Compact packings with three discs

Thomas Fernique

CNRS & Univ. Paris 13

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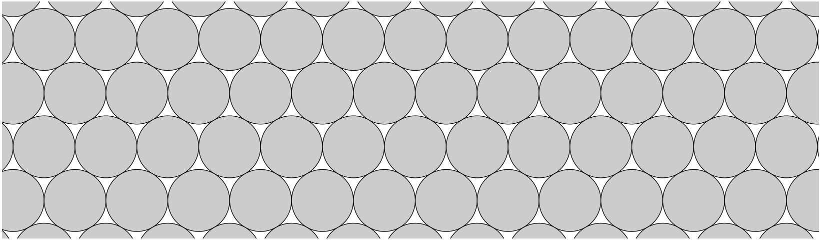
Theorem (Thue, 1910)

The densest packing in $\mathbb{R}^2$ *is the* hexagonal compact *packing.*



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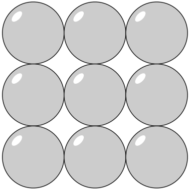
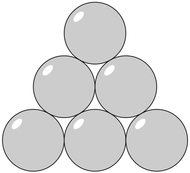
The densest packings in $\mathbb{R}^3$ *are the* close-packings*.*





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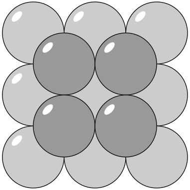

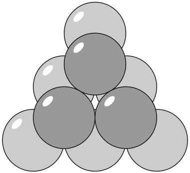

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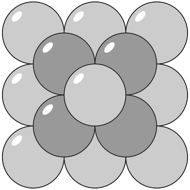

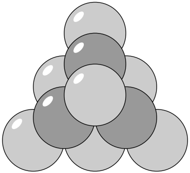

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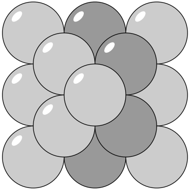

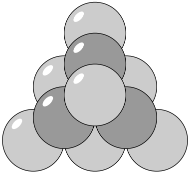

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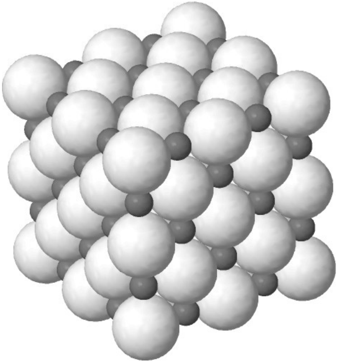
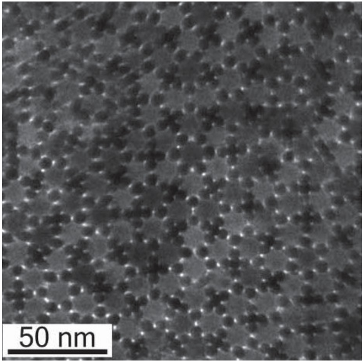
Theorem (Vyazovska *et al.*, 2017)
The densest packings are known in $\mathbb{R}^8$ *and* $\mathbb{R}^{24}$.

Unequal sphere packings

The density becomes parametrized by the ratios of sphere sizes.
Natural problem in materials science!





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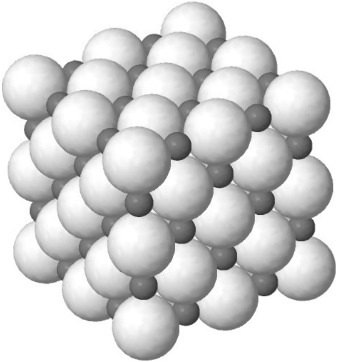
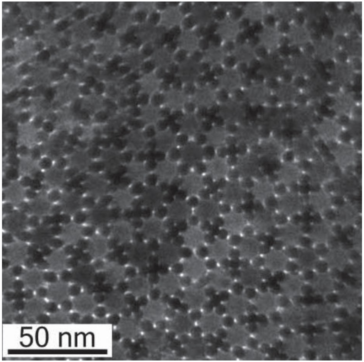
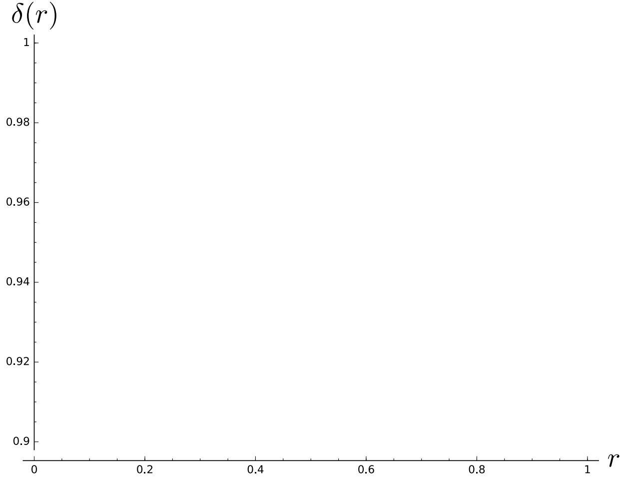




