by counting the degrees.*

Then, the conclusions of Theorem 2.6 can be revisited as follows

i) *$\mathcal{J}_{BZ}^{Z}$ (resp. $\mathcal{J}_B$) is the 1-component (resp. 0-component) of $\mathcal{J}$ i.e. $\mathcal{J}_1 = \mathcal{J}_Z + \mathcal{J}_{BZ} = \mathcal{J}_{BZ}^{Z}$ (resp. $\mathcal{J}_0 = \mathcal{J}_B$).*

ii-iv) *The remainder of the theorem is a consequence of the following general fact which says that a Lie ideal of $\mathfrak{g}$ which is $\mathbb{B}$-graded as a submodule is a $\mathbb{B}$-graded Lie ideal (i.e. the kernel of a $\mathbb{B}$-morphism, see Proposition 5.2 in Appendix 5.4.1).*

Lemma 2.19. *Let $\mathfrak{g}$ be a $\mathbb{B}$-graded Lie algebra and $\mathfrak{h}$ be a Lie ideal of $\mathfrak{g}$ which is $\{0, 1\}$-graded as a submodule. Then* *i.* $\mathfrak{h}_0$ *is a Lie ideal of* $\mathfrak{g}_0$*,* $\mathfrak{h}_1$ *is a Lie ideal of* $\mathfrak{g}$ *(and then of* $\mathfrak{g}_1$*).*

ii. We have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{g}_1 \arrow[r, "j_1"] \arrow[d, "s_1"] & \mathfrak{g} \arrow[r, dotted, shift left, "p"] \arrow[d, "s"] & \mathfrak{g}_0 \arrow[l, shift left, "j_0"] \arrow[r] \arrow[d, "s_0"] & 0 \\ 0 \arrow[r] & \mathfrak{g}_1/\mathfrak{h}_1 \arrow[r, "{[j_1]}"] & \mathfrak{g}/\mathfrak{h} \arrow[r, dotted, shift left, "{[p]}"] & \mathfrak{g}_0/\mathfrak{h}_0 \arrow[l, shift left, "{[j_0]}"] \arrow[r] & 0. \end{tikzcd}$$

Proof. This is a particular case of Proposition 5.2 with $S = \mathbb{B}$. □

ii) For implementation purposes, one can remark that the Lie ideal $\mathcal{J}_B$ *is the set of Lie polynomials on* $\{\mathrm{ad}_{(u)}(r_j)\}_{u\in B^*, j\in J_B}$.

A free functor on the category of double sets.

Our next aim is to investigate that Lazard's elimination defines a free functor from the category of double sets to the category of $\mathbb{B}$-graded Lie algebras with respect to the forgetful functor $F : \mathbb{B}\text{-}\mathbf{GrLie} \to \mathbf{Set}^2$ defined in the following way

- Given an object $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b} \in \mathbb{B}\text{-}\mathbf{GrLie}$, then $F(\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b}) \overset{def}{=} (\mathfrak{h}, \mathfrak{b})$, where $\mathfrak{h}$ and $\mathfrak{b}$ are only underlying sets (forgetting their Lie structures), so is an object in $\mathbf{Set}^2$ (the product of the category $\mathbf{Set}$ with itself);

- Given a $\mathbb{B}\text{-}\mathbf{GrLie}$-morphism $\mathfrak{g}_1 = \mathfrak{h}_1 \rtimes \mathfrak{b}_1 \xrightarrow{\varphi} \mathfrak{g}_2 = \mathfrak{h}_2 \rtimes \mathfrak{b}_2$, we define $F(\varphi)$ as

$$F(\varphi) \overset{def}{=} (\mathfrak{h}_1, \mathfrak{b}_1) \xrightarrow{F(\varphi)} (\mathfrak{h}_2, \mathfrak{b}_2)$$

  the underlying (double) set-theoretical map corresponding to
  $F(\varphi)((\mathfrak{h}_1, \mathfrak{b}_1)) = (\varphi(\mathfrak{h}_1), \varphi(\mathfrak{b}_1))$.

We first start with a pair of sets $(Z, B) \in \mathbf{Set}^2$, and then under the Lie homomorphism $\alpha : \mathcal{L}_{\mathbf{k}}(B) \to \mathfrak{Der}(\mathcal{L}_{\mathbf{k}}(B^*Z))$ (appearing in the proof of Theorem 2.3) we can construct the below classical elimination which is a SESS of Lie algebras

$$0 \longrightarrow \mathcal{L}_{\mathbf{k}}(B^*Z) \xrightarrow{j_{B|Z}} \mathcal{L}_{\mathbf{k}}(B^*Z) \rtimes \mathcal{L}_{\mathbf{k}}(B) \underset{s_{B|Z}}{\overset{p_{B|Z}}{\rightleftarrows}} \mathcal{L}_{\mathbf{k}}(B) \longrightarrow 0. \tag{2.101}$$

In this situation, we now set $\mathfrak{g}(Z, B) := \mathcal{L}_{\mathbf{k}}(B^*Z) \rtimes \mathcal{L}_{\mathbf{k}}(B) = \mathfrak{g}_1(Z, B) \oplus \mathfrak{g}_0(Z, B)$ with the grading $\mathfrak{g}_1(Z, B) := j_{B|Z}(\mathcal{L}_{\mathbf{k}}(B^*Z))$ the Lie ideal of the Lie algebra $\mathfrak{g}(Z, B)$

and $\mathfrak{g}_0(Z,B) := s_{B|Z}(\mathcal{L}_{\mathbf{k}}(B))$ the Lie subalgebra of it. One notes that $\mathfrak{g}(Z,B) = \mathfrak{g}_1(Z,B) \oplus \mathfrak{g}_0(Z,B)$ is indeed an object in $\mathbb{B}$-**GrLie**. Moreover, we define the morphism $j_{(Z,B)} = (j_Z, j_B) : (Z,B) \to (\mathfrak{g}_1(Z,B), \mathfrak{g}_0(Z,B)) = F(\mathfrak{g}(Z,B))$ in $\mathbf{Set}^2$, where $j_Z : Z \to \mathfrak{g}_1(Z,B)$ and $j_B : B \to \mathfrak{g}_0(Z,B)$ are set as the composites

$$Z \to \mathcal{L}_{\mathbf{k}}(B^*Z) \to j_{B|Z}(\mathcal{L}_{\mathbf{k}}(B^*Z)) \text{ and } B \to \mathcal{L}_{\mathbf{k}}(B) \to s_{B|Z}(\mathcal{L}_{\mathbf{k}}(B)), \text{ respectively.}$$

Amazingly, we now remark that the pair $(j_{(Z,B)}, \mathfrak{g}(Z,B))$ satisfies the following universal problem: for any object (Z,B) in $\mathbf{Set}^2$ and $\mathfrak{L} = \mathfrak{L}_1 \rtimes \mathfrak{L}_0$ in $\mathbb{B}$-**GrLie**, for each morphism $f = (f_Z, f_B) : (Z,B) \to F(\mathfrak{L}) = (\mathfrak{L}_1, \mathfrak{L}_0)$ in $\mathbf{Set}^2$ (this simply means that $f_Z : Z \to \mathfrak{L}_1$ and $f_B : B \to \mathfrak{L}_0$ are morphisms in **Set**), there exists a unique $\widehat{f} \in \mathrm{Hom}_{\mathbb{B}\text{-}\mathbf{GrLie}}(\mathfrak{g}(Z,B), \mathfrak{L})$ such that $f = F(\widehat{f}) \circ j_{(Z,B)}$, it means that the below diagram commutes

$$\begin{tikzcd}
\mathbf{Set}^2 & & \mathbb{B}\text{-}\mathbf{GrLie} \arrow[ll, dotted, "F"'] \\
(Z,B) \arrow[rr, dashed, "f=(f_Z,f_B)"] \arrow[drr, dashed, "j_{(Z,B)}"'] & & \mathfrak{L} = \mathfrak{L}_1 \rtimes \mathfrak{L}_0 \\
& & \mathfrak{g}(Z,B) = \mathfrak{g}_1(Z,B) \rtimes \mathfrak{g}_0(Z,B). \arrow[u, "\widehat{f}"']
\end{tikzcd} \tag{2.102}$$

In fact, if we take $uz = b_1 \cdots b_k z \in B^*Z$ then $[f_B(b_1), \cdots, [f_B(b_k), f_Z(z)] \cdots] \in \mathfrak{L}_1$ since $\mathfrak{L}_1$ is a Lie ideal of $\mathfrak{L}$. We thus construct a Lie morphism $\mathcal{L}_{\mathbf{k}}(B^*Z) \xrightarrow{f_1} \mathfrak{L}_1$ that is the unique extension of a map $f_{BZ} : B^*Z \to \mathfrak{L}_1, b_1 \cdots b_k z \mapsto [f_B(b_1), \cdots, [f_B(b_k), f_Z(z)] \cdots]$ by universal property of Diagram (1.30). Similarly, we can also find a Lie homomorphism $\mathcal{L}_{\mathbf{k}}(B) \xrightarrow{f_0} \mathfrak{L}_0$ as the unique extension of a map $f_B : B \to \mathfrak{L}_0$. As a result, we arrive at Lie homomorphisms $\widehat{f_1} : \mathcal{L}_{\mathbf{k}}(B^*Z) \to \mathfrak{L}$ and $\widehat{f_0} : \mathcal{L}_{\mathbf{k}}(B) \to \mathfrak{L}$ by such embedding f_1 and f_0 into $\mathfrak{L}$, respectively. Further, it is straightforward to check on generators that these Lie homomorphisms and the action $\alpha : \mathcal{L}_{\mathbf{k}}(B) \to \mathfrak{Der}(\mathcal{L}_{\mathbf{k}}(B^*Z))$ induce the following commutative diagram

$$\begin{tikzcd}
\mathcal{L}_{\mathbf{k}}(B) \otimes \mathcal{L}_{\mathbf{k}}(B^*Z) \arrow[r, "\widehat{f_0} \otimes \widehat{f_1}"] \arrow[d, "\alpha_{\otimes}"] & \mathfrak{L} \otimes \mathfrak{L} \arrow[d, "\mathrm{ad}^{\mathfrak{L}}_{\otimes}"] \\
\mathcal{L}_{\mathbf{k}}(B^*Z) \arrow[r, "\widehat{f_1}"] & \mathfrak{L}.
\end{tikzcd} \tag{2.103}$$

The equivariant diagram (2.103) say that there is a unique Lie morphism $\widehat{f} : \mathfrak{g}(Z,B) = \mathcal{L}_{\mathbf{k}}(B^*Z) \rtimes \mathcal{L}_{\mathbf{k}}(B) \to \mathfrak{L}$ extending $\widehat{f_0}$ and $\widehat{f_1}$ in the usual sense i.e. $\widehat{f} \circ s_{B|Z}(\mathcal{L}_{\mathbf{k}}(B)) =$

$\widehat{f}_0(\mathcal{L}_{\mathbf{k}}(B))$ and $\widehat{f} \circ j_{B|Z}(\mathcal{L}_{\mathbf{k}}(B^*Z)) = \widehat{f}_1(\mathcal{L}_{\mathbf{k}}(B^*Z))$. In particular, we observe that $\widehat{f}(\mathfrak{g}_1(Z,B)) = \widehat{f} \circ j_{B|Z}(\mathcal{L}_{\mathbf{k}}(B^*Z)) = \widehat{f}_1(\mathcal{L}_{\mathbf{k}}(B^*Z)) \subseteq \mathfrak{L}_1$ and $\widehat{f}(\mathfrak{g}_0(Z,B)) = \widehat{f} \circ s_{B|Z}(\mathcal{L}_{\mathbf{k}}(B)) = \widehat{f}_0(\mathcal{L}_{\mathbf{k}}(B)) \subseteq \mathfrak{L}_0$, proving that $\widehat{f} : \mathfrak{g}(Z,B) = \mathfrak{g}_1(Z,B) \rtimes \mathfrak{g}_0(Z,B) \to \mathfrak{L} = \mathfrak{L}_1 \rtimes \mathfrak{L}_0$ is a morphism in $\mathbb{B}$-**GrLie**. Moreover, the following lemma is sufficient for our investigation

Lemma 2.20. *One has* $\widehat{f} \in \mathrm{Hom}_{\mathbb{B}\text{-}\boldsymbol{GrLie}}(\mathfrak{g}(Z,B),\mathfrak{L})$ *is the unique morphism in* $\mathbb{B}$-***GrLie*** *such that* $f = F(\widehat{f}) \circ j_{(Z,B)}$.

Proof. It is immediate to verify by calculation that for any $(z,b) \in (Z,B)$ then $F(\widehat{f}) \circ j_{(Z,B)}[(z,b)] = F(\widehat{f})(j_Z(z), j_B(b)) = \big(\widehat{f}(j_Z(z)), \widehat{f}(j_B(b))\big) = (\widehat{f}_1(z), \widehat{f}_0(b)) = \big(f_{BZ}(z), f_B(b)\big) = (f_Z(z), f_B(b)) = f((z,b))$, thus clearly $f = F(\widehat{f}) \circ j_{(Z,B)}$. On the other hand, if there is a morphism $g : \mathfrak{g}(Z,B) \to \mathfrak{L}$ in $\mathbb{B}$-**GrLie** which is an another solution of the diagram (2.102), we then see that for each $b \in B$, one notes that $g \circ s_{B|Z}(b) = g(j_B(b)) = f_B(b) = \widehat{f}_0(b)$, thus $g \circ s_{B|Z}(\mathcal{L}_{\mathbf{k}}(B)) = \widehat{f}_0(\mathcal{L}_{\mathbf{k}}(B))$. Moreover, for each $uz = b_1 \cdots b_k z \in B^*Z$, then $g \circ j_{B|Z}(z) = g(j_Z(z)) = f_Z(z) = f_{BZ}(z) = \widehat{f}_1(z)$ and hence $g \circ j_{B|Z}(uz) = g((uz,0)) = g([(0,b_1), \cdots, [(0,b_k),(z,0)]\cdots]) = [g \circ s_{B|Z}(b_1), \cdots, [g \circ s_{B|Z}(b_k), g \circ j_{B|Z}(z)]\cdots] = [f_B(b_1), \cdots, [f_B(b_k), f_Z(z)]\cdots] = f_{BZ}(uz) = \widehat{f}_1(uz)$, we arrive at $g \circ j_{B|Z}(\mathcal{L}_{\mathbf{k}}(B^*Z)) = \widehat{f}_1(\mathcal{L}_{\mathbf{k}}(B^*Z))$. As a consequence, the morphism $g : \mathfrak{g}(Z,B) \to \mathfrak{L}$ extends $\widehat{f}_0$ and $\widehat{f}_1$ in the usual sense. Thus, $g = \widehat{f}$ by the uniqueness of the equivariant extension. We verified our lemma. □

As a result, the solution of the universal problem (2.102) provides a *free functor* $L : \mathbf{Set}^2 \to \mathbb{B}\text{-}\mathbf{GrLie}, (Z,B) \mapsto \mathfrak{g}(Z,B)$, so-called the Lazard elimination functor, which is left-adjoint to the forgetful functor $F : \mathbb{B}\text{-}\mathbf{GrLie} \to \mathbf{Set}^2$.

Remark 2.10. To prove directly that these functors L and F determine an adjunction
$$\begin{tikzcd} \mathbf{Set}^2 \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbb{B}\text{-}\mathbf{GrLie} \arrow[l, bend left, "F"] \end{tikzcd},$$
we can construct a pair $(\mathbf{1}_{\mathbf{Set}^2} \overset{\eta}{\Rightarrow} F \circ L, L \circ F \overset{\varepsilon}{\Rightarrow} \mathbf{1}_{\mathbb{B}\text{-}\mathbf{GrLie}})$ of natural transformations (called the unit and counit of the adjunction) satisfying the triangle identities

$$\begin{tikzcd} L \arrow[r, "L\eta"] \arrow[dr, "\mathbf{1}_L"'] & L \circ F \circ L \arrow[d, "\varepsilon L"] \\ & L \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\eta F"] \arrow[dr, "1_F"'] & F \circ L \circ F \arrow[d, "F\varepsilon"] \\ & F \end{tikzcd}$$

(then they are commutative diagrams in the functor categories $[\mathbf{Set}^2, \mathbb{B}\text{-}\mathbf{GrLie}]$ and $[\mathbb{B}\text{-}\mathbf{GrLie}, \mathbf{Set}^2]$, respectively.)

2.3.3 Drinfeld-Kohno Lie algebra with infinite number of generators: from strange to generalized gradings.

More general semigroups and strange gradings.

Now, for any integer $n \in \mathbb{N}$, let $([2, n+1], \vee)$ be the upper semi-lattice (where we use the classical supremum $i \vee j := \sup\{i, j\}$ for all $i, j \in [2, n+1]$). We remark that, $([2, n+1], \vee)$ is also a commutative semigroup. We then claim that iterated decompositions of the Drinfeld-Kohno Lie algebras are naturally graded by supremum. In fact, we now describe a $([2, n+1], \vee)$-graded structure of the Drinfeld-Kohno Lie algebra $\mathrm{DK}_{\mathbf{k},n+1} = \mathcal{L}_{\mathbf{k}}(\mathcal{T}_{n+1}) \big/ \mathcal{J}_{\mathbf{R[n+1]}}$ as follows: for each $j \in [2, n+1]$, we introduce $\mathrm{DK}^{(j)}_{\mathbf{k},n+1}$ the Lie subalgebra of $\mathrm{DK}_{\mathbf{k},n+1}$ generated by the set

$$[T_j] := \{[t_{i,j}] = t_{i,j} + \mathcal{J}_{\mathbf{R[n+1]}} \mid i \in [1, j-1]\},$$

that is in fact a free Lie algebra over $\mathbf{k}$ because $\mathrm{DK}^{(j)}_{\mathbf{k},n+1} \cong \mathcal{L}_{\mathbf{k}}(T_j) \cong \mathcal{L}_{\mathbf{k}}(X_{j-1})$ as Lie algebras, where X_{j-1} is a set of $j-1$ elements $\{x_1, \cdots, x_{j-1}\}$.

$$\mathcal{T}_{n+1} = \begin{array}{cccccc} T_2 & T_3 & T_4 & \dots & \dots & T_{n+1} \\ \hline t_{1,2} & t_{1,3} & t_{1,4} & \dots & \dots & t_{1,n+1} \\ & t_{2,3} & t_{2,4} & \dots & \dots & \dots \\ & & t_{3,4} & \dots & \dots & \dots \\ & & & \dots & \dots & \dots \\ & & & & \dots & \dots \\ & & & & & t_{n,n+1} \end{array}$$

It is not hard to show that for all $i, j \in [2, n+1]$, one has

$$[\mathrm{DK}^{(i)}_{\mathbf{k},n+1}, \mathrm{DK}^{(j)}_{\mathbf{k},n+1}] \subseteq \mathrm{DK}^{(i\vee j)}_{\mathbf{k},n+1}$$

and moreover we get that the Drinfeld-Kohno Lie algebra $\mathrm{DK}_{\mathbf{k},n+1}$ is an iterated semi-direct product of free Lie algebras (see Remark 2.6)

$$\mathrm{DK}_{\mathbf{k},n+1} \quad = \quad \mathrm{DK}^{(n+1)}_{\mathbf{k},n+1} \rtimes \left(\mathrm{DK}^{(n)}_{\mathbf{k},n+1} \rtimes (\cdots \rtimes \mathrm{DK}^{(2)}_{\mathbf{k},n+1}) \cdots \right)$$

$$\cong \quad \mathcal{L}_{\mathbf{k}}(X_n) \rtimes (\mathcal{L}_{\mathbf{k}}(X_{n-1}) \rtimes (\cdots \rtimes \mathcal{L}_{\mathbf{k}}(X_1)) \cdots). \tag{2.104}$$

In particular, $\mathrm{DK}_{\mathbf{k},n+1} = \bigoplus_{2\leq j\leq n+1} \mathrm{DK}^{(j)}_{\mathbf{k},n+1}$ is a $([2, n+1], \vee)$-graded Lie algebra.

Strange gradings allow not only to manage semi-direct products but, more complex elimination schemes like iterated decompositions. Indeed, suppose we had an elimination scheme (0.1)

$$STRUCT\langle x_1, x_2, \ldots, x_n\rangle \cong NICE\langle x_1, x_2, \ldots, x_n\rangle \diamond STRUCT_1\langle x_1, \ldots, x_{n-1}\rangle$$

where $NICE$ et $STRUCT_1$ stand for algebraic structures generated (sometimes freely) by generators x_i. Iterating it, we get

$$\begin{aligned} &STRUCT\langle x_1, x_2, \ldots, x_n\rangle \\ &\cong NICE\langle x_1, x_2, \ldots, x_n\rangle \diamond (NICE\langle x_1, x_2, \ldots, x_{n-1}\rangle \diamond (\cdots \diamond NICE\langle x_1\rangle) \cdots). \end{aligned}$$

In the next part, we can even manage infinite decompositions with $(\mathbb{N}_{\geq 2}, \vee)$ or non-linear eliminations with other semigroups.

Remark 2.11. Iterated semi-direct decompositions (i.e. formulas like Equation (2.104)) provide a natural grading by the semigroup $(I, \vee)$ where $I = \{i_1 < i_2 < \cdots < i_n\}$ for arbitrary Lie algebras such that

$$\mathfrak{g} \quad = \quad \mathfrak{g}_{i_n} \rtimes \left(\mathfrak{g}_{i_{n-1}} \rtimes (\cdots \rtimes \mathfrak{g}_{i_1}) \cdots \right). \tag{2.105}$$

The direct limit of Drinfeld-Kohno Lie algebras.

If we consider the direct system that is a chain of embeddings

$$\mathrm{DK}_{\mathbf{k},2} \xrightarrow{f_2} \mathrm{DK}_{\mathbf{k},3} \longrightarrow \cdots \longrightarrow \mathrm{DK}_{\mathbf{k},n} \xrightarrow{f_n} \mathrm{DK}_{\mathbf{k},n+1} \xrightarrow{f_{n+1}} \cdots \tag{2.106}$$

where the structure Lie monomorphisms $f_n : \mathrm{DK}_{\mathbf{k},n} \to \mathrm{DK}_{\mathbf{k},n+1}$ are defined in the following way

$$f_n(P + \mathcal{J}_{\mathbf{R[n]}}) = P + \mathcal{J}_{\mathbf{R[n+1]}}, \quad \text{where } P \in \mathcal{L}_{\mathbf{k}}(\mathcal{T}_n) \subset \mathcal{L}_{\mathbf{k}}(\mathcal{T}_{n+1}),$$

then $\varinjlim \mathrm{DK}_{\mathbf{k},n}$ the direct limit with such structure homomorphisms (2.106) has a very simple and in a sense a tame description by the following proposition

Proposition 2.21. *Consider the infinite Drinfeld-Kohno Lie algebra* $\mathrm{DK}_{\mathbf{k},\infty}$*, defined by the quotient of the free Lie algebra* $\mathcal{L}_{\mathbf{k}}(\mathcal{T}_\infty)$ *generated by an infinite set of non-commutative variables* $\mathcal{T}_\infty = \{t_{i,j}\}_{1\le i<j<+\infty}$ *modulo the Lie ideal* $\mathcal{J}_{\mathbf{R}[\infty]}$ *generated by infinitesimal pure braid relations*

$$\mathbf{R}[\infty] = \begin{cases} \mathbf{R_1}[\infty] & [t_{i,j}, t_{i,k} + t_{j,k}] \qquad \text{for } 1 \le i < j < k < +\infty, \\ \mathbf{R_2}[\infty] & [t_{i,j} + t_{i,k}, t_{j,k}] \qquad \text{for } 1 \le i < j < k < +\infty, \\ \mathbf{R_3}[\infty] & [t_{i,j}, t_{k,l}] \quad \text{for } \begin{matrix} 1 \le i < j < +\infty, \\ 1 \le k < l < +\infty, \end{matrix} \text{ and } |\{i,j,k,l\}| = 4 \end{cases} \tag{2.107}$$

then the infinite Drinfeld-Kohno Lie algebra is indeed the direct limit of such structure homomorphisms (2.106)

$$\mathrm{DK}_{\mathbf{k},\infty} = \varinjlim \mathrm{DK}_{\mathbf{k},n}.$$

Proof. We construct the structure Lie homomorphisms $\phi_n : \mathrm{DK}_{\mathbf{k},n} \to \mathrm{DK}_{\mathbf{k},\infty}$ that are determined by setting

$$\phi_n(P + \mathcal{J}_{\mathbf{R[n]}}) = P + \mathcal{J}_{\mathbf{R}[\infty]}, \quad \text{where } P \in \mathcal{L}_{\mathbf{k}}(\mathcal{T}_n) \subset \mathcal{L}_{\mathbf{k}}(\mathcal{T}_\infty),$$

these are clearly also Lie monomorphims between two presented Lie algebras. Then a pair $(\mathrm{DK}_{\mathbf{k},\infty}, \phi_n)$ is a target of such direct system $\{\mathrm{DK}_{\mathbf{k},n} \xrightarrow{f_n} \mathrm{DK}_{\mathbf{k},n+1}\}_{n\ge2}$ in category **k-Lie** because it satisfies the following property

$$\phi_n = \phi_m \circ f_{nm} \ \text{ whenever } n \le m, \ \text{ where } f_{nn} = \mathrm{Id}_{\mathrm{DK}_{\mathbf{k},n}} \text{ and } f_{nm} = f_{m-1} \circ \cdots \circ f_n.$$

Suppose we are given a target $(\mathfrak{g}, \psi_n)$ with $\psi_n : \mathrm{DK}_{\mathbf{k},n} \to \mathfrak{g}$ in **k-Lie**, then we construct a Lie homomorphism $u_0 : \mathcal{L}_{\mathbf{k}}(\mathcal{T}_\infty) \to \mathfrak{g}$ that is a unique extension of the mapping from $\mathcal{T}_\infty$ to $\mathfrak{g}$ corresponding to $t_{i,j} \mapsto \psi_j([t_{i,j}]) = \psi_j(t_{i,j} + \mathcal{J}_{\mathbf{R[j]}})$ (where $1 \le i < j < +\infty$) by universal property of Diagram (1.30). Further, we now show that for any polynomial P of type $\mathbf{R}[\infty]$ we have $u_0(P) = 0$ because we have the following properties

- for each $1 \le i < j < k < +\infty$ then $u_0([t_{i,j}, t_{i,k} + t_{j,k}]) = [u_0(t_{i,j}), u_0(t_{i,k}) + u_0(t_{j,k})] = \big[\psi_j([t_{i,j}]), \psi_k([t_{i,k}]) + \psi_k([t_{j,k}])\big] = \big[\psi_k \circ f_{jk}([t_{i,j}]), \psi_k([t_{i,k}]) + \psi_k([t_{j,k}])\big] = \big[\psi_k([t_{i,j}]), \psi_k([t_{i,k}]) + \psi_k([t_{j,k}])\big] = \psi_k\big([t_{i,j}, t_{i,k} + t_{j,k}] + \mathcal{J}_{\mathbf{R[k]}}\big) = \psi_k(0) = 0$, and then similarly one has $u_0([t_{i,j} + t_{i,k}, +t_{j,k}]) = \psi_k\big([t_{i,j} + t_{i,k}, t_{j,k}] + \mathcal{J}_{\mathbf{R[k]}}\big) = \psi_k(0) = 0$;

- for each $1 \leq i < j < +\infty, 1 \leq k < l < +\infty$ such that $|\{i, j, k, l\}| = 4$, assume that $l = \max\{i, j, k, l\}$ then we can obtain $u_0([t_{i,j}, t_{k,l}]) = [u_0(t_{i,j}), u_0(t_{k,l})] = \big[\psi_j([t_{i,j}]), \psi_l([t_{k,l}])\big] = \big[\psi_l \circ f_{jl}([t_{i,j}]), \psi_l([t_{k,l}])\big] = \big[\psi_l([t_{i,j}]), \psi_l([t_{k,l}])\big] = \psi_l\big([t_{i,j}, t_{k,l}] + \mathscr{J}_{\mathbf{R[l]}}\big) = \psi_l(0) = 0$.

We verified that $u_0(P) = 0$, thus clearly $\mathscr{J}_{\mathbf{R}[\infty]}$ is in the kernel of u_0. We arrive at a conclusion that u_0 induces a Lie homomorphism $u : \mathrm{DK}_{\mathbf{k},\infty} \to \mathfrak{g}$, and moreover $u \circ \phi_n = \psi_n$ for each $n \geq 2$ because they equal on its generators: for each $1 \leq i < j \leq n$ then $u \circ \phi_n(t_{i,j} + \mathscr{J}_{\mathbf{R[n]}}) = u(t_{i,j} + \mathscr{J}_{\mathbf{R}[\infty]}) = \psi_j(t_{i,j} + \mathscr{J}_{\mathbf{R[j]}}) = \psi_n \circ f_{jn}(t_{i,j} + \mathscr{J}_{\mathbf{R[j]}}) = \psi_n(t_{i,j} + \mathscr{J}_{\mathbf{R[n]}})$. As a result, we can take a commutative diagram of Lie algebras

$$\begin{tikzcd}
\mathrm{DK}_{\mathbf{k},n} \arrow[rr, "f_{nm}"] \arrow[dr, "\phi_n"] \arrow[ddr, "\psi_n"'] & & \mathrm{DK}_{\mathbf{k},m} \arrow[dl, "\phi_m"'] \arrow[ddl, "\psi_m"] \\
& \mathrm{DK}_{\mathbf{k},\infty} \arrow[d, "u"] & \\
& \mathfrak{g}. &
\end{tikzcd}$$

In particular, it is easy to verify that u is a unique Lie homomorphism such that $u \circ \phi_n = \psi_n$ for each $n \geq 2$. By this universal property, we thus deduce that $\mathrm{DK}_{\mathbf{k},\infty} = \varinjlim \mathrm{DK}_{\mathbf{k},n}$. $\square$

Under the universal enveloping functor

$$\mathcal{U} : \textbf{k-Lie} \to \textbf{k-AAU}, \qquad \mathfrak{g} \longmapsto \mathcal{U}(\mathfrak{g}) \tag{2.108}$$

which is a left adjoint to the Liezation functor $F : \textbf{k-AAU} \to \textbf{k-Lie}$, we have the following

Corollary 2.22. *Consider the direct system that is a chain of embeddings*

$$\mathcal{U}(\mathrm{DK}_{\mathbf{k},2}) \overset{\mathcal{U}(f_2)}{\longrightarrow} \mathcal{U}(\mathrm{DK}_{\mathbf{k},3}) \longrightarrow \cdots \longrightarrow \mathcal{U}(\mathrm{DK}_{\mathbf{k},n}) \overset{\mathcal{U}(f_n)}{\longrightarrow} \mathcal{U}(\mathrm{DK}_{\mathbf{k},n+1}) \overset{\mathcal{U}(f_{n+1})}{\longrightarrow} \cdots, \tag{2.109}$$

then the universal enveloping algebra of infinite Drinfeld-Kohno Lie algebra is indeed the direct limit of such structure morphisms (2.109) *in* $\mathbf{k}$-$\boldsymbol{AAU}$

$$\mathcal{U}(\mathrm{DK}_{\mathbf{k},\infty}) = \varinjlim \mathcal{U}(\mathrm{DK}_{\mathbf{k},n}).$$

Proof. As a left-adjoint, the universal enveloping functor preserves colimits, so in particular it sends direct limits in **k-Lie** to direct limits in **k-AAU**. We therefore obtain that

$$\mathcal{U}(\mathrm{DK}_{\mathbf{k},\infty}) = \mathcal{U}(\varinjlim \mathrm{DK}_{\mathbf{k},n}) = \varinjlim \mathcal{U}(\mathrm{DK}_{\mathbf{k},n}).$$

□

Further, we consider $(\mathbb{N}_{\geq 2}, \vee)$ the upper semi-lattice on the classical supremum $m \vee n := \sup\{m, n\}$ (for all $m, n \in \mathbb{N}_{\geq 2}$), in the same way as above we now describe a $(\mathbb{N}_{\geq 2}, \vee)$-graded structure of the infinite Drinfeld-Kohno Lie algebra $\mathrm{DK}_{\mathbf{k},\infty}$ by the following way: for all $m \in \mathbb{N}_{\geq 2}$, if we denote $\mathrm{DK}^{(m)}_{\mathbf{k},\infty}$ the Lie subalgebra of $\mathrm{DK}_{\mathbf{k},\infty}$ generated by $[T_m] := \{[t_{i,m}] = t_{i,m} + \mathcal{J}_{\mathbf{R}[\infty]} \mid i \in [1, m-1]\}$, then one verifies without difficulty that $\mathrm{DK}^{(m)}_{\mathbf{k},\infty} \cong \mathcal{L}_{\mathbf{k}}(T_m) \cong \mathcal{L}_{\mathbf{k}}(X_{m-1})$ is indeed a free Lie algebra over $\mathbf{k}$.

$$\mathcal{T}_\infty = \begin{array}{cccccccc}
T_2 & T_3 & T_4 & \dots & \dots & T_m & \dots \\
\hline
t_{1,2} & t_{1,3} & t_{1,4} & \dots & \dots & t_{1,m} & \dots \\
 & t_{2,3} & t_{2,4} & \dots & \dots & \dots & \dots \\
 & & t_{3,4} & \dots & \dots & \dots & \dots \\
 & & & \dots & \dots & \dots & \dots \\
 & & & & \dots & \dots & \dots \\
 & & & & & t_{m-1,m} & \dots \\
 & & & & & & \dots
\end{array}$$

By using infinitesimal pure braid relations $\mathbf{R}[\infty]$ (2.107), it is not hard to prove that

$$[\mathrm{DK}^{(m)}_{\mathbf{k},\infty}, \mathrm{DK}^{(n)}_{\mathbf{k},\infty}] \subseteq \mathrm{DK}^{(m\vee n)}_{\mathbf{k},\infty} (\text{for all } m, n \in \mathbb{N}_{\geq 2})$$

and moreover we have an infinite iterated semi-direct product of free Lie algebras

$$\mathrm{DK}_{\mathbf{k},\infty} = \cdots \rtimes \left(\mathrm{DK}^{(m)}_{\mathbf{k},\infty} \rtimes (\mathrm{DK}^{(m-1)}_{\mathbf{k},\infty} \rtimes (\cdots \rtimes \mathrm{DK}^{(2)}_{\mathbf{k},\infty}) \cdots)\right). \tag{2.110}$$

As a consequence, we write the infinite Drinfeld-Kohno algebra as the direct sum

$$\mathrm{DK}_{\mathbf{k},\infty} = \bigoplus_{m\geq 2} \mathrm{DK}^{(m)}_{\mathbf{k},\infty}$$

which is a $(\mathbb{N}_{\geq 2}, \vee)$-graded Lie algebra over $\mathbf{k}$. Furthermore, due to the injections $\mathrm{DK}_{\mathbf{k},n} \hookrightarrow \mathrm{DK}_{\mathbf{k},\infty}$, the set $\mathcal{T}_\infty = \{t_{i,j}\}_{1\leq i<j}$ can be considered as a subset of $\mathrm{DK}_{\mathbf{k},\infty}$ and

due to the semi-direct decompositions $\mathrm{DK}_{\mathbf{k},\infty}$ is a free module (see the formula (2.110)) and embeds within its enveloping algebra $\mathcal{U}(\mathrm{DK}_{\mathbf{k},\infty})$. We have the following

Proposition 2.23. *Under the natural projection* $s_\infty : \mathbf{k}\langle\mathcal{T}_\infty\rangle \to \mathcal{U}(\mathrm{DK}_{\mathbf{k},\infty})$*, the image of the set of words of the form*

$$t_{i_1,j_1}t_{i_2,j_2}\dots t_{i_n,j_n} \in \mathcal{T}_\infty^*$$

(where $n \geq 0$ *and* $2 \leq j_1 \leq j_2 \leq \dots \leq j_n$*) is a* $\mathbf{k}$*-linear basis of the algebra* $\mathcal{U}(\mathrm{DK}_{\mathbf{k},\infty})$.

Proof. See Proposition 5.7 in Appendix 5.4.2. □

S-graded Lie algebras and properties of them and their enveloping algebras with respect to Hilbert series.

Definition 2.3. Assume that $(S,+)$ is a commutative semigroup satisfying "Condition (D)"[18]. Let $\mathcal{A} = \bigoplus_{s\in S}\mathcal{A}_s$ be a S-graded algebra in finite dimensions, the Hilbert series of $\mathcal{A}$ is given by

$$Hilb(\mathcal{A}) = \sum_{s\in S}\dim_{\mathbf{k}}(\mathcal{A}_s).s \in \mathbb{Q}[[S]],$$

where the total semigroup algebra $\mathbb{Q}[[S]]$ is the completion of the Hausdorff topological semigroup algebra $\mathbb{Q}[S]$ (see Appendix 5.3.2), where $\mathbb{Q}[[S]]$ is a $\mathbb{Q}$-module of all infinite sum $\sum_{s\in S}\alpha_s s$ and the convolution product

$$\left(\sum_{s_1\in S}\alpha_{s_1}s_1\right)\left(\sum_{s_2\in S}\beta_{s_2}s_2\right) = \sum_{s\in S}\left(\sum_{\substack{s_1,s_2\in S\\ s_1+s_2=s}}\alpha_{s_1}\beta_{s_2}\right)s.$$

Proposition 2.24. *Suppose we are given two finitely* S*-graded algebras* $\mathcal{A} = \bigoplus_{s\in S}\mathcal{A}_s$ *and* $\mathcal{B} = \bigoplus_{s\in S}\mathcal{B}_s$*. Then, so are direct sum and tensor product, moreover two Hilbert series*

$$Hilb(\mathcal{A}\oplus\mathcal{B}) = Hilb(\mathcal{A}) + Hilb(\mathcal{B}) \text{ and } Hilb(\mathcal{A}\otimes\mathcal{B}) = Hilb(\mathcal{A}).Hilb(\mathcal{B}).$$

[18]A semigroup $(S,+)$ is said to satisfy "Condition (D)" if each $s \in S$ admits only a finite number of factorizations $s = s_1 + \dots + s_k$ (the positive integer k is fixed). To be more precise, for any $s \in S$, the set $D_2(s) = \{(s_1,s_2) \in S\times S \,|\, s_1+s_2 = s\}$ is finite i.e. the map $\mu_2 : [S]^2 \to S, (s_1,s_2) \mapsto s_1+s_2$ has finite fibers, see Bourbaki [10] Ch III § 2.10.

Proof. As a S-graded algebra $\mathcal{A}\oplus\mathcal{B}=\bigoplus_{s\in S}\mathcal{A}_s\oplus\mathcal{B}_s=\bigoplus_{s\in S}(\mathcal{A}\oplus\mathcal{B})_s$, we get that

$$\begin{aligned}
Hilb(\mathcal{A}\oplus\mathcal{B}) &= \sum_{s\in S}\dim_{\mathbf{k}}(\mathcal{A}_s\oplus\mathcal{B}_s).s\\
&= \sum_{s\in S}\dim_{\mathbf{k}}(\mathcal{A}_s).s+\sum_{s\in S}\dim_{\mathbf{k}}(\mathcal{B}_s).s\\
&= Hilb(\mathcal{A})+Hilb(\mathcal{B}).
\end{aligned}$$

Moreover, by using "Condition (D)", we arrive at a finitely S-graded structure

$$\mathcal{A}\otimes\mathcal{B}=\left(\bigoplus_{s_1\in S}\mathcal{A}_{s_1}\right)\otimes\left(\bigoplus_{s_2\in S}\mathcal{B}_{s_2}\right)=\bigoplus_{s\in S}\left(\bigoplus_{\substack{s_1,s_2\in S\\ s_1+s_2=s}}\mathcal{A}_{s_1}\otimes\mathcal{B}_{s_2}\right)=\bigoplus_{s\in S}(\mathcal{A}\otimes\mathcal{B})_s,$$

then the Hilbert series is

$$\begin{aligned}
Hilb(\mathcal{A}\otimes\mathcal{B}) &= \sum_{s\in S}\dim_{\mathbf{k}}\left(\bigoplus_{\substack{s_1,s_2\in S\\ s_1+s_2=s}}\mathcal{A}_{s_1}\otimes\mathcal{B}_{s_2}\right).s\\
&= \sum_{s\in S}\left(\bigoplus_{\substack{s_1,s_2\in S\\ s_1+s_2=s}}\dim_{\mathbf{k}}(\mathcal{A}_{s_1})\dim_{\mathbf{k}}(\mathcal{B}_{s_2})\right).s\\
&= \sum_{s_1,s_2\in S}\dim_{\mathbf{k}}(\mathcal{A}_{s_1})\dim_{\mathbf{k}}(\mathcal{B}_{s_2}).(s_1+s_2)\\
&= \left(\sum_{s_1\in S}\dim_{\mathbf{k}}(\mathcal{A}_{s_1}).s_1\right)\left(\sum_{s_2\in S}\dim_{\mathbf{k}}(\mathcal{B}_{s_2}).s_2\right)\\
&= Hilb(\mathcal{A}).Hilb(\mathcal{B}).
\end{aligned}$$

□

Furthermore, for the grading of enveloping algebras of S-graded Lie algebras see Appendix 5.4.2.

Remark 2.12. Assume that $(S,+)$ is locally finite[19] commutative semigroup and $\mathfrak{g}=\bigoplus_{s\in S}\mathfrak{g}_s$ is a S-graded Lie algebra. If $\mathfrak{g}$ is a finitely S-graded i.e. each $\mathbf{k}$-module $\mathfrak{g}_s$ is free of finite rank, so is the $S\sqcup\{0\}$-graded enveloping algebra $\mathcal{U}(\mathfrak{g})$ (5.23).

[19]In computer science, it means that each $s\in S$ admits only a finite number of factorizations $s=s_1+\cdots+s_k$ ($k\in\mathbb{N}_{\geq 1}$) i.e. the map $\mu:[S]^+=[S]^*\setminus 1_{[S]^*},(s_1,\ldots,s_k)\mapsto s_1+\ldots+s_k$ has finite fibers, see Eilenberg [43]. Remarkable that "locally finite" induces "Condition (D)", but the converse is not true in general.