Theorem (Heppes-Kennedy, 2004–2006)

The densest packings with two discs are known for seven ratios.

Two discs



The maximal density is a function $\delta(r)$ of the ratio $r \in [0, 1]$.

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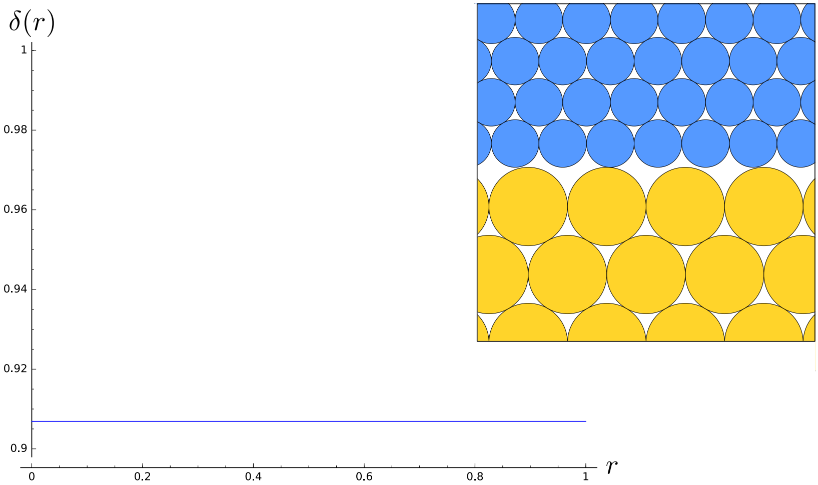
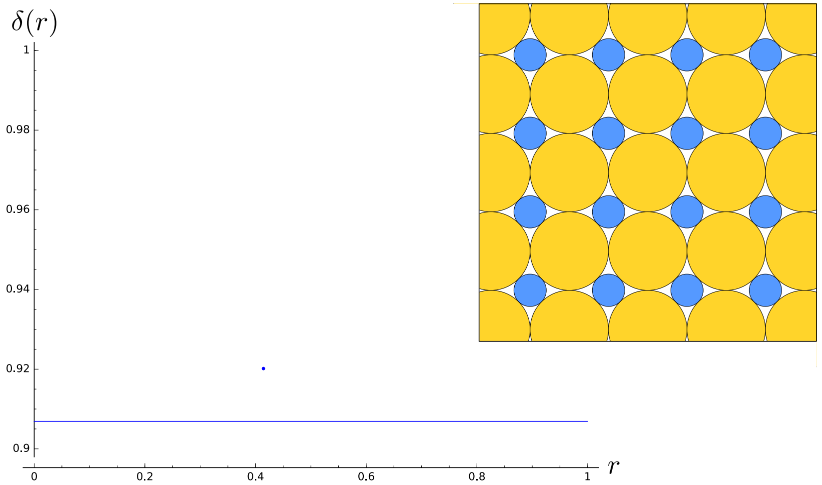


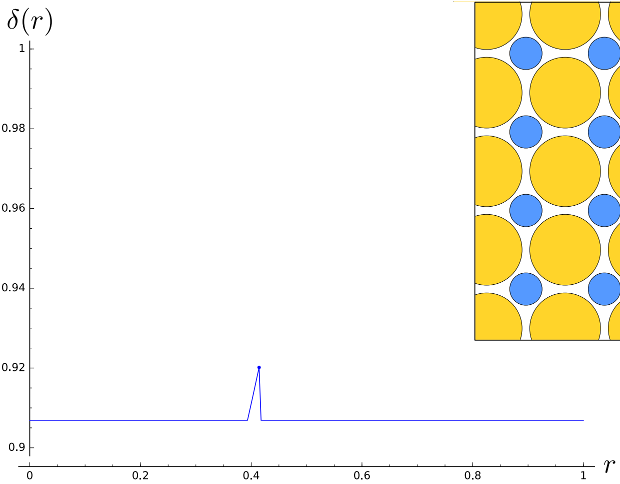
The hexagonal compact packing yields a uniform lower bound.