Now we will deal with a more general graded type of the infinite Drinfeld-Kohno Lie algebra and then its enveloping algebra as follows

Example 2.4. Let us recall that the infinite Drinfeld-Kohno Lie algebra

$$\mathrm{DK}_{\mathbf{k},\infty} = \bigoplus_{m\geq 2} \mathrm{DK}^{(m)}_{\mathbf{k},\infty}$$

is a $(\mathbb{N}_{\geq 2}, \vee)$-graded Lie algebra and moreover each component

$$\mathrm{DK}^{(m)}_{\mathbf{k},\infty} \cong \mathcal{L}_{\mathbf{k}}(X_{m-1}) = \bigoplus_{n\geq 1} \mathcal{L}_{\mathbf{k}}(X_{m-1})_n$$

in the category **k-Lie** (see Subsection 1.2.5 for more details). Let us denote by $\mathrm{DK}^{(m,n)}_{\mathbf{k},\infty}$ the set of all homogeneous Lie polynomials of total degree n of each free Lie algebra $\mathrm{DK}^{(m)}_{\mathbf{k},\infty}$, that is Lie algebra isomorphic to $\mathcal{L}_{\mathbf{k}}(X_{m-1})_n$. Furthermore, it is obviously a free $\mathbf{k}$-module with a Lyndon basis P_l, where $l \in \mathcal{L}ynX_{m-1}$ and the length $\mid l \mid= n$, here we used the indexed set $(\mathcal{L}ynX_{m-1}, <)$ which is the totally ordered set of all Lyndon words over X_{m-1}. As a consequence, under the one to one correspondence between l and P_l, one then has $\dim_{\mathbf{k}}(\mathrm{DK}^{(m,n)}_{\mathbf{k},\infty}) = \dim_{\mathbf{k}}(\mathcal{L}_{\mathbf{k}}(X_{m-1})_n) = \#\{\text{Lyndon words of length n on } X_{m-1}\}$, denoted $\mathcal{L}yn(m-1,n)$. We hence deduce that the infinite Drinfeld-Kohno Lie algebra

$$\mathrm{DK}_{\mathbf{k},\infty} = \bigoplus_{m\geq 2} \mathrm{DK}^{(m)}_{\mathbf{k},\infty} = \bigoplus_{\substack{m\geq 2\\ n\geq 1}} \mathrm{DK}^{(m,n)}_{\mathbf{k},\infty}$$

is in fact equipped with a $(\mathbb{N}_{\geq 2}, \vee)\times(\mathbb{N}_{\geq 1}, +)$-graded Lie algebra structure. Therefore, the enveloping algebra $\mathcal{U}(\mathrm{DK}_{\mathbf{k},\infty})$ inherits a $[(\mathbb{N}_{\geq 2}, \vee)\times(\mathbb{N}_{\geq 1}, +)]\sqcup\{0\}$-graded structure (5.23)

$$\mathcal{U}(\mathrm{DK}_{\mathbf{k},\infty}) = \bigoplus_{s\in S_{\mathbb{N}}\cup\{0\}} \mathcal{U}_s(\mathrm{DK}_{\mathbf{k},\infty}),$$

where $S_{\mathbb{N}} := (\mathbb{N}_{\geq 2}, \vee)\times(\mathbb{N}_{\geq 1}, +)$ and $\mathcal{U}_s(\mathrm{DK}_{\mathbf{k},\infty}) = T_s(\mathrm{DK}_{\mathbf{k},\infty})\big/_{\mathcal{J}_s}$, here we used $T_s(\mathrm{DK}_{\mathbf{k},\infty}) = \bigoplus_{\substack{w\in[S_{\mathbb{N}}]^*\\ \mu(w)=s}} T_w(\mathrm{DK}_{\mathbf{k},\infty})$ as in the formula (5.22). Note that the Hilbert series

$$Hilb(\mathcal{U}(\mathrm{DK}_{\mathbf{k},\infty})) = \sum_{s\in S_{\mathbb{N}}\cup\{0\}} \dim_{\mathbf{k}}(\mathcal{U}_s(\mathrm{DK}_{\mathbf{k},\infty})).s \in \mathbb{Q}[[S_{\mathbb{N}}\cup\{0\}]],$$

where for each $s\in S_{\mathbb{N}}\cup\{0\}$, $\dim_{\mathbf{k}}(\mathcal{U}_s(\mathrm{DK}_{\mathbf{k},\infty})) = (m-1)^n$ if $s=(m,n)\in S_{\mathbb{N}}$ and $\dim_{\mathbf{k}}(\mathcal{U}_0(\mathrm{DK}_{\mathbf{k},\infty})) = 1$ otherwise. As a final consequence, we can translate this series into the commutative algebra of formal power series of two variables $\mathbb{Q}[[t_1, t_2]]$ (endowed with the convolution product that is with the supremum w.r.t. t_1 and ordinary Cauchy product w.r.t. t_2[20]) by the following

$$Hilb(\mathcal{U}(\mathrm{DK}_{\mathbf{k},\infty}), t_1, t_2) = \prod_{m\geq 2}\left(1 + \frac{t_1^m(m-1)t_2}{1-(m-1)t_2}\right) \in \mathbb{Q}[[t_1, t_2]]$$

where, for each $m \in \mathbb{N}_{\geq 2}$,

$$1 + \frac{t_1^m(m-1)t_2}{1-(m-1)t_2} = 1 + \sum_{n\geq 1}(m-1)^n t_1^m t_2^n.$$

Let us give a calculation example for this multiplication,

$$\begin{aligned}(1+\frac{t_1^2t_2}{1-t_2})(1+\frac{t_1^3 2t_2}{1-2t_2}) &= 1 + \frac{t_1^2t_2}{1-t_2} + \frac{t_1^3 2t_2}{1-2t_2} + \frac{t_1^{2\vee 3}t_2.2t_2}{(1-t_2)(1-2t_2)}\\ &= 1 + \frac{t_1^2t_2}{1-t_2} + \frac{2t_1^3t_2}{1-2t_2} + \frac{2t_1^3t_2^2}{(1-t_2)(1-2t_2)}.\end{aligned}$$

Example 2.5. In the same manner as above, the Drinfeld-Kohno Lie algebra $\mathrm{DK}_{\mathbf{k},n+1}$ can be equipped with a $([2, n+1], \vee) \times (\mathbb{N}_{\geq 1}, +)$-graded Lie algebra structure

$$\mathrm{DK}_{\mathbf{k},n+1} = \bigoplus_{2\leq i\leq n+1} \mathrm{DK}_{\mathbf{k},\infty}^{(i)} = \bigoplus_{\substack{2\leq i\leq n+1\\ 1\leq j<+\infty}} \mathrm{DK}_{\mathbf{k},\infty}^{(i,j)}.$$

and then the enveloping algebra $\mathcal{U}(\mathrm{DK}_{\mathbf{k},n+1})$ inherits a $[([2, n+1], \vee) \times (\mathbb{N}_{\geq 1}, +)] \sqcup \{0\}$-graded structure (5.23)

$$\mathcal{U}(\mathrm{DK}_{\mathbf{k},n+1}) = \bigoplus_{s\in T_{\mathbb{N}}\cup\{0\}} \mathcal{U}_s(\mathrm{DK}_{\mathbf{k},n+1}),$$

where $T_{\mathbb{N}} := ([2, n+1], \vee) \times (\mathbb{N}_{\geq 1}, +)$. Therefore, the Hilbert series

$$Hilb(\mathcal{U}(\mathrm{DK}_{\mathbf{k},n+1}), t_1, t_2) = \prod_{2\leq i\leq n+1}\left(1 + \frac{t_1^i(i-1)t_2}{1-(i-1)t_2}\right) \in \mathbb{Q}[[t_1, t_2]].$$

When $t_1 = 1$, we can recover Kohno's formula appearing in Example 2.3.

[20] In other words, two series $\sum_{m_1,n_1\geq 0}\alpha_{m_1,n_1}t_1^{m_1}t_2^{n_1}$ and $\sum_{m_2,n_2\geq 0}\beta_{m_2,n_2}t_1^{m_2}t_2^{n_2}$ being given in $\mathbb{Q}[[t_1, t_2]]$, then their convolution product is

$$\sum_{m_i,n_i\geq 0}\alpha_{m_1,n_1}\beta_{m_2,n_2}t_1^{m_1\vee m_2}t_2^{n_1+n_2} = \sum_{m,n\geq 0}\left(\sum_{\substack{m_i,n_i\geq 0\\ m_1\vee m_2=m\\ n_1+n_2=n}}\alpha_{m_1,n_1}\beta_{m_2,n_2}\right)t_1^m t_2^n$$

the internal sum being finite because the monoid of monomials has "D" property.

2.4 Smash product algebra and Lazard's elimination.

In this section, we investigate other aspects of Lazard's elimination within **k-AAU** the category of unital associative **k**-algebras. We first introduce crossed and smash products of algebras and discuss some relevance with semi-direct products of Lie algebras and the universal enveloping functor by Example 2.7 and Proposition 2.27. In the last two examples of this section, we will study a practical application to achieve Lazard's elimination and the quotient of Lazard's elimination in **k-AAU**.

Let **k** be a commutative ring with unit. We now study the crossed product of algebras, especially a smash product of a bialgebra and an associative algebra with unit. The reader who is only interested in studying these products may turn to R.K. Molnar [84], S. Montgomery [86] and A. Borowiec, W. Marcinek [8], which can be read independently to what is presented as follows.

Definition 2.4. Let $\mathcal{A}, \mathcal{B}, \mathcal{H}$ be three objects in **k-AAU**. Assume that there are monomorphisms $i_{\mathcal{A}} : \mathcal{A} \to \mathcal{H}$ and $i_{\mathcal{B}} : \mathcal{B} \to \mathcal{H}$ in **k-AAU**. We say that $\mathcal{H}$ is a crossed (twisted) product of $\mathcal{A}$ and $\mathcal{B}$ if the canonical mapping $\Phi : \mathcal{A} \otimes \mathcal{B} \to \mathcal{H}, a \otimes b \mapsto (i_{\mathcal{A}} \otimes i_{\mathcal{B}})(a \otimes b)$ is an isomorphism in **k-Mod**.

As an immediate consequence of the above definition, a crossed product of two objects $\mathcal{A}$ and $\mathcal{B}$ is unique up to an isomorphism in **k-AAU**.

Example 2.6. The standard tensor product of algebras $\mathcal{A} \otimes \mathcal{B}$ is a crossed product of two objects $\mathcal{A}$ and $\mathcal{B}$ in **k-AAU** (here $\Phi = \mathrm{Id}$), where $i_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}, a \mapsto a \otimes 1_{\mathcal{B}}$ and $i_{\mathcal{B}} : \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}, b \mapsto 1_{\mathcal{A}} \otimes b$ are the natural monomorphisms in **k-AAU** [21].

The following definition is necessary for describing the main theorem of crossed product of algebras.

[21] If **k** is only a ring, embeddings may fail as shows the example of $\mathbb{Z}/3\mathbb{Z} = \mathcal{A}$ and $\mathbb{Z} = \mathcal{B}$ as two $\mathbb{Z}$-algebras.

Definition 2.5. Suppose given two objects $\mathcal{A}$ and $\mathcal{B}$ in **k-AAU**. A morphism $\tau : \mathcal{B} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ in **k-Mod** is called an algebra cross if it satisfies the following conditions

c1) $\tau(1_{\mathcal{B}} \otimes a) = a \otimes 1_{\mathcal{B}}$,

c2) $\tau \circ (m_{\mathcal{B}} \otimes \mathrm{Id}_{\mathcal{A}}) = (\mathrm{Id}_{\mathcal{A}} \otimes m_{\mathcal{B}}) \circ (\tau \otimes \mathrm{Id}_{\mathcal{B}}) \circ (\mathrm{Id}_{\mathcal{B}} \otimes \tau)$,

d1) $\tau(b \otimes 1_{\mathcal{A}}) = 1_{\mathcal{A}} \otimes b$,

d2) $\tau \circ (\mathrm{Id}_{\mathcal{B}} \otimes m_{\mathcal{A}}) = (m_{\mathcal{A}} \otimes \mathrm{Id}_{\mathcal{B}}) \circ (\mathrm{Id}_{\mathcal{A}} \otimes \tau) \circ (\tau \otimes \mathrm{Id}_{\mathcal{A}})$.

Theorem 2.25. *If $\tau : \mathcal{B} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ is an algebra cross then the tensor product of algebras $\mathcal{A} \otimes \mathcal{B}$ equipped with the multiplication $m_\tau = (m_{\mathcal{A}} \otimes m_{\mathcal{B}}) \circ (\mathrm{Id}_{\mathcal{A}} \otimes \tau \otimes \mathrm{Id}_{\mathcal{B}})$ is an object in* **k-*AAU***, *denoted by $\mathcal{A} \rtimes_\tau \mathcal{B} := (\mathcal{A} \otimes \mathcal{B}, m_\tau, 1_{\mathcal{A}\otimes\mathcal{B}})$. Moreover, $\mathcal{A} \rtimes_\tau \mathcal{B}$ is a crossed product of $\mathcal{A}$ and $\mathcal{B}$ if* **k** *is a field.*

Proof. We want to prove that $\mathcal{A} \rtimes_\tau \mathcal{B} \in$ **k-AAU**.
Associativity can be proved by direct computation (as below for the unit (2.111)) or found in the literature (see also Proposition 2.2 in [27] or Proposition 2.3 and Remark 2.4 (1) in [24]) or even diagrammatically (i.e. using Penrose-like calculus). In fact, the diagram

$$\begin{array}{ccc} (\mathcal{A}\otimes\mathcal{B})\otimes(\mathcal{A}\otimes\mathcal{B})\otimes(\mathcal{A}\otimes\mathcal{B}) & \xrightarrow{\mathrm{Id}_{\mathcal{A}\otimes\mathcal{B}}\otimes m_\tau} & (\mathcal{A}\otimes\mathcal{B})\otimes(\mathcal{A}\otimes\mathcal{B}) \\ \downarrow{\scriptstyle m_\tau\otimes\mathrm{Id}_{\mathcal{A}\otimes\mathcal{B}}} & & \downarrow{\scriptstyle m_\tau} \\ (\mathcal{A}\otimes\mathcal{B})\otimes(\mathcal{A}\otimes\mathcal{B}) & \xrightarrow{\quad m_\tau \quad} & \mathcal{A}\otimes\mathcal{B} \end{array}$$

commutes by using the above relations (c1),(c2), (d1),(d2) and the associative laws $m_{\mathcal{A}} \circ (\mathrm{Id}_{\mathcal{A}} \otimes m_{\mathcal{A}}) = m_{\mathcal{A}} \circ (m_{\mathcal{A}} \otimes \mathrm{Id}_{\mathcal{A}}), m_{\mathcal{B}} \circ (\mathrm{Id}_{\mathcal{B}} \otimes m_{\mathcal{B}}) = m_{\mathcal{B}} \circ (m_{\mathcal{B}} \otimes \mathrm{Id}_{\mathcal{B}})$. The only thing left to prove is that $1_{\mathcal{A}\otimes\mathcal{B}} = 1_{\mathcal{A}} \otimes 1_{\mathcal{B}}$ is a unit for the multiplication m_τ. Then, for the fact that it is a unit on the right we just need the condition (c1) as $\tau(1_{\mathcal{B}} \otimes a) = a \otimes 1_{\mathcal{A}}$, then

$$\begin{aligned} & m_\tau[(1_{\mathcal{A}} \otimes 1_{\mathcal{B}}) \otimes (y_1 \otimes y_2)] = (m_{\mathcal{A}} \otimes m_{\mathcal{B}}) \circ (\mathrm{Id}_{\mathcal{A}} \otimes \tau \otimes \mathrm{Id}_{\mathcal{B}})[(1_{\mathcal{A}} \otimes 1_{\mathcal{B}}) \otimes (y_1 \otimes y_2)] \\ & = (m_{\mathcal{A}} \otimes m_{\mathcal{B}})[\mathrm{Id}_{\mathcal{A}}(1_{\mathcal{A}}) \otimes \tau(1_{\mathcal{B}} \otimes y_1) \otimes \mathrm{Id}_{\mathcal{B}}(y_2)] \\ & = (m_{\mathcal{A}} \otimes m_{\mathcal{B}})[1_{\mathcal{A}} \otimes y_1 \otimes 1_{\mathcal{B}} \otimes y_2] = y_1 \otimes y_2. \end{aligned} \tag{2.111}$$

For the unit on the right, we must use the condition (d1) as

$$\begin{aligned} &m_\tau[(x_1 \otimes x_2) \otimes (1_{\mathcal{A}} \otimes 1_{\mathcal{B}})] = (m_{\mathcal{A}} \otimes m_{\mathcal{B}}) \circ (\mathrm{Id}_{\mathcal{A}} \otimes \tau \otimes \mathrm{Id}_{\mathcal{B}})[(x_1 \otimes x_2) \otimes (1_{\mathcal{A}} \otimes 1_{\mathcal{B}})] \\ &= (m_{\mathcal{A}} \otimes m_{\mathcal{B}})[x_1 \otimes 1_{\mathcal{A}} \otimes x_2 \otimes 1_{\mathcal{B}}] = x_1 \otimes x_2. \end{aligned}$$

We also observe that if **k** is a field then $i_{\mathcal{A}}: \ \mathcal{A} \to \mathcal{A} \rtimes_\tau \mathcal{B}, a \mapsto a \otimes 1_{\mathcal{B}}$ and $i_{\mathcal{B}}: \ \mathcal{B} \to \mathcal{A} \rtimes_\tau \mathcal{B}, b \mapsto 1_{\mathcal{A}} \otimes b$ are the natural monomorphisms in **k-AAU**. Further, the mapping $\Phi: \ \mathcal{A} \otimes \mathcal{B} \to \mathcal{A} \rtimes_\tau \mathcal{B}, a \otimes b \mapsto m_\tau \circ (i_{\mathcal{A}} \otimes i_{\mathcal{B}})(a \otimes b) = (m_{\mathcal{A}} \otimes m_{\mathcal{B}}) \circ (\mathrm{Id}_{\mathcal{A}} \otimes \tau \otimes \mathrm{Id}_{\mathcal{B}}) \circ (i_{\mathcal{A}} \otimes i_{\mathcal{B}})(a \otimes b) = a \otimes b$ is a canonical isomorphism in **k-Mod**. Thus, $\mathcal{A} \rtimes_\tau \mathcal{B}$ is a crossed product of $\mathcal{A}$ and $\mathcal{B}$. □

Remark 2.13. i) In general, when **k** is a unital commutative ring and $\tau : \mathcal{B} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ is an algebra cross, we will also say that $\mathcal{A} \rtimes_\tau \mathcal{B}$ is a crossed (twist) product of $\mathcal{A}$ and $\mathcal{B}$ if no confusion arises.

ii) The standard twist map $\tau: \ \mathcal{B} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ defined as $\tau(b \otimes a) = a \otimes b$ satisfies all conditions for the algebra cross in Definition 2.5 and then $\mathcal{A} \rtimes_\tau \mathcal{B}$ is the standard tensor product of algebras $\mathcal{A}$ and $\mathcal{B}$.

iii) (See A. Borowiec and W. Marcinek [8]) Let $f_1 \colon \mathcal{A}_1 \to \mathcal{B}_1$, $f_2 \colon \mathcal{A}_2 \to \mathcal{B}_2$ be two morphisms in **k-AAU**. Given two algebra crosses $\tau : \mathcal{A}_2 \otimes \mathcal{A}_1 \to \mathcal{A}_1 \otimes \mathcal{A}_2$ and $\sigma : \mathcal{B}_2 \otimes \mathcal{B}_1 \to \mathcal{B}_1 \otimes \mathcal{B}_2$, then $f \colon \mathcal{A}_1 \rtimes_\tau \mathcal{A}_2 \to \mathcal{B}_1 \rtimes_\sigma \mathcal{B}_2, a \otimes b \mapsto f_1(a) \otimes f_2(b)$ is a crossed product algebra homomorphism if and only if the following diagram commutes

$$\begin{tikzcd} \mathcal{A}_2 \otimes \mathcal{A}_1 \arrow[r, "f_2 \otimes f_1"] \arrow[d, "\tau"] & \mathcal{B}_2 \otimes \mathcal{B}_1 \arrow[d, "\sigma"] \\ \mathcal{A}_1 \otimes \mathcal{A}_2 \arrow[r, "f_1 \otimes f_2"] & \mathcal{B}_1 \otimes \mathcal{B}_2. \end{tikzcd}$$

iv) It can be shown that if $\tau : \mathcal{B} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ is an algebra cross, then $\tau_0 := \tau_{12}\tau\tau_{12}$ (where τ_{12} is the standard tensor flip $x \otimes y \to y \otimes x$) is an algebra cross

$$\tau_0 : \mathcal{A}^0 \otimes \mathcal{B}^0 \to \mathcal{B}^0 \otimes \mathcal{A}^0$$

where the $(-)^0$ operator means passing to the opposite algebra.

We now give the general theory of the crossed product of algebras to a particular case that is efficient in practice. Let $\mathcal{A}$ be an associative algebra with unit $1_{\mathcal{A}}$ i.e.

$\mathcal{A} \in$ **k-AAU** and $\mathcal{B}$ be a bialgebra i.e. $(\mathcal{B}, m_{\mathcal{B}}, 1_{\mathcal{B}}, \Delta_{\mathcal{B}}, e_{\mathcal{B}})$ with the usual axioms. We suppose given a left $\mathcal{B}$-module with action denoted by $\triangleright : \mathcal{B} \otimes \mathcal{A} \to \mathcal{A}$ i.e. $b_1 \triangleright (b_2 \triangleright a) = (b_1 b_2) \triangleright a$ and $1_{\mathcal{B}} \triangleright a = a$ identically. The algebra $\mathcal{A}$ is said to be a *left $\mathcal{B}$-module algebra* if it satisfies

i) $b \triangleright (a_1 a_2) = \sum_{(1)(2)} (b^{(1)} \triangleright a_1)(b^{(2)} \triangleright a_2)$, where we have used Sweedler's notation[22] $\Delta_{\mathcal{B}}(b) = \sum_{(1)(2)} b^{(1)} \otimes_{\mathbf{k}} b^{(2)}$,

ii) and $b \triangleright 1_{\mathcal{A}} = \epsilon_{\mathcal{B}}(b) 1_{\mathcal{A}}$.

It is easy to verify that the following corollary is true.

Corollary 2.26. *(Smash Product $\mathcal{A}\sharp\mathcal{B}$) With the preceding conditions and the following multiplication $m_\sharp :\ (\mathcal{A} \otimes \mathcal{B}) \otimes (\mathcal{A} \otimes \mathcal{B}) \to \mathcal{A} \otimes \mathcal{B}$,*

$$m_\sharp[(x_1 \otimes x_2) \otimes (y_1 \otimes y_2)] = \sum_{(1)(2)} x_1(x_2^{(1)} \triangleright y_1) \otimes x_2^{(2)} y_2 \tag{2.112}$$

*$\mathcal{A}\sharp\mathcal{B} = (\mathcal{A} \otimes \mathcal{B}, m_\sharp, 1_{\mathcal{A}\otimes\mathcal{B}})$ is an object in **k-AAU**. This algebra is called the smash product algebra between the bialgebra $\mathcal{B}$ and the left $\mathcal{B}$-module algebra $\mathcal{A}$.*

Proof. If the mapping $\triangleright : \mathcal{B} \otimes \mathcal{A} \to \mathcal{A}$ is a left $\mathcal{B}$-module algebra action then one can easily check through direct calculation that the mapping $\tau : \mathcal{B} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}, b \otimes a \mapsto \sum_{(1)(2)} b^{(1)} \triangleright a \otimes b^{(2)}$ is an algebra cross in **k-Mod**. According to Theorem 2.25, we deduce that $\mathcal{A}\sharp\mathcal{B} \equiv \mathcal{A} \rtimes_\tau \mathcal{B}$ is an object in **k-AAU**, where the multiplication $m_\sharp = (m_{\mathcal{A}} \otimes m_{\mathcal{B}}) \circ (\mathrm{Id}_{\mathcal{A}} \otimes \tau \otimes \mathrm{Id}_{\mathcal{B}}) = m_\tau$. $\square$

Remark 2.14. Assume that the bialgebra $\mathcal{B}$ and the left $\mathcal{B}$-module algebra $\mathcal{A}$ are endowed with Hopf structures satisfying suitable circumstances introduced in Molnar [84] Thm 2.13, then the smash product algebra $\mathcal{A}\sharp\mathcal{B}$ has a unique Hopf algebra structure in which $\mathcal{A} \otimes \mathbf{k}1_{\mathcal{B}}$ and $\mathbf{k}1_{\mathcal{A}} \otimes \mathcal{B}$ are Hopf subalgebras (where $\mathbf{k}1_{\mathcal{A}}$ and $\mathbf{k}1_{\mathcal{B}}$ are the group algebras), called the semi-direct product of Hopf algebras.

Here is an important and natural example of smash product algebras.

Example 2.7. Let $\mathfrak{g}_i,\ i = 1, 2$ be two objects in **k-Lie** and $\alpha :\ \mathfrak{g}_2 \to \mathfrak{Der}(\mathfrak{g}_1)$ be a morphism in **k-Lie**. We first extend α from $\mathfrak{g}_2$ to $\mathfrak{Der}(\mathcal{U}(\mathfrak{g}_1)) \subset \mathrm{End}(\mathcal{U}(\mathfrak{g}_1))$ as

[22] See Montgomery [86] Ch 1 §1.4.2.

in Bourbaki [13] Ch I §2.8 Prop 7. Moreover, we can also extend α as a morphism $\alpha_{\mathcal{U}} : \mathcal{U}(\mathfrak{g}_2) \to \mathrm{End}(\mathcal{U}(\mathfrak{g}_1))$ in **k-AAU** by the universal property (5.37). Together with a bialgebra structure $(\mathcal{U}(\mathfrak{g}_2), \mu_{\mathcal{U}}, 1_{\mathbf{k}}, \Delta_{\mathcal{U}}, \epsilon_{\mathcal{U}})$, we then obtain a left $\mathcal{U}(\mathfrak{g}_2)$-module algebra action $\triangleright : \mathcal{U}(\mathfrak{g}_2) \otimes \mathcal{U}(\mathfrak{g}_1) \to \mathcal{U}(\mathfrak{g}_1), b \otimes a \mapsto b \triangleright a = \alpha_{\mathcal{U}}(b)(a)$. As we already constructed above, by Corollary 2.26, the $\mathbf{k}$-module $\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$ can be endowed with a smash product structure $\mathcal{U}(\mathfrak{g}_1) \sharp \mathcal{U}(\mathfrak{g}_2) = (\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2), 1_{\mathbf{k}} \otimes 1_{\mathbf{k}})$, where the multiplication is

$$m_\sharp[(u_1 \otimes u_2) \otimes (v_1 \otimes v_2)] = \sum_{(1)(2)} u_1 \alpha_{\mathcal{U}}(u_2^{(1)})(v_1) \otimes u_2^{(2)} v_2. \tag{2.113}$$

This brings us to the following proposition

Proposition 2.27. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two objects in* **k-*Lie***. *We suppose given also a Lie $\mathbf{k}$-algebra morphism $\alpha : \mathfrak{g}_2 \to \mathfrak{Der}(\mathfrak{g}_1)$ and $f_i \colon \mathfrak{g}_i \to \mathfrak{g}$, two Lie homomorphisms into a Lie $\mathbf{k}$-algebra $\mathfrak{g}$ satisfying the equivariance* (2.13)

$$\begin{tikzcd} \mathfrak{g}_2 \otimes \mathfrak{g}_1 \arrow[r, "f_2 \otimes f_1"] \arrow[d, "\alpha_\otimes"] & \mathfrak{g} \otimes \mathfrak{g} \arrow[d, "\mathrm{ad}^{\mathfrak{g}}_\otimes"] \\ \mathfrak{g}_1 \arrow[r, "f_1"] & \mathfrak{g}, \end{tikzcd}$$

or equivalently (2.12) *i.e.*

$$f_1(\alpha(b, a)) = \mathrm{ad}^{\mathfrak{g}}(f_2(b), f_1(a))$$

for all $b \in \mathfrak{g}_2$, $a \in \mathfrak{g}_1$. Then one has

1. *There is a unique morphism $f \colon \mathfrak{g}_1 \rtimes \mathfrak{g}_2 \to \mathfrak{g}$ in* **k-*Lie*** *extending f_1 and f_2 in the usual sense.*

2. *There is a unique morphism $f_* \colon \mathcal{U}(\mathfrak{g}_1)\sharp\mathcal{U}(\mathfrak{g}_2) \to \mathcal{U}(\mathfrak{g})$ in* **k-*AAU*** *extending $f_i^* := \mathcal{U}(f_i) : \mathcal{U}(\mathfrak{g}_i) \to \mathcal{U}(\mathfrak{g})$ (for $i = 1, 2$) in the usual sense, where $\mathcal{U}$:* **k-*Lie*** $\to$ **k-*AAU***, *$\mathfrak{g} \mapsto \mathcal{U}(\mathfrak{g})$ is the universal enveloping functor.*

3. *If $f \colon \mathfrak{g}_1 \rtimes \mathfrak{g}_2 \to \mathfrak{g}$ is an isomorphism in* **k-*Lie*** *then $f_* \colon \mathcal{U}(\mathfrak{g}_1)\sharp\mathcal{U}(\mathfrak{g}_2) \to \mathcal{U}(\mathfrak{g})$ is an isomorphism in* **k-*AAU***.

Proof. 1. It is indeed a consequence of Proposition 2.2.

2. Let us define a **k**-module morphism $f_*\colon \mathcal{U}(\mathfrak{g}_1)\sharp\mathcal{U}(\mathfrak{g}_2) \to \mathcal{U}(\mathfrak{g})$ sending $u_1 \otimes u_2$ to $f_1^*(u_1)f_2^*(u_2)$. We now prove that f_* is a morphism in **k-AAU**. In fact, one has

$$\begin{aligned} &f_* \circ m_\sharp[(u_1 \otimes u_2) \otimes (v_1 \otimes v_2)] = f_*[\sum_{(1)(2)} u_1\alpha_{\mathcal{U}}(u_2^{(1)})(v_1) \otimes u_2^{(2)}v_2] \\ &= \sum_{(1)(2)} f_1^*(u_1)f_1^*(\alpha_{\mathcal{U}}(u_2^{(1)})(v_1))f_2^*(u_2^{(2)})f_2^*(v_2) \\ &= f_1^*(u_1)\big[\sum_{(1)(2)} f_1^*(\alpha_{\mathcal{U}}(u_2^{(1)})(v_1))f_2^*(u_2^{(2)})\big]f_2^*(v_2) \end{aligned}$$

and then $f_* \circ m_\sharp[(u_1 \otimes u_2) \otimes (v_1 \otimes v_2)] = f_*(u_1 \otimes u_2)f_*(v_1 \otimes v_2)$ if we have an equation $\sum_{(1)(2)} f_1^*(\alpha_{\mathcal{U}}(u_2^{(1)})(v_1))f_2^*(u_2^{(2)}) = f_2^*(u_2)f_1^*(v_1)$ in $\mathcal{U}(\mathfrak{g})$. This formula can be obtained from inductive processes and by the following extension of the equivariance (2.13) (where the below right-hand side is the right-normed bracketing in $\mathcal{U}(\mathfrak{g})$)

$$f_1^*(\alpha_{\mathcal{U}}(v)(u)) = \mathrm{ad}^{\mathcal{U}(\mathfrak{g})}_{f_2(b_1)} \circ \cdots \circ \mathrm{ad}^{\mathcal{U}(\mathfrak{g})}_{f_2(b_k)}[f_1^*(u)] \tag{2.114}$$

for all $v = b_1 \cdots b_k \in \mathcal{U}(\mathfrak{g}_2)$ and $u \in \mathcal{U}(\mathfrak{g}_1)$, where b_i in $\mathfrak{g}_2$ and $f_2^*(v) = f_2(v_1)\cdots f_2(v_k)$. More precise, if $v_1 \in \mathfrak{g}_1$ and $u_2 \in \mathfrak{g}_2$ then $\Delta_{\mathcal{U}}(u_2) = u_2 \otimes 1_{\mathbf{k}} + 1_{\mathbf{k}} \otimes u_2$, thus one has

$$\begin{aligned} &\sum_{(1)(2)} f_1^*(\alpha_{\mathcal{U}}(u_2^{(1)})(v_1))f_2^*(u_2^{(2)}) = f_1(\alpha_{\mathcal{U}}(u_2)(v_1)) + f_1(v_1)f_2(u_2) \\ &= \mathrm{ad}^{\mathcal{U}(\mathfrak{g})}_{f_2(u_2)}[f_1(v_1)] + f_1(v_1)f_2(u_2) \\ &= f_2(u_2)f_1(v_1) = f_2^*(u_2)f_1^*(v_1); \end{aligned}$$

in general case if $v_1 = av_1' \in \mathcal{U}(\mathfrak{g}_1)$ and $u_2 = bu_2' \in \mathcal{U}(\mathfrak{g}_2)$, where $(a, b) \in \mathfrak{g}_1 \times \mathfrak{g}_2$ and $(v_1', u_2') \in \mathcal{U}(\mathfrak{g}_1) \times \mathcal{U}(\mathfrak{g}_2)$, we leave it as a small exercise to interested readers (hints: we notice that $\Delta_{\mathcal{U}}(u_2) = \Delta_{\mathcal{U}}(b)\Delta_{\mathcal{U}}(u_2') = (b \otimes 1_{\mathbf{k}} + 1_{\mathbf{k}} \otimes b)(\sum_{(1)(2)} u_2'^{(1)} \otimes u_2'^{(2)}) = \sum_{(1)(2)} \big[bu_2'^{(1)} \otimes u_2'^{(2)} + u_2'^{(1)} \otimes bu_2'^{(2)}\big]$). Further, the uniqueness of the algebra homomorphism f_* comes from the definition of the mapping because

$$\begin{aligned} f_*(u_1 \otimes u_2) &= f_* \circ m_\sharp[(u_1 \otimes 1_{\mathbf{k}}) \otimes (1_{\mathbf{k}} \otimes u_2)] \\ &= f_*(u_1 \otimes 1_{\mathbf{k}})f_*(1_{\mathbf{k}} \otimes u_2) = f_1^*(u_1)f_2^*(u_2). \end{aligned}$$

3. If $g : \mathfrak{g} \to \mathfrak{g}_1 \rtimes \mathfrak{g}_2$ is an inverse Lie homomorphim of f. Then, the **k**-algebra morphism f_* can be reversed by constructing an algebra homomorphism g_* :

$\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}_1)\sharp\mathcal{U}(\mathfrak{g}_2)$ as an extension of $g : \mathfrak{g} \to \mathfrak{g}_1 \rtimes \mathfrak{g}_2 \hookrightarrow \mathcal{U}(\mathfrak{g}_1)\sharp\mathcal{U}(\mathfrak{g}_2)$ by the universal property (5.37). It immediately implies that $f_* \circ g_* = \mathrm{Id}_{\mathcal{U}(\mathfrak{g})}$ and $g_* \circ f_* = \mathrm{Id}_{\mathcal{U}(\mathfrak{g}_1)\sharp\mathcal{U}(\mathfrak{g}_2)}$.

□

It turns out that this result is very useful, in examples below a lot more algebra isomorphism structures are presented, which are just a little bit harder to state immediately to the reader. More precisely, we now treat the first application of the scheme to the free associative algebra $(\mathbf{k}\langle X\rangle, \mathrm{conc}, 1_{X^*})$.

Example 2.8. Let $X = B + Z$ be a graded set and $\alpha : \mathcal{L}_{\mathbf{k}}(B) \to \mathfrak{Der}(\mathcal{L}_{\mathbf{k}}(B^*Z))$ be a morphism in **k-Lie** defined immediately by the extension of the left translation $t_b^{(0)} : B^*Z \to B^*Z, uz \mapsto buz$ (for all $b \in B$) to the derivation $t_b^{(1)} \in \mathfrak{Der}(\mathcal{L}_{\mathbf{k}}(B^*Z))$ and then, by universal property, the map $b \mapsto t_b^{(1)} : B \to \mathfrak{Der}(\mathcal{L}_{\mathbf{k}}(B^*Z))$ can be extended to α that is indeed a morphism in **k-Lie** between the free Lie algebra $\mathcal{L}_{\mathbf{k}}(B)$ and the usual Lie algebra $\mathfrak{Der}(\mathcal{L}_{\mathbf{k}}(B^*Z))$ (the reader can review the above construction in the proof of Theorem 2.3). We consider the pairs of Lie homomorphisms $f_1 : \mathcal{L}_{\mathbf{k}}(B^*Z) \to \mathcal{L}_{\mathbf{k}}(X)$ and $f_2 : \mathcal{L}_{\mathbf{k}}(B) \to \mathcal{L}_{\mathbf{k}}(X)$ defined by $f_1(uz) = rn(uz)$ (where rn is the right-normed bracketing i.e. $rn(uz) = \mathrm{ad}_{b_1}^{\mathcal{L}_{\mathbf{k}}(X)} \circ \cdots \circ \mathrm{ad}_{b_k}^{\mathcal{L}_{\mathbf{k}}(X)}(z) = \mathrm{ad}_{(u)}^{\mathcal{L}_{\mathbf{k}}(X)}(z)$ for each $u = b_1 \cdots b_k \in B^*$ and $z \in Z$) and $f_2(b) = b$ satisfying the equivariance condition of (2.13) with respect to α. Then, the basic results of Proposition 2.27 are applied and summarized in the following properties

i) one has a morphism of Lie algebras $f : \mathcal{L}_{\mathbf{k}}(B^*Z) \rtimes \mathcal{L}_{\mathbf{k}}(B) \to \mathcal{L}_{\mathbf{k}}(X)$ extending f_i $(i = 1, 2)$ in the usual sense by Proposition 2.27 point (1).

ii) the algebra homomorphism f_* between the smash product $\mathcal{U}(\mathcal{L}_{\mathbf{k}}(B^*Z))\sharp\mathcal{U}(\mathcal{L}_{\mathbf{k}}(B))$ and the enveloping algebra $\mathcal{U}(\mathcal{L}_{\mathbf{k}}(X))$ which sends $u_1 \otimes u_2 \mapsto f_1^*(u_1)f_2^*(u_2)$ extending $f_1^* : \mathcal{U}(\mathcal{L}_{\mathbf{k}}(B^*Z)) \to \mathcal{U}(\mathcal{L}_{\mathbf{k}}(X))$ and $f_2^* : \mathcal{U}(\mathcal{L}_{\mathbf{k}}(B)) \to \mathcal{U}(\mathcal{L}_{\mathbf{k}}(X))$ is an isomorphism in **k-AAU** by obtaining from Proposition 2.27 points (2),(3) and the fact that f is a Lie isomorphism as a consequence of Theorem 2.3 which constructed Lazard's elimination in **k-Lie**.

iii) then, recall that the free associative algebra $\mathbf{k}\langle X\rangle$ is identified with $\mathcal{U}(\mathcal{L}_{\mathbf{k}}(X))$

the universal enveloping algebra of the free Lie algebra $\mathcal{L}_{\mathbf{k}}(X)$ [23], thus one has $\mathbf{k}\langle B^*Z\rangle = \mathcal{U}(\mathcal{L}_{\mathbf{k}}(B^*Z))$ and $\mathbf{k}\langle B\rangle = \mathcal{U}(\mathcal{L}_{\mathbf{k}}(B))$. We call these phenomenons $\mathbf{k}\langle B^*Z\rangle\sharp\mathbf{k}\langle B\rangle \xrightarrow{\cong} \mathbf{k}\langle X\rangle$ in **k-AAU** by Lazard's elimination in **k-AAU**.

This example deals with a more general type of the above example for quotients of Lazard's eliminations in **k-Lie**.

Example 2.9. As in Subsection 2.2.2, we suppose given a set $X = B + Z$ (partitioned in two blocks) and a relator $\mathbf{r} = \{r_j\}_{j\in J} \subset \mathcal{L}_{\mathbf{k}}(X)$ which is compatible with the alphabet partition i.e. there exists a partition of the set of indices $J = J_Z \sqcup J_B$ such that $\mathbf{r}_B = \{r_j\}_{j\in J_B} = \mathbf{r}\cap\mathcal{L}_{\mathbf{k}}(X)_B$ and $\mathbf{r}_Z = \{r_j\}_{j\in J_Z} = \mathbf{r}\cap\mathcal{L}_{\mathbf{k}}(X)_{BZ}$. As we have seen in Subsection 2.2.2, we considered that $\mathcal{J}_B$ is the Lie ideal of $\mathcal{L}_{\mathbf{k}}(X)_B$ generated by $\{r_j\}_{j\in J_B}$, $\mathcal{J}, \mathcal{J}_Z$ and $\mathcal{J}_{BZ}$ are the Lie ideals of $\mathcal{L}_{\mathbf{k}}(X)$ generated by $\mathbf{r}, \mathbf{r}_Z$ and $\mathbf{r}_{BZ} = \{\mathrm{ad}_Q\, z\}_{Q\in\mathcal{J}_B, z\in Z}$ respectively. With these constructions above, by Theorem 2.6, we have the following

i) a morphism $[\alpha]$: $\mathcal{L}_{\mathbf{k}}(X)_B\big/\mathcal{J}_B \to \mathfrak{Der}(\mathcal{L}_{\mathbf{k}}(X)_{BZ}\big/\mathcal{J}_{BZ}^Z)$ of Lie algebras, where $\mathcal{J}_{BZ}^Z = \mathcal{J}_Z + \mathcal{J}_{BZ}$ is the Lie ideal of $\mathcal{L}_{\mathbf{k}}(X)_{BZ}$.

ii) $g_1 : \mathcal{L}_{\mathbf{k}}(X)_{BZ}\big/\mathcal{J}_{BZ}^Z \to \mathcal{L}_{\mathbf{k}}(X)\big/\mathcal{J}$ and $g_2 : \mathcal{L}_{\mathbf{k}}(X)_B\big/\mathcal{J}_B \to \mathcal{L}_{\mathbf{k}}(X)\big/\mathcal{J}$ are two morphisms in **k-Lie** satisfying the equivariant property (2.13) w.r.t. $[\alpha]$.

iii) a morphism of Lie algebras (2.28)

$$\beta_{33} : \mathcal{L}_{\mathbf{k}}(X)_{BZ}\big/\mathcal{J}_{BZ}^Z \rtimes \mathcal{L}_{\mathbf{k}}(X)_B\big/\mathcal{J}_B \to \mathcal{L}_{\mathbf{k}}(X)\big/\mathcal{J}$$

extending g_1 and g_2 in the usual sense is an isomorphism in **k-Lie** (meaning the quotient of Lazard's elimination in **k-Lie**).

We are now ready to give an another important application of Proposition 2.27 point (2) and point (3). More precisely, the algebra homomorphism $(\beta_{33})_*$ from the smash product algebra $\mathcal{U}(\mathcal{L}_{\mathbf{k}}(X)_{BZ}\big/\mathcal{J}_{BZ}^Z)\sharp\mathcal{U}(\mathcal{L}_{\mathbf{k}}(X)_B\big/\mathcal{J}_B)$ to the universal enveloping algebra $\mathcal{U}(\mathcal{L}_{\mathbf{k}}(X)\big/\mathcal{J})$ extending $g_1^* : \mathcal{U}(\mathcal{L}_{\mathbf{k}}(X)_{BZ}\big/\mathcal{J}_{BZ}^Z) \to \mathcal{U}(\mathcal{L}_{\mathbf{k}}(X)\big/\mathcal{J})$ and $g_2^* : \mathcal{U}(\mathcal{L}_{\mathbf{k}}(X)_B\big/\mathcal{J}_B) \to \mathcal{U}(\mathcal{L}_{\mathbf{k}}(X)\big/\mathcal{J})$ in the usual sense i.e. $(\beta_{33})_*(u_1\otimes u_2) = g_1^*(u_1)g_2^*(u_2)$ is an isomorphism in **k-AAU**. We call these phenomenons by the quotient of Lazard

[23] See Appendix 5.4.4, Remark 5.6 point (iii).

elimination in **k-AAU**.

We would now like to treat this consequence to special cases:

- Firstly, applying the quotient of Lazard's elimination (in **k-AAU**) to Lazard's Partially Commutative Elimination i.e. Corollary 2.16, if X is a set equipped with a commutation relation θ and B is a subset of X such that $Z = X - B$ is totally non-commutative, then there is an isomorphism in **k-AAU**

$$\mathcal{U}(\mathcal{L}_{\mathbf{k}}(X,\theta)) \cong \mathcal{U}(\mathcal{L}_{\mathbf{k}}(C))\sharp\mathcal{U}(\mathcal{L}_{\mathbf{k}}(B,\theta_B)). \tag{2.115}$$

- Furthermore, if we consider the decomposition of Drinfeld-Kohno Lie algebra in Corollary 2.17, one has an isomorphism

$$\mathcal{U}(\mathrm{DK}_{\mathbf{k},n+1}) \cong \mathcal{U}(\mathcal{L}_{\mathbf{k}}(X_n))\sharp\mathcal{U}(\mathrm{DK}_{\mathbf{k},n}) \tag{2.116}$$

  in **k-AAU**, where X_n is any alphabet of cardinality n.

2.5 Table of Lazard elimination formulas.

We summarize Lazard's elimination principle (2.2)

$$\mathrm{Free}(B+Z) = \mathrm{Free}(C_B[Z]) \rtimes \mathrm{Free}(B)$$

in each category formed (2.1)

$$\mathbf{Mon}, \mathbf{Grp}, \mathbf{k\text{-}Lie} \text{ and } \mathbf{k\text{-}AAU}$$

by the following table

Category	Abbreviation	Elimination formula (free)
Monoids	**Mon**	$X^* = (B^*Z)^*B^*$
Groups	**Grp**	$\Gamma(X) = \Gamma(C_B(Z)) \rtimes \Gamma(B)$
Lie **k**-algebras	**k-Lie**	$\mathcal{L}_{\mathbf{k}}(X) \cong \mathcal{L}_{\mathbf{k}}(B^*Z) \rtimes \mathcal{L}_{\mathbf{k}}(B)$
Unital associative **k**-algebras	**k-AAU**	$\mathbf{k}\langle X\rangle \cong \mathbf{k}\langle B^*Z\rangle \sharp \mathbf{k}\langle B\rangle$

Table 2: Lazard's elimination for the list of categories.

Chapter 3

Characters

Contents

3.1 Introduction.

This chapter is about some characters, their convolutions and their extensions[1]. Taking a bialgebra $(\mathcal{B}, \mu, 1_{\mathcal{B}}, \Delta, \epsilon)$ it is known that the set $\Xi(\mathcal{B}) = \mathrm{Hom}_{\mathbf{k}\text{-}\mathbf{AAU}}(\mathcal{B}, \mathbf{k})$ of characters of the algebra part $(\mathcal{B}, \mu, 1_{\mathcal{B}})$ is a monoid under convolution[2] and, if an antipode

[1] The readers can find definitions about bi- and Hopf algebras in [56].

[2] Will be defined below Section 3.2.

is at hand (then $\mathcal{B}$ is a Hopf algebra), this monoid is a group (inversion being provided by precomposition with the antipode, see [34]).
Here will be dealt mainly with two types of characters

- Shuffle characters which will be provided by the algebra of polylogarithms (at first indexed with noncommutative polynomials $\mathbb{C}\langle x_0, x_1\rangle$ and extended to series of $\mathrm{Dom}(\mathrm{Li}) \subsetneq \mathbb{C}\langle\langle x_0, x_1\rangle\rangle$ (as in Figure 3.2).

- Stuffle characters that are provided by Harmonic sums defined on a word $w = y_{s_1} \cdots y_{s_r}$ by

$$\mathrm{H}_w = \mathrm{H}_{s_1,\cdots,s_r} = \sum_{N \geq n_1 > \cdots > n_r > 0} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}.$$

- The link between Polylogarithms, Harmonic sums and then to MZV which are (some of) their limits being provided (classically) by Taylor expansion around zero (see Proposition 3.13) of some Polylogarithms, the transfer of the extended domain Dom(Li) to Harmonic sums is warranted by *normal families* of functions (a.k.a. such in the work [85]).

Shuffle and Stuffle characters are particular cases of convolution characters on enveloping algebras, the link with Lazard elimination and, more generally, with semidirect products is highlighted by formula (3.7).

On the side of indexation, one remarks that the Stuffle product is a perturbation[3] of the Shuffle product. Many such perturbations can be found in the literature (see a non exhaustive table of them in [33]) as well as their shifted variants ([9, 42, 57, 47]), some of them directly linked to conjectures about MZV ([47]).

The structure of the chapter is the following:
In the next part (i.e. Section 3.2), we define what is the convolution product in general (i.e. between linear maps, not only characters and infinitesimal ones). This allows to express the generating series of the identity (i.e. $(\mathrm{Id}_{\mathrm{End}})^{\mathrm{gen}}$) as an infinite product of exponentials of rank one infinitesimal characters (this formula is a resolution of the identity). To this end, we first provide two examples (in Example 3.1), one with

[3] In the sense of [42].

the shuffle and one with the stuffle product. We end this paragraph with the general formula (which holds for enveloping algebras of linearly $\mathbf{k}$-free Lie algebras, $\mathbf{k}$ being itself a $\mathbb{Q}$-algebra).

We saw that a crucial step is this expression of $(\mathrm{Id}_{\mathrm{End}})^{\mathrm{gen}}$ is to express it as a summable series of tensor products obtained through dual bases. A remarkable combinatorial realization of such bases in duality is provided within the next section (see Section 3.3) and is performed through half-shuffle and Zinbiel algebras.

The end of the chapter is devoted to extensions of some characters relating to special functions: Polylogarithms for the shuffle product (a character with values in holomorphic functions) and Harmonic functions for the Stuffle. The transfer of the properties of the extended domain from Li to Harmonic sums is performed by row inequalities and columns limits (see Lemma 3.17).

3.2 Convolution algebra and factorization.

Throughout this section $\mathbf{k}$ is assumed to be a $\mathbb{Q}$-algebra.

Let us first give a famous example of the aimed factorization (see [91, 94]) for the special case when the Lie algebra $\mathfrak{g}$ is the free Lie algebra (i.e. $\mathfrak{g} = \mathcal{L}_{\mathbf{k}}(X)$).
We consider the usual Hopf enveloping algebra $\mathcal{U}(\mathcal{L}_{\mathbf{k}}(X))$, this algebra inherits the following Hopf structure $\mathcal{H}_{\mathrm{conc}}(X) = (\mathbf{k}\langle X\rangle, \mathrm{conc}, 1_{X^*}, \Delta_{⧢}, \epsilon)$[4]. As we will see below, formulas (3.1) (with their proper indexation) hold true for multihomogeneous bases, in particular bases like Hall, Lyndon, Viennot, Schützenberger which are monomial i.e. obtained by bracketting. For simplicity, we return here to the Lyndon basis[5] $(P_l)_{l\in\mathcal{L}ynX}$ of the free Lie algebra $\mathcal{L}_{\mathbf{k}}(X)$ and $(S_w)_{w\in X^*}$, computed from $(P_w)_{w\in X^*}$ by duality (where $(P_w)_{w\in X^*}$ is the Poincaré-Birkhoff-Witt - PBW for short - basis associated to $(P_l)_{l\in\mathcal{L}ynX}$ for the standard lexicographic order) is such that, by restriction, $(S_l)_{l\in\mathcal{L}ynX}$ is[6] a tran-

[4] See also Remark 5.6, Appendix 5.4.4.

[5] For this basis, we have $P_x = x$ if $x \in X$ and $P_l = [P_{l_1}, P_{l_2}]$ if $st(l) = (l_1, l_2)$ (a pair of Lyndon words (l_1, l_2) is called the standard factorization of a Lyndon word $l \notin X$ if $l = l_1 l_2$ and l_2 is the longest Lyndon proper right factor of l).

[6] Here, $S_x = x$ if $x \in X$ and $S_l = xS_u$ if $l = xu \in \mathcal{L}ynX \setminus X$.

scendence basis of the unital shuffle algebra $(\mathbf{k}\langle X\rangle, ⧢, 1_{X^*})$[7], indexed by $(\mathcal{L}ynX, <)$ the totally ordered set of all Lyndon words over X. The PBW basis $(P_w)_{w\in X^*}$ of the free associative algebra $\mathbf{k}\langle X\rangle$ and its dual family[8] $(S_w)_{w\in X^*}$ can be expressed as follows: each word $w \in X^*$ can be written uniquely as a decreasing product of Lyndon words $w = l_1^{i_1}\cdots l_k^{i_k}$ (where $l_i \in \mathcal{L}ynX, l_1 > \cdots > l_k$ and $i_1, \cdots, i_k \in \mathbb{N}$), then

$$P_w = P_{l_1}^{i_1}\cdots P_{l_k}^{i_k} \text{ and } S_w = \frac{S_{l_1}^{⧢ i_1} ⧢ \cdots ⧢ S_{l_k}^{⧢ i_k}}{i_1!\cdots i_k!}. \tag{3.1}$$

Example 3.1. In this case $\mathcal{U}(\mathfrak{g}) = \mathbf{k}\langle X\rangle$ and the generating function of Id (a.k.a $(\mathrm{Id}_{\mathrm{End}})^{\mathrm{gen}}$, see also [91], Def 3.8 and Prop 5.25), expressed in the complete tensor product $\mathbf{k}\langle\langle X\rangle\rangle\widehat{\otimes}\mathbf{k}\langle X\rangle$ reads (as Equation (3.6))

$$(\mathrm{Id}_{\mathrm{End}})^{\mathrm{gen}} = \sum_{w\in X^*} w\otimes w = \sum_{w\in X^*} S_w\otimes P_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(S_l\otimes P_l), \tag{3.2}$$

where the first equality follows Equation (5.13). This decomposition is called MRS[9] factorization.

Furthermore, note that the Lyndon basis $(P_l)_{l\in\mathcal{L}ynX}$ (as any Hall or monomial basis) is multi-homogeneous with respect to the $\mathbb{N}^{(X)}$-grading, where by Subsection 1.2.5, a Lie polynomial $P \in \mathcal{L}_{\mathbf{k}}(X) = \bigoplus_{\alpha\in\mathbb{N}^{(X)}} \mathcal{L}_{\mathbf{k}}(X)_\alpha$ is called multi-homogeneous if there exists $\alpha = (\alpha_x)_{x\in X} \in \mathbb{N}^{(X)}$ so that $P \in \mathcal{L}_{\mathbf{k}}(X)_\alpha$. Now, let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a Lie $\mathbf{k}$-algebra decomposed (internal) into two Lie subalgebras. Assume that $\mathfrak{g}$ is free $\mathbf{k}$-module then we say that a basis $\mathcal{B}$ of $\mathfrak{g}$ is compatible with this decomposition if

$$\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2, \qquad \text{where } \mathcal{B}_1 = \mathcal{B}\cap\mathfrak{g}_1, \mathcal{B}_2 = \mathcal{B}\cap\mathfrak{g}_2.$$

Proposition 3.1. *Every basis of the free Lie algebra $\mathcal{L}_{\mathbf{k}}(X)$ which is multi-homogeneous* [10] *is compatible with the classical Lazard elimination $\mathcal{L}_{\mathbf{k}}(X) = \mathcal{L}_{\mathbf{k}}(X)_{BZ} \rtimes \mathcal{L}_{\mathbf{k}}(X)_B$.*

Proof. Assume that $\mathcal{B}$ is a multi-homogeneous basis of the free Lie algebra $\mathcal{L}_{\mathbf{k}}(X)$. We

[7] See Reutenauer's [94] §5.2, Thm 5.3, Cors 5.5 and 5.6 for more details.

[8] In fact, the coordinate family, in this case a basis of the multihomogneous graded dual, a subspace of the dual space $\mathbf{k}\langle\langle X\rangle\rangle$.

[9] After Mélançon, Reutenauer and Schützenberger (see [94]).

[10] e.g. monomial bases, Hall, Lyndon, Viennot, Schützenberger [79, 94, 98, 104].

now put

$$\mathcal{B}_1 := \mathcal{B} \cap \mathcal{L}_{\mathbf{k}}(X)_{BZ} = \mathcal{B} \cap \left(\bigoplus_{\substack{\alpha \in \mathbb{N}^{(X)} \\ |\alpha|_Z > 0}} \mathcal{L}_{\mathbf{k}}(X)_\alpha \right)$$

and

$$\mathcal{B}_2 := \mathcal{B} \cap \mathcal{L}_{\mathbf{k}}(X)_{B} = \mathcal{B} \cap \left(\bigoplus_{\substack{\alpha \in \mathbb{N}^{(X)} \\ |\alpha|_Z = 0}} \mathcal{L}_{\mathbf{k}}(X)_\alpha \right).$$

Now, for each $P \in \mathcal{B}$ then $P \in \mathcal{L}_{\mathbf{k}}(X)_\alpha$, where $\alpha = (\alpha_x)_{x \in X} \in \mathbb{N}^{(X)}$. Clearly, if $|\alpha|_Z > 0$ then $P \in \mathcal{B} \cap \mathcal{L}_{\mathbf{k}}(X)_{BZ} = \mathcal{B}_1$, otherwise if $|\alpha|_Z = 0$ one has $P \in \mathcal{B} \cap \mathcal{L}_{\mathbf{k}}(X)_B = \mathcal{B}_2$. We then obtain that $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$. We verified our result. □

General case. – Given a Lie algebra $\mathfrak{g}$ over $\mathbf{k}$ a commutative ring with unit and its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, let us consider the usual Hopf algebra $(\mathcal{U}(\mathfrak{g}), \mu_{\mathcal{U}}, 1_{\mathbf{k}}, \Delta_{\mathcal{U}}, \epsilon_{\mathcal{U}})$ as mentioned in Appendix 5.4.4. On the $\mathbf{k}$-module $\mathrm{End}(\mathcal{U}(\mathfrak{g}))$, there is a structure of unital associative algebra, called the *convolution algebra*, in which the product of two $\mathbf{k}$-linear maps $f, g \in \mathrm{End}(\mathcal{U}(\mathfrak{g}))$ is defined by the formula

$$f \star g = \mu_{\mathcal{U}} \circ (f \otimes g) \circ \Delta_{\mathcal{U}} \in \mathrm{End}(\mathcal{U}(\mathfrak{g})),$$

whose unit element is $1_{\mathrm{End}} = 1_{\mathbf{k}} \circ \epsilon_{\mathcal{U}} \in \mathrm{End}(\mathcal{U}(\mathfrak{g}))$. The inverse element of the identity map $\mathrm{Id}_{\mathrm{End}}$ with respect to the convolution product is given by the antipode $S_{\mathcal{U}}$, the antiautomorphism of $\mathcal{U}(\mathfrak{g})$ characterized by $S_{\mathcal{U}}(x) = -x$ for any $x \in \mathfrak{g}$.

At $(\mathcal{U}(\mathfrak{g}), \Delta_{\mathcal{U}}, \epsilon_{\mathcal{U}})$, the (full) dual space $\mathcal{U}^*(\mathfrak{g}) = \mathrm{Hom}(\mathcal{U}(\mathfrak{g}), \mathbf{k})$ is a unital associative algebra induced from the transposes of $\Delta_{\mathcal{U}}$ and $\epsilon_{\mathcal{U}}$, respectively. More precisely, the convolution product $m_{\mathcal{U}^*} : \mathcal{U}^*(\mathfrak{g}) \otimes \mathcal{U}^*(\mathfrak{g}) \to \mathcal{U}^*(\mathfrak{g})$ (resp. a unit $\lambda_{\mathcal{U}^*} : \mathbf{k} \to \mathcal{U}^*(\mathfrak{g})$) is the composite map $\mathcal{U}^*(\mathfrak{g}) \otimes \mathcal{U}^*(\mathfrak{g}) \to (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))^* \xrightarrow{\Delta^*_{\mathcal{U}}} \mathcal{U}^*(\mathfrak{g})$ (resp. $\mathbf{k} \to \mathbf{k}^* \xrightarrow{\epsilon^*_{\mathcal{U}}} \mathcal{U}^*(\mathfrak{g})$) (cf. Grinberg and Reiner [56] § 1.6 Exercise 1.6.1 (a)).
We suppose now that $\mathfrak{g}$ is free as a $\mathbf{k}$-module. Then, due to the fact that $\mathcal{U}(\mathfrak{g})$ is also $\mathbf{k}$-linearly free, the canonical map $\Phi : \mathcal{U}^* \otimes \mathcal{U} \to \mathrm{End}(\mathcal{U})$, is into. One can show that the image of Φ is dense in $\mathrm{End}(\mathcal{U})$ when endowed with the topology of pointwise convergence (see Example 5.3).

Suppose further that $\mathbf{k}$ is a $\mathbb{Q}$-algebra and fix a totally ordered basis $\mathcal{B} = (b_i)_{i\in I}$ of the Lie algebra $\mathfrak{g}$ (which is therefore supposed to be is a free $\mathbf{k}$-module). This datum leads to an associated PBW basis of $\mathcal{U}(\mathfrak{g})$, denoted by $(\mathcal{B}^\alpha)_{\alpha\in\mathbb{N}^{(I)}}$, which is constructed by the following multi-index notation: for any $\alpha = \sum_{i\in I}\alpha_i e_i \in \mathbb{N}^{(I)}$, where the elementary multi-indices $e_i \in \mathbb{N}^{(I)}$ is defined for all $i \in I$ by $e_i(j) = \delta_i^j$ (Kronecker delta) and $\mathrm{supp}(\alpha) \subseteq \{i_1 < \cdots < i_k\}$, we set

$$\mathcal{B}^\alpha = b_{i_1}^{\alpha_{i_1}} \cdots b_{i_k}^{\alpha_{i_k}} = (\mathcal{B}^{e_{i_1}})^{\alpha_{i_1}} \cdots (\mathcal{B}^{e_{i_k}})^{\alpha_{i_k}} \in \mathcal{U}(\mathfrak{g}). \tag{3.3}$$

Notice that $\mathcal{B}^{\alpha+\beta} \neq \mathcal{B}^\alpha\mathcal{B}^\beta$ if $\mathrm{supp}(\alpha)\cap\mathrm{supp}(\beta) \neq \emptyset$ and that the standard order considered here is, in the free case, the opposite lexicographic order. At the present stage, it is straightforward to transform the above PBW basis of $\mathcal{U}(\mathfrak{g})$ to its dual (so-called) basis[11] $(\mathcal{B}_\alpha)_{\alpha\in\mathbb{N}^{(I)}}$ of $\mathcal{U}^*(\mathfrak{g})$ in terms of the duality given by

$$\langle \mathcal{B}_\alpha \mid \mathcal{B}^\beta\rangle = \delta_\alpha^\beta \text{ (the Kronecker delta), for all } \alpha,\beta \in \mathbb{N}^{(I)},$$

where the pairing $\langle \bullet \mid \bullet\rangle : \mathcal{U}^*(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \to \mathbf{k}, \varphi\otimes u \mapsto \varphi(u)$ is the usual one. More precisely, for these constructions, we have

Theorem 3.2. *One has the following*

$$\begin{aligned}
\mathcal{B}_\alpha \star \mathcal{B}_\beta &= \frac{(\alpha+\beta)!}{\alpha!\beta!}\mathcal{B}_{\alpha+\beta},\\
\mathcal{B}_\alpha &= \frac{(b_{i_1}^\star)^{\star\alpha_{i_1}} \star\cdots\star (b_{i_k}^\star)^{\star\alpha_{i_k}}}{\alpha_{i_1}!\cdots\alpha_{i_k}!}\\
&= \frac{(\mathcal{B}_{e_{i_1}})^{\star\alpha_{i_1}} \star\cdots\star (\mathcal{B}_{e_{i_k}})^{\star\alpha_{i_k}}}{\alpha_{i_1}!\cdots\alpha_{i_k}!},
\end{aligned} \tag{3.4}$$

where any $\alpha \in \mathbb{N}^{(I)}$ is expressed in the above notation and with the formula $\alpha! := \prod_{i\in I}\alpha_i!$.

Proof. We can easily check that the coproduct $\Delta_\mathcal{U}$ of any PBW basis is given by

$$\Delta_\mathcal{U}(\mathcal{B}^\gamma) = \sum_{\substack{\alpha,\beta\in\mathbb{N}^{(I)}\\ \alpha+\beta=\gamma}} \frac{\gamma!}{\alpha!\beta!}\mathcal{B}^\alpha\otimes\mathcal{B}^\beta, \text{ for any } \gamma\in\mathbb{N}^{(I)} \tag{3.5}$$

because $\Delta_\mathcal{U}$ is a morphism of algebras and the elements of $\mathfrak{g}$ are primitive.
Now, for each $\alpha,\beta \in \mathbb{N}^{(I)}$, it follows that

$$\langle \mathcal{B}_\alpha \star \mathcal{B}_\beta \mid \mathcal{B}^\gamma\rangle = \langle m_{\mathcal{U}^*}(\mathcal{B}_\alpha\otimes\mathcal{B}_\beta) \mid \mathcal{B}^\gamma\rangle$$

[11] In fact, the family of coordinate forms.

$$
\begin{aligned}
&= (m_{\mathcal{U}^*}(\mathcal{B}_\alpha \otimes \mathcal{B}_\beta))(\mathcal{B}^\gamma) \\
&= (\mathcal{B}_\alpha \otimes \mathcal{B}_\beta)(\Delta_{\mathcal{U}}(\mathcal{B}^\gamma)) \\
&= (\mathcal{B}_\alpha \otimes \mathcal{B}_\beta)\big(\sum_{\substack{\alpha_1,\beta_1 \in \mathbb{N}^{(I)} \\ \alpha_1+\beta_1=\gamma}} \frac{\gamma!}{\alpha_1!\beta_1!} \mathcal{B}^{\alpha_1} \otimes \mathcal{B}^{\beta_1} \big) \\
&= \sum_{\substack{\alpha_1,\beta_1 \in \mathbb{N}^{(I)} \\ \alpha_1+\beta_1=\gamma}} \frac{\gamma!}{\alpha_1!\beta_1!} \mathcal{B}_\alpha(\mathcal{B}^{\alpha_1}) \mathcal{B}_\beta(\mathcal{B}^{\beta_1}) \\
&= \sum_{\substack{\alpha_1,\beta_1 \in \mathbb{N}^{(I)} \\ \alpha_1+\beta_1=\gamma}} \frac{\gamma!}{\alpha_1!\beta_1!} \delta_\alpha^{\alpha_1} \delta_\beta^{\beta_1} \\
&= \delta_{\alpha+\beta}^{\gamma} \frac{\gamma!}{\alpha!\beta!}.
\end{aligned}
$$

This leads to the equation $\mathcal{B}_\alpha \star \mathcal{B}_\beta = \frac{(\alpha+\beta)!}{\alpha!\beta!}\mathcal{B}_{\alpha+\beta}$. By telescoping the products, for each $\alpha_1, \cdots, \alpha_k \in \mathbb{N}^{(I)}$ we obtain

$$
\begin{aligned}
\mathcal{B}_{\alpha_1} \star \cdots \star \mathcal{B}_{\alpha_k} &= \frac{(\alpha_1+\alpha_2)!}{\alpha_1!\alpha_2!} \frac{(\alpha_1+\alpha_2+\alpha_3)!}{(\alpha_1+\alpha_2)!\alpha_3!} \cdots \frac{(\alpha_1+\cdots+\alpha_k)!}{(\alpha_1+\cdots+\alpha_{k-1})!\alpha_k!} \mathcal{B}_{\alpha_1+\cdots+\alpha_k} \\
&= \frac{(\alpha_1+\cdots+\alpha_k)!}{\alpha_1!\cdots\alpha_k!} \mathcal{B}_{\alpha_1+\cdots+\alpha_k}.
\end{aligned}
$$

In particular, considering $\alpha_i e_i = \underbrace{e_i + \cdots + e_i}_{\alpha_i \text{ times}} \in \mathbb{N}^{(I)}$, where $i \in I, \alpha_i \in \mathbb{N}$ and the elementary multi-index $e_i \in \mathbb{N}^{(I)}$, the dual basis element $\mathcal{B}_{\alpha_i e_i}$ can then be expressed as

$$
\mathcal{B}_{\alpha_i e_i} = \frac{(e_i!)^{\alpha_i}}{(\alpha_i e_i)!} \underbrace{\mathcal{B}_{e_i} \star \cdots \star \mathcal{B}_{e_i}}_{\alpha_i \text{ times}} = \frac{(\mathcal{B}_{e_i})^{\star\alpha_i}}{\alpha_i!}.
$$

Thus, for any multi-index $\alpha = \sum_{i\in I} \alpha_i e_i = \alpha_{i_1} e_{i_1} + \cdots + \alpha_{i_k} e_{i_k} \in \mathbb{N}^{(I)}$, where $\mathrm{supp}(\alpha) \subseteq \{i_1 < \cdots < i_k\}$, one has

$$
\begin{aligned}
\mathcal{B}_\alpha &= \frac{(\alpha_{i_1} e_{i_1})! \cdots (\alpha_{i_k} e_{i_k})!}{(\alpha_{i_1} e_{i_1} + \cdots + \alpha_{i_k} e_{i_k})!} \mathcal{B}_{\alpha_{i_1} e_{i_1}} \star \cdots \star \mathcal{B}_{\alpha_{i_k} e_{i_k}} \\
&= \frac{\alpha_{i_1}! \cdots \alpha_{i_k}!}{\alpha!} \mathcal{B}_{\alpha_{i_1} e_{i_1}} \star \cdots \star \mathcal{B}_{\alpha_{i_k} e_{i_k}} \\
&= \mathcal{B}_{\alpha_{i_1} e_{i_1}} \star \cdots \star \mathcal{B}_{\alpha_{i_k} e_{i_k}} \text{ (since } \alpha! = \alpha_{i_1}! \cdots \alpha_{i_k}!) \\
&= \frac{(\mathcal{B}_{e_{i_1}})^{\star\alpha_{i_1}} \star \cdots \star (\mathcal{B}_{e_{i_k}})^{\star\alpha_{i_k}}}{\alpha_{i_1}! \cdots \alpha_{i_k}!}.
\end{aligned}
$$

We just proved our theorem. □

We now work out what is written in Reutenauer's Prop 1.10 [94] by the following

Proposition 3.3. *The linear isomorphism* $\overline{\Phi}^{-1}$: $\mathrm{End}(\mathcal{U}(\mathfrak{g})) \to \mathcal{U}^*(\mathfrak{g})\widehat{\otimes}\mathcal{U}(\mathfrak{g})$ *sending any* f *to* $\sum_{\alpha\in\mathbb{N}^{(I)}} \mathcal{B}_\alpha \otimes f(\mathcal{B}^\alpha)$. *It is a morphism of rings from the convolution algebra* $\mathrm{End}(\mathcal{U}(\mathfrak{g}))$ *to the complete tensor product* $\mathcal{U}^*(\mathfrak{g})\widehat{\otimes}\mathcal{U}(\mathfrak{g})$ *as in Example 5.3.*

Proof. Proof omitted (general fact, see Example 5.3 or see [35]). □

Correspondingly, the image of the identity map $\mathrm{Id}_{\mathrm{End}}$ under the ring isomorphism $\overline{\Phi}^{-1}$ is denoted by $(\mathrm{Id}_{\mathrm{End}})^{\mathrm{gen}}$. We can carry out a systematic formulation of the generalized identity as follows

Theorem 3.4 ([35])**.** *Let* $\mathbf{k}$ *be a* $\mathbb{Q}$*-algebra and* $\mathfrak{g}$ *be a Lie* $\mathbf{k}$*-algebra endowed with a totally ordered basis* $\mathcal{B} = (b_i)_{i\in I}$ *of the Lie algebra* $\mathfrak{g}$ *(hence free as a* $\mathbf{k}$*-module) then*
i) The following infinite product identity holds in the complete topological associative algebra $\mathcal{U}^*(\mathfrak{g})\widehat{\otimes}\mathcal{U}(\mathfrak{g})$

$$(\mathrm{Id}_{\mathrm{End}})^{\mathrm{gen}} = \sum_{\alpha\in\mathbb{N}^{(I)}} \mathcal{B}_\alpha \otimes \mathcal{B}^\alpha = \prod_{i\in I}^{\nearrow} \exp(\mathcal{B}_{e_i} \otimes \mathcal{B}^{e_i}). \tag{3.6}$$

ii) Moreover, if the ordered basis $\mathcal{B} = (b_i)_{i\in I}$ *is split in two successive parts* $\mathcal{B}_1 = (b_i)_{i\in I_1}$ *and* $\mathcal{B}_2 = (b_i)_{i\in I_2}$ *where* $I = I_1 \dotplus I_2$ *is an ordinal sum, we have*

$$(\mathrm{Id}_{\mathrm{End}})^{\mathrm{gen}} = \prod_{i\in I}^{\nearrow} \exp(\mathcal{B}_{e_i} \otimes \mathcal{B}^{e_i}) = \prod_{i\in I_1}^{\nearrow} \exp(\mathcal{B}_{e_i} \otimes \mathcal{B}^{e_i}) \times \prod_{i\in I_2}^{\nearrow} \exp(\mathcal{B}_{e_i} \otimes \mathcal{B}^{e_i}). \tag{3.7}$$

Proof. i) First equality follows from (5.13). The last right-hand side of (3.6) is

$$\prod_{i\in I}^{\nearrow} \exp(\mathcal{B}_{e_i} \otimes \mathcal{B}^{e_i}) = \prod_{i\in I}^{\nearrow} \left(\sum_{n\geq 0} \frac{1}{n!} (\mathcal{B}_{e_i})^{\star n} \otimes (\mathcal{B}^{e_i})^n \right)$$

that is equal to

$$\sum_{\substack{i_1<\cdots\cdots<i_k \\ 0\leq \alpha_{i_1},\cdots,\alpha_{i_k}}} \frac{1}{\alpha_{i_1}!\cdots\alpha_{i_k}!} (\mathcal{B}_{e_{i_1}})^{\star\alpha_{i_1}} \star\cdots\star (\mathcal{B}_{e_{i_k}})^{\star\alpha_{i_k}} \otimes (\mathcal{B}^{e_{i_1}})^{\alpha_{i_1}} \cdots (\mathcal{B}^{e_{i_k}})^{\alpha_{i_k}}$$

by the usual expression. Let us however say a word about convergence. We consider the nets of finite partial sums and products

$$S_F = \sum_{\substack{\alpha\in\mathbb{N}^{(I)} \\ supp(\alpha)\subset F}} \mathcal{B}_\alpha \otimes \mathcal{B}^\alpha \qquad P_F = \prod_{i\in F}^{\nearrow} \exp(\mathcal{B}_{e_i} \otimes \mathcal{B}^{e_i})$$

for all $F \subset_{finite} I$. Convergence of these nets to $(\mathrm{Id}_{\mathrm{End}})^{\mathrm{gen}}$ goes as follows.
The nets $\Phi(S_F)$ and $\Phi(P_F)$ converge pointwise to $\mathrm{Id}_{\mathrm{End}}$ due to the following

Lemma 3.5. *Let M be a free module with basis $\mathcal{B} = (\beta_i)_{i\in I}$, endowed with the discrete topology, then, for a net*
$\mathcal{N} = (\varphi_\alpha)_{\alpha\in A}$ *within* $\mathrm{End}(M)$*, TFAE*

1. *$\mathcal{N}$ converges pointwise to φ_{lim} .*

2. *For all $i \in I$, $\varphi_\alpha(\beta_i)$ converges to $\varphi_{lim}(\beta_i)$.*

Proof. A net $\mathcal{N} = (\varphi_\alpha)_{\alpha\in A}$ within $\mathrm{End}(M)$ converges pointwise to $\ell \in \mathrm{End}(M)$ if and only if

$$(\forall F \subset_{finite} M)(\exists B \in A)(\forall \alpha \geq B)(\varphi_\alpha|_F = \ell|_F) \tag{3.8}$$

□

Now, due to the equation (3.3) and Theorem 3.2, we deduce our result.
ii) **First proof.** – Second equality is a particular case of *ordinal partition of indices* for infinite ordered products.
Now, for the reader who wants to feel what is happening inside (3.7), we give a second - combinatorial - proof.
Second proof. – By using $I = I_1 \dotplus I_2$, the last right-hand side of (3.7) is

$$\begin{aligned}
&\prod_{i\in I_1}^{\nearrow} \exp(\mathcal{B}_{e_i} \otimes \mathcal{B}^{e_i}) \times \prod_{i\in I_2}^{\nearrow} \exp(\mathcal{B}_{e_i} \otimes \mathcal{B}^{e_i}) \\
&= \prod_{i\in I_1}^{\nearrow} \left(\sum_{n\geq 0} \frac{1}{n!} (\mathcal{B}_{e_i})^{\star n} \otimes (\mathcal{B}^{e_i})^n \right) \times \prod_{i\in I_2}^{\nearrow} \left(\sum_{n\geq 0} \frac{1}{n!} (\mathcal{B}_{e_i})^{\star n} \otimes (\mathcal{B}^{e_i})^n \right) \\
&= \left(\sum_{\substack{i_1 < \cdots\cdots < i_k \in I_1 \\ 0 \leq \alpha_{i_1}, \cdots, \alpha_{i_k}}} \frac{1}{\alpha_{i_1}! \cdots \alpha_{i_k}!} (\mathcal{B}_{e_{i_1}})^{\star\alpha_{i_1}} \star \cdots \star (\mathcal{B}_{e_{i_k}})^{\star\alpha_{i_k}} \otimes (\mathcal{B}^{e_{i_1}})^{\alpha_{i_1}} \cdots (\mathcal{B}^{e_{i_k}})^{\alpha_{i_k}} \right) \\
&\times \left(\sum_{\substack{i_1 < \cdots\cdots < i_k \in I_2 \\ 0 \leq \alpha_{i_1}, \cdots, \alpha_{i_k}}} \frac{1}{\alpha_{i_1}! \cdots \alpha_{i_k}!} (\mathcal{B}_{e_{i_1}})^{\star\alpha_{i_1}} \star \cdots \star (\mathcal{B}_{e_{i_k}})^{\star\alpha_{i_k}} \otimes (\mathcal{B}^{e_{i_1}})^{\alpha_{i_1}} \cdots (\mathcal{B}^{e_{i_k}})^{\alpha_{i_k}} \right) \\
&= \sum_{\substack{i_1 < \cdots\cdots < i_k \in I \\ 0 \leq \alpha_{i_1}, \cdots, \alpha_{i_k}}} \frac{1}{\alpha_{i_1}! \cdots \alpha_{i_k}!} (\mathcal{B}_{e_{i_1}})^{\star\alpha_{i_1}} \star \cdots \star (\mathcal{B}_{e_{i_k}})^{\star\alpha_{i_k}} \otimes (\mathcal{B}^{e_{i_1}})^{\alpha_{i_1}} \cdots (\mathcal{B}^{e_{i_k}})^{\alpha_{i_k}} \\
&= \prod_{i\in I}^{\nearrow} \left(\sum_{n\geq 0} \frac{1}{n!} (\mathcal{B}_{e_i})^{\star n} \otimes (\mathcal{B}^{e_i})^n \right) = \prod_{i\in I}^{\nearrow} \exp(\mathcal{B}_{e_i} \otimes \mathcal{B}^{e_i}) = (\mathrm{Id}_{\mathrm{End}})^{\mathrm{gen}}.
\end{aligned}$$

We have then completed the second proof.

QED

□

Example 3.2. Two applications. –

i) Formula (3.7) generalizes without difficulty to the case when the totally ordered set I splits as $I = I_1 \nrightarrow \cdots \nrightarrow I_m$ (m-factor ordinal sum) as

$$(\mathrm{Id}_{\mathrm{End}})^{\mathrm{gen}} = \prod_{i\in I_1}^{\nearrow} \exp(\mathcal{B}_{e_i} \otimes \mathcal{B}^{e_i}) \times \cdots \times \prod_{i\in I_m}^{\nearrow} \exp(\mathcal{B}_{e_i} \otimes \mathcal{B}^{e_i}). \tag{3.9}$$

Recall that the Drinfeld-Kohno Lie algebra $\mathrm{DK}_{\mathbf{k},n+1} = \bigoplus_{2\leq j\leq n+1} \mathrm{DK}^{(j)}_{\mathbf{k},n+1}$ is a $([2, n+1], \vee)$-graded Lie algebra in which each factor $\mathrm{DK}^{(j)}_{\mathbf{k},n+1}$ is a free Lie algebra $\mathcal{L}_{\mathbf{k}}(T_j)$ (see Equation (2.104)). Thus, under the n-factor ordinal sum $\mathcal{T}_{n+1} = T_2 \nrightarrow \cdots \nrightarrow T_{n+1}$, the Lie algebra $\mathrm{DK}_{\mathbf{k},n+1}$ has a $\mathbf{k}$-linear basis $\mathcal{B} = \bigcup_{l\in I_{\mathrm{DK}}} P_l$, where we used the totally ordered set $I_{\mathrm{DK}} := \mathcal{L}yn(T_2) \nrightarrow \cdots \nrightarrow \mathcal{L}yn(T_{n+1})$. As a consequence, we have the following infinite product identity in the complete topological associative algebra $\mathcal{U}^*(\mathrm{DK}_{\mathbf{k},n+1})\widehat{\otimes}\mathcal{U}(\mathrm{DK}_{\mathbf{k},n+1})$ (over $\mathbf{k}$ a $\mathbb{Q}$-algebra)

$$(\mathrm{Id}_{\mathrm{End}})^{\mathrm{gen}} = \prod_{l\in I_{\mathrm{DK}}}^{\searrow} \exp(S_l \otimes P_l) = \prod_{l\in \mathcal{L}yn(T_{n+1})}^{\searrow} \exp(S_l \otimes P_l) \times \cdots \times \prod_{l\in \mathcal{L}yn(T_2)}^{\searrow} \exp(S_l \otimes P_l).$$

ii) We now define a deformation of the shuffle product (in fact an interpolation between shuffle and stuffle products[12]).

Let Y be the infinite alphabet $\{y_k \mid \ k \geq 1\}$ and take a parameter q in $\mathbf{k}$. If the ground ring was R and q is seen as formal, take $\mathbf{k} = R[q]$, later, but not now, we will require that $\mathbf{k}$ be a $\mathbb{Q}$-algebra. The q-deformed stuffle product, noted $\uplus_q$, is defined by the following recursion

$$u \uplus_q 1_{Y^*} = 1_{Y^*} \uplus_q u = u, \tag{3.10}$$

$$y_{k_1}u \uplus_q y_{k_2}v = y_{k_1}(u \uplus_q y_{k_2}v) + y_{k_2}(y_{k_1}u \uplus_q v) + q y_{k_1+k_2}(u \uplus_q v) \tag{3.11}$$

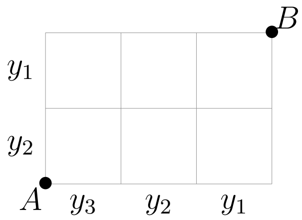
for all $y_{k_1}, y_{k_2} \in Y$ and $u, v \in Y^*$.

[12]Deformation is in the sense of [83] Ch 4 i.e. such that, at $q = 0$ we get the shuffle product and at $q = 1$ we get the stuffle product.

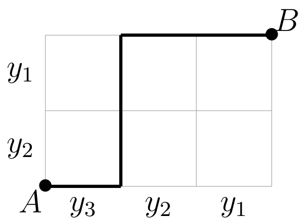
i) As was said, when $q = 0$, we get the shuffle product:

$$\begin{aligned} y_2y_1 \sqcup\!\sqcup\, y_3y_1y_2 &= y_2(y_1 \sqcup\!\sqcup\, y_3y_1y_2) + y_3(y_2y_1 \sqcup\!\sqcup\, y_1y_2) \\ &= y_2y_1y_3y_1y_2 + 2y_2y_3y_1^2y_2 + y_2y_3y_1y_2y_1 + 2y_3y_2y_1^2y_2 \\ &\quad +y_3y_2y_1y_2y_1 + y_3y_1y_2y_1y_2 + 2y_3y_1y_2^2y_1. \end{aligned}$$

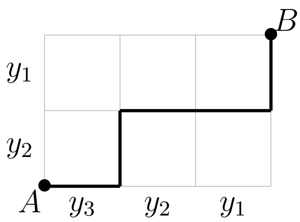
One can also see this product as indexed by paths (with North and East steps) from A to B as below



For instance,

evaluation of the path 

is $y_3y_2y_1y_2y_1$;

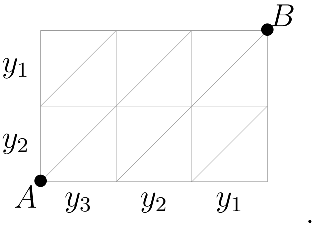
and evaluation of the path 

is $y_3y_2^2y_1^2$; etc.

ii) When $q = 1$, one gets the stuffle product:

$$\begin{aligned} y_2y_1 \uplus y_3y_1y_2 &= y_2(y_1 \uplus y_3y_1y_2) + y_3(y_2y_1 \uplus y_1y_2) + y_5(y_1 \uplus y_1y_2) \\ &= y_2y_1y_3y_1y_2 + 2y_2y_3y_1^2y_2 + y_2y_3y_1y_2y_1 + y_2y_3y_1y_3 + y_2y_3y_2^2 \\ &\quad +y_2y_4y_1y_2 + 2y_3y_2y_1^2y_2 + y_3y_2y_1y_2y_1 + y_3y_2y_1y_3 + y_3y_2^3 \\ &\quad +y_3y_1y_2y_1y_2 + 2y_3y_1y_2^2y_1 + y_3y_1y_2y_3 + y_3y_1y_4y_1 + y_3^2y_1y_2 \\ &\quad +y_3^2y_2y_1 + y_3^2y_3 + 2y_5y_1^2y_2 + y_5y_1y_2y_1 + y_5y_1y_3 + y_5y_2^2. \end{aligned}$$

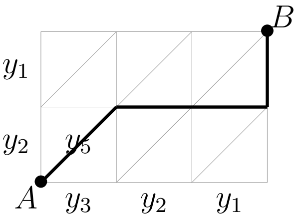
iii) In general, one can also see the q-stuffle product as indexed by paths (with North, North-East and East steps, each North-East path provides a q) from A to B as below



.

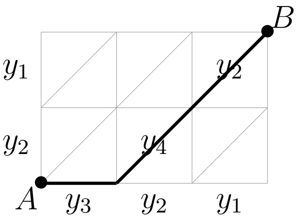
For instance,

the path



evaluates as $q.y_5y_2y_1^2$;

and the path



evaluates as $q^2.y_3y_4y_2$; etc.

iv) The complete q-stuffle product containing the two terms above is

$$\begin{aligned} y_2y_1 \uplus_q y_3y_1y_2 &= y_2(y_1 \uplus_q y_3y_1y_2) + y_3(y_2y_1 \uplus_q y_1y_2) + qy_5(y_1 \uplus_q y_1y_2) \\ &= y_2y_1y_3y_1y_2 + 2y_2y_3y_1^2y_2 + y_2y_3y_1y_2y_1 + qy_2y_3y_1y_3 + qy_2y_3y_2^2 \\ &\quad +qy_2y_4y_1y_2 + 2y_3y_2y_1^2y_2 + y_3y_2y_1y_2y_1 + qy_3y_2y_1y_3 + qy_3y_2^3 \\ &\quad +y_3y_1y_2y_1y_2 + 2y_3y_1y_2^2y_1 + qy_3y_1y_2y_3 + qy_3y_1y_4y_1 + qy_3^2y_1y_2 \\ &\quad +qy_3^2y_2y_1 + q^2y_3^2y_3 + 2qy_5y_1^2y_2 + qy_5y_1y_2y_1 + q^2y_5y_1y_3 + q^2y_5y_2^2. \end{aligned}$$

In fact, $\uplus_q$ is the dual of the comultiplication

$$\Delta_{\uplus_q} : \ \mathbf{k}\langle Y\rangle \to \mathbf{k}\langle Y\rangle \otimes \mathbf{k}\langle Y\rangle$$

defined on letters by

$$\Delta_{\uplus_q}(y_s) = y_s \otimes 1_{Y^*} + 1_{Y^*} \otimes y_s + q. \sum_{\substack{p+q=s \\ p,q\geq 1}} y_p \otimes y_q$$

and, with the grading $||w|| = ||y_{i_1} \cdots y_{i_k}|| = i_1 + \cdots + i_k$, one checks easily that the bialgebra

$$\mathcal{H}_{\uplus_q}(Y) = (\mathbf{k}\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\uplus_q}, \epsilon) \tag{3.12}$$

is connected, cocommutative and $\mathbb{N}$-graded.
Then, due to the fact that $\frac{(-1)^{n-1}}{n}.(I_+)^{\star n}$ is summable (see [13] Ch II §1.6 or Cartier [21] Thm 3.6.1), $\mathcal{H}_{\uplus_q}(Y) = \mathcal{U}(\mathrm{Prim}(\mathcal{H}))$ and using the Eulerian projector

$$\pi_1 = \log(I) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n}(I_+)^{\star n}$$

(i.e. the Eulerian projector as in [94]), one constructs two bases (of $\mathbf{k}\langle Y\rangle$) in duality Π_w and Σ_w. Then

$$(\mathrm{Id})^{\mathrm{gen}} = \sum_{w\in Y^*} w\otimes w = \sum_{w\in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(\Sigma_l\otimes\Pi_l),$$

for details, see [17].