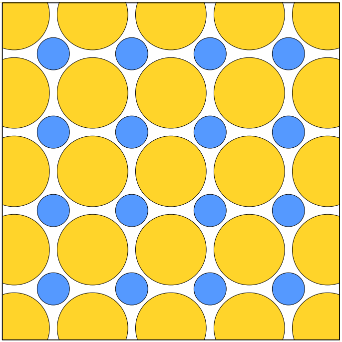
Two discs



Any given packing yields a lower bound for a specific r.

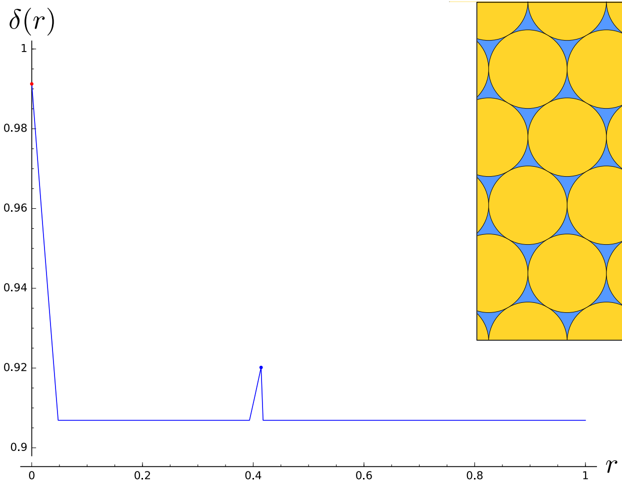
Two discs

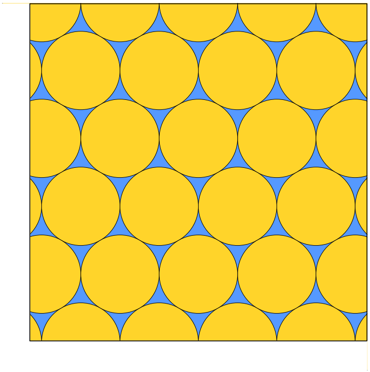




It actually yields a lower bound in a neighborhood of r.

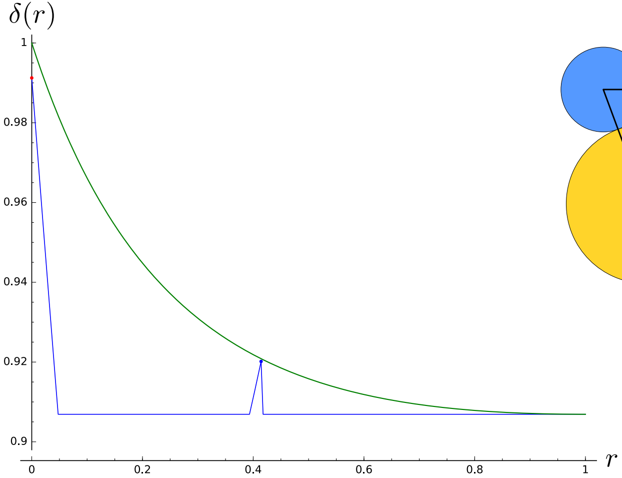
Two discs

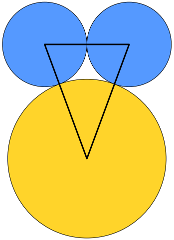




$$\lim_{r \to 0} \delta(r) = \delta(1) + (1 - \delta(1))\delta(1) \simeq 0.99133.$$