Example 3.3. A motivation. –
In this example, we show an application of Lazard's elimination to the combinatorics of polylogarithms.
Among multihomogeneous bases (Hall, Schützenberger-Viennot-Hall, Shirshov, etc.), one is particularly interesting, the Lyndon basis and its dual family (which is this case admits a nice combinatorial resursion). With the same notations as in Section 3.1 and as in the beginning of Section 3.2, we first construct $\{P_w\}_{w\in X^*}$ from the Lyndon basis (see [79] and [94]) and its dual family $\{S_w\}_{w\in X^*}$ which satisfies the following recursion ([94] Thm 5.3)

$$\begin{array}{rcll} S_l = & xS_u, & \text{for } l = xu \in \mathcal{L}ynX\setminus X, \\ S_w = & \dfrac{S_{l_1}^{\sqcup\!\sqcup i_1}\sqcup\!\sqcup\ldots\sqcup\!\sqcup S_{l_k}^{\sqcup\!\sqcup i_k}}{i_1!\ldots i_k!} & \text{for } w = l_1^{i_1}\ldots l_k^{i_k}, l_1 > \ldots > l_k\ . \end{array} \tag{3.13}$$

A table of the first elements, up to length 6, can be found in Appendix 5.5.1.

The polylogarithmic function is a shuffle character on $(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})$ $(X = \{x_0, x_1\})$ with values in $\mathcal{H}(\Omega)$[13]

$$(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}) \xhookrightarrow{\mathrm{Li}} (\mathcal{H}(\Omega), \times, 1_\Omega) \tag{3.14}$$

[13] $\Omega\subset\mathbb{C}$ is a simply-connected domain convenient for the system (3.38) as, for instance, $\Omega = \mathbb{C}\setminus(]-\infty,0]\cup[1,+\infty[)$.

built from (3.38).

Product formula (3.2) provides a factorization of Li (in fact of any shuffle character)

$$Ł = \sum_{w \in X^*} \mathrm{Li}_w \, w = \sum_{w \in X^*} \mathrm{Li}_{S_w} \, P_w = \prod_{l \in \mathcal{L}ynX}^{\searrow} \exp(\mathrm{Li}_{S_l} \, P_l). \tag{3.15}$$

Using Lazard's elimination with $X = \{x_0, x_1\} = B \dotplus Z$ where $B = \{x_0\}$ and $Z = \{x_1\}$ and this, by (3.7), automatically splits the product as

$$Ł = \prod_{\substack{l \in \mathcal{L}ynX \\ |l|_{x_1} \geq 1}}^{\searrow} \exp(\mathrm{Li}_{S_l} \, P_l) \times \prod_{\substack{l \in \mathcal{L}ynX \\ |l|_{x_1} = 0}}^{\searrow} \exp(\mathrm{Li}_{S_l} \, P_l) = Ł_+ \times \exp(x_0 \log(z)). \tag{3.16}$$

Now, using recursion (3.13), one checks that all terms Li_{S_l} in the exponent of $Ł_+$ tend to zero as $z \in \Omega$ tends to zero. This entails that $\lim_{z \in \Omega, z \to 0} Ł(z) e^{-x_0 \log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X \rangle\rangle}$ which is precisely the *asymptotic initial condition* of (3.38).

3.3 Bases in duality: Zinbiel algebra and Magnus basis.

Here, we deal with Magnus basis on the free associative algebra arising from Lazard's elimination in **k-AAU**. We then focus on Zinbiel algebras and present a systematic method to study Magnus duality in Theorem 3.9. In the last part of this section, there are several applications of the Magnus duality introduced in Remark 3.2.

For convenience, in this section we suppose that $\mathbf{k}$ is a commutative ring with unit and $X = B + Z$ is a finite set partitioned in two blocks, where $B = \{b_1, \cdots, b_M\}$ and $Z = \{z_1, \cdots, z_N\}$. Now we formulate and prove the following corollary:

Corollary 3.6. *(Magnus basis) Let us consider the free associative algebra*

$$\mathbf{k}\langle X \rangle = \mathbf{k}\langle b_1, \cdots, b_M, z_1, \cdots, z_N \rangle.$$

The collection of polynomials (called by Magnus polynomials, cf. [81] Ch V §5.6)

$$rn(w_1 z_{i_1}) \cdots rn(w_k z_{i_k}).w, \tag{3.17}$$

where $k \geq 0, w_1, \cdots, w_k, w \in B^*; i_1, \cdots, i_k \in [1, N]$ (if $k = 0$ then $rn(w_1 z_{i_1}) \cdots rn(w_k z_{i_k})$ will be denoted by 1_{X^*}), is a $\mathbf{k}$*-linear basis of* $\mathbf{k}\langle X \rangle$*.*

Proof. It follows from Example 2.8 points (ii) and (iii) that the natural algebra homomorphism

$$f_*\colon \mathbf{k}\langle B^*Z\rangle\sharp\mathbf{k}\langle B\rangle \to \mathbf{k}\langle X\rangle, u_1 \otimes u_2 \mapsto f_1^*(u_1)f_2^*(u_2)$$

is an isomorphism in **k-AAU**. Moreover, the families of monomials $\{w_1z_{i_1}\cdots w_kz_{i_k} \mid k \geq 0, w_1, \cdots, w_k \in B^*; i_1, \cdots, i_k \in [1, N]\}$ and $\{w \mid w \in B^*\}$ form $\mathbf{k}$-linear bases of, respectively, $\mathbf{k}\langle B^*Z\rangle$ and $\mathbf{k}\langle B\rangle$. Under the algebra isomorphism $f_*(u_1 \otimes u_2) = f_1^*(u_1)f_2^*(u_2)$, we thus deduce that the collection of Magnus polynomials (3.17) is a $\mathbf{k}$-linear basis of $\mathbf{k}\langle X\rangle$. We verified our corollary by applying Lazard's elimination in **k-AAU**. □

Our next aim is to describe the dual of Magnus basis under the standard pairing

$$\langle \bullet \mid \bullet\rangle : \mathbf{k}\langle X\rangle \otimes \mathbf{k}\langle X\rangle \to \mathbf{k} \tag{3.18}$$

classically defined by, for $T \in \mathbf{k}\langle X\rangle$ and $P \in \mathbf{k}\langle X\rangle$, $\langle T \mid P\rangle = \sum_{w\in X^*}\langle T \mid w\rangle\langle P \mid w\rangle$, where, when w is a word, $\langle T \mid w\rangle$ stands for the coefficient of w in T (see Subsection 1.2.4). More precisely, in this case, there is a unique dual of these polynomials because the Magnus basis (3.17) is multi-homogeneous with respect to the $\mathbb{N}^{(X)}$-grading [14].

We start with a notion of Zinbiel algebras that were introduced by Jean-Louis Loday in [77] (see also in Loday [78]) as the Koszul dual operad to Leibniz algebras (hence the name coined by the topologist J.M. Lemaire), which is more than adequate for our purpose which means to construct the dual of Magnus basis.

Definition 3.1. A *(left) Zinbiel algebra* over $\mathbf{k}$ (a unital commutative ring) is a $\mathbf{k}$-module $\mathcal{A}$ equipped with a bilinear map $\prec$ satisfying the following relation

$$(x \prec y) \prec z = x \prec (y \prec z) + x \prec (z \prec y), \text{ for all } x, y, z \in \mathcal{A}. \tag{3.19}$$

Moreover, an element $1_{\mathcal{A}}$ of a (left) Zinbiel algebra $\mathcal{A}$ is called the unit if for any $1_{\mathcal{A}} \neq x \in \mathcal{A}$,

$$1_{\mathcal{A}} \prec x = 0, \qquad x \prec 1_{\mathcal{A}} = x,$$

and $1_{\mathcal{A}} \prec 1_{\mathcal{A}}$ is not define.

[14] We recall that a polynomial $T \in \mathbf{k}\langle X\rangle$ is called multi-homogeneous if $T \in \mathbf{k}_\alpha\langle X\rangle$, for $\alpha = (\alpha_x)_{x\in X} \in \mathbb{N}^{(X)}$ (see also Subsection 1.2.4).

However, we can replace the usual (left) Zinbiel algebra by its opposite algebra, called (right) Zinbiel algebra. Namely,

Definition 3.2. A $\mathbf{k}$-module $\mathcal{A}$ equipped with a bilinear map $\succ$ is called *(right) Zinbiel algebra* if it satisfies the identity

$$x \succ (y \succ z) = (x \succ y) \succ z + (y \succ x) \succ z, \text{ for all } x, y, z \in \mathcal{A}. \tag{3.20}$$

The behavior of $\succ$ with respect to the unit $1_{\mathcal{A}} \in \mathcal{A}$ is given by (for any $1_{\mathcal{A}} \neq x \in \mathcal{A}$)

$$1_{\mathcal{A}} \succ x = x, \qquad x \succ 1_{\mathcal{A}} = 0,$$

note that $1_{\mathcal{A}} \succ 1_{\mathcal{A}}$ is not defined.
Unital (right) Zinbiel algebras form a category denoted by **k-Zinb** (the category of unital (right) Zinbiel $\mathbf{k}$-algebras). Moreover, for any (right) Zinbiel algebra $\mathcal{A}$, we can construct a corresponding commutative associative algebra by using the symmetrized product

$$x * y = x \succ y + y \succ x. \tag{3.21}$$

In other word, $(\mathcal{A}, *)$ is an associative and commutative algebra.

Starting from the paper of M. P. Schüzenberger [98] in 1958, the author originally introduced the concept of *(left) half-shuffle* on the free associative algebra $\mathbf{k}\langle X\rangle$, it also was studied in the recent works of E. Burgunder [18], L. Foissy and F. Patras [51], H. Nakamura [89]. In our context, we now introduce a notion of *(right) half-shuffle* on $\mathbf{k}\langle X\rangle$ as the linear extension of the binary product on words given by

$$\begin{aligned}
&(x_1 \cdots x_p) \overset{r}{\sqcup\!\sqcup} (x_{p+1} \cdots x_n) = (x_1 \cdots x_p \sqcup\!\sqcup x_{p+1} \cdots x_{n-1}) x_n,\\
&1_{X^*} \overset{r}{\sqcup\!\sqcup} (x_{p+1} \cdots x_n) = x_{p+1} \cdots x_n,\\
&(x_1 \cdots x_p) \overset{r}{\sqcup\!\sqcup} 1_{X^*} = 0.
\end{aligned}$$

In terms of the (right) half-shuffle product, we can easily verify without difficulty that $(\mathbf{k}\langle X\rangle, \overset{r}{\sqcup\!\sqcup}, 1_{X^*})$ is a unital (right) Zinbiel algebra. Moreover, by the symmetric product $x * y = x \overset{r}{\sqcup\!\sqcup} y + y \overset{r}{\sqcup\!\sqcup} x = x \sqcup\!\sqcup y$, the commutative algebra associated to the unital

Zinbiel algebra $(\mathbf{k}\langle X\rangle, \overset{r}{\sqcup\!\sqcup}, 1_{X^*})$ is the unital shuffle algebra $(\mathbf{k}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})$.

Before introducing the Magnus's duality theorem on the free associative algebra $(\mathbf{k}\langle X\rangle, \mathrm{conc}, 1_{X^*})$ and the Zinbiel algebra $(\mathbf{k}\langle X\rangle, \overset{r}{\sqcup\!\sqcup}, 1_{X^*})$, we proceed to the following propositions that are necessary for describing our general picture how they arise.

Proposition 3.7. *Let us consider* $\mathbf{k}\langle X\rangle Z \oplus \mathbf{k}.1_{X^*} \cong \mathbf{k}\langle B^*Z\rangle$ *(resp.* $\mathbf{k}\langle X\rangle_B \cong \mathbf{k}\langle B\rangle$*) the subalgebra of the free associative algebra* $\mathbf{k}\langle X\rangle$ *generated by* $B^*Z \cup \{1_{X^*}\}$ *(resp.* B^**). Then*

i) $\mathbf{k}\langle X\rangle_B$ *and* $\mathbf{k}\langle X\rangle Z \oplus \mathbf{k}.1_{X^*}$ *are unital shuffle subalgebras of* $(\mathbf{k}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})$.

ii) $\mathbf{k}\langle X\rangle_B$ *and* $\mathbf{k}\langle X\rangle Z \oplus \mathbf{k}.1_{X^*}$ *are unital Zinbiel subalgebras of* $(\mathbf{k}\langle X\rangle, \overset{r}{\sqcup\!\sqcup}, 1_{X^*})$.

iii) We have $\mathbf{k}\langle X\rangle = (\mathbf{k}\langle X\rangle Z \oplus \mathbf{k}.1_{X^*}) \sqcup\!\sqcup \mathbf{k}\langle X\rangle_B$*, as shuffle algebras.*

Proof. It is straightforward to show that $\mathbf{k}\langle X\rangle_B$ and $\mathbf{k}\langle X\rangle Z \oplus \mathbf{k}.1_{X^*}$ are close under the shuffle (resp. Zinbiel) product $\sqcup\!\sqcup$ (resp. $\overset{r}{\sqcup\!\sqcup}$). One derives (i) and (ii). To prove (iii), as a consequence of Radford's results [92], one can write down a basis of any free shuffle algebra in terms of Lyndon words over X, where $X = B \nrightarrow Z$ is an ordinal sum. This implies that any polynomial $P \in \mathbf{k}\langle X\rangle$ can be written uniquely as a linear combination of the shuffle of words in B^* with polynomials in $\mathbf{k}\langle X\rangle Z \oplus \mathbf{k}.1_{X^*}$

$$P = \sum_{k\geq 0} P_k \sqcup\!\sqcup w_k, \qquad P_k \in \mathbf{k}\langle X\rangle Z \oplus \mathbf{k}.1_{X^*} \text{ and } w_k \in B^*, \tag{3.22}$$

where the families $(P_k)_{k\geq 0}$ being finitely supported. We verified our claim and then we omit the obvious proof of the point (iii). □

Let us consider multi-homogeneous polynomials of $\mathbf{k}\langle X\rangle Z \oplus \mathbf{k}.1_{X^*}$ arising by Zinbiel product of B^*Z, so-call Zinbiel polynomials, defined by

$$(\cdots((w_1 z_{i_1} \overset{r}{\sqcup\!\sqcup} w_2 z_{i_2}) \overset{r}{\sqcup\!\sqcup} w_3 z_{i_3}) \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup} w_{k-1} z_{i_{k-1}}) \overset{r}{\sqcup\!\sqcup} w_k z_{i_k}, \tag{3.23}$$

where $k \geq 0, w_1, \cdots, w_k \in B^*$ and $i_1, \cdots, i_k \in [1, N]$ (if $k = 0$ then (3.23) will be denoted by 1_{X^*}). Henceforth we write simply $w_1 z_{i_1} \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup} w_k z_{i_k}$ instead of (3.23) (please don't mistake this for an associative expression).

The above formulas lead to the next assertion

Proposition 3.8. *The Zinbiel algebra* $\mathbf{k}\langle X\rangle Z \oplus \mathbf{k}.1_{X^*}$ *is a free* $\mathbf{k}$*-module with a natural monomial basis formed* $w_1 z_{i_1}\cdots w_k z_{i_k}$*, where* $w_1,\cdots,w_k$ *are words in* B^* *and* $i_1,\cdots,i_k \in [1,N]$*. Moreover, the Zinbiel algebra has a second* $\mathbf{k}$*-linear basis formed as* (3.23)*.*

Proof. The first item is trivial. To prove the next statement, one claims that any monomial basis element $w_1 z_{i_1}\cdots w_k z_{i_k}$ of the free $\mathbf{k}$-module $\mathbf{k}\langle X\rangle Z\oplus\mathbf{k}.1_{X^*}$ is a $\mathbf{k}$-linear combination of multi-homogeneous polynomials formed as (3.23). In fact, we now prove the claim by induction on k : if $k=1$ then we first observe that $w_1 z_{i_1} = w_1 z_{i_1}$ satisfies our claim. Now, for any word $w_1,\cdots,w_k,w_{k+1}$ in B^*, where $i_1,\cdots,i_k,i_{k+1}\in[1,N]$ then, by the formula (3.22), there is $u_l \in B^*$ and $P_l \in \mathbf{k}\langle X\rangle Z\oplus\mathbf{k}.1_{X^*}(\forall l\in\mathbb{N})$ such that

$$w_1 z_{i_1}\cdots w_k z_{i_k} w_{k+1} = \sum_{l\geq 0} P_l \sqcup\!\sqcup\, u_l,$$

where $(P_l)_{l\geq 0}$ being finitely supported and each $P_l\in\mathbf{k}\langle X\rangle Z\oplus\mathbf{k}.1_{X^*}$ is $\mathbf{k}$-linear combination of multi-homogeneous polynomials arising by Zinbiel product (the right half-shuffle) of B^*Z

$$v_1 z_{i_1} \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup} v_k z_{i_k},\ \text{where } v_1,\cdots,v_k \text{ are words in } B^*$$

by inductive hypothesis. Therefore, let us consider the concatenation product of the above equation with the letter $z_{i_{k+1}}$ on the right, one has

$$w_1 z_{i_1} w_2 z_{i_2}\cdots w_k z_{i_k} w_{k+1} z_{i_{k+1}} \;=\; \Big(\sum_{l\geq 0} P_l \sqcup\!\sqcup\, u_l\Big) z_{i_{k+1}} = \sum_{l\geq 0} P_l \overset{r}{\sqcup\!\sqcup} u_l z_{i_{k+1}}$$

and so is a $\mathbf{k}$-linear combination of multi-homogeneous polynomials formed as (3.23) because in general in terms of the non associative identity (3.2) on Zinbiel product, it is straightforward that for any $v_1,\cdots,v_k,v_{k+1},\cdots,v_n\in B^*$ and $i_1,\cdots,i_n\in[1,N]$ then $(v_1 z_{i_1} \overset{r}{\sqcup\!\sqcup}\cdots\overset{r}{\sqcup\!\sqcup} v_k z_{i_k}) \overset{r}{\sqcup\!\sqcup} (v_{k+1} z_{i_{k+1}} \overset{r}{\sqcup\!\sqcup}\cdots\overset{r}{\sqcup\!\sqcup} v_n z_{i_n})$ is a $\mathbf{k}$-linear combination of elements (3.23). We verified our claim. On the other hand, as may be easily verified, under the totally ordered set X splits as $X = B \dotplus Z$, the largest monomial in $w_1 z_{i_1} \overset{r}{\sqcup\!\sqcup}\cdots\overset{r}{\sqcup\!\sqcup} w_k z_{i_k}$ will be the following

$$w_1 z_{i_1}\cdots w_k z_{i_k}, \tag{3.24}$$

they will occur with coefficient 1. It is obvious that distinct elements in (3.23) have distinct largest terms (3.24), hence the linearly independent of the elements (3.23) is then verified by taking and evaluating any **k**-linear expression of elements (3.23). We then deduce our result. □

As a consequence of Proposition 3.7 and Proposition 3.8, the collection of Zinbiel polynomials

$$(w_1 z_{i_1} \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup} w_k z_{i_k}) \sqcup\!\sqcup\, w \tag{3.25}$$

(where $k \geq 0, w_1, \cdots, w_k, w \in B^*$ and $i_1, \cdots, i_k \in [1, N]$) is a **k**-linear basis of $(\mathbf{k}\langle X\rangle, \overset{r}{\sqcup\!\sqcup}, 1_{X^*})$.

The next theorem shows that for the descriptions of the above propositions it is sufficient to consider the duality of the Magnus basis in the free associative algebra $(\mathbf{k}\langle X\rangle, \mathrm{conc}, 1_{X^*})$.

Theorem 3.9. *(Magnus duality, cf. Nakamura [89]) The collections of Magnus polynomials* (3.17) *and Zinbiel polynomials* (3.25) *are dual bases of, respectively* $(\mathbf{k}\langle X\rangle, \mathrm{conc}, 1_{X^*})$ *and* $(\mathbf{k}\langle X\rangle, \overset{r}{\sqcup\!\sqcup}, 1_{X^*})$.

Remark 3.1. It is interesting to consider the following useful discussions

i). Our main technical idea to solve the above theorem is similar to the tool of constructing and proving the PBW basis P_w and its dual S_w (w words on X) presented in Reutenauer's book [94] §5.2. But, our approach is in generalized graded bialgebra type structures studied in Appendix 5.4.5.

ii). We survey briefly in the Reutenauer's book [94] §1.5 (more precisely, see Appendix 5.4.4) that the **k**-module $\mathbf{k}\langle X\rangle$ of the noncommutative polynomials has two natural graded Hopf algebra structures which are dual to each other (in the graded sense), that is $\mathcal{H}_{\mathrm{conc}}(X) = (\mathbf{k}\langle X\rangle, \mathrm{conc}, 1_{X^*}, \Delta_{\sqcup\!\sqcup}, \epsilon)$ and its graded dual $\mathcal{H}_{\sqcup\!\sqcup}(X) = (\mathbf{k}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}, \Delta_{\mathrm{conc}}, \epsilon)$.

iii) Our approach appeared in Appendix 5.4.5 is indeed the (right) Zinbiel bialgebra $\mathcal{Z}_{\overset{r}{\sqcup\!\sqcup}}(X) := (\mathbf{k}\langle X\rangle, \overset{r}{\sqcup\!\sqcup}, 1_{X^*}, \Delta_{\mathrm{conc}}, \epsilon)$ and its graded dual the *Zinb*^{c}-*As*-bialgebra

$\mathcal{Z}_{\mathrm{conc}}(X) := (\mathbf{k}\langle X\rangle, \mathrm{conc}, 1_{X^*}, \Delta_{\overset{r}{\sqcup\!\sqcup}}, \epsilon)$. This mathematical framework provides useful computing techniques that will be presented to the reader as below.

Recall the Lazard's elimination in the category **k-AAU** meaning there exists the natural algebra isomorphism $f_* : \mathbf{k}\langle B^*Z\rangle\sharp\mathbf{k}\langle B\rangle \to \mathbf{k}\langle X\rangle$ which has been completely studied in Section 2.4. Let us denote $\mathbf{k}\langle X\rangle_{BZ} \oplus \mathbf{k}.1_{X^*} := f_*(\mathbf{k}\langle B^*Z\rangle)$ the subalgebra of the free associative algebra $\mathbf{k}\langle X\rangle$ generated by $\{rn(wz)\}_{w\in B^*, z\in Z} \cup \{1_{X^*}\}$. The following proposition will be the first step on the way of presenting our proof of Theorem 3.9.

Proposition 3.10. *(Conc & Half shuffle duality) The families of Magnus polynomials $rn(w_1 z_{i_1}) \cdots rn(w_k z_{i_k})$ and Zinbiel polynomials $w_1 z_{i_1} \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup} w_k z_{i_k}$ (where $k \geq 0, w_1, \cdots, w_k \in B^*$ and $i_1, \cdots, i_k \in [1, N]$) are dual bases of, respectively* $(\mathbf{k}\langle X\rangle_{BZ} \oplus \mathbf{k}.1_{X^*}, \mathrm{conc}, 1_{X^*})$ *and* $(\mathbf{k}\langle X\rangle Z \oplus \mathbf{k}.1_{X^*}, \overset{r}{\sqcup\!\sqcup}, 1_{X^*})$.

Proof. We assume that $B = \{b_1, \cdots, b_M\}$ and $Z = \{z\}$ only one element. To show that any standard pairing $\langle w_1 z \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup} w_k z \mid rn(u_1 z) \cdots rn(u_k z)\rangle = \delta_{(w_1,\cdots,w_k)}^{(u_1,\cdots,u_k)}$ (that is the Kronecker delta, over two chains of words $(w_1, \cdots, w_k)$ and $(u_1, \cdots, u_k)$ $(k \geq 1)$ over B, equal to 0 or 1 if the chains coincide not respectively), we now have computing processes

$$\begin{aligned}
&\langle w_1 z \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup} w_k z \mid rn(u_1 z) \cdots rn(u_k z)\rangle \\
&\overset{(1)}{=} \langle [w_1 z \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup} w_{k-1} z] \otimes w_k z \mid \Delta_{\overset{r}{\sqcup\!\sqcup}}[rn(u_1 z) \cdots rn(u_k z)]\rangle \\
&\overset{(2)}{=} \langle [w_1 z \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup} w_{k-1} z] \otimes w_k z \mid \Delta_{\sqcup\!\sqcup}[rn(u_1 z) \cdots rn(u_{k-1} z)] \Delta_{\overset{r}{\sqcup\!\sqcup}}[rn(u_k z)]\rangle.
\end{aligned}$$

where, the first equality passing from the adjoint (for the scalar product $\langle \bullet \mid \bullet\rangle$ above) $\langle T \overset{r}{\sqcup\!\sqcup} P \mid Q\rangle = \langle T \otimes P \mid \Delta_{\overset{r}{\sqcup\!\sqcup}}(Q)\rangle$ $(T, P, Q$ are polynomials such that $\langle Q \mid 1_{X^*}\rangle = 0)$ studied in Appendix 5.4.5, and the second equality passing from the compatible relation $\Delta_{\overset{r}{\sqcup\!\sqcup}}(TP) = \Delta_{\sqcup\!\sqcup}(T)\Delta_{\overset{r}{\sqcup\!\sqcup}}(P)$ (where T, P polynomials) in Example 5.8 point (ii), Appendix 5.4.5. We also have the expression $\Delta_{\sqcup\!\sqcup}[rn(u_i z)] = rn(u_i z) \otimes 1 + 1 \otimes rn(u_i z)$ for each primitive elements $rn(u_i z)$ because we have the fragment $\{rn(u_i z)\}_{1\leq i\leq k-1} \subseteq \mathcal{L}_{\mathbf{k}}(X) \subseteq \mathrm{Prim}\, \mathcal{H}_{\mathrm{conc}}(X)$, see more details in Remark 5.6 point (ii), Appendix 5.4.4.

The following lemma is necessary for the proof.

Lemma 3.11. *For any polynomials $T_1, T_2 \in \mathbf{k}\langle X\rangle Z$ and double polynomial $P = \sum_{i=1}^m P_1^i \otimes P_2^i \in \mathbf{k}\langle X\rangle \otimes \mathbf{k}\langle X\rangle$, then $\langle T_1 \otimes T_2 \mid P.\Delta_{\underset{\sqcup\!\sqcup}{r}}[rn(wz)]\rangle = \langle T_1 \otimes T_2 \mid P.[1 \otimes rn(wz)]\rangle$ for each word w over the alphabet B.*

Proof. If $w = 1_{X^*}$ then $\langle T_1 \otimes T_2 \mid P.\Delta_{\underset{\sqcup\!\sqcup}{r}}[rn(wz)]\rangle = \langle T_1 \otimes T_2 \mid P.\Delta_{\underset{\sqcup\!\sqcup}{r}}(z)\rangle = \langle T_1 \otimes T_2 \mid P.(1 \otimes z)\rangle$ by the definition of $\Delta_{\underset{\sqcup\!\sqcup}{r}}$ in Appendix 5.4.5. Assume that for any word w over B of length k, then $\langle T_1 \otimes T_2 \mid P.\Delta_{\underset{\sqcup\!\sqcup}{r}}[rn(wz)]\rangle = \langle T_1 \otimes T_2 \mid P.[1 \otimes rn(wz)]\rangle$ for any polynomials $T_1, T_2 \in \mathbf{k}\langle X\rangle Z$ and double polynomial $P = \sum_{i=1}^m P_1^i \otimes P_2^i \in \mathbf{k}\langle X\rangle \otimes \mathbf{k}\langle X\rangle$. Now, in case $w = b_{m_1} \cdots b_{m_k} b_{m_{k+1}} \in B^*$, where $m_1, \cdots, m_{k+1} \in [1, M]$, it follows from the compatible relation of $\Delta_{\underset{\sqcup\!\sqcup}{r}}$ in Appendix 5.4.5 that one has

$$\begin{aligned}
&\Delta_{\underset{\sqcup\!\sqcup}{r}}[rn(wz)] = \Delta_{\underset{\sqcup\!\sqcup}{r}}[b_{m_1}.rn(b_{m_2} \cdots b_{m_{k+1}} z) - rn(b_{m_2} \cdots b_{m_{k+1}} z).b_{m_1}] \\
&= \Delta_{\sqcup\!\sqcup}(b_{m_1})\Delta_{\underset{\sqcup\!\sqcup}{r}}[rn(b_{m_2} \cdots b_{m_{k+1}} z)] - \Delta_{\sqcup\!\sqcup}[rn(b_{m_2} \cdots b_{m_{k+1}} z)]\Delta_{\underset{\sqcup\!\sqcup}{r}}(b_{m_1}) \\
&= (b_{m_1} \otimes 1 + 1 \otimes b_{m_1})\Delta_{\underset{\sqcup\!\sqcup}{r}}[rn(b_{m_2} \cdots b_{m_{k+1}} z)] - \Delta_{\sqcup\!\sqcup}[rn(b_{m_2} \cdots b_{m_{k+1}} z)](1 \otimes b_{m_1}) \\
&= (b_{m_1} \otimes 1 + 1 \otimes b_{m_1})\Delta_{\underset{\sqcup\!\sqcup}{r}}[rn(b_{m_2} \cdots b_{m_{k+1}} z)] \\
&-[rn(b_{m_2} \cdots b_{m_{k+1}} z) \otimes b_{m_1} + 1 \otimes rn(b_{m_2} \cdots b_{m_{k+1}} z)b_{m_1}].
\end{aligned}$$

Thus, by the above formula and the inductive hypothesis, we arrive at the following

$$\begin{aligned}
\langle T_1 \otimes T_2 \mid P.\Delta_{\underset{\sqcup\!\sqcup}{r}}[rn(wz)]\rangle &= \langle T_1 \otimes T_2 \mid P.(b_{m_1} \otimes 1 + 1 \otimes b_{m_1})\Delta_{\underset{\sqcup\!\sqcup}{r}}[rn(b_{m_2} \cdots b_{m_{k+1}} z)]\rangle \\
&- \langle T_1 \otimes T_2 \mid P.[rn(b_{m_2} \cdots b_{m_{k+1}} z) \otimes b_{m_1}]\rangle \\
&- \langle T_1 \otimes T_2 \mid P.[1 \otimes rn(b_{m_2} \cdots b_{m_{k+1}} z)b_{m_1}]\rangle \\
&= \langle T_1 \otimes T_2 \mid P.(b_{m_1} \otimes 1 + 1 \otimes b_{m_1})[1 \otimes rn(b_{m_2} \cdots b_{m_{k+1}} z)]\rangle \\
&- \langle T_1 \otimes T_2 \mid P.[rn(b_{m_2} \cdots b_{m_{k+1}} z) \otimes b_{m_1}]\rangle \\
&- \langle T_1 \otimes T_2 \mid P.[1 \otimes rn(b_{m_2} \cdots b_{m_{k+1}} z)b_{m_1}]\rangle \\
&= \langle T_1 \otimes T_2 \mid P.[b_{m_1} \otimes rn(b_{m_2} \cdots b_{m_{k+1}} z)]\rangle \\
&- \langle T_1 \otimes T_2 \mid P.[rn(b_{m_2} \cdots b_{m_{k+1}} z) \otimes b_{m_1}]\rangle \\
&+ \langle T_1 \otimes T_2 \mid P.[1 \otimes rn(b_{m_1} b_{m_2} \cdots b_{m_{k+1}} z)]\rangle \\
&= \langle T_1 \otimes T_2 \mid P.[1 \otimes rn(b_{m_1} b_{m_2} \cdots b_{m_{k+1}} z)]\rangle,
\end{aligned}$$

where we obtained $\langle T_1 \otimes T_2 \mid P.[b_{m_1} \otimes rn(b_{m_2} \cdots b_{m_{k+1}} z)]\rangle = \sum_{i=1}^m \langle T_1 \otimes T_2 \mid P_1^i.b_{m_1} \otimes P_2^i.rn(b_{m_2} \cdots b_{m_{k+1}} z)\rangle = 0$ by using $T_1 \in \mathbf{k}\langle X\rangle Z$, and similarly one has $\langle T_1 \otimes T_2 \mid P.[rn(b_{m_2} \cdots b_{m_{k+1}} z) \otimes b_{m_1}]\rangle = 0$. We verified our lemma. □

End of the proof of Proposition 3.10. –
By the above lemma, one has

$$\begin{aligned}
&\langle [w_1z \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup}\, w_{k-1}z] \otimes w_kz \mid \Delta_{\sqcup\!\sqcup}[rn(u_1z)\cdots rn(u_{k-1}z)]\Delta_{\overset{r}{\sqcup\!\sqcup}}[rn(u_kz)]\rangle \\
&= \langle [w_1z \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup}\, w_{k-1}z] \otimes w_kz \mid \Delta_{\sqcup\!\sqcup}[rn(u_1z)\cdots rn(u_{k-1}z)][1 \otimes rn(u_kz)]\rangle .
\end{aligned}$$

Thus, from the equation above and a standard formula $\langle w_kz \mid rn(u_kz).T\rangle = 0$ for each non empty monomial T of the set $\{rn(u_iz)\}_{1\le i\le k-1}$, we arrive at the following computational processes

$$\begin{aligned}
&\langle w_1z \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup}\, w_kz \mid rn(u_1z)\cdots rn(u_kz)\rangle \\
&= \langle [w_1z \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup}\, w_{k-1}z] \otimes w_kz \mid \Delta_{\sqcup\!\sqcup}[rn(u_1z)\cdots rn(u_{k-1}z)][1 \otimes rn(u_kz)]\rangle \\
&= \langle [w_1z \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup}\, w_{k-1}z] \otimes w_kz \mid \\
&[rn(u_1z)\otimes 1 + 1\otimes rn(u_1z)]\cdots[rn(u_{k-1}z)\otimes 1 + 1\otimes rn(u_{k-1}z)][1 \otimes rn(u_kz)]\rangle \\
&= \langle [w_1z \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup}\, w_{k-1}z] \otimes w_kz \mid rn(u_1z)\cdots rn(u_{k-1}z) \otimes rn(u_kz)\rangle \\
&= \langle w_1z \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup}\, w_{k-1}z \mid rn(u_1z)\cdots rn(u_{k-1}z)\rangle.\langle w_kz \mid rn(u_kz)\rangle \\
&= \langle w_1z \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup}\, w_{k-1}z \mid rn(u_1z)\cdots rn(u_{k-1}z)\rangle.\delta_{w_k}^{u_k} \\
&= \delta_{w_1}^{u_1}\cdots\delta_{w_{k-1}}^{u_{k-1}}\delta_{w_k}^{u_k} = \delta_{(w_1,\cdots,w_k)}^{(u_1,\cdots,u_k)} \text{ (by the inductive progresses) .}
\end{aligned}$$

Moreover, we now state to the general case $B = \{b_1, \cdots, b_M\}$ and $Z = \{z_1, \cdots, z_N\}$. Let k be a positive integer. For convenience, we will borrow and extend some of the notations and techniques appeared in Nakamura's paper [89] which the author used to prove the Magnus duality in case $X = B + Z$ where $B = \{x_0\}$ and $Z = \{x_\lambda\}_{\lambda\in\Lambda}$ (Λ: a nonempty index set). We now denote $\iota = (i_1, \cdots, i_k) \in [1, N]^k$ and for $w_1, \cdots, w_k, w \in B^* = \{b_1, \cdots, b_M\}^*$ we shall denote

$$\begin{aligned}
P^{(\iota)}_{(w_1,\cdots,w_k)} &= rn(w_1z_{i_1})\cdots rn(w_kz_{i_k}), \\
S^{(\iota)}_{(w_1,\cdots,w_k)} &= w_1z_{i_1} \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup}\, w_kz_{i_k}, \\
w^{(\iota)}_{(w_1,\cdots,w_k,w)} &= w_1z_{i_1}\cdots w_kz_{i_k}.w.
\end{aligned}$$

We now prove that the pairing $\langle S^{(\iota)}_{(w_1,\cdots,w_k)} \mid P^{(\lambda)}_{(u_1,\cdots,u_h)}\rangle$ is equal to the Kronecker delta $\delta^{(\lambda;u_1,\cdots,u_h)}_{(\iota;w_1,\cdots,w_k)}$, where $\iota = (i_1, \cdots, i_k) \in [1, N]^k, w_1, \cdots w_k \in B^*$ and $\lambda = (\lambda_1, \cdots, \lambda_h) \in [1, N]^h, u_1, \cdots, u_h \in B^*$. Given a fix $\iota = (i_1, \cdots, i_k) \in [1, N]^k$. Let us consider V_ι a submodule of $\mathbf{k}\langle X\rangle$ generated by the monomials $\{w^{(\iota)}_{(w_1,\cdots,w_k,w)} : w_1,\cdots,w_k,w \in B^*\}$. We easily observe that if $\iota = (i_1,\cdots,i_k) \neq \lambda = (\lambda_1,\cdots,\lambda_h)$ then V_ι and V_λ are orthogonal under the standard pairing $\langle \bullet \mid \bullet\rangle$. One also observes that

- Given a fix λ, then $P^{(\lambda)}_{(u_1,\cdots,u_h)} \in V_\lambda$, for any $u_1,\cdots,u_h \in B^*$.
- Given a fix ι, then $S^{(\iota)}_{(w_1,\cdots,w_k)} \in V_\iota$, for any $w_1,\cdots,w_k \in B^*$.

Thus, if $\iota \neq \lambda$, then $\langle S^{(\iota)}_{(w_1,\cdots,w_k)} \mid P^{(\lambda)}_{(u_1,\cdots,u_h)}\rangle = 0 = \delta^{(\lambda;u_1,\cdots,u_h)}_{(\iota;w_1,\cdots,w_k)}$. We only consider the case $\iota = \lambda = (i_1,\cdots,i_k)$. We now introduce the following notations

$$\begin{aligned}
P_{(w_1,\cdots,w_k)} &= rn(w_1z)\cdots rn(w_kz),\\
S_{(w_1,\cdots,w_k)} &= w_1z \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup}\, w_kz,\\
w_{(w_1,\cdots,w_k,w)} &= w_1z\cdots w_kz.w,
\end{aligned}$$

and then V_k a submodule of $\mathbf{k}\langle B,\{z\}\rangle$ generated by the monomials $\{w_{(w_1,\cdots,w_k,w)} : w_1,\cdots,w_k,w \in B^*\}$. The mapping $\phi_\iota : V_k \to V_\iota, w_{(w_1,\cdots,w_k,w)} \mapsto w^{(\iota)}_{(w_1,\cdots,w_k,w)}$ is an isomorphism of $\mathbf{k}$-modules and preserves the standard pairing $\langle \bullet \mid \bullet\rangle$ (for this tool, see [89] Thm 3.2). Thus, one has the following

$$\begin{aligned}
&\langle S^{(\iota)}_{(w_1,\cdots,w_k)} \mid P^{(\iota)}_{(u_1,\cdots,u_k)}\rangle = \langle \phi_\iota(S_{(w_1,\cdots,w_k)}) \mid \phi_\iota(P_{(u_1,\cdots,u_k)})\rangle = \langle S_{(w_1,\cdots,w_k)} \mid P_{(u_1,\cdots,u_k)}\rangle\\
&= \langle w_1z \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup}\, w_kz \mid rn(u_1z)\cdots rn(u_kz)\rangle = \delta^{(u_1,\cdots,u_k)}_{(w_1,\cdots,w_k)} = \delta^{(\iota;u_1,\cdots,u_k)}_{(\iota;w_1,\cdots,w_k)}.
\end{aligned}$$

Consequently, we verified the "Conc & Half shuffle duality" that means $\langle S^{(\iota)}_{(w_1,\cdots,w_k)} \mid P^{(\lambda)}_{(u_1,\cdots,u_h)}\rangle = \delta^{(\lambda;u_1,\cdots,u_h)}_{(\iota;w_1,\cdots,w_k)}$. The proof is complete.

QED

□

Proof of Theorem 3.9 : We recall the graded set $X = B + Z$, where $B = \{b_1,\cdots,b_M\}$ and $Z = \{z_1,\cdots,z_N\}$. Let k be a positive integer. To approach the notations already used when proving Proposition 3.10, although in a more sophisticated way, if $\iota = (i_1,\cdots,i_k) \in [1,N]^k$ and for each $w_1,\cdots,w_k,w \in B^* = \{b_1,\cdots,b_M\}^*$, we consider useful notations

$$P^{(\iota)}_{(w_1,\cdots,w_k,w)} = rn(w_1z_{i_1})\cdots rn(w_kz_{i_k}).w,$$

$$S^{(\iota)}_{(w_1,\cdots,w_k,w)} = (w_1 z_{i_1} \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup} w_k z_{i_k}) \sqcup\!\sqcup w,$$
$$w^{(\iota)}_{(w_1,\cdots,w_k,w)} = w_1 z_{i_1} \cdots w_k z_{i_k}.w.$$

We now prove that $\langle S^{(\iota)}_{(w_1,\cdots,w_k,w)} \mid P^{(\lambda)}_{(u_1,\cdots,u_h,u)}\rangle$ is equal to the Kronecker delta $\delta^{(\lambda;u_1,\cdots,u_h,u)}_{(\iota;w_1,\cdots,w_k,w)}$. Given a fix $\iota = (i_1,\cdots,i_k) \in [1,N]^k$. According to the proof of Proposition 3.10, let us recall V_ι that is a submodule of $\mathbf{k}\langle X\rangle$ generated by the monomials $\{w^{(\iota)}_{(w_1,\cdots,w_k,w)} : w_1,\cdots,w_k,w \in B^*\}$. We also recall that if $\iota = (i_1,\cdots,i_k) \neq \lambda = (\lambda_1,\cdots,\lambda_h)$ then V_ι and V_λ are orthogonal under the standard pairing $\langle \bullet \mid \bullet\rangle$. Then, we observe certain properties

- Given a fix λ, then $P^{(\lambda)}_{(u_1,\cdots,u_h,u)} \in V_\lambda$, for any $u_1,\cdots,u_h,u \in B^*$.
- Given a fix ι, then $S^{(\iota)}_{(w_1,\cdots,w_k,w)} \in V_\iota$, for any $w_1,\cdots,w_k,w \in B^*$.

As a consequence, if $\iota \neq \lambda$, one has $\langle S^{(\iota)}_{(w_1,\cdots,w_k,w)} \mid P^{(\lambda)}_{(u_1,\cdots,u_h,u)}\rangle = 0 = \delta^{(\lambda;u_1,\cdots,u_h,u)}_{(\iota;w_1,\cdots,w_k,w)}$. We finally consider the case $\iota = \lambda = (i_1,\cdots,i_k)$. According to the adjoint $\langle T \sqcup\!\sqcup P \mid Q\rangle = \langle T \otimes P \mid \Delta_{\sqcup\!\sqcup}(Q)\rangle$ (T, P, Q are polynomials) successfully achieved from constructing the graded Hopf algebra $\mathcal{H}_{\mathrm{conc}}(X) = (\mathbf{k}\langle X\rangle, \mathrm{conc}, 1_{X^*}, \Delta_{\sqcup\!\sqcup}, \epsilon)$ and its graded dual $\mathcal{H}_{\sqcup\!\sqcup}(X) = (\mathbf{k}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}, \Delta_{\mathrm{conc}}, \epsilon)$ (see Appendix 5.4.4), we shall present

$$\begin{aligned}
&\langle S^{(\iota)}_{(w_1,\cdots,w_k,w)} \mid P^{(\iota)}_{(u_1,\cdots,u_k,u)}\rangle = \langle (w_1 z_{i_1} \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup} w_k z_{i_k}) \sqcup\!\sqcup w \mid rn(u_1 z_{i_1}) \cdots rn(u_k z_{i_k}).u\rangle \\
&= \langle w_1 z_{i_1} \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup} w_k z_{i_k} \otimes w \mid \Delta_{\sqcup\!\sqcup}\big[rn(u_1 z_{i_1}) \cdots rn(u_k z_{i_k}).u\big]\rangle \\
&= \langle w_1 z_{i_1} \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup} w_k z_{i_k} \otimes w \mid [rn(u_1 z_{i_1}) \otimes 1 + 1 \otimes rn(u_1 z_{i_1})] \\
&\cdots [rn(u_k z_{i_k}) \otimes 1 + 1 \otimes rn(u_k z_{i_k})](b_{m_1} \otimes 1 + 1 \otimes b_{m_1}) \cdots (b_{m_{|u|}} \otimes 1 + 1 \otimes b_{m_{|u|}})\rangle,
\end{aligned}$$

where we putted $u = b_{m_1} \cdots b_{m_{|u|}} \in B^*$. It follows from the fact that $\langle T \mid P\rangle = 0$ if $T \in \mathbf{k}\langle X\rangle Z$ and $P \in \mathbf{k}\langle X\rangle B$, after expressing the right-hand side of the last pairing above as a linear sum of $Q_1 \otimes Q_2$ ($Q_i \in \mathbf{k}\langle X\rangle$), we clearly have that the last equality is equal to $\langle w_1 z_{i_1} \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup} w_k z_{i_k} \otimes w \mid rn(u_1 z_{i_1}) \cdots rn(u_k z_{i_k}) \otimes u\rangle$. Then, Proposition 3.10 is used to verify the following equation

$$\begin{aligned}
&\langle w_1 z_{i_1} \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup} w_k z_{i_k} \otimes w \mid rn(u_1 z_{i_1}) \cdots rn(u_k z_{i_k}) \otimes u\rangle \\
&= \langle w_1 z_{i_1} \overset{r}{\sqcup\!\sqcup} \cdots \overset{r}{\sqcup\!\sqcup} w_k z_{i_k} \mid rn(u_1 z_{i_1}) \cdots rn(u_k z_{i_k})\rangle \langle w \mid u\rangle \\
&= \delta^{(u_1,\cdots,u_k)}_{(w_1,\cdots,w_k)} \delta^u_w = \delta^{(u_1,\cdots,u_k,u)}_{(w_1,\cdots,w_k,w)} = \delta^{(\iota;u_1,\cdots,u_k,u)}_{(\iota;w_1,\cdots,w_k,w)},
\end{aligned}$$

and then $\langle S^{(\iota)}_{(w_1,\cdots,w_k,w)} \mid P^{(\iota)}_{(u_1,\cdots,u_k,u)}\rangle = \delta^{(\iota;u_1,\cdots,u_k,u)}_{(\iota;w_1,\cdots,w_k,w)}$. We give a complete proof of the Magnus's duality and then our theorem. $\square$

Remark 3.2. Our constructions and results above for the Magnus basis and their duality in case the finite graded set $X = B + Z$ (where $B = \{b_1, \cdots, b_M\}$ and $Z = \{z_1, \cdots, z_N\}$) can be automatically approached to any graded set $X = B + Z$ where $B = \{b_\gamma\}_{\gamma\in\Gamma}$ and $Z = \{z_\lambda\}_{\lambda\in\Lambda}$ (Γ, Λ are nonempty index sets). In case $X = B+Z$ where $B = \{x_0\}$ and $Z = \{x_\lambda\}_{\lambda\in\Lambda}$ (Λ: a nonempty index set, for example $\mathbb{N}_+$), the Magnus duality also appeared in [89] Thm 3.2 to derive a formula of Le-Murakami [75], Furusho type [52] that expresses arbitrary coefficients of a group-like series $J \in \mathbf{k}\langle\langle x_0, x_1\rangle\rangle$ ($\mathbf{k}$ is a field of characteristic zero) in terms of the "regular" coefficients of J ([89] Thm 4.1). On the other hand, images of the Magnus polynomial and its dual under the anti-automorphism Φ of $\mathbf{k}\langle X\rangle$ (which sends $w \mapsto \widetilde{w}$ for all words on X, where $\widetilde{w}$ reverses the order of letters in the word w) belong in the (left) Zinbiel bialgebra and its dualisation framework [18], Appendix: Associative-Zinbiel bialgebras. The images also appeared in [60] Prop 5.10 to describe the coefficients of the complete generating series (3.37)

$$Ł(z) = \sum_{w\in X^*} \mathrm{Li}_w(z) w$$

(where $X = \{x_0, x_1\}$) in terms of the indeterminates are monomials of the set $\{rn(x_0^k x_1) = \mathrm{ad}^k_{x_0} x_1\}_{k\geq 0}$, where $Ł(z)$ is the group-like solution of the following first order noncommutative differential equation (3.38) (the Knizhnik-Zamolodchikov equation KZ_3 due to Drinfeld [30, 31])

$$\begin{cases} \mathbf{d}(S) = (\omega_0(z)x_0 + \omega_1(z)x_1)S, & (NCDE) \\ \lim\limits_{z\in\Omega, z\to 0} S(z)e^{-x_0\log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle x_0,x_1\rangle\rangle}, & \text{asymptotic initial condition,} \end{cases}$$

where, $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$ are two differential forms on the simply-connected domain $\Omega = \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$ and for any series $S \in \mathcal{H}(\Omega)\langle\langle x_0, x_1\rangle\rangle$ over $\mathcal{H}(\Omega)$ the algebra (for the pointwise product) of complex-valued functions which are holomorphic on Ω (for more details see in [60] Prop 5.10 and Subsection 3.4.1).

3.4 Extension of characters: A theory of Domains for Harmonic Functions and its Symbolic Counterpart.

In this section, we begin by reviewing the calculus induced by the framework of [39]. In there, we extended Polylogarithm functions over a subalgebra of noncommutative rational power series, recognizable by finite state (multiplicity) automata over the alphabet $X = \{x_0, x_1\}$. The stability of this calculus under shuffle products relies on the nuclearity of the target space [97]. We also concentrated on algebraic and analytic aspects of this extension allowing to index polylogarithms, at non positive multi-indices, by rational series and also allowing to regularize divergent polyzetas, at non positive multi-indices [39]. As a continuation of works in [39] and in order to understand the bridge between the extension of this "polylogarithmic calculus" and the world of harmonic sums, we propose a local theory, adapted to a full calculus on indices of Harmonic Sums based on the Taylor expansions, around zero, of polylogarithms with index x_1 on the rightmost end. This theory is not only compatible with Stuffle products but also with the Analytic Model. In this respect, it provides a stable and fully algorithmic model for Harmonic calculus. Examples by computer are also provided.

3.4.1 Introduction.

Riemann's zeta function is defined by the series

$$\zeta(s) \;:=\; \sum_{n\geq 1} \frac{1}{n^s} \tag{3.26}$$

where s is a complex number. It is absolutely convergent for $\Re(s) > 1$ (for any $s \in \mathbb{C}$, $\Re(s)$ stands for the real part of s).

It can be extended to a meromorphic function on the complex plane $\mathbb{C}$ with a single pole at $s = 1$ [95][15].). In fact, the story began with Euler's works to find the solution

[15] Whence the famous sum $\zeta(-1) = 1 + 2 + 3 + \cdots = -\frac{1}{12}$ by which, among other "results", S. Ramanujan was noticed by G. H. H. Hardy (see [1]).

of Basel problem. In these works, Euler proved that [45]

$$\zeta(2) = \sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{3.27}$$

Moreover, for suitable s_1, s_2, Euler gave an important identity as follows[16]:

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_1 + s_2) + \zeta(s_2, s_1), \tag{3.28}$$

where

$$\zeta(s_1, s_2) := \sum_{n_1 > n_2 \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}}. \tag{3.29}$$

The numbers $\zeta(s_1, s_2)$ were called "double zeta values" at (s_1, s_2). More generally, for any $r \in \mathbb{N}_+$ and $s_1, \cdots, s_r \in \mathbb{C}$, we denote

$$\zeta(s_1, \cdots, s_r) := \sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}. \tag{3.30}$$

Then H. Furusho et al. [53] showed that the series $\zeta(s_1, \cdots, s_r)$ converges absolutely for $s \in \mathcal{H}_r$ where

$$\mathcal{H}_r := \{\mathbf{s} = (s_1, \cdots, s_r) \in \mathbb{C}^r | \forall m = 1, \cdots, r;\ \Re(s_1) + \cdots + \Re(s_m) > m\}. \tag{3.31}$$

In the convergent cases, $\zeta(s_1, \cdots, s_r)$ they are called "polyzeta values" (or MZV[17]) at multi-index $\mathbf{s} = (s_1, \cdots, s_r)$. Indeed $\mathbf{s} \mapsto \zeta(\mathbf{s})$ is holomorphic on $\mathcal{H}_r$ and has been extended to $\mathbb{C}^r$ as a meromorphic function (see [55, 107]).

Then, for any r-uplet $(s_1, \cdots, s_r) \in \mathbb{N}_+^r \cap \mathcal{H}_r$, $r \in \mathbb{N}_+$ i.e. with $s_1 \geq 2$, the polyzeta $\zeta(s_1, \cdots, s_r)$ is also the limit at $z = 1$ of the *polylogarithmic function*, defined by:

$$\mathrm{Li}_{s_1,\cdots,s_r}(z) := \sum_{n_1 > \cdots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \cdots n_r^{s_r}} \tag{3.32}$$

for any $z \in \mathbb{C}$ such that $|z| < 1$. It is easily seen that, for any $s_i \in \mathbb{N}_+,\ r > 1$,

$$z\frac{d}{dz}\mathrm{Li}_{s_1,\cdots,s_r}(z) = \mathrm{Li}_{s_1-1,\cdots,s_r}(z) \text{ if } s_1 > 1$$

[16]In fact, in Euler's formula, $s_1, s_2 \in \mathbb{N}_+$. This identity appeared under the name "Prima Methodus ..." (see [46] pp 141-144).

[17]Multiple Zeta Values.

$$(1-z)\frac{d}{dz}\operatorname{Li}_{1,s_2,\cdots,s_r}(z) \quad = \quad \operatorname{Li}_{s_2,\cdots,s_r}(z) \text{ if } r>1 \tag{3.33}$$

and this formulas will be ended at the "seed" $\operatorname{Li}_1(z) = \log\left(\frac{1}{1-z}\right)$.

Moreover, if X^* is the free monoid of rank two (generators, or the alphabet, $X = \{x_0, x_1\}$ and the neutral 1_{X^*}) then the polylogarithms indexed by a list

$$(s_1, \cdots, s_r) \in \mathbb{N}_+^r \text{ can be reindexed by the word } x_0^{s_1-1}x_1 \cdots x_0^{s_r-1}x_1 \in X^*x_1 \tag{3.34}$$

In order to reverse the recursion introduced in Equations (3.33) (two equations), we introduce two differential forms

$$\omega_0(z) = z^{-1}dz \quad \text{and} \quad \omega_1(z) = (1-z)^{-1}dz, \tag{3.35}$$

on Ω[18]. We then get an integral representation[19] of the functions (3.32) as follows[20] (see Def 3.2 [39] and Figure 3.1)

$$\operatorname{Li}_w(z) = \begin{cases} 1_{\mathcal{H}(\Omega)} & \text{if} \quad w = 1_{X^*} \\ \int_0^z \omega_1(s)\operatorname{Li}_u(s) & \text{if} \quad w = x_1 u \\ \int_1^z \omega_0(s)\operatorname{Li}_u(s) & \text{if} \quad w = x_0 u \quad \text{and} \quad |u|_{x_1} = 0, i.e.\ w \in x_0^* \\ \int_0^z \omega_0(s)\operatorname{Li}_u(s) & \text{if} \quad w = x_0 u \quad \text{and} \quad |u|_{x_1} > 0, i.e.\ w \notin x_0^*, \end{cases} \tag{3.36}$$

the upper bound z belongs to Ω (as $\Omega = \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$ is a simply-connected domain, the integrals, which can be proved to be convergent in all cases, depend only on their bounds). The neutral element of the algebra of analytic functions $\mathcal{H}(\Omega)$, a constant function, will be here denoted $1_{\mathcal{H}(\Omega)}$.

This provides not only the analytic continuation of (3.32) to Ω but also extends the indexation to the whole monoid X^*, allowing to study the complete generating series

$$\text{Ł}(z) = \sum_{w \in X^*} \operatorname{Li}_w(z) w \tag{3.37}$$

---

[18] Ω is the simply-connected domain $\mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$.

[19] In here, we code the moves $z\frac{d}{dz}$ (resp. $(1-z)\frac{d}{dz}$) - or more precisely sections $\int_0^z \frac{f(s)}{s} ds$ (resp. $\int_0^z \frac{f(s)}{1-s} ds$) - with x_0 (resp. x_1).

[20] Given a word $w \in X^*$, we note $|w|_{x_1}$ the number of occurrences of x_1 within w.

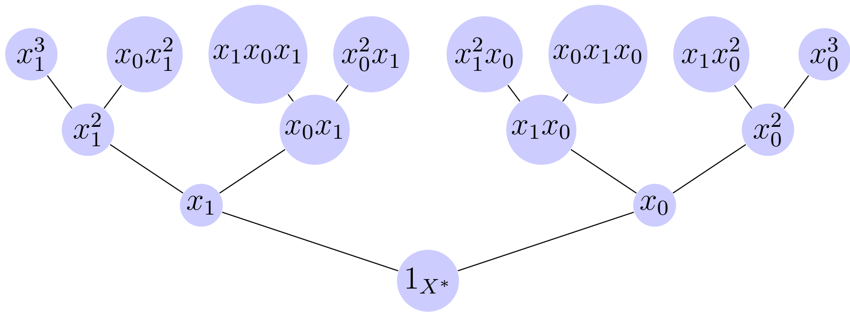


Figure 3.1: Tree of addresses and corresponding coefficients below.

and show that it is the solution of the following first order noncommutative differential system (see [31])

$$\begin{cases} \mathbf{d}(S) = (\omega_0(z)x_0 + \omega_1(z)x_1)S, & (NCDE) \\ \lim\limits_{z\in\Omega, z\to 0} S(z)e^{-x_0\log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle}, & \text{asymptotic initial condition,} \end{cases} \tag{3.38}$$

where, for any $S \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle$ and $\mathbf{d}$ stands for the term by term derivation $\mathbf{d}(S) = \sum_{w\in X^*} \frac{d}{dz}(\langle S \mid w\rangle)w$.

This differential system allows to show that Ł is a $\shuffle$-character[21] i.e.

$$\forall u, v \in X^*, \quad \langle Ł \mid u \shuffle v\rangle = \langle Ł \mid u\rangle\langle Ł \mid v\rangle \quad \text{and} \quad \langle Ł \mid 1_{X^*}\rangle = 1_{\mathcal{H}(\Omega)}. \tag{3.39}$$

$$\begin{array}{rcl} \langle Ł \mid x_0^n\rangle = \dfrac{\log(z)^n}{n!} & ; & \langle Ł \mid x_1^n\rangle = \dfrac{(-\log(1-z))^n}{n!} \\ \langle Ł \mid x_0x_1\rangle = \mathrm{Li}_2(z) = \displaystyle\sum_{n\geq 1}\frac{z^n}{n^2} & ; & \langle Ł \mid x_1x_0\rangle = \langle Ł \mid x_1 \shuffle x_0 - x_0x_1\rangle(z) \\ \langle Ł \mid x_0^2x_1\rangle = \mathrm{Li}_3(z) = \displaystyle\sum_{n\geq 1}\frac{z^n}{n^3} & ; & \langle Ł \mid x_1x_0\rangle = (-\log(1-z))\log(z) - \mathrm{Li}_2(z) \\ \langle Ł \mid x_0^{r-1}x_1\rangle = \mathrm{Li}_r(z) = \displaystyle\sum_{n\geq 1}\frac{z^n}{n^r} & ; & \langle Ł \mid x_1^2x_0\rangle = \langle Ł \mid \dfrac{1}{2}(x_1 \shuffle x_1 \shuffle x_0) - (x_1 \shuffle x_0x_1) + x_0x_1^2\rangle. \end{array}$$

Note that, in what precedes, we used the pairing $\langle \bullet \mid \bullet\rangle$ between series and polynomials, classically defined by, for $S \in \mathbf{k}\langle\langle X\rangle\rangle$ and $P \in \mathbf{k}\langle X\rangle$

$$\langle S \mid P\rangle = \sum_{w\in X^*} \langle S \mid w\rangle\langle P \mid w\rangle, \tag{3.40}$$

[21] Here, the shuffle product is denoted by $\shuffle$. Its definition is classical and recalled in the equation (3.66) of Subsection 3.4.5.

where, when w is a word, $\langle S \mid w\rangle$ stands for the coefficient of w in S and $\mathbf{k}$ any commutative ring with unit (as here $\mathcal{H}(\Omega)$). With this at hand, we extend at once the indexation of Li from X^* to $\mathbb{C}\langle X\rangle$ by

$$\mathrm{Li}_P := \sum_{w\in X^*} \langle P \mid w\rangle \, \mathrm{Li}_w = \sum_{n\geq 0}\Big(\sum_{|w|=n} \langle P \mid w\rangle \, \mathrm{Li}_w\Big). \tag{3.41}$$

In [39], it has been established that the polylogarithm, well defined locally by (3.32), could be extended to some series (with conditions) by the last part of formula (3.41) where the polynomial P is replaced by some series. A complete theory of global domains was presented in [39], the present work concerns the whole project of extending $\mathrm{H}_\bullet$ [38, 63] over stuffle subalgebras of rational power series on the alphabet Y, in particular the stars of letters and some explicit combinatorial consequences of this extension.

In fact, we focus on what happens in (well chosen) neighborhoods of zero (see Section 3.4.3), therefore, the aim of this work is manyfold.

(a) Use the extension to local Taylor expansions[22] as in (3.32) and the coefficients of their quotients by $1-z$, namely the harmonic sums, denoted $\mathrm{H}_\bullet$ and defined, for any $w \in X^*x_1$, as follows[23] (and also related literature [3, 63])

$$\frac{\mathrm{Li}_w(z)}{1-z} = \sum_{N\geq 0} \mathrm{H}_{\pi_Y(w)}(N) z^N, \tag{3.42}$$

by a suitable theory of local domains which assures to carry over the computation of these Taylor coefficients and preserves the stuffle indentity, again true for polynomials over the alphabet $Y = \{y_n\}_{n\geq 1}$, *i.e.*[24]

$$\forall S, T \in \mathbb{C}\langle Y\rangle, \qquad \mathrm{H}_{S \uplus T} = \mathrm{H}_S \mathrm{H}_T \text{ and } \mathrm{H}_{1_{\mathbb{C}\langle Y\rangle}} = 1_{\mathbb{C}^{\mathbb{N}}}, \tag{3.43}$$

note that $1_{\mathbb{C}\langle Y\rangle}$ is identified with 1_{Y^*} and $1_{\mathbb{C}^{\mathbb{N}}}$ is the constant (to one) function[25] $\mathbb{N} \to \mathbb{C}$. This means that

$$\mathrm{H}_\bullet : (\mathbb{C}\langle Y\rangle, \uplus, 1_{Y^*}) \longrightarrow (\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, \times, 1)$$

[22] Around zero.

[23] Here, the conc-morphism $\pi_X : (\mathbb{C}\langle Y\rangle, \mathrm{conc}, 1_{Y^*}) \to (\mathbb{C}\langle X\rangle, \mathrm{conc}, 1_{X^*})$ is defined by $\pi_X(y_n) = x_0^{n-1}x_1$ and π_Y is its inverse on $\mathrm{Im}(\pi_X)$. See [39] for more details and a full definition of π_Y.

[24] Here, $\uplus$ stands for the stuffle product which will be recalled as in the subsection 3.4.5.

[25] In fact, it could be $\mathbb{Q}$ but we will use afterwards $\mathbb{C}$-linear combinations.

mapping any word $w = y_{s_1} \cdots y_{s_r} \in Y^*$ to

$$\mathrm{H}_w = \mathrm{H}_{s_1,\cdots,s_r} = \sum_{N \geq n_1 > \cdots > n_r > 0} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}} \tag{3.44}$$

is a $⧢$ (unital) morphism[26].

(b) Extend these correspondences (*i.e.* $\mathrm{Li}_\bullet, \mathrm{H}_\bullet$) to some series (over X and Y, respectively) in order to preserve the identity[27]

$$\frac{\mathrm{Li}_{\pi_X(S)}(z)}{1-z} \odot \frac{\mathrm{Li}_{\pi_X(T)}(z)}{1-z} = \frac{\mathrm{Li}_{\pi_X(S ⧢ T)}(z)}{1-z} \tag{3.45}$$

true for polynomials $S, T \in \mathbb{C}\langle Y \rangle$.

To this end, we use the explicit parametrization of the conc-characters obtained in [39] and the fact that, under the stuffle products, they form a group.

3.4.2 Polylogarithms: from global to local domains.

Now we are facing the following constraint:

In order that the results given by symbolic computation reflect the reality with complex numbers (and analytic functions), we have to introduce some topology[28].

Let $\mathcal{H}(\Omega) = C^\omega(\Omega; \mathbb{C})$ be the algebra (for the pointwise product) of complex-valued functions which are holomorphic on Ω. Endowed with the topology of compact convergence [29], it is a nuclear space[30].

Definition 3.3. (i) Let $S \in \mathbb{C}\langle\langle X \rangle\rangle$ be a series decomposed in its homogeneous (w.r.t. the length) components

$$S_n = \sum_{|w|=n} \langle S \mid w \rangle \, w$$

[26]In fact, it was shown that this morphism is into, see [62] Thm 4.

[27]Here $\odot$ stands for the Hadamard product [59].

[28]Readers who are not keen on topology or functional analysis may skip the details of this section and hold its conclusions.

[29]This topology is defined by the seminorms (where $K \subset \Omega$ is compact) $p_K(f) = \sup_{s \in K} |f(s)|$.

[30]Space where commutatively convergent and absolutely convergent series are the same. This will allow the domain of the polylogarithm to be closed by shuffle products (i.e. the possiblity to compute legal polylogarithms through shuffle products).

(so that $S = \sum_{n\geq 0} S_n$) is in the *domain of* Li if and only if the family $(\mathrm{Li}_{S_n})_{n\geq 0}$ is summable in $\mathcal{H}(\Omega)$ in other words, due to the fact that the space is complete (see [97]), if and only if one has

$$(\forall\ W \in \mathcal{B}_{\mathcal{H}(\Omega)})(\exists F \subset_{finite} \mathbb{N})(\forall F' \subset_{finite} (\mathbb{N} \setminus F)),\ \left(\sum_{j\in F'} \mathrm{Li}_{S_j} \in W\right) \tag{3.46}$$

where $\mathcal{B}_{\mathcal{H}(\Omega)}$ is the set of neighbourhoods of 0 in $\mathcal{H}(\Omega)$.

(ii) The set of these series will be noted Dom(Li) and, for $S \in \mathrm{Dom(Li)}$, the sum $\sum_{n\geq 0} \mathrm{Li}_{S_n}$ will be noted Li_S.

Of course, the criterium (3.46) is only a theoretical tool to establish properties of the domain of Li. In further calculations (i.e. in practice), we will not use it but the stability of the domain under certain operations.

Example 3.4 ([39])**.** For example, the classical polylogarithms: dilogarithm Li_2, trilogarithm Li_3, etc... are defined and obtained through the coding (3.34) by

$$\mathrm{Li}_k(z) = \sum_{n\geq 1} \frac{z^n}{n^k} = \mathrm{Li}_{x_0^{k-1}x_1}(z) = \langle \mathrm{L}(z) \mid x_0^{k-1}x_1 \rangle$$

(where $\mathrm{L}(z)$ is as in Equation (3.37)) but, one can check that, for $t \geq 0$ (real), the series $(tx_0)^*x_1$ belongs to $\mathrm{Dom}(\mathrm{Li}_\bullet)$ (see Definition 3.3. (ii)) iff $0 \leq t < 1$. In fact, in this case,

$$\mathrm{Li}_{(tx_0)^*x_1}(z) = \sum_{n\geq 1} \frac{z^n}{n-t}.$$

This opens the door to Hurwitz polyzetas [61].

The map $\mathrm{Li}_\bullet$ is now extended to a subdomain of $\mathbb{C}\langle\langle X \rangle\rangle$, called $\mathrm{Dom}(\mathrm{Li}_\bullet)$ (see Def. 3.3, and also [39]).

Example 3.5. For any $\alpha, \beta \in \mathbb{C}$, $(\alpha x_0)^*$, $(\beta x_1)^*$ and $(\alpha x_0 + \beta x_1)^* = (\alpha x_0)^* \sqcup\!\sqcup (\beta x_1)^*$. We have

$$\mathrm{Li}_{(\alpha x_0)^*}(z) = z^\alpha\ ;\ \mathrm{Li}_{(\beta x_1)^*}(z) = (1-z)^{-\beta}\ ;\ \mathrm{Li}_{(\alpha x_0 + \beta x_1)^*}(z) = z^\alpha (1-z)^{-\beta}$$

where $z \in \Omega$.

Proposition 3.12. *(i) The domain* Dom(Li) *is a shuffle subalgebra of* $(\mathbb{C}\langle\langle X \rangle\rangle, \sqcup\!\sqcup, 1_{X^*})$.

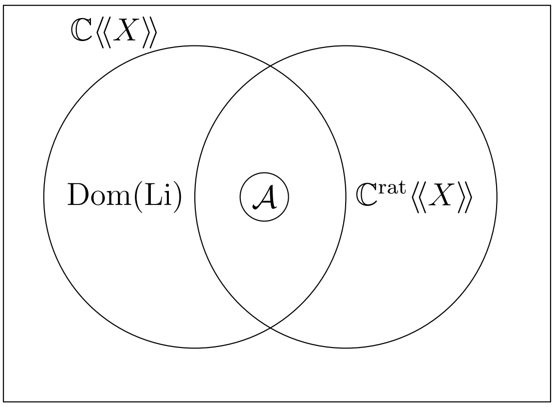


Figure 3.2: Domain of Polylogarithms and the algebra $\mathcal{A}$.

(ii) The extended polylogarithm $\mathrm{Li}: \ \mathrm{Dom(Li)} \to \mathcal{H}(\Omega)$ *is a shuffle morphism, i.e.* $S, T \in \mathrm{Dom(Li)}$*, we still have*

$$\mathrm{Li}_{S \sqcup\!\sqcup T} = \mathrm{Li}_S \, \mathrm{Li}_T \ \ \textit{and} \ \ \mathrm{Li}_{1_{X^*}} = 1_{\mathcal{H}(\Omega)} \ . \tag{3.47}$$

Proof. This proof has been done in [39]. □

The picture about Dom(Li) within the algebra $(\mathbb{C}\langle\langle X\rangle\rangle, \sqcup\!\sqcup, 1_{X^*})$, the positioning of $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ (rational series, see [2, 39]) and shuffle subalegbras as, for example, $\mathcal{A} = \mathbb{C}\langle X\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$ read as follows:

3.4.3 From Polylogarithms to Harmonic sums.

Definition of Dom(Li) has many merits[31] and can easily be adapted to arbitrary (open and connected) domains. However this definition, based on a global condition over a fixed domain $\Omega \subset \mathbb{C} \setminus ([0, +\infty[)$ with $0 \in \overline{\Omega}$, does not provide a sufficiently clear interpretation of the stable symbolic computations around a point, in particular at $z = 0$. One needs to consider a sort of "symbolic local germ" worked out explicitely. Indeed, as the harmonic sums (or MZV) are the coefficients of the Taylor expansion at zero of the convergent polylogarithms divided by $1 - z$, we only need to know locally these functions. In order to gain more indexing series and to describe the local situation at zero, we reshape and define a new domain of Li around zero to $\mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet)$.

[31] As the fact that, due to special properties of $\mathcal{H}(\Omega)$ (it is a nuclear space [97]), one can show that Dom(Li) is closed by shuffle products.

The first step will be to characterize the polylogarithms having a removable singularity at zero. The following proposition helps us characterize their indices.

Proposition 3.13. *Let* $P \in \mathbb{C}\langle X\rangle$ *and* $f(z) = \langle Ł \mid P\rangle = \sum_{w\in X^*} \langle P \mid w\rangle \operatorname{Li}_w$.

1. *The following conditions are equivalent*

   (i) f *can be analytically extended around zero.*

   (ii) $P \in \mathbb{C}\langle X\rangle x_1 \oplus \mathbb{C}.1_{X^*}$.

2. *In this case* Ω *itself*[32] *can be extended to* $\Omega_1 = \mathbb{C} \setminus (]-\infty,-1] \cup [1,+\infty[)$.

Sketch. (ii) $\Longrightarrow$ (i) being straightforward, it remains to prove that (ii) $\Longrightarrow$ (i). Let then $P \in \mathbb{C}\langle X\rangle$ such that $f(z) = \langle Ł \mid P\rangle$ has a removable singularity at zero. As a consequence of Proposition 3.7 point (iii) for the set partitioned $X = x_0 + x_1$, this implies that our polynomial reads

$$P = \sum_{k\geq 0} P_k \sqcup\!\sqcup x_0^{\sqcup\!\sqcup k} \text{ with } P_k \in \mathbb{C}\langle X\rangle x_1 \oplus \mathbb{C}.1_{X^*} \tag{3.48}$$

the family $(P_k)_{k\geq 0}$ being unique and finitely supported. Using (3.48) and (3.39), we get

$$\operatorname{Li}_P(z) = \sum_{k\geq 0} \operatorname{Li}_{P_k}(z)\log(z)^k.$$

We can see that only the term with $k = 0$ survives using monodromy, for example, as follows.

We suppose that $\operatorname{Li}_P(z) = \sum_{k\geq 0} \operatorname{Li}_{P_k}(z)\log(z)^k$ can be analytically extended in a neighbourhood of zero (say $B_r(0)$). Let $z \in \Omega \cap B_r(0)$. Using the path $\gamma_n(t) = z.e^{2in\pi.t}$ (starting and ending at z winding n times around zero), we get

$$\operatorname{Li}_P(z) = \operatorname{Li}_{P_0}(z) + \sum_{k\geq 1} \operatorname{Li}_{P_k}(z)(\log(z) + 2in\pi)^k$$

for all $n \in \mathbb{Z}$ which entails $\operatorname{Li}_{P_k}(z) = 0$ for all $k \geq 1$. This holds for all $z \in \Omega \cap B_r(0)$ and hence we must have $P_k = 0$ for all $k \geq 1$. $\square$

The second step will be provided by the following Proposition which says that, for appropriate series, the Taylor coefficients behave nicely.

[32] The domain, for z of Li_P.

Proposition 3.14. *Let $S \in \mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}1_{X^*}$ such that $S = \sum_{n\geq 0}[S]_n$ where*

$$[S]_n = \sum_{w\in X^*, |w|=n} \langle S \mid w\rangle w, \text{ ($[S]_n$ are the homogeneous components of S)},$$

we suppose that $0 < R \leq 1$ and that $\sum_{n\geq 0} \mathrm{Li}_{[S]_n}$ is unconditionally convergent (for the standard topology) within the open disk $\mid z \mid < R$[33]. Remarking that $\frac{1}{1-z}\sum_{n\geq 0} \mathrm{Li}_{[S]_n}(z)$ is unconditionally convergent in the same disk, we set

$$\frac{1}{1-z}\sum_{n\geq 0} \mathrm{Li}_{[S]_n}(z) = \sum_{N\geq 0} a_N z^N \ .$$

Then, for all $N \geq 0$, $\sum_{n\geq 0} \mathrm{H}_{\pi_Y([S]_n)}(N) = a_N$.

Proof. Let us recall that, for any $w \in X^*x_1$, the function $(1-z)^{-1}\,\mathrm{Li}_w(z)$ is analytic in the open disk $|z| < R$. Moreover, one has

$$\frac{1}{1-z}\,\mathrm{Li}_w(z) = \sum_{N\geq 0} \mathrm{H}_{\pi_Y(w)}(N) z^N.$$

Since $[S]_n = \sum_{w\in X^*, |w|=n} \langle S \mid w\rangle w$ and $(1-z)^{-1}\sum_{n\geq 0} \mathrm{Li}_{[S]_n}$ absolutely converges (for the standard topology[34]) within the open disk $D_{<R}$, one obtains, for all $|z| < R$

$$\begin{aligned}
\frac{1}{1-z}\sum_{n\geq 0} \mathrm{Li}_{[S]_n}(z) &= \frac{1}{1-z}\sum_{n\geq 0}\sum_{w\in X^*, |w|=n} \langle S \mid w\rangle\,\mathrm{Li}_w(z) \\
&= \sum_{n\geq 0}\sum_{w\in X^*, |w|=n} \langle S \mid w\rangle \frac{\mathrm{Li}_w(z)}{1-z} \\
&= \sum_{n\geq 0}\sum_{w\in X^*, |w|=n} \langle S \mid w\rangle \sum_{N\geq 0} \mathrm{H}_{\pi_Y(w)}(N) z^N \\
&\overset{(*)}{=} \sum_{N\geq 0}\sum_{n\geq 0}\sum_{w\in X^*, |w|=n} \langle S \mid w\rangle \mathrm{H}_{\pi_Y(w)}(N) z^N \\
&= \sum_{N\geq 0}\sum_{n\geq 0} \mathrm{H}_{\pi_Y([S]_n)}(N) z^N \ ,
\end{aligned}$$

[33]With the definition given later (3.49) this amounts to say that

$$S \in \mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}1_{X^*} \cap \mathrm{Dom}_R(\mathrm{Li}) \ .$$

[34]For this topology, unconditional and absolute convergence coincide [97].

($*$) being possible because $\sum_{w\in X^*,|w|=n}$ is finite. This implies that, for any $N \geq 0$,

$$a_N = \sum_{n\geq 0} \mathrm{H}_{\pi_Y([S]_n)}(N).$$

$\square$

To prepare the construction of the "symbolic local germ" around zero, let us set, in the same manner as in [39],

$$\mathrm{Dom}_R(\mathrm{Li}) := \{S \in \mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}1_{X^*} | \sum_{n\geq 0} \mathrm{Li}_{[S]_n} \text{ is unconditionally convergent in } \mathcal{H}(D_{<R})\} \quad (3.49)$$

and prove the following:

Proposition 3.15. *With the notations as above, we have:*

1. *The map given by* $R \mapsto \mathrm{Dom}_R(\mathrm{Li})$ *from* $]0,1]$ *to* $2^{\mathbb{C}\langle\langle X\rangle\rangle}$ *(the target is the set of subsets*[35] *of* $\mathbb{C}\langle\langle X\rangle\rangle$ *ordered by inclusion) is strictly decreasing*

2. *Each* $\mathrm{Dom}_R(\mathrm{Li})$ *is a shuffle (unital) subalgebra of* $\mathbb{C}\langle\langle X\rangle\rangle$.

Proof. 1. For $0 < R_1 < R_2 \leq 1$ it is straightforward that $\mathrm{Dom}_{R_2}(\mathrm{Li}) \subset \mathrm{Dom}_{R_1}(\mathrm{Li})$. Let us prove that the inclusion is strict. Take $|\,z\,|< 1$ and let us, be it finite or infinite, evaluate the sum

$$M(z) = \sum_{n\geq 0} |\,\mathrm{Li}_{[S]_n(t)}(z)\,|= \sum_{n\geq 0} \langle S(t) \mid x_1^n\rangle \;|\,\mathrm{Li}_{x_1^n}(z)\,|$$

then, by means of Lemma 3.16, with $x_1^+ = x_1x_1^* = x_1^*{-}1$ and $S(t) = \sum_{m\geq 0} t^m (x_1^+)^{\sqcup\!\sqcup m}$, we have

$$\begin{aligned} M(z) &= \sum_{n\geq 0} |\,S(t)|\; x_1^n\; |\mathrm{Li}_{x_1^n}(z)\,|= \sum_{n\geq 0}\sum_{m\geq 0} |\,t^m (x_1^+)^{\sqcup\!\sqcup m}\,|\; x_1^n\; |\mathrm{Li}_{x_1^n}(z)\,| \\ &= \sum_{m\geq 0} m!t^m \sum_{n\geq 0} S_2(n,m) \frac{|\,\mathrm{Li}_{x_1}(z)\,|^n}{n!} \leq \sum_{m\geq 0} m!t^m \sum_{n\geq 0} S_2(n,m) \frac{\mathrm{Li}^n_{x_1}(|\,z\,|)}{n!}, \end{aligned}$$

due to the fact that $|\,\mathrm{Li}_{x_1}(z)\,|\leq \mathrm{Li}_{x_1}(|\,z\,|)$ (Taylor series with positive coefficients). Finally, in view of equation (3.52), we get, on the one hand, for $|\,z\,|< (t+1)^{-1}$,

$$M(z) \leq \sum_{m\geq 0} t^m (e^{\mathrm{Li}_{x_1}(|z|)} - 1)^m = \sum_{m\geq 0} t^m (\frac{|\,z\,|}{1-|\,z\,|})^m = \frac{1-|\,z\,|}{1-(t+1)\,|\,z\,|}.$$

[35] For any set E, the set of its subsets is noted 2^E.

This proves that, for all $r \in]0, \dfrac{1}{t+1}[$, $\sum_{n\geq 0} \| \operatorname{Li}_{[S]_n(t)}(z)\|_r < +\infty$.

On the other hand, if $(t+1)^{-1} \leq |z| < 1$, one has $M(|z|) = +\infty$, and the preceding calculation shows that, with t choosen such that

$$0 \leq \frac{1}{R_2} - 1 < t < \frac{1}{R_1} - 1,$$

we have $S(t) \in \mathrm{Dom}_{R_1}(\mathrm{Li})$ but $S(t) \notin \mathrm{Dom}_{R_2}(\mathrm{Li})$ whence, for $0 < R_1 < R_2 \leq 1$, $\mathrm{Dom}_{R_2}(\mathrm{Li}) \subsetneq \mathrm{Dom}_{R_1}(\mathrm{Li})$.

2. One has (proofs as in [39])

   (a) $1_{X^*} \in \mathrm{Dom}_R(\mathrm{Li})$ (because $1_{X^*} \in \mathbb{C}\langle X\rangle$) and $\mathrm{Li}_{1_{X^*}} = 1_{\mathcal{H}(\Omega)}$.

   (b) Taking $S, T \in \mathrm{Dom}_R(\mathrm{Li})$ we have, by absolute convergence, $S \sqcup\!\sqcup T \in \mathrm{Dom}_R(\mathrm{Li})$. It is easily seen that $S \sqcup\!\sqcup T \in \mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}1_{X^*}$ and, moreover, that[36]

$$\mathrm{Li}_S\,\mathrm{Li}_T = \mathrm{Li}_{S \sqcup\!\sqcup T}\,.$$

□

The combinatorial Lemma needed in the Theorem 3.18 is the following:

Lemma 3.16. *For a letter "a", one has*

$$\left\langle (a^+)^{\sqcup\!\sqcup m} | a^n \right\rangle = m! S_2(n,m), \tag{3.50}$$

($S_2(n,m)$ being the Stirling numbers of the second kind). The exponential generating series of R.H.S. in equation (3.50) *(w.r.t. n) is given by*

$$\sum_{n\geq 0} m! S_2(n,m) \frac{x^n}{n!} = (e^x - 1)^m. \tag{3.51}$$

Proof. The expression $(a^+)^{\sqcup\!\sqcup m}$ is the specialization of

$$L_m = a_1^+ \sqcup\!\sqcup a_2^+ \sqcup\!\sqcup \cdots \sqcup\!\sqcup a_m^+$$

to $a_j \to a$ (for all $j = 1, 2 \cdots m$). The words of L_m are in bijection with the surjections $[1 \cdots n] \to [1 \cdots m]$, therefore the coefficient $\langle (a^+)^{\sqcup\!\sqcup m} | a^n\rangle$ is exactly the number of such

[36] Proof by absolute convergence as in [39].

surjections namely $m!S_2(n,m)$. A classical formula[37] says that

$$\sum_{n\geq 0} m!S_2(n,m)\frac{x^n}{n!} = (e^x-1)^m. \tag{3.52}$$

□

In Theorem 3.18 below, we study, for series taken in $\mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}.1_{X^*}$, the correspondence $\mathrm{Li}_\bullet$ to some $\mathcal{H}(D_{<R})$, first (point 1) establishes its surjectivity (in a certain sense) and then (points 2 and 3) examine the relation between summability of the functions and that of their Taylor coefficients. For that, let us begin with a very general Lemma on sequences of Taylor series which adapts, for our needs, the notion of *normal families* as in [85].

Lemma 3.17. *Let $\tau = (a_{n,N})_{n,N\geq 0}$ be a double sequence of complex numbers. Setting*

$$I(\tau) := \{r \in]0,+\infty[\mid \sum_{n,N\geq 0} |a_{n,N}r^N| < +\infty\},$$

one has

1. *$I(\tau)$ is an interval of $]0,+\infty[$, it is not empty iff there exists $z_0 \in \mathbb{C}\setminus\{0\}$ such that*

$$\sum_{n,N\geq 0} |a_{n,N}z_0^N| < +\infty. \tag{3.53}$$

   In this case, we set $R(\tau) := \sup(I(\tau)) > 0$ and one has,

   (a) *For all N, the series $\sum_{n\geq 0} a_{n,N}$ converges absolutely (in $\mathbb{C}$). Let us note a_N - with one subscript - its limit.*

   (b) *For all n, the convergence radius of the Taylor series $T_n(z) = \sum_{N\geq 0} a_{n,N}z^N$ is at least $R(\tau)$ and $\sum_{n\in\mathbb{N}} T_n$ is summable for the standard topology of $\mathcal{H}(D_{<R(\tau)})$ with sum $T(z) = \sum_{N\geq 0} a_N z^N$.*

2. *Conversely, we suppose that there exists $R > 0$ such that*

   (a) *Each Taylor series $T_n(z) = \sum_{N\geq 0} a_{n,N}z^N$ converges in $\mathcal{H}(D_{<R})$.*

[37] See [100], the twelvefold way, formula (1.94b)(pp. 74) for instance.

(b) The series $\sum_{n\in\mathbb{N}} T_n$ *converges unconditionnally in* $\mathcal{H}(D_{<R})$.

Then $I(\tau) \neq \emptyset$ *and* $R(\tau) \geq R$.

Proof. 1. The fact that $I(\tau) \subset]0,+\infty[$ is straightforward from the Definition. If there exists $z_0 \in \mathbb{C}\setminus\{0\}$ such that $\sum_{n,N\geq 0} \mid a_{n,N} z_0^N \mid < +\infty$ then, for all $r \in]0, |z_0|[$, we have

$$\sum_{n,N\geq 0} \mid a_{n,N} r^N \mid = \sum_{n,N\geq 0} \mid a_{n,N} z_0^N \mid \left(\frac{r}{\mid z_0 \mid}\right)^N \leq \sum_{n,N\geq 0} |a_{n,N} z_0^N| < +\infty$$

in particular $I(\tau) \neq \emptyset$ and it is an interval of $]0,+\infty[$ with lower bound zero.

(a) Take $r \in I(\tau)$ (hence $r \neq 0$) and $N \in \mathbb{N}$, then we get the expected result as

$$r^N \sum_{n\geq 0} \mid a_{n,N} \mid = \sum_{n\geq 0} \mid a_{n,N} r^N \mid \leq \sum_{n,N\geq 0} \mid a_{n,N} r^N \mid < +\infty.$$

(b) Again, take any $r \in I(\tau)$ and $n \in \mathbb{N}$, then $\sum_{N\geq 0} \mid a_{n,N} r^N \mid < +\infty$ which proves that $R(T_n) \geq r$, hence the result[38]. We also have

$$\mid \sum_{N\geq 0} a_N r^N \mid \leq \sum_{N\geq 0} r^N \mid \sum_{n\geq 0} a_{n,N} \mid \leq \sum_{n,N\geq 0} \mid a_{n,N} r^N \mid < +\infty$$

and this proves that $R(T) \geq r$, hence $R(T) \geq R(\tau)$.

2. Let $0 < r < r_1 < R$ and consider the path $\gamma(t) = r_1 e^{2i\pi t}$, by Cauchy's formula, we have

$$\mid a_{n,N} \mid = \mid \frac{1}{2i\pi} \int_\gamma \frac{T_n(z)}{z^{N+1}} dz \mid \leq \frac{2\pi}{2\pi} \frac{r_1 \|T_n\|_K}{r_1^{N+1}} \leq \frac{\|T_n\|_K}{r_1^N}$$

with $K = \gamma([0, 2\pi])$, hence

$$\sum_{n,N\geq 0} \mid a_{n,N} r^N \mid \leq \sum_{n,N\geq 0} \|T_n\|_K (\frac{r}{r_1})^N \leq \frac{r_1}{r_1 - r} \sum_{n\geq 0} \|T_n\|_K < +\infty.$$

□

Remark 3.3. (i) First point says that every function analytic at zero can be represented around zero as $\mathrm{Li}_S(z)$ for some $S \in \mathbb{C}\langle\langle x_1 \rangle\rangle$.