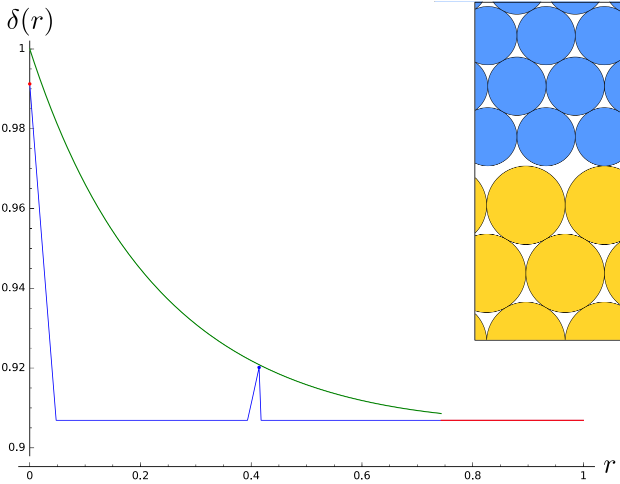
Two discs

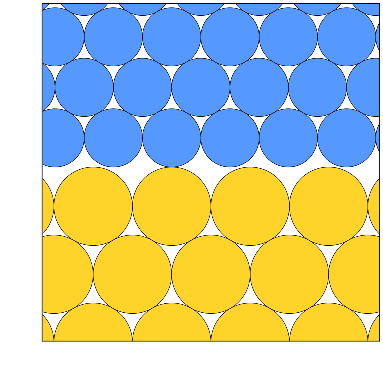




The density in the r1r triangle is an upper bound (Florian, 1960).

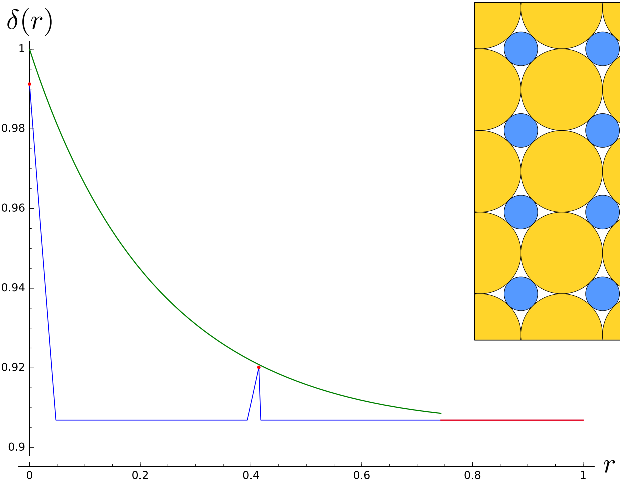
Two discs

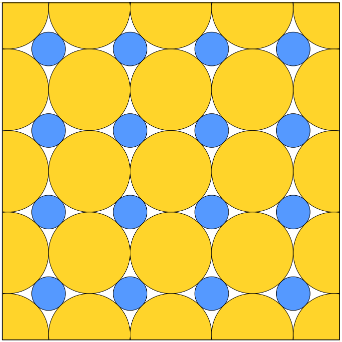




For $r \geq 0.74$, two discs do not pack better than one (Blind, 1969).

The seven "magic" ratios





0.41, root of $X^2 + 2X - 1$.

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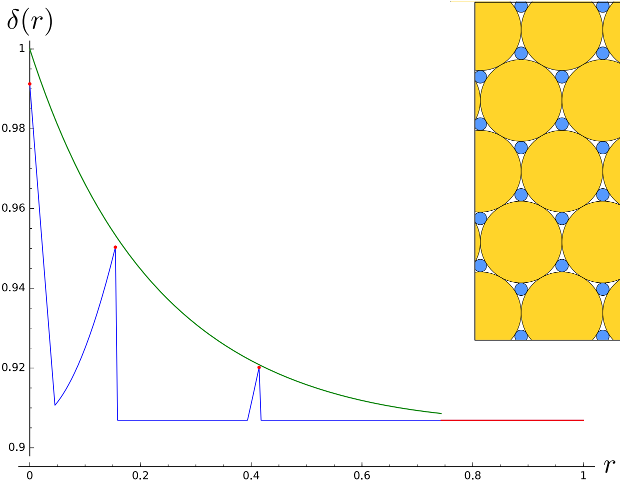

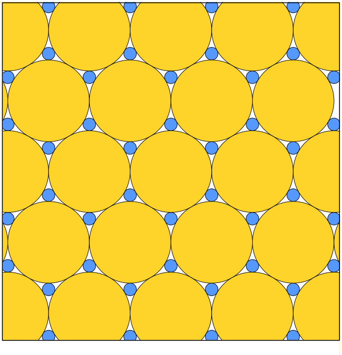

0.15, root of $3X^2 + 6X - 1$.