[38]For a Taylor series T, we note $R(T)$ the radius of convergence of T.

(ii) In point 2, the arithmetic functions $\mathrm{H}_{\pi_Y(S)} \in \mathbb{Q}^{\mathbb{N}}$, for $S \in \mathrm{Dom}_R(\mathrm{Li})$ are quickly defined (and in a way extending the old definition) and we draw a very important bound saying that, in this condition, for some $r > 0$ the array $\left(\mathrm{H}_{\pi_Y([S]_n)}(N)r^N\right)_{n,N}$ converges (then, in particular, horizontally and vertically).

(iii) Point 3 establishes the converse.

Theorem 3.18. *1. Let $T(z) = \sum_{N\geq 0} a_N z^N$ be a Taylor series converging on some non-empty disk centered at zero i.e. such that $\limsup_{N\to+\infty} |\, a_N \,|^{\frac{1}{N}} = B < +\infty$, then the series*

$$S = \sum_{N\geq 0} a_N(-(-x_1)^+)^{⧢N} \tag{3.54}$$

is summable in $\mathbb{C}\langle\langle X\rangle\rangle$ (with sum in $\mathbb{C}\langle\langle x_1\rangle\rangle$), $S \in \mathrm{Dom}_R(\mathrm{Li})$ with $R = (B+1)^{-1}$ and $\mathrm{Li}_S = T$.

2. Let $S \in \mathrm{Dom}_R(\mathrm{Li})$ and $S = \sum_{n\geq 0} [S]_n$ (homogeneous decomposition), we define $N \longmapsto \mathrm{H}_{\pi_Y(S)}(N)$ by[39]

$$\frac{\mathrm{Li}_S(z)}{1-z} = \sum_{N\geq 0} \mathrm{H}_{\pi_Y(S)}(N) z^N. \tag{3.55}$$

Then,

$$\forall r \in]0, R[, \qquad \sum_{n,N\geq 0} |\, \mathrm{H}_{\pi_Y([S]_n)}(N) r^N \,| < +\infty. \tag{3.56}$$

In particular, for all $N \in \mathbb{N}$, the series (of complex numbers), $\sum_{n\geq 0} \mathrm{H}_{\pi_Y([S]_n)}(N)$ converges absolutely to $\mathrm{H}_{\pi_Y(S)}(N)$.

3. Conversely, let $Q \in \mathbb{C}\langle\langle Y\rangle\rangle$ with $Q = \sum_{n\geq 0} Q_n$ (decomposition by weights), we suppose that there exists $r \in]0, 1]$ such that

$$\sum_{n,N\geq 0} |\, \mathrm{H}_{Q_n}(N) r^N \,| < +\infty, \tag{3.57}$$

[39]This definition is compatible with the old one when S is a polynomial.

in particular, for all $N \in \mathbb{N}$, $\sum_{n\geq 0} \mathrm{H}_{Q_n}(N) = \ell(N) \in \mathbb{C}$ unconditionally (= absolutely) converges (in $\mathbb{C}$). Under such circumstances, $S := \pi_X(Q) \in \mathrm{Dom}_r(\mathrm{Li})$ and, for all $z \in \mathbb{C}$ such that $|z| < r$,

$$\frac{\mathrm{Li}_S(z)}{1-z} = \sum_{N\geq 0} \ell(N) z^N. \tag{3.58}$$

Proof. 1. The fact that the series (3.54) is comes from the fact that

$$\omega(a_N(-(-x_1)^+)^{ш N}) \geq N.$$

Now from the Lemma 3.16, we get

$$(S)_n = \sum_{N\geq 0} (a_N(-(-x_1)^+)^{ш N})_n = (-1)^{N+n} a_N N! S_2(n,N) x_1^n \sum_{N\geq 0} (-1)^{N+n} a_N N! S_2(n,N) x_1^n.$$

Then, with $r = \sup_{z\in K} |z|$ (we have indeed $r = ||\mathrm{Id}||_K$) and taking into account that

$$\| \mathrm{Li}_{x_1} \|_K \leq \log(\frac{1}{1-r}),$$

we have

$$\begin{aligned}
\sum_{n\geq 0} \| \mathrm{Li}_{(S)_n} \|_K &\leq \sum_{n\geq 0}\sum_{N\geq 0} |a_N| N! S_2(n,N) \| \mathrm{Li}_{x_1^n} \|_K \\
&\leq \sum_{n\geq 0}\sum_{N\geq 0} |a_N| N! S_2(n,N) \frac{\| \mathrm{Li}_{x_1} \|_K^n}{n!} \\
&\leq \sum_{N\geq 0} |a_N| \sum_{n\geq 0} N! S_2(n,N) \frac{\| \mathrm{Li}_{x_1} \|_K^n}{n!} \\
&\leq \sum_{N\geq 0} |a_N| (e^{\log(\frac{1}{1-r})} - 1)^N \\
&= \sum_{N\geq 0} |a_N| \left(\frac{r}{1-r}\right)^N.
\end{aligned}$$

Now, if we suppose that $r \leq (B+1)^{-1}$, we have $r(1-r)^{-1} \leq \dfrac{1}{B}$ and this shows that the last sum is finite. Moreover, by Lemma 3.16 and $\mathrm{Li}_{x_1^n} = \frac{\log^n(\frac{1}{1-z})}{n!}$ for all $n \geq 0$, one clearly has $\mathrm{Li}_S = T$.

2. This point and next point are consequences of Lemma 3.17. Now, considering the homogeneous decomposition $S = \sum_{n\geq 0} [S]_n \in \mathrm{Dom}_R(\mathrm{Li})$. We first establish

inequation (3.56). Let $0 < r < r_1 < R$ and consider the path $\gamma(t) = r_1 e^{2i\pi t}$, we have

$$|\,\mathrm{H}_{\pi_Y([S]_n)}(N)\,|=|\,\frac{1}{2i\pi}\int_\gamma \frac{\mathrm{Li}_{[S]_n}(z)}{(1-z)z^{N+1}}dz\,|\leq \frac{2\pi}{2\pi}\frac{\|\,\mathrm{Li}_{[S]_n}\,\|_K}{(1-r_1)r_1^{N+1}},$$

$K = \gamma([0,1])$ being the circle of center 0 and radius r_1. Taking into account that, for $K \subset_{compact} D_{<R}$, we have a decomposition $\sum_{n\in\mathbb{N}} \|\,\mathrm{Li}_{[S]_n}\,\|_K = M < +\infty$, we get

$$\begin{aligned}
\sum_{n,N\geq 0} |\,\mathrm{H}_{\pi_Y([S]_n)}(N)r^N\,| &= \sum_{n,N\geq 0} |\,\mathrm{H}_{\pi_Y([S]_n)}(N)r_1^N\,|\,(\frac{r}{r_1})^N \\
&= \sum_{N\geq 0}(\frac{r}{r_1})^N \sum_{n\geq 0} |\,\mathrm{H}_{\pi_Y([S]_n)}(N)r_1^N\,| \\
&\leq \sum_{N\geq 0}(\frac{r}{r_1})^N \frac{M}{(1-r_1)r_1} \\
&\leq \frac{M}{(1-r_1)(r_1-r)} < +\infty.
\end{aligned}$$

The series $\sum_{n\geq 0}\mathrm{Li}_{[S]_n}(z)$ converges to $\mathrm{Li}_S(z)$ in $\mathcal{H}(D_{<R})$ ($D_{<R}$ is the open disk defined by $|z| < R$). For any $N \geq 0$, by Cauchy's formula, one has,

$$\begin{aligned}
\mathrm{H}_{\pi_Y(S)}(N) &= \frac{1}{2i\pi}\int_\gamma \frac{\mathrm{Li}_S(z)}{(1-z)z^{N+1}}dz \\
&= \frac{1}{2i\pi}\int_\gamma \frac{\sum_{n\geq 0}\mathrm{Li}_{[S]_n}(z)}{(1-z)z^{N+1}}dz \\
&= \frac{1}{2i\pi}\sum_{n\geq 0}\int_\gamma \frac{\mathrm{Li}_{[S]_n}(z)}{(1-z)z^{N+1}}dz \\
&= \sum_{n\geq 0}\mathrm{H}_{\pi_Y([S]_n)}(N)
\end{aligned}$$

the exchange of sum and integral being due to the compact convergence. The absolute convergence comes from the fact that the convergence of $\sum_{n\geq 0}\mathrm{Li}_{[S]_n}(z)$ is unconditional [97].

3. Fixing $N \in \mathbb{N}$, from inequation (3.57), we get $\sum_{n\geq 0} |\,\mathrm{H}_{Q_n}(N)\,| < +\infty$ which proves the absolute convergence. Remark now that $(\pi_X(Q))_n = \pi_X(Q_n)$ and $\pi_Y(\pi_X(Q_n)) = Q_n$, one has, for all $|\,z\,|\leq r$ $|\,z\,|< r$,

$$|\,\mathrm{Li}_{\pi_X(Q_n)}(z)\,|=|\,1-z\,||\sum_{N\in\mathbb{N}}\mathrm{H}_{Q_n}(N)z^N\,|\leq 2\,|\sum_{N\in\mathbb{N}}\mathrm{H}_{Q_n}(N)r^N\,|\,.$$

Thus, for all $K \subset_{compact} D_{<r}$ and $z \in K$, we arrive at

$$|\,\mathrm{Li}_{\pi_X(Q_n)}(z)\,|\leq 2\,|\sum_{N\in\mathbb{N}}\mathrm{H}_{Q_n}(N)r^N\,|,$$

in other words,

$$\| \operatorname{Li}_{\pi_X(Q_n)} \|_K \leq 2 \,| \sum_{N\in\mathbb{N}} \operatorname{H}_{Q_n}(N) r^N |$$

and

$$\sum_{n\in\mathbb{N}} \| \operatorname{Li}_{\pi_X(Q_n)} \|_K \leq 2 \,| \sum_{n,N\in\mathbb{N}} \operatorname{H}_{Q_n}(N) r^N | < +\infty$$

which shows that $\pi_X(Q) \in \operatorname{Dom}_r(\operatorname{Li})$. The equation (3.58) is a consequence of point 2, taking $S = \pi_X(Q)$.

□

Now, we have have a better understanding of what can (and will) be the domain, $\operatorname{Dom}(\operatorname{H}_\bullet)$, of harmonic sums.

Definition 3.4. We set $\operatorname{Dom}^{\mathrm{loc}}(\operatorname{Li}) = \bigcup_{0<R\leq 1} \operatorname{Dom}_R(\operatorname{Li})$; $\operatorname{Dom}(\operatorname{H}_\bullet) = \pi_Y(\operatorname{Dom}^{\mathrm{loc}}(\operatorname{Li}))$ and, for $S \in \operatorname{Dom}^{\mathrm{loc}}(\operatorname{Li})$,

$$\operatorname{Li}_S(z) = \sum_{n\geq 0} \operatorname{Li}_{[S]_n}(z) \text{ and } \frac{\operatorname{Li}_S(z)}{1-z} = \sum_{N\geq 0} \operatorname{H}_{\pi_Y(S)}(N) z^N.$$

3.4.4 Applications.

We remark that formula (3.32), *i.e.*,

$$\operatorname{Li}_{s_1,\cdots,s_r}(z) := \sum_{n_1>\cdots>n_r>0} \frac{z^{n_1}}{n_1^{s_1}\cdots n_r^{s_r}},$$

still makes sense for $|z| < 1$ and $(s_1, \cdots, s_r) \in \mathbb{C}^r$ so that we will freely use the indexing list to get index lists with $s_i \in \mathbb{Z}$ for any $i = 1, \cdots, r$ and $r \in \mathbb{N}^+$.

Recall that for any $s_1, \cdots, s_r \in \mathbb{N}$, we can express $\operatorname{Li}_{-s_1,\cdots,-s_r}(z)$ as a polynomial of $\frac{1}{1-z}$ with integer coefficients. Then, using (3.47) and $(kx_1)^* = [(x_1)^*]^{⧢ k}$, we get

$$\frac{1}{(1-z)^k} = \operatorname{Li}_{(kx_1)^*}(z),\ \forall k \in \mathbb{N}^+$$

and we obtain a polynomial $P \in \operatorname{Dom}(\operatorname{Li}) \cap \mathbb{C}[x_1^*] = \mathbb{C}[x_1^*]$ such that $\operatorname{Li}_{-s_1,\cdots,-s_r} = \operatorname{Li}_P$ (see [39]). Using Theorem 3.18, we have

$$\frac{\operatorname{Li}_P(z)}{1-z} = \sum_{N\geq 0} \operatorname{H}_{\pi_Y(P)}(N) z^N.$$

This means that we can provide a class of elements of $\mathrm{Dom}(\mathrm{H}_\bullet)$ (as in Definition 3.4) relative to the set of indices of harmonic sums at negative integer multiindices. Here are some examples.

Example 3.6. For any $|z| < 1$, we have

$$\mathrm{Li}_{x_1^*}(z) = \frac{1}{1-z} \ ; \ \mathrm{Li}_{x_1^*-1_{X^*}}(z) = \frac{z}{1-z} = \mathrm{Li}_0(z) \ ; \ \mathrm{Li}_{(2x_1)^*-x_1^*}(z) = \frac{z}{(1-z)^2} = \mathrm{Li}_{-1}(z);$$

$$\mathrm{Li}_{(2x_1)^*-2x_1^*+1_{X^*}}(z) = \frac{z^2}{(1-z)^2} = \mathrm{Li}_{0,0}(z);$$

$$\mathrm{Li}_{12(5x_1)^*-33(4x_1)^*+31(3x_1)^*-11(2x_1)^*+x_1^*}(z) = \frac{z^4+7z^3+4z^2}{(1-z)^5} = \mathrm{Li}_{-2,-1}(z);$$

$$\mathrm{Li}_{40(6x_1)^*-132(5x_1)^*+161(4x_1)^*-87(3x_1)^*+19(2x_1)^*-x_1^*}(z) = \frac{z^5+14z^4+21z^3+4z^2}{(1-z)^6} = \mathrm{Li}_{-2,-2}(z);$$

$$\mathrm{Li}_{1260(8x_1)^*-5400(7x_1)^*+9270(6x_1)^*-8070(5x_1)^*+3699(4x_1)^*-829(3x_1)^*+71(2x_1)^*-x_1^*}(z)$$
$$= \frac{z^7+64z^6+424z^5+584z^4+179z^3+8z^2}{(1-z)^8} = \mathrm{Li}_{-3,-3}(z);$$

$$\mathrm{Li}_{10(6x_1)^*-38(5x_1)^*+55(4x_1)^*-37(3x_1)^*+11(2x_1)^*-x_1^*}(z) = \frac{z^5+6z^4+3z^3}{(1-z)^6} = \mathrm{Li}_{-1,0,-2}(z);$$

$$\mathrm{Li}_{280(8x_1)^*-1312(7x_1)^*+2497(6x_1)^*-2457(5x_1)^*+1310(4x_1)^*-358(3x_1)^*+41(2x_1)^*-x_1^*}(z)$$
$$= \frac{z^7+34z^6+133z^5+100z^4+12z^3}{(1-z)^8} = \mathrm{Li}_{-1,-2,-2}(z).$$

Thus, for any $N \in \mathbb{N}$, for readability, below 1 stands for 1_{X^*}

$$\mathrm{H}_{\pi_Y(x_1^*)}(N) = N+1,$$

$$\mathrm{H}_{\pi_Y(x_1^*-1)}(N) = N = \sum_{n=1}^{N} n^0,$$

$$\mathrm{H}_{\pi_Y((2x_1)^*-x_1^*)}(N) = \frac{1}{2}N^2 + \frac{1}{2}N = \sum_{n=1}^{N} n^1;$$

$$\mathrm{H}_{\pi_Y((2x_1)^*-2x_1^*+1)}(N) = \frac{1}{2}N^2 - \frac{1}{2}N = \sum_{n_1=1}^{N} n_1^0 \sum_{n_2=1}^{n_1-1} n_2^0;$$

$$\mathrm{H}_{\pi_Y(12(5x_1)^*-33(4x_1)^*+31(3x_1)^*-11(2x_1)^*+x_1^*)}(N) = \frac{1}{10}N^5 + \frac{1}{8}N^4 - \frac{1}{12}N^3 - \frac{1}{60}N - \frac{1}{8}N^2;$$

$$\mathrm{H}_{\pi_Y(40(6x_1)^*-132(5x_1)^*+161(4x_1)^*-87(3x_1)^*+19(2x_1)^*-x_1^*)}(N) = \frac{1}{15}N^5 + \frac{1}{18}N^6 - \frac{5}{72}N^4 + \frac{1}{72}N^2$$
$$+\frac{1}{60}N - \frac{1}{12}N^3 = \sum_{n_1=1}^{N} n_1^2 \sum_{n_2=1}^{n_1-1} n_2^2;$$

$$\mathrm{H}_{\pi_Y(10(6x_1)^*-38(5x_1)^*+55(4x_1)^*-37(3x_1)^*+11(2x_1)^*-x_1^*)}(N) = -\frac{1}{40}N^5 + \frac{1}{72}N^6 - \frac{1}{36}N^4 + \frac{1}{72}N^2$$

$$+\frac{1}{24}N^3 - \frac{1}{60}N = \sum_{n_1=1}^{N} n_1^1 \sum_{n_2=1}^{n_1-1} \sum_{n_3=1}^{n_2-1} n_3^2;$$

$$\mathrm{H}_{\pi_Y(280(8x_1)^*-1312(7x_1)^*+2497(6x_1)^*-2457(5x_1)^*+1310(4x_1)^*-358(3x_1)^*+41(2x_1)^*-x_1^*)}(N) = -\frac{13}{1260}N^7$$
$$+\frac{1}{144}N^8 - \frac{7}{240}N^6 + \frac{1}{24}N^4 - \frac{7}{360}N^2 + \frac{23}{720}N^5 + \frac{1}{210}N - \frac{19}{720}N^3 = \sum_{n_1=1}^{N} n_1^1 \sum_{n_2=1}^{n_1-1} n_2^2 \sum_{n_3=1}^{n_2-1} n_3^2.$$

Observe that, from Definition 3.4, Theorem 3.19 will show us that $\mathrm{Dom}(\mathrm{H}_\bullet)$ is a stuffle subalgebra of $\mathbb{C}\langle\langle Y\rangle\rangle$. Let us however remark that some series are not in this domain as shown below

(i) The series $T = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{y_n}{n} \in \mathbb{C}\langle\langle Y\rangle\rangle$ is not in $\mathrm{Dom}(\mathrm{H}_\bullet)$ because we see that its decomposition by weights ($T = \sum_{n=1}^{\infty} T_n$ as in (3.57)) provides $T_n = \frac{(-1)^{n-1}}{n}y_n$ for $n \geq 1$ and $T_0 = 0$. Direct calculation gives, for $n \geq 1$,

$$\mathrm{H}_{y_n}(N) = \sum_{k=1}^{N} \frac{1}{k^n},$$

so that we have $\mathrm{H}_{y_n}(N) \geq 1, \forall n \in \mathbb{N}^+; N \in \mathbb{N}^+$, because $\mathrm{H}_{y_n}(0) = 0$, for all $0 < r < 1$, one has

$$\sum_{n,N} |\mathrm{H}_{T_n}(N)r^N| = \sum_{N\geq 0}\sum_{n\geq 1} |\frac{1}{n}\mathrm{H}_{y_n}(N)r^N| \geq \left(\sum_{n\geq 0}\frac{1}{n}\right)\frac{r}{1-r} = +\infty. \quad (3.59)$$

However one can get unconditional convergence using a sommation by pairs (odd + even).

(ii) For all $s \in]1,+\infty[$, the series $T(s) = \sum_{n=1}^{\infty}(-1)^{n-1}y_n n^{-s} \in \mathbb{C}\langle\langle Y\rangle\rangle$ is in $\mathrm{Dom}(\mathrm{H}_\bullet)$.

We can now state the

Theorem 3.19. *Let* $S, T \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li})$, *then* $S \sqcup\!\sqcup T \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}), \pi_X(\pi_Y(S) \uplus \pi_Y(T)) \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li})$ *and for all* $N \geq 0$,

$$\mathrm{Li}_{S\sqcup\!\sqcup T} = \mathrm{Li}_S\,\mathrm{Li}_T; \quad \mathrm{Li}_{1_{X^*}} = 1_{\mathcal{H}(\Omega)}, \quad (3.60)$$

$$\mathrm{H}_{\pi_Y(S)\uplus\pi_Y(T)}(N) = \mathrm{H}_{\pi_Y(S)}(N)\mathrm{H}_{\pi_Y(T)}(N). \quad (3.61)$$

$$\frac{\mathrm{Li}_S(z)}{1-z} \odot \frac{\mathrm{Li}_T(z)}{1-z} = \frac{\mathrm{Li}_{\pi_X(\pi_Y(S)\uplus\pi_Y(T))}(z)}{1-z}. \quad (3.62)$$

Proof. For equation (3.60), we get, from Proposition 3.15 that $\mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li})$ is the union of an increasing set of shuffle subalgebras of $\mathbb{C}\langle\langle X\rangle\rangle$ (the map $R \to \mathrm{Dom}_R(\mathrm{Li})$ is strictly decreasing). It is therefore a shuffle subalgebra of the latter.

For equation (3.61), suppose $S \in \mathrm{Dom}_{R_1}(\mathrm{Li})$ (resp. $T \in \mathrm{Dom}_{R_2}(\mathrm{Li})$). By [59] and Theorem 3.18 point (3.55), for $|z| < R_1R_2$, one has

$$f(z) := \frac{\mathrm{Li}_S(z)}{1-z} \odot \frac{\mathrm{Li}_T(z)}{1-z} = \sum_{N\geq 0} \mathrm{H}_{\pi_Y(S)}(N)\mathrm{H}_{\pi_Y(T)}(N)z^N, \tag{3.63}$$

where $\odot$ stands for the Hadamard product [59]. Now, due to Theorem 3.18 point (2), for all N, $\sum\limits_{p\geq 0} \mathrm{H}_{\pi_Y(S_p)}(N) = \mathrm{H}_{\pi_Y(S)}(N)$ and $\sum\limits_{q\geq 0} \mathrm{H}_{\pi_Y(T_q)}(N) = \mathrm{H}_{\pi_Y(T)}(N)$ (absolute convergence) then, as the product of two absolutely convergent series is absolutely convergent (w.r.t. the Cauchy product), one has, for all N,

$$\begin{aligned}
\mathrm{H}_{\pi_Y(S)}(N)\mathrm{H}_{\pi_Y(T)}(N) &= \Big(\sum_{p\geq 0} \mathrm{H}_{\pi_Y(S_p)}(N)\Big)\Big(\sum_{q\geq 0} \mathrm{H}_{\pi_Y(T_q)}(N)\Big) \\
&= \sum_{p,q\geq 0} \mathrm{H}_{\pi_Y(S_p)}(N)\mathrm{H}_{\pi_Y(T_q)}(N) = \sum_{n\geq 0}\sum_{p+q=n} \mathrm{H}_{\pi_Y(S_p) \uplus \pi_Y(T_q)}(N) \\
&= \sum_{n\geq 0} \mathrm{H}_{(\pi_Y(S) \uplus \pi_Y(T))_n}(N).
\end{aligned} \tag{3.64}$$

Remains to prove that condition of Theorem 3.18, *i.e.* inequation (3.57) is fulfilled. To this end, we use the well-known fact that if $\sum\limits_{m\geq 0} c_m\, z^m$ has radius of convergence $R > 0$, then $\sum\limits_{m\geq 0} |\, c_m \,|\, z^m$ has the same radius of convergence (use $1/R = \limsup_{m\geq 1} |\, c_m\, |^{1/m}$), then from the fact that $S \in \mathrm{Dom}_{R_1}(\mathrm{Li})$ (resp. $T \in \mathrm{Dom}_{R_2}(\mathrm{Li})$), we have (3.56) for each of them and, using the Hadamard product of these expressions, we get

$$\forall r \in]0, R_1.R_2[, \textstyle\sum_{p,q,N\geq 0} |\mathrm{H}_{\pi_Y(S_p)}(N)\mathrm{H}_{\pi_Y(T_q)}(N)\, r^N| < +\infty,$$

and this assures, for $|z| < R_1R_2$, the convergence of

$$f(z) = \sum_{n,N\geq 0} \mathrm{H}_{(\pi_Y(S) \uplus \pi_Y(T))_n}(N)z^N. \tag{3.65}$$

Applying Theorem 3.18 point (3) to $Q = \pi_Y(S) \uplus \pi_Y(T)$ (with any $r < R_1R_2$), we get $\pi_X(Q) = \pi_X(\pi_Y(S) \uplus \pi_Y(T)) \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li})$ and

$$f(z) = \sum_{N\geq 0}\Big(\sum_{n\geq 0} \mathrm{H}_{(\pi_Y(S) \uplus \pi_Y(T))_n}(N)\Big)z^N = \frac{\mathrm{Li}_{\pi_X(\pi_Y(S) \uplus \pi_Y(T))}(z)}{1-z}.$$

Hence we obtain (3.62) and then (3.61). □

3.4.5 Stuffle products, usage of one-parameter subgroups within stuffle characters and their symbolic computations.

It is well-known that, a Hopf algebra $(\mathcal{H}, \mu, 1_{\mathcal{H}}, \Delta, \epsilon, S)$ and $\mathcal{A} \in$ **k-CAAU** (a **k**-commutative and associative algebra with unit), the set $\Xi(\mathcal{H}, \mathcal{A}) := \mathrm{Hom}_{\mathbf{k\text{-}AAU}}(\mathcal{H}, \mathcal{A})$ is a group for convolution (and inverse performed through precomposition with S). When $\mathbf{k}$ is a $\mathbb{Q}$-algebra and under the usual condition that the reduced coproduct[40] $\Delta_+ : \mathcal{H}_+ \to \mathcal{H}_+ \otimes \mathcal{H}_+$ is (locally) conilpotent, this group can be considered as a Lie group (an infinite-dimensional pro-unipotent[41] one) with a nice log-exp correspondence. This feature is used in combinatorial physics [37] and one-parameter groups is a nice tool to get new combinatorial identities (see [36]) as in the sequel.

For the some reader's convenience, we recall here the definitions of shuffle and stuffle products. As regards shuffle, the alphabet X is arbitrary and $\sqcup\!\sqcup$ is defined by the following recursion (for $x, y \in X$ and $u, v \in X^*$)

$$u \sqcup\!\sqcup 1_{X^*} = 1_{X^*} \sqcup\!\sqcup u = u;\ xu \sqcup\!\sqcup yv = x(u \sqcup\!\sqcup yv) + y(xu \sqcup\!\sqcup v). \tag{3.66}$$

As regards stuffle, the alphabet is $Y = Y_{\mathbb{N}_+} = \{y_s\}_{s \in \mathbb{N}_+}$ and $\uplus$ is defined by the following recursion

$$u \uplus 1_{Y^*} = 1_{Y^*} \uplus u = u, \tag{3.67}$$

$$y_s u \uplus y_t v = y_s(u \uplus y_t v) + y_t(y_s u \uplus v) + y_{s+t}(u \uplus v). \tag{3.68}$$

Be it for stuffle or shuffle, the noncommutative[42] polynomials equipped with this product form an associative commutative and unital algebra namely $(\mathbb{C}\langle X \rangle, \sqcup\!\sqcup, 1_{X^*})$ (resp. $(\mathbb{C}\langle Y \rangle, \uplus, 1_{Y^*})$).

Example 3.7. As examples of characters, we have already seen

- $\mathrm{Li}_\bullet$ from $(\mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet), \sqcup\!\sqcup, 1_{X^*})$ to $\mathcal{H}(\Omega)$, where $X = \{x_0, x_1\}$.

[40]Here, $\mathcal{H}_+$ denote the kernel of ϵ and for all $x \in \mathcal{H} = \mathcal{H}_+ \oplus \mathbf{k}.1_{\mathcal{H}}$, then $\Delta_+(x - \epsilon(x).1_{\mathcal{H}}) = \Delta(x) - x \otimes 1_{\mathcal{H}} - 1_{\mathcal{H}} \otimes x + \epsilon(x).1_{\mathcal{H}} \otimes 1_{\mathcal{H}}$, see Bourbaki [13] Ch II §1.1.

[41]In this context, this means that for any element g in this group, the family $((g - 1_{\Xi})^{\star n})_{n \geq 0}$ is summable in $\Xi(\mathcal{H}, \mathcal{A})$, where $1_{\Xi} = 1_{\mathcal{A}} \circ \epsilon$ denotes the unit element in the group $\Xi(\mathcal{H}, \mathcal{A})$.

[42]For concatenation.

- $\mathrm{H}_\bullet$ from $(\mathrm{Dom}(\mathrm{H}_\bullet), ш, 1_{Y^*})$ to $\mathbb{C}^\mathbb{N}$ (arithmetic functions $\mathbb{N} \to \mathbb{C}$).

In general, a character from a $\mathbf{k}$-algebra[43] $(\mathcal{A}, *_1, 1_\mathcal{A})$ with values in $(\mathcal{B}, *_2, 1_\mathcal{B})$ is none other than a morphism between the $\mathbf{k}$-algebras $\mathcal{A}$ and a commutative algebra[44] $\mathcal{B}$. The algebra $(\mathcal{A}, *_1, 1_\mathcal{A})$ does not have to be commutative, for example characters of $(\mathbb{C}\langle X\rangle, \mathrm{conc}, 1_{X^*})$ - *i.e.* conc-characters - can be easily proved to be all of the form

$$\left(\sum_{x\in X} \alpha_x x\right)^* \tag{3.69}$$

They are closed under shuffle and stuffle and endowed with these laws, they form a group. Expressions like the infinite sum within brackets in (3.69) (*i.e.* homogeneous series of degree 1) form a vector space noted $\widehat{\mathbb{C}.Y}$.

As a consequence, given $P = \sum_{i\geq 1} \alpha_i y_i$ and $Q = \sum_{j\geq 1} \beta_j y_j$, we know in advance that their stuffle is a conc-character *i.e.* of the form $(\sum_{n\geq 1} c_n y_n)^*$. Examining the effect of this stuffle on each letter (which suffices), we get the identity

$$\left(\sum_{i\geq 1} \alpha_i y_i\right)^* ш \left(\sum_{j\geq 1} \beta_j y_j\right)^* = \left(\sum_{i\geq 1} \alpha_i y_i + \sum_{j\geq 1} \beta_j y_j + \sum_{i,j\geq 1} \alpha_i\beta_j y_{i+j}\right)^* \tag{3.70}$$

This suggests to take an auxiliary variable, say q, and code "the plane" $\widehat{\mathbb{C}.Y}$, *i.e.* expressions like (3.69), in the style of Umbral calculus by

$$\pi_Y^{\mathrm{Umbra}} : \sum_{n\geq 1} \alpha_n\, q^n \longmapsto \sum_{n\geq 1} \alpha_n y_n$$

which is linear and bijective[45] from $\mathbb{C}_+[[q]]$ to $\widehat{\mathbb{C}.Y}$.
With this coding at hand and for $S, T \in \mathbb{C}_+[[q]]$, identity (3.70) reads

$$(\pi_Y^{\mathrm{Umbra}}(S))^* ш (\pi_Y^{\mathrm{Umbra}}(T))^* = (\pi_Y^{\mathrm{Umbra}}((1+S)(1+T)-1))^*. \tag{3.71}$$

This shows that if one sets, for $z \in \mathbb{C}$ and $T \in \mathbb{C}_+[[q]]$, $G(z) = (\pi_Y^{\mathrm{Umbra}}(e^{zT}-1))^*$, we get a one-parameter stuffle group[46] such that every coefficient is polynomial in z. Differentiating it we get

$$\frac{d}{dz}(G(z)) = (\pi_Y^{\mathrm{Umbra}}(T))G(z) \tag{3.72}$$

[43] Here we will use $\mathbf{k} = \mathbb{Q}$ or $\mathbb{C}$.
[44] In this context all algebras are associative and unital.
[45] Its inverse will be naturally noted π_q^{Umbra}.
[46] *i.e.* $G(z_1+z_2) = G(z_1) ш G(z_2); G(0) = 1_{Y^*}$.

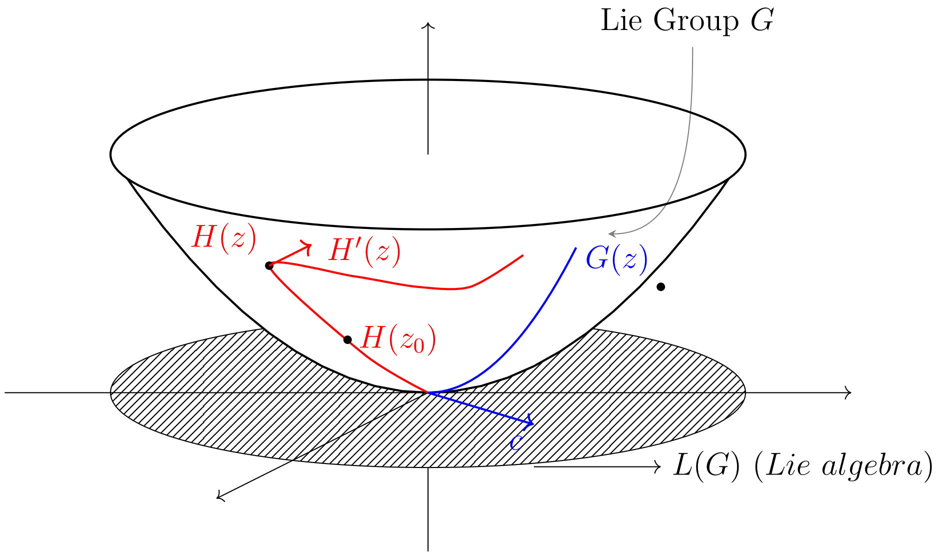


Figure 3.3: A path $z \to H(z)$ with left multiplier $H'(z)$ and the one-parameter group $G(z)$ with infinitesimal generator $c = \pi_Y^{\mathrm{Umbra}}(T)$.

and (3.72) with the initial condition $G(0) = 1_{Y^*}$ integrates as

$$G(z) = \exp_{\uplus}(z\pi_Y^{\mathrm{Umbra}}(T)) \tag{3.73}$$

where the exponential map for the stuffle product is defined, for any $P \in \mathbb{C}\langle\langle Y \rangle\rangle$ such that $\langle P \mid 1_{Y^*}\rangle = 0$, is defined by

$$\exp_{\uplus}(P) := 1_{Y^*} + \frac{P}{1!} + \frac{P \uplus P}{2!} + \cdots + \frac{P^{\uplus n}}{n!} + \cdots .$$

In particular, from (3.73), one gets, for $k \geq 1$, the identity,

$$(zy_k)^* = \exp_{\uplus}\Big(-\sum_{n\geq 1} y_{nk}\frac{(-z)^n}{n}\Big). \tag{3.74}$$

Chapter 4

Conclusions and future directions

Starting from groups and their associated Lie algebras in semi-direct product form, we investigated the presented premiages analogs which are precisely Lazard's elimination process. This study took place principally within the category **k-Lie** of Lie algebras over a ring **k** where a practical sufficient condition was found in order to recognize, at first sight, whether the presented structure could be split in semi-direct form. To this end was developed, successively, the notion of $\mathbb{B} = (\{0,1\}, \vee)$-graded structures and for iterated such, a larger notion of S-graded structures (Lie and their enveloping algebras) where $(S, +)$ is a commutative semigroup.

Moreover, conditions on S where investigated in order to transfer the classical machinery to explicit formulas and the specialized notion of the direct sum and tensor products for the Hilbert series of any S-graded algebra in finite dimensions, S being an additive commutative semigroup. Essentially the links with "Condition (D)" (Bourbaki [10] Ch III § 2.10.), and locally finite semigroups of computer science (Eilenberg [43]) are made precisely. Then, we allow ourselves to examine this definition by Example 2.4 and Example 2.5.

Questions (Q1)[1] and (Q2)[2] have been reformulated in terms of algebraic structures and "free functors", this elusive "free functor" has been completely worked out (source,

[1]What are the expressions of Lazard elimination (LE) in several categories where there are sort of semi-direct products?

[2]Are these universal? (i.e. is every semi-direct product the image of some Lazard eliminations?) and is LE a free object?

target and formulas) in the case of the category of Lie **k**-algebras. This question is parallel to a similar one with "Free Partially Commutative Structures"[41] for which all targets of the functor are known[3]. The (unpublished) resolution of this "free functor" has been completely worked out and reformulated in Section 1.3. Remains to complete the same work for LE i.e.

1. Consider "Free looking" elimination formulas (see Table 2.5).
2. Find the correct enrichment for Monoids and Groups (because $\mathbb{B}$-grading works also for **k-AAU**, the category of unital associative **k**-algebras).
3. Find the sources and functors (if possible, the source could be "double sets", as for partially commutative structures for which the source is unique whatever the target category).

The hope is that this technique based on a filtration of the alphabet of generators in conjunction with an appropriate filtration of the relators could apply to other algebras coming from combinatorics or geometry. Let us say a word about this last point. For many fibered spaces (in particular configuration spaces), the fundamental group of the base space acts on the fibers by automorphisms and then acts on the fundamental group of the fibers. Then appear natural semi-direct products of groups. In general, the lower central series (see Chapter 2) goes too fast for transferring these semi-direct products to Lie algebras by the associated graded algebra mechanism gr (see equation (2.4)), but other appropriate filtrations are usually considered by Nakamura and Takao [88] and recently by Sawada [96]. We think that our elimination techniques can ease the understanding and calculations of these semi-direct products.

[3]And in which LE plays a crucial tool.

Chapter 5

Appendixes

Contents

5.1 Miscellaneous facts.

5.1.1 About central extensions.

We have the following well known result about central extensions

Lemma 5.1. *Let*

$$0 \longrightarrow \mathfrak{g}_1 \xrightarrow{j} \mathfrak{g}_3 \underset{s}{\overset{p}{\rightleftarrows}} \mathfrak{g}_2 \longrightarrow 0 \tag{5.1}$$

be a split (i.e. $ps = \mathrm{Id}_{\mathfrak{g}_2}$) and central (i.e. $j(\mathfrak{g}_1) \subset Z(\mathfrak{g}_3) = \{x \in \mathfrak{g}_3 \mid [x, \mathfrak{g}_3] = 0\}$) extension of Lie algebras. Then, if $\mathfrak{g}_2$ is commutative, so is $\mathfrak{g}_3$.

Proof. Let $x, y \in \mathfrak{g}_3$. Then, since $ps = \mathrm{Id}_{\mathfrak{g}_2}$, we have $x - s(p(x)) \in \mathrm{Ker}\, p = \mathrm{Im}\, j \subseteq Z(\mathfrak{g}_3)$, so that $[x - s(p(x)),\ y] = 0$. In other words, $[x, y] = [s(p(x)), y]$. Similarly, we find $y - s(p(y)) \in Z(\mathfrak{g}_3)$, thus $[s(p(x)),\ y - s(p(y))] = 0$. In other words,

$$\begin{aligned} [s(p(x)), y] &= [s(p(x)), s(p(y))] && (5.2)\\ &= s([p(x),\ p(y)]) \qquad \text{(since } s \text{ is a Lie morphism)} && (5.3)\\ &= s(0) \qquad \text{(since } \mathfrak{g}_2 \text{ is abelian)}\,. && (5.4) \end{aligned}$$

Hence, $[x, y] = [s(p(x)), y] = 0$. □

5.1.2 Factorization of characters.

This section deals with applications of MRS formula (3.6) to the factorization of characters.

As a result of equations (3.4), we have seen that, a Lie $\mathbf{k}$-algebra (free as a module and $\mathbf{k}$ being a $\mathbb{Q}$-algebra) $\mathfrak{g}$ together with a totally ordered basis $\mathcal{B} = (b_i)_{i\in I}$ basis of it being given[1], the space $span\{\mathcal{B}_\alpha\}_{\alpha\in\mathbb{N}^{(I)}}$ is a unital $\star$-subalgebra of $\mathcal{U}^*(\mathfrak{g})$. Let us then set

$$\mathcal{U}^\vee(\mathfrak{g}) := span\{\mathcal{B}_\alpha\}_{\alpha\in\mathbb{N}^{(I)}}. \tag{5.5}$$

We suppose now that

1. $\mathfrak{g} = \bigoplus_{s\in S} \mathfrak{g}_s$ is a S-graded Lie algebra (where S is an additive commutative semigroup),

[1] Throughout this paragraph, notations will be those of Section 3.2.

2. the basis $\mathcal{B} = (b_i)_{i \in I}$ is S-graded (we set $\deg(b_i) := s$ for the unique $s \in S$ such that $b_i \in \mathfrak{g}_s$),

3. the basis $\mathcal{B}$ is graded "in finite ranks" which means that, for all $s \in S$, the set
$$I_s := |\{i \in I \mid \deg(b_i) = s\}|$$
is finite i.e. each $\mathfrak{g}_s$ is a free $\mathbf{k}$-module of finite rank with basis $(b_i)_{i \in I_s}$,

4. the semigroup $(S, +)$ is locally finite.

This case encompasses all free partially commutative Lie algebras and, in particular, all free Lie algebras (with $S = \mathbb{N}^{(X)}$) and Drinfeld-Kohno Lie algebras (with $S = ([2, n+1], \vee) \times (\mathbb{N}_{\geq 1}, +)$ or $S = (\mathbb{N}_{\geq 2}, \vee) \times (\mathbb{N}_{\geq 1}, +)$) and many other combinatorial Lie algebras.

By Appendix 5.4.2 and the assumption that "$(S, +)$ is locally finite", the universal enveloping $\mathcal{U}(\mathfrak{g}) = \bigoplus_{s \in S \sqcup \{0\}} \mathcal{U}_s(\mathfrak{g})$ (see formula (5.23)) is a finitely[2] $S \sqcup \{0\}$-graded Hopf algebra. Thus, we can consider $\mathcal{U}^\vee(\mathfrak{g})$ as the graded dual of $\mathcal{U}(\mathfrak{g})$ i.e. $\mathcal{U}^\vee(\mathfrak{g}) = \bigoplus_{s \in S \sqcup \{0\}} \mathcal{U}_s^*(\mathfrak{g})$ (for the case when $S = (\mathbb{N}, +)$, see Grinberg and Reiner [56] § 1.6). Then one can check (as an exercise) that

1. the bases $\{\mathcal{B}^\alpha\}_{\alpha \in \mathbb{N}^{(I)}}$ of $\mathcal{U}(\mathfrak{g})$ and $\{\mathcal{B}_\alpha\}_{\alpha \in \mathbb{N}^{(I)}}$ of $\mathcal{U}^\vee(\mathfrak{g})$ are $S \sqcup \{0\}$-graded in finite ranks,

2. one can dualize the $S \sqcup \{0\}$-graded Hopf algebra structure of
$$(\mathcal{U}(\mathfrak{g}), \mu_{\mathcal{U}}, 1_{\mathcal{U}}, \Delta_{\mathcal{U}}, \epsilon_{\mathcal{U}}, S_{\mathcal{U}}) \tag{5.6}$$
by setting

   (a) $\mu_{\mathcal{U}^\vee} = \star$ (the convolution restricted to $\mathcal{U}^\vee(\mathfrak{g})$),

   (b) $1_{\mathcal{U}^\vee} = \epsilon_{\mathcal{U}}$ (the counit restricted to $\mathcal{U}^\vee(\mathfrak{g})$),

   (c) $\Delta_{\mathcal{U}^\vee} : \mathcal{U}^\vee(\mathfrak{g}) \to \mathcal{U}^\vee(\mathfrak{g}) \otimes \mathcal{U}^\vee(\mathfrak{g})$ is the dual of $\mu_{\mathcal{U}}$ by
$$\Delta_{\mathcal{U}^\vee}(\mathcal{B}_\alpha) = \sum_{\alpha_1, \alpha_2 \in \mathbb{N}^{(I)}} \langle \mathcal{B}_\alpha \mid \mathcal{B}^{\alpha_1} \mathcal{B}^{\alpha_2} \rangle \mathcal{B}_{\alpha_1} \otimes \mathcal{B}_{\alpha_2} \tag{5.7}$$
   due to the hypotheses about the S-grading one can check (exercise) that this sum is finitely supported,

[2] It means that each $\mathbf{k}$-module $\mathcal{U}_s(\mathfrak{g})$ is of finite rank.

(d) $\epsilon_{\mathcal{U}^\vee} = \delta_1$ the Dirac evaluation which means that, for $f \in \mathcal{U}^\vee(\mathfrak{g})$, $\epsilon_{\mathcal{U}^\vee}(f) = \langle f \mid 1_{\mathcal{U}}\rangle$,

(e) $S_{\mathcal{U}^\vee}(f) = \sum_{\alpha\in\mathbb{N}^{(I)}} \langle f \mid S_{\mathcal{U}}(\mathcal{B}^\alpha)\rangle \mathcal{B}_\alpha$ again, due to the hypotheses about the S-grading one can check (exercise) that this sum is finitely supported,

3. with this at hand, one can check that $(\mathcal{U}^\vee(\mathfrak{g}), \mu_{\mathcal{U}^\vee}, 1_{\mathcal{U}^\vee}, \Delta_{\mathcal{U}^\vee}, \epsilon_{\mathcal{U}^\vee}, S_{\mathcal{U}^\vee})$ is a Hopf algebra.

Remark 5.1. In the case when $\mathfrak{g}$ is the free Lie algebra $\mathcal{L}_{\mathbf{k}}(X) = \bigoplus_{\alpha\in\mathbb{N}^{(X)}} \mathcal{L}_{\mathbf{k}}(X)_\alpha$ (1.31) (then the enveloping algebra $\mathcal{U}(\mathfrak{g}) = (\mathbf{k}\langle X\rangle, \mathrm{conc}, 1_{X^*}, \Delta_{\sqcup\!\sqcup}, \epsilon, S_{\mathcal{U}}))$ and $\{\mathcal{B}^\alpha\}_{\alpha\in\mathbb{N}^{(I)}}$ is a multihomogeneous basis, we have $\mathcal{U}^\vee(\mathfrak{g}) = (\mathbf{k}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}, \Delta_{\mathrm{conc}}, \epsilon, S_{\mathcal{U}^\vee})$ (due to the fine grading by $S = \mathbb{N}^{(X)}$ (1.27)).

Now we have the following factorization of characters

1. If χ is a $\mathbf{k}$-valued $\star$-character on $\mathcal{U}^\vee(\mathfrak{g})$ i.e. $\chi \in \Xi(\mathcal{U}^\vee(\mathfrak{g}), \mathbf{k})$, then the operator $\chi \otimes \mathrm{Id}$ is continuous on $\mathcal{U}^\vee(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ for the limiting process due to finite pointwise convergence[3] and then extends to $\mathcal{U}^\vee(\mathfrak{g})\widehat{\otimes}\mathcal{U}(\mathfrak{g})$ i.e. one has a morphism of topological rings $\chi\widehat{\otimes}\,\mathrm{Id} : \mathcal{U}^\vee(\mathfrak{g})\widehat{\otimes}\mathcal{U}(\mathfrak{g}) \to (\mathcal{U}^\vee(\mathfrak{g}))^*$. Thus, by denoting $(\chi)^{\mathrm{gen}} = (\chi\widehat{\otimes}\,\mathrm{Id})\big(\sum_{\alpha\in\mathbb{N}^{(I)}} \mathcal{B}_\alpha \otimes \mathcal{B}^\alpha\big)$, we deduce that

$$(\chi)^{\mathrm{gen}} = \sum_{\alpha\in\mathbb{N}^{(I)}} \chi(\mathcal{B}_\alpha)\mathcal{B}^\alpha = \prod_{i\in I}^{\nearrow} \exp(\chi(\mathcal{B}_{e_i})\mathcal{B}^{e_i}). \tag{5.8}$$

2. If χ is a $\mathbf{k}$-valued $\mu_{\mathcal{U}}$-character on $\mathcal{U}(\mathfrak{g})$ i.e. $\chi \in \Xi(\mathcal{U}(\mathfrak{g}), \mathbf{k})$, then the operator $\mathrm{Id}\otimes\chi$ is continuous on $\mathcal{U}^\vee(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ for the limiting process due to finite poinwise convergence and then extends to $\mathcal{U}^\vee(\mathfrak{g})\widehat{\otimes}\mathcal{U}(\mathfrak{g})$ i.e. one has a morphism of topological rings $\mathrm{Id}\,\widehat{\otimes}\chi : \mathcal{U}^\vee(\mathfrak{g})\widehat{\otimes}\mathcal{U}(\mathfrak{g}) \to \mathcal{U}^*(\mathfrak{g})$. Thus, by denoting $(\chi)^{\mathrm{gen}} = (\mathrm{Id}\,\widehat{\otimes}\chi)\big(\sum_{\alpha\in\mathbb{N}^{(I)}} \mathcal{B}_\alpha \otimes \mathcal{B}^\alpha\big)$, we then have (due to the fact that $\mathbf{k}$ is

[3] By Example 5.3, the pair $\big(\mathcal{U}^*(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g}), \Phi, \mathrm{End}_{\mathbf{k}}(\mathcal{U}(\mathfrak{g}))\big)$ is a completion triplet of topological rings. Then we can consider $\mathcal{U}^\vee(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$ as a topological subring of the complete tensor product $\mathcal{U}^*(\mathfrak{g})\widehat{\otimes}\mathcal{U}(\mathfrak{g})$ or, through the standard embedding, of $\mathrm{End}(\mathcal{U}(\mathfrak{g}))$. As $\mathcal{U}^*(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$, we remark that $\mathcal{U}^\vee(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$ is also a dense subset of $\mathcal{U}^*(\mathfrak{g})\widehat{\otimes}\mathcal{U}(\mathfrak{g})$, then we can complete the tensor product as $\mathcal{U}^\vee(\mathfrak{g})\widehat{\otimes}\mathcal{U}(\mathfrak{g})$ which is equal to $\mathcal{U}^*(\mathfrak{g})\widehat{\otimes}\mathcal{U}(\mathfrak{g})$.

commutative, its action can be noted on the right)

$$(\chi)^{\mathrm{gen}} = \sum_{\alpha\in\mathbb{N}^{(I)}} \mathcal{B}_\alpha \chi(\mathcal{B}^\alpha) = \prod_{i\in I}^{\nearrow} \exp(\mathcal{B}_{e_i}\chi(\mathcal{B}^{e_i})). \tag{5.9}$$

On the other hand, it is well known[4] that, a Hopf algebra $(\mathcal{H}, \mu_{\mathcal{H}}, 1_{\mathcal{H}}, \Delta, \epsilon, S)$[5] and a *commutative* (associative with unit) algebra $(\mathcal{A}, \mu_{\mathcal{A}}, 1_{\mathcal{A}})$ being given (all over the same commutative ring $\mathbf{k}$), then the set $\Xi(\mathcal{H},\mathcal{A}) = \mathrm{Hom}_{\mathbf{k}\text{-}\mathbf{AAU}}(\mathcal{H},\mathcal{A})$ is a group under convolution (the inverse being performed through precomposition with S, proofs are essentially the same as for $\mathbf{k}$-valued characters).
If $\mathcal{H} = \mathcal{U}(\mathfrak{g})$ (where the Lie algebra $\mathfrak{g}$ satisfies all above assumptions), formulas (5.8) and (5.9) still hold true when χ is a $\mathcal{A}$-valued character, $\mathcal{A}$ being a commutative (associative with unit) algebra.

We must warn the reader that this is no longer the case if $\mathcal{A}$ can be non-commutative as shows the following counterexample. By Appendix 5.4.4, let us recall the construction of a cocommutative Hopf algebra $\mathcal{H}_{\mathrm{conc}}(X) = (\mathbf{k}\langle X\rangle, \mathrm{conc}, 1_{X^*}, \Delta_{⊔⊔}, \epsilon)$ where $X = \{a,b\}$ and of a non-commutative algebra (indeed its algebra part) $\mathcal{A}_{\mathrm{conc}} = (\mathbf{k}\langle X\rangle, \mathrm{conc}, 1_{X^*})$. We then define an algebra morphism $f \in \Xi(\mathcal{H}_{\mathrm{conc}}, \mathcal{A}_{\mathrm{conc}})$ by the universal property (1.22) as follows: $f(a) = a, f(b) = b$. One observes that

- $\Delta_{⊔⊔}(a) = a\otimes 1 + 1\otimes a$, and $\Delta_{⊔⊔}(b) = b\otimes 1 + 1\otimes b$, then
- $\Delta_{⊔⊔}(ab) = \Delta_{⊔⊔}(a)\Delta_{⊔⊔}(b) = ab\otimes 1 + a\otimes b + b\otimes a + 1\otimes ab.$

Thus, we arrive at

$$\begin{aligned}
& f\star f(a).f\star f(b) = \mathrm{conc}\circ(f\otimes f)\circ\Delta_{⊔⊔}(a).\,\mathrm{conc}\circ(f\otimes f)\circ\Delta_{⊔⊔}(b)\\
& = \big(f(a)f(1) + f(1)f(a)\big)\big(f(b)f(1) + f(1)f(b)\big)\\
& = (a+a)(b+b) = 4ab
\end{aligned}$$

and

$$f\star f(ab) \quad = \quad \mathrm{conc}\circ(f\otimes f)\circ\Delta_{⊔⊔}(ab)$$

[4] See e.g. Kassel [67] Ch III §8 Exercise 11.

[5] The order is always (space, product, unit, coproduct, counit, antipode).

$$\begin{aligned} &= f(ab)f(1) + f(a)f(b) + f(b)f(a) + f(1)f(ab) \\ &= ab + ab + ba + ab = 3ab + ba. \end{aligned}$$

As above, $f \star f(a).f \star f(b) \neq f \star f(ab)$, therefore $f \star f$ is not an algebra homomorphism i.e. $f \star f \notin \Xi(\mathcal{H}_{\text{conc}}, \mathcal{A}_{\text{conc}})$.

5.2 Appendix A: Limits and Colimits.

In category theory [80, 66], a *limit* of a diagram[6] $F : \mathcal{I} \to \mathcal{C}$, if exists, is an object

$$\varprojlim_{i \in \mathcal{I}} F_i \qquad \text{or } \varprojlim_{\mathcal{I}} F \text{ for short}$$

in $\mathcal{C}$ together with morphisms

$$\phi_i : \varprojlim_{\mathcal{I}} F \to F_i$$

such that

i) for all morphisms $\alpha : i \to j$ in $\mathcal{I}$ the triangle

$$\begin{tikzcd} & \varprojlim_{\mathcal{I}} F \arrow[dl, "\phi_i"] \arrow[dr, "\phi_j"] & \\ F_i \arrow[rr, "F_\alpha"'] & & F_j \end{tikzcd} \tag{5.10}$$

commutes.

ii) moreover, the limit $\varprojlim_{\mathcal{I}} F$ is the (initial) universal object with this property

$$\begin{tikzcd} & X \arrow[d, "u"] \arrow[ddl, "\psi_i"'] \arrow[ddr, "\psi_j"] & \\ & \varprojlim_{\mathcal{I}} F \arrow[dl, "\phi_i"] \arrow[dr, "\phi_j"'] & \\ F_i \arrow[rr, "F_\alpha"'] & & F_j. \end{tikzcd} \tag{5.11}$$

Notice that limits $(\varprojlim_{\mathcal{I}} F, (\phi_i)_{i \in \mathcal{I}})$ are (if they exist) unique up to isomorphism by the uniqueness requirement (5.11) of the point (ii) in the above definition.

[6] i.e. a functor from a small category $\mathcal{I}$ to an arbitrary category $\mathcal{C}$. We say that $\mathcal{C}$ is a diagram of the index category $\mathcal{I}$.

Example 5.1. If an index category $\mathcal{I}$ is discrete (resp. a category with two objects and two parallel morphisms from one object to the other), then a diagram $F : \mathcal{I} \to \mathcal{C}$ is a family of objects (resp. a pair of parallel morphisms) in $\mathcal{C}$, a limit of the diagram F is called a *product* (resp. an *equalizer*) of these objects (resp. morphisms). Moreover, a limit of a contravariant functor $F : \mathcal{I}^{op} \to \mathcal{C}$, where $\mathcal{I}$ is a directed set[7] that is considered as a small category in which the morphisms consist of arrows $\alpha : i \to j$ if and only if $i \leq j$, is called an *inverse limit* of the inverse system $((F_i)_{i\in\mathcal{I}}, (F_{ij})_{i\leq j\in\mathcal{I}})$, where $F_{ij} : F_j \to F_i$ are morphisms in $\mathcal{C}$.

A *colimit* of a diagram $F : \mathcal{I} \to \mathcal{C}$ is the limit of the opposite diagram $F^{op} : \mathcal{I}^{op} \to \mathcal{C}^{op}$, where $\mathcal{I}^{op}$ and $\mathcal{C}^{op}$ are the opposite categories of $\mathcal{I}$ and $\mathcal{C}$ respectively, denoted by the following

$$\varinjlim_{i\in\mathcal{I}} F_i \qquad \text{or } \varinjlim_{\mathcal{I}} F.$$

The *coproducts, coequalizers* and *direct limits* are respectively the dual concept of products, equalizers and inverse limits, they are examples of colimits in category theory.

5.3 Appendix B: Topological rings, their completions and combinatorics.

5.3.1 Topological rings.

Definition 5.1. A ring $\mathcal{R}$ endowed with a topology $\mathcal{T}_{\mathcal{R}}$ is called a topological ring if and only if sum and product are continuous operations. Precisely, the following maps $\mathcal{R} \times \mathcal{R} \to \mathcal{R}$, $(x, y) \to x + y$ and $(x, y) \to x.y$ are continuous.

The topology of a topological ring is uniquely determined by the filter of neighbourhoods of zero $\mathcal{B}(0)$ (or a base of it). For conditions on $\mathcal{B}(0)$, see [14] Ch III §6.3. A topological ring $(\mathcal{R}, \mathcal{T}_{\mathcal{R}})$ is Hausdorff if and only if $\bigcap_{\mathfrak{m}\in\mathcal{B}(0)} \mathfrak{m} = \{0\}$ and is said to

[7] i.e. a poset such that for any elements $i, j \in \mathcal{I}$, there exists an element $k \in \mathcal{I}$ such that $i \leq k$ and $j \leq k$.

be complete if every Cauchy net[8] converges to a unique limit (this entails in particular that $(\mathcal{R}, \mathcal{T}_{\mathcal{R}})$ is Hausdorff).

Topological rings $(\mathcal{R}, \mathcal{T}_{\mathcal{R}})$ (resp. Hausdorff topological rings, complete topological rings) and continuous morphisms of rings form categories : **TopRng**, **HausTopRng**, **CompHausTopRng**.

Let now $(\mathcal{R}, \mathcal{T}_{\mathcal{R}})$ be a topological ring (i.e. $(\mathcal{R}, \mathcal{T}_{\mathcal{R}}) \in$ **TopRng**) and F be the inclusion functor **CompHausTopRng** $\to$ **TopRng** ("Complete Hausdorff Topological Rings" to "Topological Rings"), then, we can state

Definition 5.2. Let $(\mathcal{R}, \mathcal{T}_{\mathcal{R}})$ be a topological ring. A completion of $\mathcal{R}$ is any pair $(j_{\mathcal{R}}, \widehat{\mathcal{R}})$ such that

1. $(\widehat{\mathcal{R}}, \mathcal{T}_{\widehat{\mathcal{R}}}) \in$ **CompHausTopRng**
2. and $j_{\mathcal{R}} \in \mathrm{Hom}_{\mathbf{TopRng}}(\mathcal{R}, F(\widehat{\mathcal{R}}))$

fulfilling the following property

- for each morphism $f \in \mathrm{Hom}_{\mathbf{TopRng}}(\mathcal{R}, F(\mathcal{S}))$, there exists a unique morphism $\widehat{f} \in \mathrm{Hom}_{\mathbf{CompHausTopRng}}(\widehat{\mathcal{R}}, \mathcal{S})$ such that $f = F(\widehat{f}) \circ j_{\mathcal{R}}$.

In other words, $(j_{\mathcal{R}}, \widehat{\mathcal{R}})$ is a solution of the following universal problem

$$\begin{tikzcd} \mathbf{TopRng} & & \mathbf{CompHausTopRng} \arrow[ll, dotted, "F"'] \\ (\mathcal{R}, \mathcal{T}_{\mathcal{R}}) \arrow[rr, dashed, "f"] \arrow[drr, dashed, "j_{\mathcal{R}}"'] & & (\mathcal{S}, \mathcal{T}_{\mathcal{S}}) \\ & & (\widehat{\mathcal{R}}, \mathcal{T}_{\widehat{\mathcal{R}}}). \arrow[u, "\widehat{f}"'] \end{tikzcd} \tag{5.12}$$

There are many ways to introduce a completion but in our case, we can use the following simpler characterization.

Characterization. – Let $(\mathcal{R}, \mathcal{T}_{\mathcal{R}}) \in$ **HausTopRng**, $(\mathcal{S}, \mathcal{T}_{\mathcal{S}}) \in$ **CompHausTopRng** and $i : \mathcal{R} \to \mathcal{S}$ is a topological embedding (of type **HausTopRng**), which means that i is a homeomorphism between $\mathcal{R}$ (with the original topology) and $i(\mathcal{R})$ endowed with the induced topology from $\mathcal{S}$, such that $i(\mathcal{R})$ is dense in $\mathcal{S}$. Then, the pair $(F \circ i, \mathcal{S})$ is a completion of $\mathcal{R}$.

[8]It is a net [102, 110] $(x_\alpha)_{\alpha \in A}$ (A is a directed set) such that for all $\mathfrak{m} \in \mathcal{B}(0)$ there exists $B \in A$ such that, for all $\alpha, \beta \geq B$, $x_\alpha - x_\beta \in \mathfrak{m}$ (this means that the set $\{x_\alpha\}_{\alpha \geq B}$ is $\mathfrak{m}$-small).

Example 5.2. Let us consider $\mathbf{k}\langle\langle X\rangle\rangle$ (i.e. the $\mathbf{k}$-total algebra of X^*, see 5.3.2) the associative algebra of formal power series over X (with coefficients in $\mathbf{k}$) and the decreasing sequence of ideals

$$\mathfrak{m}_n := \{S \in \mathbf{k}\langle\langle X\rangle\rangle \mid (\forall w \in X^{<n})(\langle S \mid w\rangle = 0)\}.$$

One can check that the ring $\mathbf{k}\langle\langle X\rangle\rangle$ is topologized by the filter base $\mathcal{B} = \{\mathfrak{m}_n\}_{n\geq 0}$ and that its topology is defined by the ultrametric distance

$$d(T, S) = 2^{-\omega(T-S)},$$

where, for each non zero series R,

$$\omega(R) = \max_{n\in\mathbb{N}}(R \in \mathfrak{m}_n)$$

is the length of the shortest words w such that $\langle R \mid w\rangle \neq 0$ and we set $\omega(0) = +\infty$.

With this distance, one can check that the completion of the Hausdorff topological ring $\mathbf{k}\langle X\rangle$ is $\mathbf{k}\langle\langle X\rangle\rangle$ if and only if X is finite. The reason for this is that the closure of $\mathbf{k}\langle X\rangle$ (for this topology) is the set of series for which each isobaric component is a polynomial. In other words, writing a series $S = \sum_{n\geq 0} S_n$ where $S_n := \sum_{|w|=n}\langle S \mid w\rangle\, w$ we have (left as an exercise)

$$\overline{\mathbf{k}\langle X\rangle} = \{S \in \mathbf{k}\langle\langle X\rangle\rangle \,|\, (\forall n)(S_n \in \mathbf{k}\langle X\rangle)\}$$

This explains, in particular, why the sum of all variables $\sum_{x\in X} x$, which is a polynomial in the case when X is finite, does not even belong to the $\mathcal{B}$-completion of $\mathbf{k}\langle\langle X\rangle\rangle$ in the case when X is infinite (see discussion in [113]). In the latter case a finer topology has to be defined to recover the series as the completion of the polynomials. It is (in any cases, but when X is infinite, this topology is different) the topology of pointwise convergence for which a fundamental system of neighbourhoods of zero is given by the system of two-sided ideals $(\mathfrak{m}_F)_{F\subset_{finite} X^*}$

$$\mathfrak{m}_F = \{S \in \mathbf{k}\langle\langle X\rangle\rangle \,|\, (\forall w \in F)(\forall u \text{ s.t. } w \in X^*uX^*)(\langle S \mid u\rangle = 0)\}\ .$$

Another example which will be in used for finding the generating function of an endomorphism is the following in particular $(\mathrm{Id})^{\mathrm{gen}}$ for enveloping algebras, see Section 3.2).

Example 5.3. Let M be a free $\mathbf{k}$-module with a fixed basis $\mathcal{B} = (\beta_i)_{i\in I}$. One can show the following facts

1. The usual morphism $\Phi : \ M^* \otimes M \to \mathrm{End}_{\mathbf{k}}(M)$ is into.

2. The image of Φ is closed for composition or, when M is a bialgebra, for convolution.

3. The image of Φ is dense in $\mathrm{End}_{\mathbf{k}}(M)$ endowed with the topology of pointwise convergence (which is compatible with the ring structure, be it for composition or, when M is a bialgebra, for convolution).

4. Therefore this topology induces a topological ring structure on $M^* \otimes M$, for which we can complete the tensor product as $M^*\widehat{\otimes}M$ and the pair $(M^* \otimes M, \Phi, \mathrm{End}_{\mathbf{k}}(M))$ is a completion triplet.

5. For any basis $\mathcal{B} = (\beta_i)_{i\in I}$ and every $f \in \mathrm{End}_{\mathbf{k}}(M)$, the family $(\beta^i \otimes f(\beta_i))_{i\in I}$ (where $(\beta^i)_{i\in I}$ is the coordinate family) is summable (see [56], Def 1.7.2). Its sum realises the inverse of $\overline{\Phi}$, therefore we can state.

6. $\overline{\Phi} : \ M^*\widehat{\otimes}M \to \mathrm{End}_{\mathbf{k}}(M)$ is an isomorphism and the inverse isomorphism of rings is given by
$$\overline{\Phi}^{-1}(f) = \sum_{i\in I} \beta^i \otimes f(\beta_i) \tag{5.13}$$
(we recall that $(\beta^i)_{i\in I}$ is the coordinate family of forms defined by $\langle \beta^i \mid \beta_j\rangle = \delta_{ij}$).

7. We will denote $\overline{\Phi}^{-1}(f)$ by $(f)^{\mathrm{gen}}$ and call it the *generating series* of f.

5.3.2 Towards series: the threefold way.

Total algebras (i.e. series without topology).

Given $\mathbf{k}$ a commutative ring (with unit) and M a multiplicative semigroup which satisfies "Condition (D)"[9] i.e. for all $m \in M$ the set

$$D_2(m) = \{(m_1, m_2) \in M \times M \,|\, m_1 m_2 = m\}$$

[9] Bourbaki [10] Ch III § 2.10.

is finite (for example, $M = X^*$ where X is a set). Let us define the total semigroup algebra $\mathbf{k}[[M]]$ [10] which is the $\mathbf{k}$-module of all infinite sums

$$\sum_{m \in M} \alpha_m m$$

and the convolution product

$$\left(\sum_{m_1 \in M} \alpha_{m_1} m_1\right)\left(\sum_{m_2 \in M} \beta_{m_2} m_2\right) = \sum_{m \in M}\left(\sum_{\substack{m_1, m_2 \in M \\ m_1 m_2 = m}} \alpha_{m_1}\beta_{m_2}\right) m. \tag{5.14}$$

Endowed with this product $\mathbf{k}[[M]]$ is a $\mathbf{k}$-AA and a $\mathbf{k}$-AAU when M is a monoid.

Completion by inverse limits.

In the following part, let us consider that $\mathcal{R}$ is a ring and $\mathcal{B}$ is a filter base of ideals of $\mathcal{R}$ i.e.

$$(\forall \mathfrak{m}_1, \mathfrak{m}_2 \in \mathcal{B})(\exists \mathfrak{m}_3 \in \mathcal{B})(\mathfrak{m}_3 \subseteq \mathfrak{m}_1 \cap \mathfrak{m}_2). \tag{5.15}$$

Now, due to (5.15) and as a consequence of Prop 2 §1.2 [14], we can define a unique topology $\mathcal{T}_\mathcal{R}$ on $\mathcal{R}$ such that $(\mathcal{R}, \mathcal{T}_\mathcal{R})$ is a topological ring in which $\mathcal{B}$ is a fundamental set of neighborhoods of zero for $\mathcal{T}_\mathcal{R}$. Assume that the topology $\mathcal{T}_\mathcal{R}$ is Hausdorff i.e. $\bigcap_{\mathfrak{m} \in \mathcal{B}} \mathfrak{m} = \{0\}$, then the inverse limit

$$\widetilde{\mathcal{R}} = \varprojlim_{\mathfrak{m} \in \mathcal{B}} \mathcal{R}/\mathfrak{m} \subseteq \prod_{\mathfrak{m} \in \mathcal{B}} \mathcal{R}/\mathfrak{m}$$

in the category **TopRng** is a completion $\widehat{\mathcal{R}}$ of the Hausdorff topological ring $(\mathcal{R}, \mathcal{T}_\mathcal{R})$.

Case of the total algebras. –

In the preceding case (here 5.3.2), for all $m \in M$, we denote the set of factors of M

$$Fact(m) := \{u \in M \mid m \in MuM\} \subset M.$$

it is straightforward, using "Condition (D)", that $Fact(m)$ is a finite set and that for all $m_1, m_2 \in M$

$$Fact(m_1) \cup Fact(m_2) \subset Fact(m_1 m_2) \tag{5.16}$$

then

$$\mathfrak{m}_m := \mathrm{span}_{\mathbf{k}}(M \setminus Fact(m))$$

[10] This associative algebra is unital if M admits a unit i.e. M is a monoid.

is the largest two-sided ideal of the semigroup algebra $\mathbf{k}[M]$ [11] which avoids m (i.e. J such that $m \notin J$). Now, due to (5.16), we can define a unique topology $\mathcal{T}_M$ on $\mathbf{k}[M]$ such that $(\mathbf{k}[M], \mathcal{T}_M)$ is a Hausdorff topological ring in which $\mathcal{B} = (\mathfrak{m}_m)_{m\in M}$ is a fundamental set of neighborhoods of zero for $\mathcal{T}_M$ (every set of two sided ideals closed by intersection does that, here finite intersections of elements picked in $\mathcal{B}$, and moreover $\bigcap_{m\in M} \mathfrak{m}_m = \{0\}$). See also discussion in [109].

Furthermore, we can make explicit the topology of this particular completion as follows, the total semigroup algebra $\mathbf{k}[[M]]$ (endowed with a topology in which $\widehat{\mathcal{B}} = (\widehat{\mathfrak{m}}_m)_{m\in M}$ is a fundamental set of neighborhoods of zero, where

$$\widehat{\mathfrak{m}}_m = \{\text{all infinite sums} \sum_{u\in M\setminus Fact(m)} \alpha_u u\}$$

is the largest two-sided ideal of the total semigroup algebra $\mathbf{k}[[M]]$ avoiding m) is the completion $\widehat{\mathbf{k}[M]}$ of the Hausdorff topological semigroup algebra $(\mathbf{k}[M], \mathcal{T}_M)$.

Classical completion.

As above, assume that that $\mathcal{R}$ is a ring and $\mathcal{B}$ is a filter base of ideals of $\mathcal{R}$ (5.15), we then arrive to the fact that $(\mathcal{R}, \mathcal{T}_{\mathcal{R}})$ is a topological ring in which $\mathcal{B}$ is a fundamental set of neighborhoods of zero for $\mathcal{T}_{\mathcal{R}}$. In this case, $\mathcal{R}$ is said to be *linearly topologized* and $\mathcal{T}_{\mathcal{R}}$ is called a *linear topology* (see Bourbaki [11] Ch III §4.2).

In particular, if $\mathcal{B} = \{\mathfrak{m}_n\}_{n\geq 0}$ where $(\mathfrak{m}_n)_{n\geq 0}$ is a decreasing sequence of ideals, then in this case (decreasing sequence and, moreover, the topology is Hausdorff), setting

$$d(s,t) = 2^{-\omega(s-t)} \text{ where } \omega(r) := \max_{n\in\mathbb{N}}(r \in \mathfrak{m}_n) \text{ and } \omega(0) := +\infty \qquad (5.17)$$

(for all $r, s, t \in \mathcal{R}$), we can prove that d is an ultrametric distance and the linear topology $\mathcal{T}_{\mathcal{R}}$ can be defined by the ultrametric $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_{\geq 0}$. In this way, we can get the completion $\widehat{\mathcal{R}}$ of the ultrametric space $(\mathcal{R}, d)$. For example when $\mathcal{R}$ is the free associative algebra $\mathbf{k}\langle X\rangle$ (with X finite), the completion is $\mathbf{k}\langle\langle X\rangle\rangle$, see Example 5.2.

[11] For this algebra, see Clifford and Preston [26].

5.4 Appendix C: Structures of Bialgebra type.

5.4.1 Gradings.

The idea of a graded set or structure [54] is to combine an algebraic structure with a (simpler) discrete/combinatorial structure with another one which "follows" the computations within it.

Let $\mathcal{A} \in$ **k-AA** and $(S, \times)$ be a (commutative or not) semigroup. We will say that $\mathcal{A} = \bigoplus_{u \in S} \mathcal{A}_u$, $[S]$-graded as a module, is a S-graded algebra if the usual condition $\mathcal{A}_u \mathcal{A}_v \subset \mathcal{A}_{uv}$ holds for all $u, v \in S$. A morphism between S-graded algebras $\varphi : \mathcal{A} \to \mathcal{B}$ is just a morphism between the underlying algebra structures which preserves the gradation (i.e. for all $u \in S$, $\varphi(\mathcal{A}_u) \subset \mathcal{B}_u$). We then have the following proposition which generalizes the similar classical ones for usual (commutative) gradings.

Proposition 5.2. *Let $\mathcal{J} \subset \mathcal{A}$, TFAE*

i) $\mathcal{J} = \mathrm{Ker}(\varphi)$ for some morphism between S-graded algebras.

ii) $\mathcal{J}$ is a two-sided ideal of $\mathcal{A}$ which is $[S]$-graded as a module.

Proof. The proof is *mutatis mutandis* the same as in classical treatises. We sketch it there.

(i)$\Longrightarrow$ (ii) being straightforward, remains to prove the converse, now (ii) being assumed, we consider the canonical surjection $s : \mathcal{A} \to \mathcal{A}/\mathcal{J}$. We set $(\mathcal{A}/\mathcal{J})_u := s(\mathcal{A}_u)$ and, as graded modules, we have

$$\mathcal{A}/\mathcal{J} = \bigoplus_{u \in S} (\mathcal{A}/\mathcal{J})_u = \bigoplus_{u \in S} \mathcal{A}_u/\mathcal{J}_u$$

with $\mathcal{J}_u = \mathcal{J} \cap \mathcal{A}_u$. So $\mathcal{A}/\mathcal{J}$ is naturally endowed with a structure such that s is a morphism of S-graded algebras and $\mathrm{Ker}(s) = \mathcal{J}$. □

Remark 5.2. The treatment here is very similar to what can be found in Bourbaki [10] Ch II §11 and [13] Ch II §2.6 save that, for our purpose, we need that a Lie algebra be graded on a semigroup rather than a monoid. In particular, in [58], one reads the sentence "If we do not require that the ring have an identity element, semigroups may replace monoids" shows that additive semigroups of degrees is probably a good working notion for Lie algebras and their enveloping algebras.

5.4.2 Enveloping algebra of S-graded Lie algebras.

For this section, the semigroup of degrees of a Lie algebra is commutative (because it must "follow" antisymmetry). Now, given $(S, +)$ an additive (that is, commutative and noted additively) semigroup and $A = [S]$ its underlying set, we now pass to the construction of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a S-graded Lie algebra $\mathfrak{g} = \bigoplus_{s\in S} \mathfrak{g}_s$ by the following: the tensor algebra inherits a A^*-gradation (see paragraph 5.4.1) from $\mathfrak{g}$ in the standard way, for $w = [s_1]\cdots[s_n] \in A^+$, we set

$$T_w(\mathfrak{g}) = \mathfrak{g}_{s_1} \otimes \cdots \otimes \mathfrak{g}_{s_n} \tag{5.18}$$

and

$$T_{1_{A^*}} = T^{(0)}(\mathfrak{g}) = \mathbf{k}.1_{T(\mathfrak{g})} \tag{5.19}$$

then

$$T(\mathfrak{g}) = \bigoplus_{w\in A^*} T_w(\mathfrak{g}). \tag{5.20}$$

One can check easily that this constitutes a (yet non-commutative) grading as, for all $u, v \in A^*$,

$$T_u(\mathfrak{g}).T_v(\mathfrak{g}) \subset T_{uv}(\mathfrak{g}). \tag{5.21}$$

In the sequel we will use the following regrading lemma.

Lemma 5.3. *Let S, T be two semigroups (commutative or not) and $\varphi : \ S \to T$ be a morphism of semigroups. Let $\mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s$ be a S-graded algebra (associative or not). We set, for $t \in T$, $\mathcal{A}_{\varphi,t} := \bigoplus_{\varphi(s)=t} \mathcal{A}_s$. Then $\mathcal{A}_{\varphi,-} := \bigoplus_{t\in T} \mathcal{A}_{\varphi,t}$ is a T-graded algebra.*

Proof. The proof is left to the reader. $\square$

Now, $\mu : A^* \to S \sqcup \{0\}, s_1 \ldots s_k \mapsto s_1 + \ldots + s_k, 1_{A^*} \mapsto 0$ is a morphism of monoids ($S \sqcup \{0\}$ is the monoid with neutral 0 constructed from $(S, +)$), the tensor algebra can be regraded (through Lemma 5.3) with a $S \sqcup \{0\}$-graded structure

$$T(\mathfrak{g}) = \bigoplus_{s\in S\sqcup\{0\}} T_s(\mathfrak{g}), \text{ where } T_s(\mathfrak{g}) = \bigoplus_{\substack{w\in A^* \\ \mu(w)=s}} T_w(\mathfrak{g}). \tag{5.22}$$

Now, let us consider the ideal $\mathcal{J}$ of $T(\mathfrak{g})$ which is generated by all elements of the form $a \otimes b - b \otimes a - [a, b]$ where a, b are homogeneous, say $a \in \mathfrak{g}_s, b \in \mathfrak{g}_t$. As $[\mathfrak{g}_s, \mathfrak{g}_t] \subseteq \mathfrak{g}_{s+t}$

$(s,t\in S)$, the ideal J is homogeneous with respect to the gradation (5.22). Therefore the enveloping algebra $\mathcal{U}(\mathfrak{g}) = {}^{T(\mathfrak{g})}/_{\mathcal{J}}$ can be equipped with an induced $S\sqcup\{0\}$-gradation

$$\mathcal{U}(\mathfrak{g}) = \bigoplus_{s\in S\sqcup\{0\}} \mathcal{U}_s(\mathfrak{g}),\ \text{where}\ \mathcal{U}_s(\mathfrak{g}) = {}^{T_s(\mathfrak{g})}/_{\mathcal{J}_s}. \tag{5.23}$$

Moreover, it is easily checked that $\mathcal{U}(\mathfrak{g})$ is a $S\sqcup\{0\}$-graded Hopf algebra i.e. that $\Delta_{\mathcal{U}}, \epsilon_{\mathcal{U}}$ and $1_{\mathcal{U}}:\ \mathbf{k}\to\mathcal{U}(\mathfrak{g})$ and the antipode $S_{\mathcal{U}}$ are graded morphisms ($\mathbf{k}$ being $S\sqcup\{0\}$-graded with $(\mathbf{k})_0=\mathbf{k}$ and $(\mathbf{k})_s=\{0_{\mathbf{k}}\}$ for $s\in S_+=S\setminus\{0_S\}$).

5.4.3 Iterated smash products and sup-gradings.

In order to formulate a theorem about iterated smash products, we start with $(A,<)$ a totally ordered alphabet. Let $S_A := \{A,\vee\}$ be the corresponding max-semigroup (i.e. $a\vee b=\max\{a,b\}$ for all $a,b\in A$) and $\mathfrak{g}=\bigoplus_{a\in A}\mathfrak{g}_a$ a S_A-graded Lie algebra (i.e. for all $a,b\in A$, $[\mathfrak{g}_a,\mathfrak{g}_b]\subseteq\mathfrak{g}_{\max\{a,b\}}$). Let us consider

1. the formal direct sum $M=\bigoplus_{a\in A}\mathcal{U}_+(\mathfrak{g}_a)$ (where $\mathcal{U}_+(\mathfrak{g}_a)$ is the augmentation ideal of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_a)$). In this module, all $\mathcal{U}_+(\mathfrak{g}_a)$ has degree α, i.e. $(M)_\alpha=\mathcal{U}_+(\mathfrak{g}_a)$.

2. the language of strictly increasing words $SI(A)\subset A^*$, formally

$$SI(A) := \{w\in A^*\mid \text{for all } j<|w|,\ w[j]<w[j+1]\}$$

where, for a word w and $1\le j\le|w|$, $w[j]$ is the letter of w at place j.

3. the decomposition $T(M)=\bigoplus_{w\in A^*} T_w(M)$

4. the space $T_{SI(A)} := \bigoplus_{w\in SI(A)} T_w(M)$ where $SI(A)\subset A^*$ is the language of strictly increasing words

5. the language of (weakly) increasing words $WI(A)\subset A^*$, formally

$$WI(A) := \{w\in A^*\mid \text{for all } 1\le j<|w|,\ w[j]\le w[j+1]\}.$$

The following theorem states two things. Firstly that $T_{SI(A)}$ is a section of the natural morphism $T(M) \to \mathcal{U}(\mathfrak{g})$ and secondly that rearranging the tensors in increasing form converges towards the projector on $T_{SI(A)}$ parallel to the kernel of the natural morphism. To this end, we must define what is "rearranging the tensors" and will use the structure of paths of computations through appropriate labeled graphs in the spirit of [48][12].
We need the following remark and definitions

Remark 5.3. Given $a, b \in A$ with $a < b$, by using the fact that $[\mathfrak{g}_a, \mathfrak{g}_b] \subset \mathfrak{g}_b$, then the mapping $\alpha : \ \mathfrak{g}_a \to \mathfrak{Der}(\mathfrak{g}_b), x \mapsto (\mathrm{ad}_x : y \mapsto [x, y])$ is a morphism in **k-Lie**. By Example 2.7, one has a left $\mathcal{U}(\mathfrak{g}_a)$-module algebra action $\triangleright : \mathcal{U}(\mathfrak{g}_a) \otimes \mathcal{U}(\mathfrak{g}_b) \to \mathcal{U}(\mathfrak{g}_b), x \otimes y \mapsto x \triangleright y = \alpha_{\mathcal{U}}(x)(y)$. Due to the proof of Lemma 2.26, the mapping $\tau : \mathcal{U}(\mathfrak{g}_a) \otimes \mathcal{U}(\mathfrak{g}_b) \to \mathcal{U}(\mathfrak{g}_b) \otimes \mathcal{U}(\mathfrak{g}_a), x \otimes y \mapsto \sum_{(1)(2)} x^{(1)} \triangleright y \otimes x^{(2)}$ is an algebra cross in **k-Mod**. Note that we have

$$\tau(\mathcal{U}_+(\mathfrak{g}_a) \otimes \mathcal{U}_+(\mathfrak{g}_b)) \subset \mathcal{U}_+(\mathfrak{g}_b) \otimes \mathcal{U}_+(\mathfrak{g}_a) + \mathcal{U}_+(\mathfrak{g}_b) \otimes 1_{\mathbf{k}}.$$

Thus, if we denote $\tau_0 = (S_a \otimes S_b) \circ \tau_{12} \circ \tau \circ \tau_{12} \circ (S_b \otimes S_a) : \mathcal{U}(\mathfrak{g}_b) \otimes \mathcal{U}(\mathfrak{g}_a) \to \mathcal{U}(\mathfrak{g}_a) \otimes \mathcal{U}(\mathfrak{g}_b)$, where τ_{12} is the standard twist map which interchanges the two factors in the tensor product and S_a (resp. S_b) is the antipode of the Hopf algebra $\mathcal{U}(\mathfrak{g}_a)$ (resp. $\mathcal{U}(\mathfrak{g}_b)$), then one has

$$\tau_0(\mathcal{U}_+(\mathfrak{g}_b) \otimes \mathcal{U}_+(\mathfrak{g}_a)) \subset \mathcal{U}_+(\mathfrak{g}_a) \otimes \mathcal{U}_+(\mathfrak{g}_b) + 1_{\mathbf{k}} \otimes \mathcal{U}_+(\mathfrak{g}_b).$$

We now have to build a transition structure similar to what is defined in [101] p.200 Fig 6.4, here the set of states will be infinite.

Definition 5.3. With the preceding notations, we define
The graph of transitions Γ_{trans}

(a) **Vertices**: All finite sets of words $2^{(A^*)}$.

(b) **Elementary Steps**: Their set will be noted ES. These steps are of three types: **First type** (Reduction of inversions) $\alpha = (\{ubav\}, \varphi_\alpha, \{uabv, ubv\})$ with $a < b$ and

$$\varphi_\alpha \ : \ x_u \otimes x_b \otimes x_a \otimes x_v \ \to \ x_u \otimes \tau_0(x_b \otimes x_a) \otimes x_v \qquad (5.24)$$

[12] For a modern version, see [49].

where τ_0 is the "twist" of the smash product (see Remark 5.3). It can be shown that

$$\tau_0(\mathcal{U}_+(\mathfrak{g}_b)\otimes\mathcal{U}_+(\mathfrak{g}_a))\subset\mathcal{U}_+(\mathfrak{g}_a)\otimes\mathcal{U}_+(\mathfrak{g}_b)+1_{\mathbf{k}}\otimes\mathcal{U}_+(\mathfrak{g}_b) \tag{5.25}$$

therefore the result of this reduction process belongs to $T_{uabv}(M)\oplus T_{ubv}(M)$.

Second type (Reduction of powers) $\alpha=(\{ua^pv\},\varphi_\alpha,\{uav\})$ with $p\geq 2$, by

$$\varphi_\alpha\ :\ x_u\otimes\overbrace{x_a^{(1)}\otimes\cdots\otimes x_a^{(p)}}^{p\text{ factors in }\mathcal{U}_+(\mathfrak{g}_a)}\otimes x_v\ \to\ x_u\otimes\underbrace{x_a^{(1)}\cdots x_a^{(p)}}_{\text{multiplication}}\otimes x_v \tag{5.26}$$

the result of this reduction process is in $T_{uav}(M)$.

Third type (Loops) $\alpha=(\{w\},\varphi_\alpha,\{w\})$ for $w\in SI(A)$ with $\varphi_\alpha=\mathrm{Id}_{T_w}$.

All the preceding (linear) maps φ_α (of first, second and third types) are extended by 0 outside of their definition domains ($T_{ubav}(M)$ for the first type $T_{ua^pv}(M)$ for the second and $T_w(M),\ w\in SI(A)$ for the third).

Summarizing, all φ_α belong to $\mathrm{End}(T(M))$.

(c) **General arrows** i.e. all arrows of Γ_{trans}. Their set is denoted GA. It is the set of triplets (F_1,Φ,F_2), with $F_i\in 2^{(A^*)}$, $\Phi\in 2^{(ES)}$ (finite sets of elementary steps) such that

   (a) for all $w\in F_1$ exists one and only one elementary step in $\alpha\in\Phi$ with $t(\alpha)=\{w\}$ (its tail).

   (b) $F_2=\cup_{\alpha\in\Phi}h(\alpha)$ (union of their heads).

(d) **Tail and Head**: For every general arrow $\alpha=(F_1,\Phi,F_2)$, we set $t(\alpha)=F_1$ and $h(\alpha)=F_2$. This definition is extended for elementary arrows by (for $\alpha=(F_1,\varphi,F_2)$) the same projections (i.e. $t(\alpha)=F_1$ and $h(\alpha)=F_2$).