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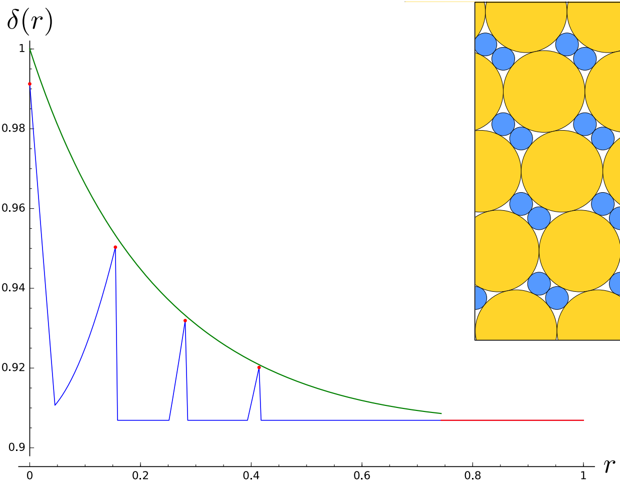

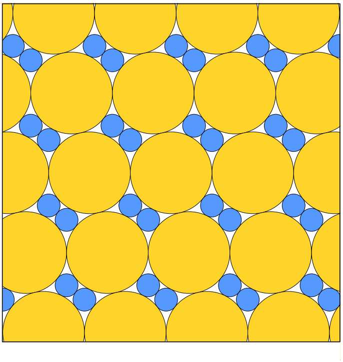

0.28, root of $2X^2 + 3X - 1$.

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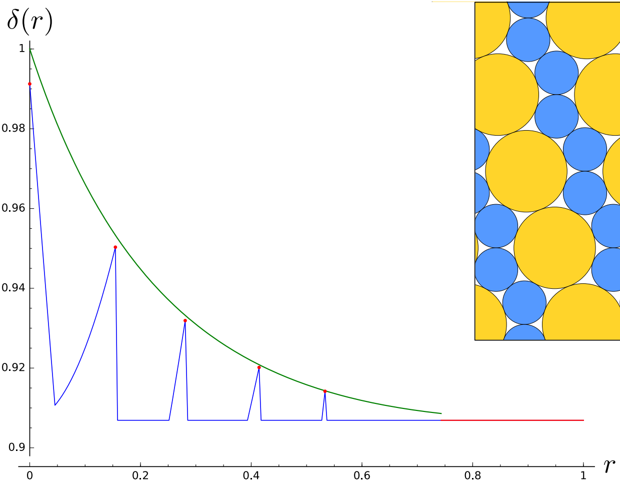

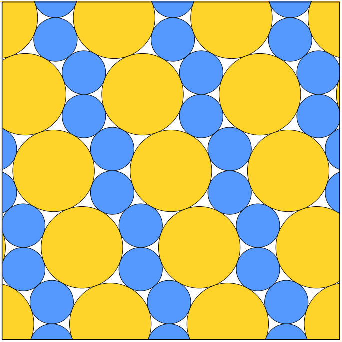

0.53, root of $8X^3 + 3X^2 - 2X - 1$.

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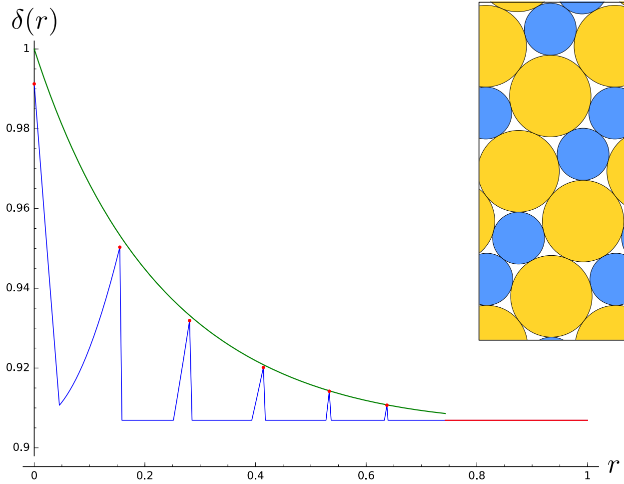

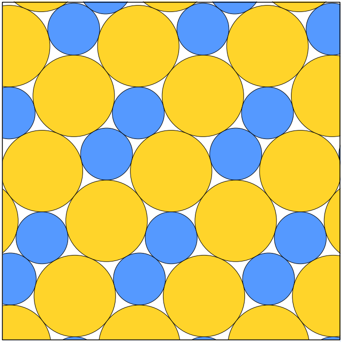

0.64, root of $X^4 - 10X^2 - 8X + 9$.

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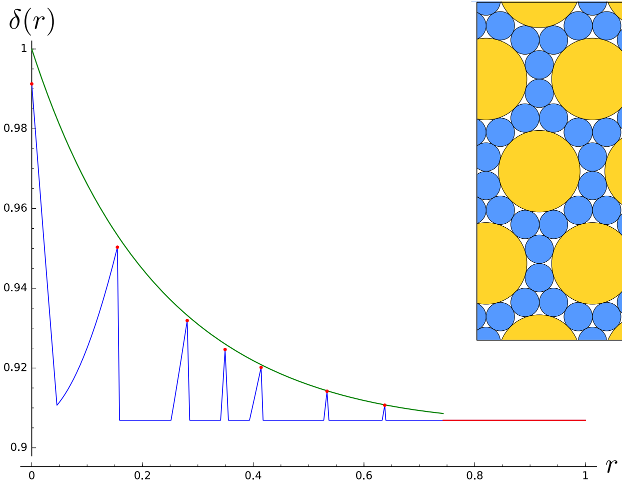

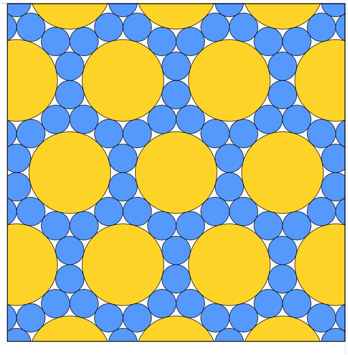

0.35, root of $X^4 - 28X^3 - 10X^2 + 4X + 1$.

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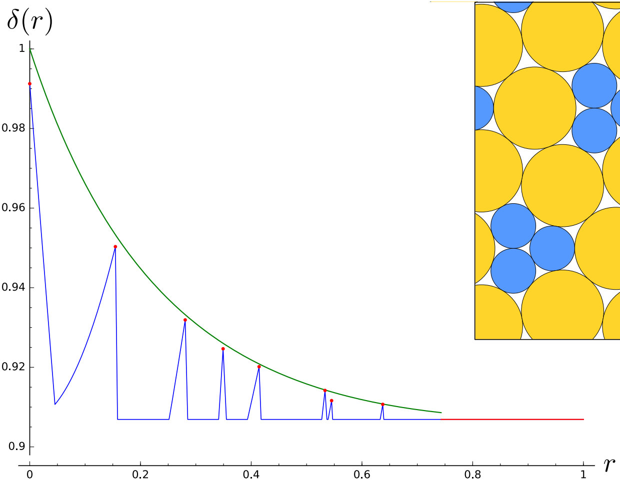

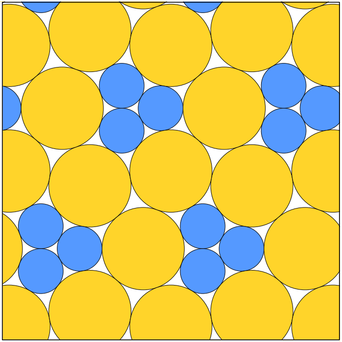

0.55, root of X^8-$8X^7$-$44X^6$-$232X^5$-$482X^4$-$24X^3+388X^2$-$120X+9$.

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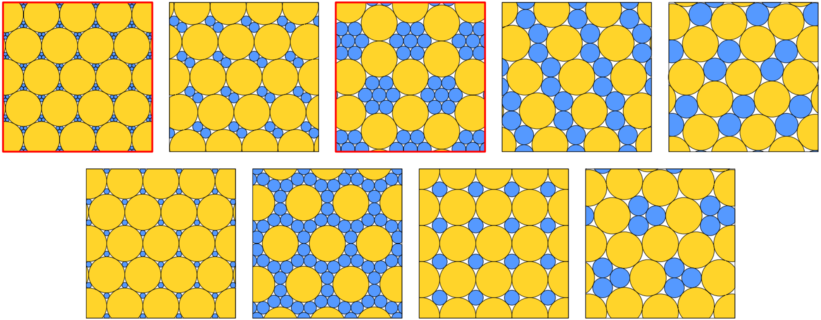
Compact packings are candidates to provably maximize the density.