(e) **Composition of Arrows**: Composition of (F_1,Φ_1,F_2) and (F_2,Φ_2,F_3) is $(F_1,\Phi_2\circ\Phi_1,F_3)$ where

$$\Phi_2\circ\Phi_1=\{pr_2(\beta)\circ pr_2(\alpha)\,|\,\beta\in\Phi_2,\alpha\in\Phi_1,t(\beta)\subseteq h(\alpha)\}.$$

(f) **Paths**: A path in Γ_{trans} is a word $P=\alpha_1\cdots\alpha_n\in GA^*$ such that, for all $j<|P|\,(=n)$, $h(\alpha_j)=t(\alpha_{j+1})$, we classically have $t(P)=t(\alpha_1)$ and $h(P)=h(\alpha_n)$.

The evaluation of P, $Ev(P)$ is the composition of all the linear maps of its arrows i.e. with $P = \alpha_1 \cdots \alpha_n$,

$$Ev(P) = pr_2(\alpha_n) \circ \cdots \circ pr_2(\alpha_1) \tag{5.27}$$

Norm: For all $w \in A^*$, we set $\mathrm{norm}(w) = 3^{(|w|+Inv(w))}$ (where $Inv(w) = \sharp\{(i,j)|1 \leq i < j \leq |w| \text{ and } w[i] > w[j]\}$). This definition is at once extended to finite subsets of $F \subset A^*$ by $\mathrm{norm}(F) = \sum_{w\in F} \mathrm{norm}(w)$. We remark that, for all elementary arrow α of the two first types, $\mathrm{norm}(t(\alpha)) > \mathrm{norm}(h(\alpha))$ and equality is got for the third type. Hence, for any general arrow $\alpha = (F_1, \Phi, F_2)$, $\mathrm{norm}(t(\alpha)) > \mathrm{norm}(h(\alpha))$ unless $F_1 = F_2 \subset SI(A)$ in which case we have equality and all arrows of Φ are of third type.

(g) **Aperiodic paths**: An aperiodic path is a path whose last arrow has identical head and tail i.e. $\alpha_n = (F, \Phi, F)$, this entails that $F \subset SI(A)$ and that all arrows of Φ are of third type.

(h) **Remark.** – Conditions (b.i) and (b.ii) above say respectively that there is no outgoing computation fork (i.e. two different elementary steps) from one $w \in F_1$ and that F_2 is the image of F_1 through the arrows of Φ.

Theorem 5.4. *We consider the canonical morphism defined by multiplication of factors*

$$\mathrm{can}: \; T(M) \to \mathcal{U}(\mathfrak{g}) \tag{5.28}$$

$$\textit{i.e. } \mathrm{can}: \; x_{a_1} \otimes \cdots \otimes x_{a_k} \mapsto x_{a_1} \cdots x_{a_k}.$$

Then, with the notations and constructions above, one has

1. *Every sufficiently long path in Γ_{trans} with origin F_1 is aperiodic and ends with a subset $\mathrm{F}_n \subset SI(A)$. More precisely, let F_1 be a finite subset of A^* (i.e. $F_1 \in 2^{(A^*)}$) and*

$$N = norm(F_1) = \sum_{w\in F_1} 2^{(Inv(w)+|w|)}$$

(where, again, $Inv(w) = \sharp\{(i,j)|1 \leq i < j \leq |w|$ and $w[i] > w[j]\}$).

We consider a path of Γ_{trans} originating from F_1

$$\mathrm{F}_1 \xrightarrow{\Phi_1} \mathrm{F}_2 \xrightarrow{\Phi_2} \cdots \qquad\qquad \cdots \xrightarrow{\Phi_{n-1}} \mathrm{F}_n \xrightarrow{\Phi_n} \mathrm{F}_{n+1} \tag{5.29}$$

Then, if $n > N+1$,

i) This "sufficiently long" path (5.29) *is aperiodic.*
This entails $\mathrm{F}_{n+1} = \mathrm{F}_n \subset SI(A)$ *and* $\Phi_n \subset ES_3$ *(where* $ES_3 \subset ES$ *is the set of elementary steps of the third type).*

ii) The canonical morphism, can, *restricted to* $T_{SI(A)}$, *is onto and*

$$T(M) = T_{SI(A)} \oplus \mathrm{Ker}(\mathrm{can}) \tag{5.30}$$

in other words $T_{SI(A)} := \bigoplus_{w\in SI(A)} T_w(M)$ *is a section of* can.

2. *Let us call* **proj**, *the projection on* $T_{SI(A)}$ *parallel to* $\mathrm{Ker}(\mathrm{can})$ *and, for all* $\mathrm{F} \subset A^*$, *let us denote by* $\mathbf{proj}|_F$ *the restriction* $\mathbf{proj}|_{\oplus_{w\in F} T_w(M)}$, *then*

   i) For any chain of tensors $(t_1, \cdots, t_n)$ *such that* $supp(t_n) \subset SI(A)$ *and such that, for all* $1 \le j < n$, $\mathrm{can}(t_j) = \mathrm{can}(t_{j+1})$, *we have* $t_n = \mathbf{proj}(t_1)$.

   ii) The evaluation[13] *of the path* (5.29) *is* $\mathbf{proj}_{F_1}$.

Proof. 1. i) It suffices to remark that, for every step (F, Φ, F')

1. either $\Phi \not\subset ES_3$ and $\mathrm{norm}(F') < \mathrm{norm}(F)$
2. or $\Phi \subset ES_3$ and then
   (a) $F = F' \subset SI(A)$
   (b) any further step of the path is a loop.

ii) As, in the preceding preceding point, the steps are preserving can-evaluation i.e. for any for $1 \le j < n$, step $F_j \xhookrightarrow{\Phi_j} F_{j+1}$ and tensor $t \in \bigoplus_{w\in F_j} T_w(M)$, we have $\mathrm{can}(t) = \mathrm{can}(\sum_{\varphi\in\Phi_j} \varphi(t))$, by composition, we get that for the "sufficiently long" path $P =$(5.29) and $t \in \bigoplus_{w\in F_1} T_w(M)$, $\mathrm{can}(t) = \mathrm{can}(\sum_{\varphi\in Ev(P)} \varphi(t))$. As $F_n \subset SI(A)$, we can use the following

Lemma 5.5. *Let* $f : M \to N$ *be a linear morphism of two modules. We suppose given* $(M_i)_{i\in I}$ *a directed family of submodules (of* M*) such that, setting* $M_\infty := \bigcup_{i\in I} M_i$ *the submodule of* M,

[13] See the equation (5.27).

(a) For all $i \in I$, $f|_{M_i}$ is injective.

(b) $f|_{M_\infty}$ is onto.

Then $M = M_\infty \oplus \mathrm{Ker}(f)$.

and one gets the result.

2. i) Condition $\mathrm{can}(t_j) = \mathrm{can}(t_{j+1})$ means that, for every step $t_j - t_{j+1} \in \mathrm{Ker}(\mathrm{can})$, then $\mathbf{proj}(t_j) = \mathbf{proj}(t_{j+1})$. Hence $\mathbf{proj}(t_1) = \mathbf{proj}(t_n) = t_n$.

ii) Is a direct consequence of (2.i) above. □

Example 5.4. A computation scheme, starting from 3222 with a swap episode followed by reduction of powers, is as follows (note that last step is aperiodic with $F = \{23, 3\}$)

$$\begin{tikzcd}
3222 \arrow[r] \arrow[dr] & 2322 \arrow[r] \arrow[dr] & 2232 \arrow[r] \arrow[dr] & 2223 \arrow[dr] & & \\
& 322 \arrow[r] \arrow[dr] & 232 \arrow[r] \arrow[dr] & 223 \arrow[r] & 23 \arrow[r] & 23 \\
& & 32 \arrow[r] \arrow[dr] & 23 \arrow[ur] & & \\
& & & 3 \arrow[r] & 3 \arrow[r] & 3
\end{tikzcd} \tag{5.31}$$

Remark 5.4. i) Any computation

$$t_1 \to t_2 \to \cdots \to t_n$$

such that, for all $1 \le j < n$, $t_j - t_{j+1} \in \mathrm{Ker}(\mathrm{can})$ and $\mathrm{support}(t_n) \in SI(A)$ gives the same result (from t_1) which is $\mathbf{proj}(t_1)$.

ii) As a consequence of Theorem 5.4 (1.ii), in case when $\mathfrak{g}$ admits a A-graded linear basis (then as a A-graded module, the original theorem is more general), the canonical morphism $\mathrm{can} : T(M) \to \mathcal{U}(\mathfrak{g})$ preserves a linear basis of the tensor subspace $T_{SI(A)} = \bigoplus_{w \in SI(A)} T_w(M)$ (recall that $SI(A) \subset A^*$ is the language of strictly increasing words) to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

We now pass to the following application of the graded structure (5.23) of $S \sqcup \{0\}$-graded Hopf algebras $\mathcal{U}(\mathfrak{g})$ in case $\mathfrak{g}$ being the infinite Drinfeld-Kohno Lie algebra $\mathrm{DK}_{\mathbf{k},\infty}$ (for the definition of this Lie algebra, see Proposition 2.21) and moreover we have studied that $\mathrm{DK}_{\mathbf{k},\infty} = \bigoplus_{m \ge 2} \mathrm{DK}_{\mathbf{k},\infty}^{(m)}$ is a $(\mathbb{N}_{\ge 2}, \vee)$-graded Lie algebra over $\mathbf{k}$ (see the formula (2.110)).

Let us recall that $\mathcal{T}_\infty = \{t_{i,j}\}_{1\le i<j<+\infty}$ be a set of non commutative variables and then, the universal enveloping algebra of the infinite Drinfeld-Kohno Lie algebra is presented as

$$\mathcal{U}(\mathrm{DK}_{\mathbf{k},\infty}) = \langle\, \mathcal{T}_\infty \,|\, \mathbf{R}[\infty]\,\rangle_{\mathbf{k}\text{-}\mathbf{AAU}} = {}^{\mathbf{k}\langle\mathcal{T}_\infty\rangle}\Big/_{\mathcal{J}_{\mathbf{R}[\infty]}}$$

which is the quotient of the free associative algebra $\mathbf{k}\langle\mathcal{T}_\infty\rangle$ modulo the ideal $\mathcal{J}_{\mathbf{R}[\infty]}$ generated by infinitesimal pure braid relations $\mathbf{R}[\infty] = \mathbf{R_1}[\infty]\cup\mathbf{R_2}[\infty]\cup\mathbf{R_3}[\infty]$ (2.107), where

- $\mathbf{R_1}[\infty] : [t_{i,j}, t_{i,k} + t_{j,k}]$ for $1 \le i < j < k < +\infty$,
- $\mathbf{R_2}[\infty] : [t_{i,j} + t_{i,k}, t_{j,k}]$ for $1 \le i < j < k < +\infty$,
- $\mathbf{R_3}[\infty] : [t_{i,j}, t_{k,l}]$ for $1 \le i < j < +\infty, 1 \le k < l < +\infty$ and $|\{i, j, k, l\}| = 4$.

Definition 5.4. i) We consider the natural projection $s_\infty :\ \mathbf{k}\langle\mathcal{T}_\infty\rangle \to \mathcal{U}(\mathrm{DK}_{\mathbf{k},\infty})$ and, for convenience, for all polynomials $P, Q \in \mathbf{k}\langle\mathcal{T}_\infty\rangle$, we will denote by $P \equiv Q$ the fact that $s_\infty(P) = s_\infty(Q)$ i.e. $P - Q \in \mathcal{J}_{\mathbf{R}[\infty]}$.

ii) A pair of adjacent letters $t_{i,j}t_{k,l}$ is called *an inversion* if $j > l$.

iii) A word of form $t_{i_1,j_1}t_{i_2,j_2}\ldots t_{i_n,j_n} \in \mathcal{T}_\infty^*$, for $n \ge 0$ and $2 \le j_1 \le j_2 \le \ldots \le j_n$ (it means it has no inversion), is called in *normal form*, and the set of all such words named $Inc(\infty)$.

Due to this definition, we can expand the above relations as follows:

1. $\mathbf{R_1}[\infty]$: For $1 \le i < j < k < +\infty$,

$$\begin{array}{rrcl} & [t_{i,j}, t_{i,k} + t_{j,k}] & \equiv & 0 \\ \iff & t_{i,j}t_{i,k} - t_{i,k}t_{i,j} + t_{i,j}t_{j,k} - t_{j,k}t_{i,j} & \equiv & 0 \\ \iff & t_{i,k}t_{i,j} & \equiv & t_{i,j}t_{i,k} + t_{i,j}t_{j,k} - t_{j,k}t_{i,j}. \end{array} \tag{5.32}$$

Observing the positions of the letters on the table

$$\mathcal{T}_\infty = \begin{array}{cccccccc} T_2 & \dots & T_j & T_{j+1} & \dots & T_k & & \dots \\ \hline t_{1,2} & \dots & t_{1,j} & t_{1,j+1} & \dots & t_{1,k} & & \dots \\ & \dots & \dots & \dots & \dots & \dots & & \dots \\ & \dots & t_{i,j} & t_{i,j+1} & \dots & t_{i,k} & & \dots \\ & & \dots & \dots & \dots & \dots & & \dots \\ & & & t_{j,j+1} & \dots & t_{j,k} & & \dots \\ & & & & \dots & t_{j+1,k} & & \dots \end{array}$$

we can model the last equality of (5.32) as follows

$$\longleftarrow \equiv \longrightarrow + \searrow - \nwarrow . \qquad (I_1)$$

2. $\mathbf{R_2}[\infty]$: For $1 \leq i < j < k < +\infty$. Similarly as above, we have

$$\begin{array}{rrcl} & [t_{i,j} + t_{i,k}, t_{j,k}] & \equiv & 0 \\ \Longleftrightarrow & t_{i,j}t_{j,k} - t_{j,k}t_{i,j} + t_{i,k}t_{j,k} - t_{j,k}t_{i,k} & \equiv & 0 \\ \Longleftrightarrow & t_{j,k}t_{i,j} & \equiv & t_{i,j}t_{j,k} + t_{i,k}t_{j,k} - t_{j,k}t_{i,k} \end{array} \qquad (5.33)$$

and then

$$\nwarrow \equiv \searrow + \downarrow - \uparrow . \qquad (I_2)$$

Replacing (I_2) to (I_1) we have that

$$\longleftarrow \equiv \longrightarrow + \uparrow - \downarrow . \qquad (I_1')$$

3. $\mathbf{R_3}[\infty]$: For the case $|\{i, j, k, l\}| = 4$,

$$\begin{array}{rrcl} & [t_{i,j}, t_{k,l}] & \equiv & 0 \\ \Leftrightarrow & t_{i,j}t_{k,l} & \equiv & t_{k,l}t_{i,j} \end{array} \qquad (5.34)$$

we can model as

$$\nwarrow \equiv \searrow \quad \mathit{or} \quad \swarrow \equiv \nearrow . \qquad (I_3)$$

Note that moving $\searrow$, $\nwarrow$ between two positions having row-index of this position equal to column-index of the other position, but $\searrow$, $\nwarrow$ having different indices one by one. The three modeling equalities (I_1'), (I_2) and (I_3) help to represent any word in combination of normal forms. Firstly we have the following lemma.

Lemma 5.6. *Let w be a word having inversion $t_{i,j}t_{k,l}$ (i.e. $w = w_1 t_{i,j} t_{k,l} w_2$ such that $j > l$). We can represent this inversion part (only the inverted factor) by a combination of normal forms.*

Proof. In this case $k < l < j$. We then have the following cases.

- If $|\{i,j,k,l\}| = 4$, by (I_3) we have $w \equiv w_1 t_{k,l} t_{i,j} w_2$.
- If $|\{i,j,k,l\}| = 3$, it is one in the two subcases:
  - if $i = k$, it means $w = w_1 t_{i,j} t_{i,l} w_2$. By (I_1') we have $w \equiv w_1 t_{i,l} t_{i,j} w_2 + w_1 t_{l,j} t_{i,j} w_2 - w_1 t_{i,j} t_{l,j} w_2$.
  - if $i = l$, it means $w = w_1 t_{i,j} t_{k,i} w_2$. By (I_2) (note that $k < i < j$) we have $w \equiv w_1 t_{k,i} t_{i,j} w_2 + w_1 t_{k,j} t_{i,j} w_2 - w_1 t_{i,j} t_{k,j} w_2$.

$\square$

We have the following simple examples.

Example 5.5. i) For $w = t_{1,3}t_{1,2}t_{2,5}t_{3,4} \in \mathcal{T}_\infty^*$, then

$$\begin{aligned} w &\equiv \overbrace{t_{1,3}t_{1,2}}^{\text{inversion}} t_{2,5}t_{3,4} \\ &\overset{I_1'}{\equiv} (t_{1,2}t_{1,3} + t_{2,3}t_{1,3} - t_{1,3}t_{2,3})t_{2,5}t_{3,4} \\ &\equiv t_{1,2}t_{1,3}\overbrace{t_{2,5}t_{3,4}}^{\text{inversion}} + t_{2,3}t_{1,3}\overbrace{t_{2,5}t_{3,4}}^{\text{inversion}} - t_{1,3}t_{2,3}\overbrace{t_{2,5}t_{3,4}}^{\text{inversion}} \\ &\overset{I_3}{\equiv} t_{1,2}t_{1,3}t_{3,4}t_{2,5} + t_{2,3}t_{1,3}t_{3,4}t_{2,5} - t_{1,3}t_{2,3}t_{3,4}t_{2,5}. \end{aligned}$$

ii) For $w = t_{2,3}t_{1,2}t_{2,5}t_{3,4} \in \mathcal{T}_\infty^*$, then

$$\begin{aligned} w &= \overbrace{t_{2,3}t_{1,2}}^{\text{inversion}} t_{2,5}t_{3,4} \\ &\overset{(I_2)}{\equiv} (t_{1,2}t_{2,3} + t_{1,3}t_{2,3} - t_{2,3}t_{1,3})t_{2,5}t_{3,4} \end{aligned}$$

$$\begin{aligned} &= t_{1,2}t_{2,3}\overbrace{t_{2,5}t_{3,4}}^{\text{inversion}} + t_{1,3}t_{2,3}\overbrace{t_{2,5}t_{3,4}}^{\text{inversion}} - t_{2,3}t_{1,3}\overbrace{t_{2,5}t_{3,4}}^{\text{inversion}} \\ &\overset{(I_3)}{\equiv} t_{1,2}t_{2,3}t_{3,4}t_{2,5} + t_{1,3}t_{2,3}t_{3,4}t_{2,5} - t_{2,3}t_{1,3}t_{3,4}t_{2,5}. \end{aligned}$$

Note that Lemma 5.6 can be used to remove an inversion between some two successive positions of a word w over $\mathcal{T}_\infty$ but, sometimes solving this inversion can make a new inversion, however this procedure terminates. In fact, at this stage we are going to study a linear basis in the universal enveloping algebra of the infinite Drinfeld-Kohno Lie algebra $\mathrm{DK}_{\mathbf{k},\infty}$.

Proposition 5.7. *The image of the set of normal forms, $s_\infty(Inc(\infty))$, is a $\mathbf{k}$-linear basis of the associative $\mathbf{k}$-algebra $\mathcal{U}(\mathrm{DK}_{\mathbf{k},\infty})$.*

Proof. This proposition is a direct application of Remark 5.4 (ii) to the alphabet $A = \mathbb{N}_{\geq 2}$ and the $(\mathbb{N}_{\geq 2}, \vee)$-graded Lie algebra $\mathrm{DK}_{\mathbf{k},\infty} = \bigoplus_{m\geq 2} \mathrm{DK}^{(m)}_{\mathbf{k},\infty}$ (2.110) (we remark that each augmentation ideal $\mathcal{U}_+(\mathrm{DK}^{(m)}_{\mathbf{k},\infty}) \cong \mathbf{k}_+\langle T_m\rangle$ in **k-AA**). $\square$

We now arrive to write an algorithm to implement that any element $w = t_{i_1,j_1}\dots t_{i_n,j_n}$ of $\mathcal{T}^*_\infty$ can be expressed as a linear combination of normal forms as follows.

Algorithm 1 Algorithm represents a word in form of combination of words in $s_\infty(Inc(\infty))$

Input: A word $w = t_{i_1,j_1}\dots t_{i_{n-1},j_{n-1}}t_{i_n,j_n} \in \mathcal{T}^*_\infty$

Output: Representation of w by combination of elements in $s_\infty(Inc(\infty))$

Assign $C := w = t^1 \dots t^n$ (re-indexed for more simplicity)

for k form 2 to n **do**

  m=k

  for u in C **do**

    while $m > 1$ and $t^{m-1}t^m$ is an inversion **do**

      Dispose the two letters due to (I_1'), (I_2) and (I_3) and then update to C

      $m := m - 1$

    end while

  end for

end for

Example 5.6. The following are obtained by program Algorithm 1 on Maple :

$$\begin{aligned}
t_{1,3}t_{1,3}t_{1,3}t_{1,2}t_{1,2} \equiv\ & -2t_{1,2}t_{1,3}t_{1,3}t_{1,3}t_{2,3} - 2t_{2,3}t_{1,3}t_{1,3}t_{1,3}t_{2,3} \\
& + t_{1,3}t_{1,3}t_{1,3}t_{2,3}t_{2,3} - t_{1,3}t_{1,3}t_{1,3}t_{1,3}t_{2,3} \\
& + t_{1,3}t_{1,3}t_{1,3}t_{2,3}t_{1,3} + 2t_{1,2}t_{2,3}t_{1,3}t_{1,3}t_{1,3} \\
& + t_{1,3}t_{2,3}t_{1,3}t_{1,3}t_{1,3} - t_{2,3}t_{1,3}t_{1,3}t_{1,3}t_{1,3} \\
& + t_{2,3}t_{2,3}t_{1,3}t_{1,3}t_{1,3} + t_{1,2}t_{1,2}t_{1,3}t_{1,3}t_{1,3}; \\
t_{4,6}t_{7,9}t_{6,7}t_{3,6}t_{8,9} \equiv\ & -t_{4,6}t_{6,7}t_{3,7}t_{7,9}t_{8,9} + t_{4,6}t_{3,7}t_{6,7}t_{7,9}t_{8,9} + t_{4,6}t_{3,6}t_{6,7}t_{7,9}t_{8,9} \\
& - t_{4,6}t_{6,9}t_{3,9}t_{7,9}t_{8,9} + t_{4,6}t_{3,9}t_{6,9}t_{7,9}t_{8,9} + t_{4,6}t_{3,6}t_{6,9}t_{7,9}t_{8,9} \\
& + t_{4,6}t_{7,9}t_{6,9}t_{3,9}t_{8,9} - t_{4,6}t_{7,9}t_{3,9}t_{6,9}t_{8,9} - t_{4,6}t_{3,6}t_{7,9}t_{6,9}t_{8,9}.
\end{aligned}$$

5.4.4 Hopf structures of the k-module of noncommutative polynomials.

We first review in [94] §1.5 that the $\mathbf{k}$-module $\mathbf{k}\langle X\rangle$ of the noncommutative polynomials has two natural graded Hopf algebra structures which are dual to each other (in the graded sense).

The first one is $\mathcal{H}_{\mathrm{conc}}(X) = (\mathbf{k}\langle X\rangle, \mathrm{conc}, 1_{X^*}, \Delta_{⧢}, \epsilon)$, where the cocommutative coproduct $\Delta_{⧢}$ (so called the co-shuffle coproduct) defined as

$$\Delta_{⧢} : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle X\rangle \otimes \mathbf{k}\langle X\rangle \tag{5.35}$$

which is a unique algebra homomorphism for which the words $x \in X$, are primitive elements

$$\Delta_{⧢}(x) = x \otimes 1_{X^*} + 1_{X^*} \otimes x. \tag{5.36}$$

More precisely, for any polynomial T,

$$\Delta_{⧢}(T) = \sum_{u,v \in X^*} \langle T \mid u ⧢ v\rangle u \otimes v.$$

The counit $\epsilon : \mathbf{k}\langle X\rangle \to \mathbf{k}$ is given by $\epsilon(T) = \langle T \mid 1_{X^*}\rangle$ for all polynomials T. The antipode $S : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle X\rangle$ is the $\mathbf{k}$-linear map defined, over words $w \in X^*$, by

$S(w) = (-1)^{|w|}\widetilde{w}$, where $\widetilde{w}$ reverses the order of letters in the word w.

We now state to the following remarks

Remark 5.5. i) We recall that the universal enveloping algebra of a Lie $\mathbf{k}$-algebra $\mathfrak{g}$ is a pair $(\sigma, \mathcal{U}(\mathfrak{g}))$, where $\mathcal{U}(\mathfrak{g})$ is an object in **k-AAU** and $\sigma : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is a morphism in **k-Lie**, which is a solution of the following universal problem:

$$\begin{tikzcd} \mathbf{k\text{-}Lie} & & \mathbf{k\text{-}AAU} \arrow[ll, dotted, "F"'] \\ \mathfrak{g} \arrow[rr, dashed, "f"] \arrow[drr, dashed, "\sigma"'] & & \mathcal{A} \\ & & \mathcal{U}(\mathfrak{g}). \arrow[u, "\widehat{f}"'] \end{tikzcd} \tag{5.37}$$

This arises that there exists the universal enveloping functor

$$\mathcal{U} : \mathbf{k\text{-}Lie} \to \mathbf{k\text{-}AAU}, \qquad \mathfrak{g} \longmapsto \mathcal{U}(\mathfrak{g}) \tag{5.38}$$

which is a left-adjoint to the Liezation functor F.

ii) We also recall a noncommutative cocommutative Hopf structure of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie $\mathbf{k}$-algebra $\mathfrak{g}$. We construct a Lie algebra morphism

$$\delta : \mathfrak{g} \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$$

by the formula

$$\delta(x) = x \otimes 1_{\mathbf{k}} + 1_{\mathbf{k}} \otimes x, \tag{5.39}$$

for any $x \in \mathfrak{g}$. By the universal property (5.37), δ extends to a unique algebra homomorphism

$$\Delta_{\mathcal{U}} : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}). \tag{5.40}$$

Moreover, there exists a $\mathbf{k}$-linear map (called the antipode) $S_{\mathcal{U}} : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ characterized by $S_{\mathcal{U}}(x) = -x$ for any $x \in \mathfrak{g}$, and an algebra homomorphism (that is the counit) $\epsilon_{\mathcal{U}} : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(0) = \mathbf{k}$ induced from the projection $\mathfrak{g} \to 0$ (where 0 is the trivial Lie algebra). The reader can easily verify that $(\mathcal{U}(\mathfrak{g}), \mu_{\mathcal{U}}, 1_{\mathbf{k}}, \Delta_{\mathcal{U}}, \epsilon_{\mathcal{U}})$ satisfying the axioms of a Hopf algebra.

Remark 5.6. With the notations mentioned as above, we have

i) In a general Hopf algebra $\mathcal{H}$ with the coproduct $\Delta_{\mathcal{H}}$, we call an element $a \in \mathcal{H}$ is *primitive* if $\Delta_{\mathcal{H}}(a) = a \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes a$. The set of all primitive elements in $\mathcal{H}$ forms a Lie algebra (with a usual Lie bracket $[x, y] = xy - yx$) denoted by $\operatorname{Prim}\mathcal{H}$.

ii) In the case $\mathcal{H} = \mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of a Lie $\mathbf{k}$-algebra $\mathfrak{g}$, by the above Hopf structure of $\mathcal{U}(\mathfrak{g})$ one observes that any element in $\mathfrak{g}$ is primitive i.e. $\mathfrak{g} \subseteq \operatorname{Prim}\mathcal{U}(\mathfrak{g})$ (and coincides if $\mathbf{k}$ is a field of characteristic zero, cf. Bourbaki [13] Ch II §1.5 Corollary to Prop 9 or Cartier [21] Thm 3.6.1).

iii) Let us consider now from the first part of Subsection 1.2.5, it provides that the free Lie algebra $\mathcal{L}_{\mathbf{k}}(X)$ is the Lie subalgebra of $\mathbf{k}\langle X\rangle$ generated by X. Together with its canonical map $j : \mathcal{L}_{\mathbf{k}}(X) \hookrightarrow \mathbf{k}\langle X\rangle$, it is not hard to see that the pair $(j, \mathbf{k}\langle X\rangle)$ is a solution of the universal problem (5.37) in the above remark. We thus deduce that the free associative algebra $\mathbf{k}\langle X\rangle$ is the universal enveloping algebra $\mathcal{U}(\mathcal{L}_{\mathbf{k}}(X))$ of the free Lie algebra $\mathcal{L}_{\mathbf{k}}(X)$, these algebras are inherited the same Hopf structure $\mathcal{H}_{\mathrm{conc}}(X) = (\mathbf{k}\langle X\rangle, \mathrm{conc}, 1_{X^*}, \Delta_{\sqcup\!\sqcup}, \epsilon)$ (cf. Bourbaki [13] Ch II §3.1 Cor 1). Then, it is used to obtain the embedding $\mathcal{L}_{\mathbf{k}}(X) \subseteq \operatorname{Prim}\mathcal{H}_{\mathrm{conc}}(X)$ (they are equal if $\mathbf{k}$ is a field of characteristic zero, cf. Bourbaki [13] Ch II §3.1 Cor 2).

The second Hopf algebra structure will be the graded dual of $\mathcal{H}_{\mathrm{conc}}(X)$, namely $\mathcal{H}_{\sqcup\!\sqcup}(X) = (\mathbf{k}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}, \Delta_{\mathrm{conc}}, \epsilon)$. That is a commutative graded Hopf algeba with the shuffle product and the deconcatenation coproduct $\Delta_{\mathrm{conc}} : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle X\rangle \otimes \mathbf{k}\langle X\rangle$ given by, over the words $w \in X^*$,

$$\Delta_{\mathrm{conc}}(w) = \sum_{w=uv} u \otimes v. \tag{5.41}$$

More precisely, for any polynomial T, we can write

$$\Delta_{\mathrm{conc}}(T) = \sum_{u,v \in X^*} \langle T \mid uv\rangle u \otimes v.$$

The counit and the antipode are respectively given by ϵ and S as previously.

5.4.5 Zinbiel bialgebra and its dualisation.

According to [18], Burgunder constructed completely Zinbiel bialgebras and its dualisation in case the left ($\prec$) Zinbiel product. In view of later developments, we will similarly study the fundamental results of Burgunder's framework in case the right ($\succ$) Zinbiel product with these notations translate into $x \prec y := y \succ x$.

We recall in Definition 3.2 that a $\mathbf{k}$-module $\mathcal{A}$ over $\mathbf{k}$ a unital commutative ring equipped with a bilinear map $\succ$ is called (right) Zinbiel algebra (or dual Leibniz algebra) if it satisfies the identity

$$x \succ (y \succ z) = (x \succ y) \succ z + (y \succ x) \succ z, \text{ for all } x, y, z \in \mathcal{A}. \tag{5.42}$$

The behavior of $\succ$ with respect to the unit $1_{\mathcal{A}} \in \mathcal{A}$ is given by (for any $1_{\mathcal{A}} \neq x \in \mathcal{A}$) $1_{\mathcal{A}} \succ x = x, x \succ 1_{\mathcal{A}} = 0$ and $1_{\mathcal{A}} \succ 1_{\mathcal{A}}$ is not define. The corresponding algebra $(\mathcal{A}, *, 1_{\mathcal{A}})$ is the unital commutative associative algebra obtained by the symmetrized product

$$x * y = x \succ y + y \succ x. \tag{5.43}$$

The tensor product of a Zinbiel algebra $(\mathcal{A}, \succ)$ with itself will be a Zinbiel algebra equipped with following Zinbiel structure:

$$\succ : \ (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A}) \otimes (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A}) \to \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A}, \tag{5.44}$$

where $(x_1 \otimes_{\mathbf{k}} x_2) \succ (y_1 \otimes_{\mathbf{k}} y_2) = (x_1 \succ y_1) \otimes_{\mathbf{k}} (x_2 * y_2)$. In particular, for unital setting,

$$(1_{\mathcal{A}} \otimes_{\mathbf{k}} x) \succ (1_{\mathcal{A}} \otimes_{\mathbf{k}} y) := 1_{\mathcal{A}} \otimes_{\mathbf{k}} (x \succ y).$$

Definition 5.5. In the unital frame work, an As^c-$Zinb$-bialgebra, or a *Zinbiel bialgebra* is a (right) Zinbiel algebra with unit $(\mathcal{Z}, \succ, 1_{\mathcal{Z}})$ endowed with a structure of (counitary coassociative) coalgebra $\Delta : \mathcal{Z} \to \mathcal{Z} \otimes_{\mathbf{k}} \mathcal{Z},\ e :\ \mathcal{Z} \to \mathbf{k}$, whose compatibility relation is unital semi-Hopf

$$\Delta(x \succ y) \quad = \quad \Delta(x) \succ \Delta(y).$$

More precisely,

$$\Delta(x) \succ \Delta(y) \quad = \quad (x^{(1)} \otimes_{\mathbf{k}} x^{(2)}) \succ (y^{(1)} \otimes_{\mathbf{k}} y^{(2)})$$

$$= \quad (x^{(1)} \succ y^{(1)}) \otimes_{\mathbf{k}} (x^{(2)} * y^{(2)}),$$

here we used a shorthand Sweedler's notation $\Delta(x) = \sum_{(1)(2)} x^{(1)} \otimes_{\mathbf{k}} x^{(2)} \equiv x^{(1)} \otimes_{\mathbf{k}} x^{(2)}$, for any $x \in \mathcal{Z}$ (this is similar to Einstein's summation convention in mathematical physics, where the dummy summation index (1)(2) is dropped).

Example 5.7. $\mathcal{Z}_{\overset{r}{\sqcup\!\sqcup}}(X) := (\mathbf{k}\langle X\rangle, \overset{r}{\sqcup\!\sqcup}, 1_{X^*}, \Delta_{\text{conc}}, \epsilon)$ is a Zinbiel bialgebra, where the coproduct Δ_{conc} and the counit ϵ were studied in Appendix 5.4.4 above and the unital semi-Hopf relation $\Delta_{\text{conc}}(x \overset{r}{\sqcup\!\sqcup} y) = \Delta_{\text{conc}}(x) \overset{r}{\sqcup\!\sqcup} \Delta_{\text{conc}}(y)$.

We now study a dualisation of the above Zinbiel bialgebra.

Definition 5.6. A *(right) Zinbiel coalgebra* is a $\mathbf{k}$-module $\mathcal{C}$ equipped with a cooperation

$$\Delta_{\succ} : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$$

satisfying

$$(\mathrm{Id} \otimes \Delta_{\succ}) \circ \Delta_{\succ} = (\Delta_{\succ} \otimes \mathrm{Id}) \circ \Delta_{\succ} + (\tau\Delta_{\succ} \otimes \mathrm{Id}) \circ \Delta_{\succ},$$

here $\tau : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ is the standard twist map which interchanges the two factors. A (right) Zinbiel coalgebra is said to be counital if it admits a linear map $e : \mathcal{C} \to \mathbf{k}$ which satisfies

$$(e \otimes \mathrm{Id}) \circ \Delta_{\succ} = \mathrm{Id},$$
$$(\mathrm{Id} \otimes e) \circ \Delta_{\succ} = 0,$$

note that $(e \otimes e) \circ \Delta_{\succ}$ is not define. This notion is dual to the notion of (right) Zinbiel algebra in Definition 3.2. Moreover, the cooperation $\Delta_* := \tau\Delta_{\succ} + \Delta_{\succ} : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ is coassociative cocommutative and counital.

Definition 5.7. In the counital frame work, a $Zinb^c$-As-bialgebra is a unital associative algebra $(\mathcal{Z}, \mu, 1_{\mathcal{Z}})$ endowed with a structure of a counital (right) Zinbiel coalgebra $(\mathcal{Z}, \Delta_{\succ}, e)$ verifying the following compatible relation

$$\Delta_{\succ} \circ \mu = (\mu \otimes \mu) \circ (\mathrm{Id} \otimes \tau \otimes \mathrm{Id}) \circ (\Delta_* \otimes \Delta_{\succ}).$$

Let us consider the co-half-shuffle coproduct $\Delta_{\overset{r}{\sqcup\!\sqcup}} : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle X\rangle \otimes \mathbf{k}\langle X\rangle$ which is a $\mathbf{k}$-linear map defined over words as follows

$$
\begin{aligned}
\Delta_{\overset{r}{\sqcup\!\sqcup}}(1_{X^*}) &= 1_{X^*} \otimes 1_{X^*},\\
\Delta_{\overset{r}{\sqcup\!\sqcup}}(x) &= 1_{X^*} \otimes x, x \in X,\\
\Delta_{\overset{r}{\sqcup\!\sqcup}}(x_1x_2) &= (x_1 \otimes 1_{X^*} + 1_{X^*} \otimes x_1)(1 \otimes x_2), x_1, x_2 \in X,\\
&\ \ \cdots\\
\Delta_{\overset{r}{\sqcup\!\sqcup}}(x_1 \cdots x_n) &= \Delta_{\sqcup\!\sqcup}(x_1 \cdots x_{n-1})\Delta_{\overset{r}{\sqcup\!\sqcup}}(x_n), x_1 \cdots x_n \in X^*,
\end{aligned}
$$

or, equivalently, for each polynomial T such that $\langle T \mid 1_{X^*}\rangle = 0$, then

$$\Delta_{\overset{r}{\sqcup\!\sqcup}}(T) = \sum_{u,v \in X^*} \langle T \mid u \overset{r}{\sqcup\!\sqcup} v\rangle u \otimes v, \tag{5.45}$$

here we used the notation $\langle w \mid 1_{X^*} \overset{r}{\sqcup\!\sqcup} 1_{X^*}\rangle = 0$ for any non empty word w.

Example 5.8. We have the following

i) It is easy to verify that $(\mathrm{Id} \otimes \Delta_{\overset{r}{\sqcup\!\sqcup}}) \circ \Delta_{\overset{r}{\sqcup\!\sqcup}} = (\Delta_{\overset{r}{\sqcup\!\sqcup}} \otimes \mathrm{Id}) \circ \Delta_{\overset{r}{\sqcup\!\sqcup}} + (\tau\Delta_{\overset{r}{\sqcup\!\sqcup}} \otimes \mathrm{Id}) \circ \Delta_{\overset{r}{\sqcup\!\sqcup}}$ and

$$(\epsilon \otimes \mathrm{Id}) \circ \Delta_{\overset{r}{\sqcup\!\sqcup}} = \mathrm{Id},$$
$$(\mathrm{Id} \otimes \epsilon) \circ \Delta_{\overset{r}{\sqcup\!\sqcup}} = 0.$$

Thus, $(\mathbf{k}\langle X\rangle, \Delta_{\overset{r}{\sqcup\!\sqcup}}, \epsilon)$ is a counital (right) Zinbiel coalgebra. Furthermore, the cooperation $\Delta_* = \tau\Delta_{\overset{r}{\sqcup\!\sqcup}} + \Delta_{\overset{r}{\sqcup\!\sqcup}} = \Delta_{\sqcup\!\sqcup}$.

ii) The $\mathbf{k}$-module $\mathbf{k}\langle X\rangle$ of noncommutative polynomials on X has two natural generalized graded bialgebra type structures which are dual to each other (in the graded sense). The first one is the Zinbiel bialgebra (or As^c-$Zinb$-bialgebra) $\mathcal{Z}_{\overset{r}{\sqcup\!\sqcup}}(X) = (\mathbf{k}\langle X\rangle, \overset{r}{\sqcup\!\sqcup}, 1_{X^*}, \Delta_{\mathrm{conc}}, \epsilon)$ discussed in Example 5.7, and its graded dual is a $Zinb^c$-As-bialgebra $\mathcal{Z}_{\mathrm{conc}}(X) := (\mathbf{k}\langle X\rangle, \mathrm{conc}, 1_{X^*}, \Delta_{\overset{r}{\sqcup\!\sqcup}}, \epsilon)$, where the co-half-shuffle $\Delta_{\overset{r}{\sqcup\!\sqcup}}$ satisfies the compatible relation

$$\Delta_{\overset{r}{\sqcup\!\sqcup}} \circ \mathrm{conc} = (\mathrm{conc} \otimes \mathrm{conc}) \circ (\mathrm{Id} \otimes \tau \otimes \mathrm{Id}) \circ (\Delta_{\sqcup\!\sqcup} \otimes \Delta_{\overset{r}{\sqcup\!\sqcup}}),$$

i.e. for any polynomials T, P then $\Delta_{\overset{r}{\sqcup\!\sqcup}}(TP) = \Delta_{\sqcup\!\sqcup}(T)\Delta_{\overset{r}{\sqcup\!\sqcup}}(P)$.

5.5 Appendix D: Maple Outputs.

5.5.1 Lyndon basis and its dual.

Table of families $\{P_w\}_{w\in X^*}$ (Lyndon basis) and its dual $\{S_w\}_{w\in X^*}$.

Let $X = \{x_0, x_1\}$ with $x_0 < x_1$.

l	P_l	S_l
x_0	x_0	x_0
x_1	x_1	x_1
x_0x_1	$[x_0, x_1]$	x_0x_1
$x_0^2x_1$	$[x_0, [x_0, x_1]]$	$x_0^2x_1$
$x_0x_1^2$	$[[x_0, x_1], x_1]$	$x_0x_1^2$
$x_0^3x_1$	$[x_0, [x_0, [x_0, x_1]]]$	$x_0^3x_1$
$x_0^2x_1^2$	$[x_0, [[x_0, x_1], x_1]]$	$x_0^2x_1^2$
$x_0x_1^3$	$[[[x_0, x_1], x_1], x_1]$	$x_0x_1^3$
$x_0^4x_1$	$[x_0, [x_0, [x_0, [x_0, x_1]]]]$	$x_0^4x_1$
$x_0^3x_1^2$	$[x_0, [x_0, [[x_0, x_1], x_1]]]$	$x_0^3x_1^2$
$x_0^2x_1x_0x_1$	$[[x_0, [x_0, x_1]], [x_0, x_1]]$	$2x_0^3x_1^2 + x_0^2x_1x_0x_1$
$x_0^2x_1^3$	$[x_0, [[[x_0, x_1], x_1], x_1]]$	$x_0^2x_1^3$
$x_0x_1x_0x_1^2$	$[[x_0, x_1], [[x_0, x_1], x_1]]$	$3x_0^2x_1^3 + x_0x_1x_0x_1^2$
$x_0x_1^4$	$[[[[x_0, x_1], x_1], x_1], x_1]$	$x_0x_1^4$
$x_0^5x_1$	$[x_0, [x_0, [x_0, [x_0, [x_0, x_1]]]]]$	$x_0^5x_1$
$x_0^4x_1^2$	$[x_0, [x_0, [x_0, [[x_0, x_1], x_1]]]]$	$x_0^4x_1^2$
$x_0^3x_1x_0x_1$	$[x_0, [[x_0, [x_0, x_1]], [x_0, x_1]]]$	$2x_0^4x_1^2 + x_0^3x_1x_0x_1$
$x_0^3x_1^3$	$[x_0, [x_0, [[[x_0, x_1], x_1], x_1]]]$	$x_0^3x_1^3$
$x_0^2x_1x_0x_1^2$	$[x_0, [[x_0, x_1], [[x_0, x_1], x_1]]]$	$3x_0^3x_1^3 + x_0^2x_1x_0x_1^2$
$x_0^2x_1^2x_0x_1$	$[[x_0, [[x_0, x_1], x_1]], [x_0, x_1]]$	$6x_0^3x_1^3 + 3x_0^2x_1x_0x_1^2 + x_0^2x_1^2x_0x_1$
$x_0^2x_1^4$	$[x_0, [[[[x_0, x_1], x_1], x_1], x_1]]$	$x_0^2x_1^4$
$x_0x_1x_0x_1^3$	$[[x_0, x_1], [[[x_0, x_1], x_1], x_1]]$	$4x_0^2x_1^4 + x_0x_1x_0x_1^3$
$x_0x_1^5$	$[[[[[x_0, x_1], x_1], x_1], x_1], x_1]$	$x_0x_1^5$

Chapter 6

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