Compact packings with two discs



Theorem (Kennedy, 2006)

There are <u>nine</u> ratios allowing a compact packing with two discs.

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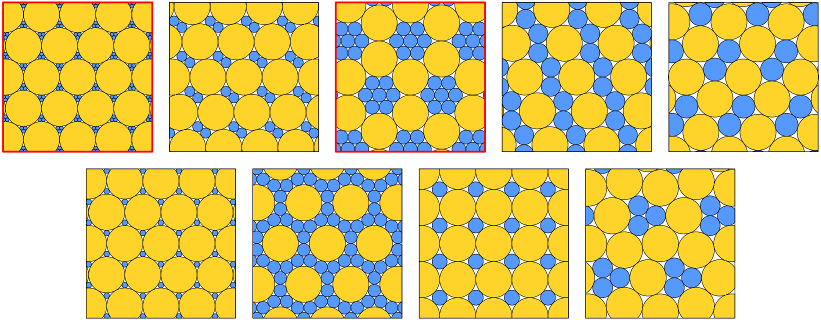

Theorem (Kennedy, 2006)

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Remark: two have (still?) not been proven to maximize the density.

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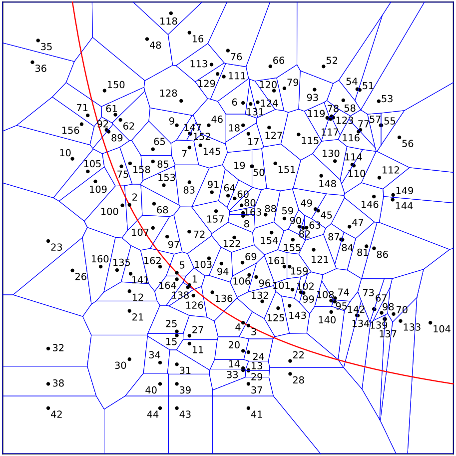

Theorem (F.-Hashemi-Sizova)

There are 164 *ratios allowing a compact packing with three discs.*

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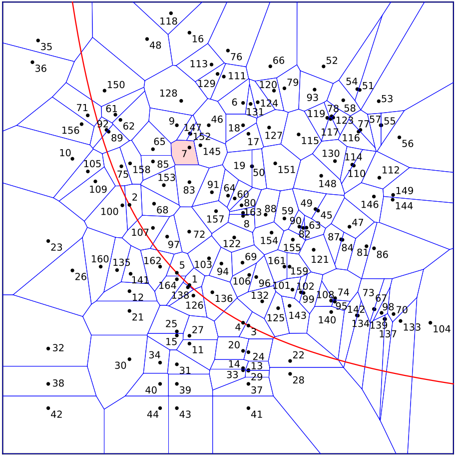

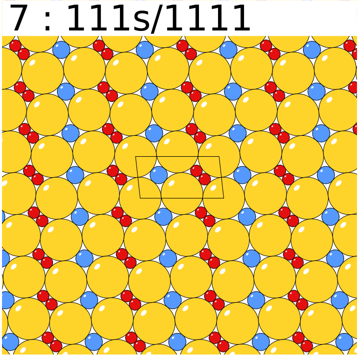

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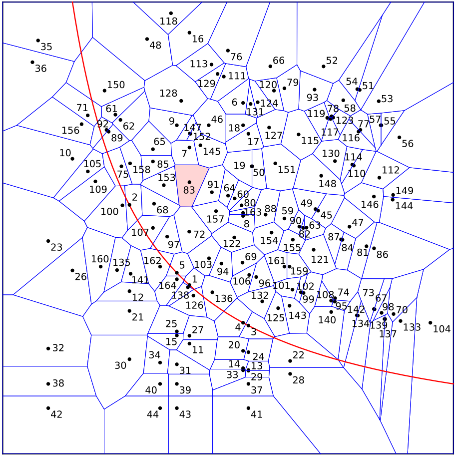

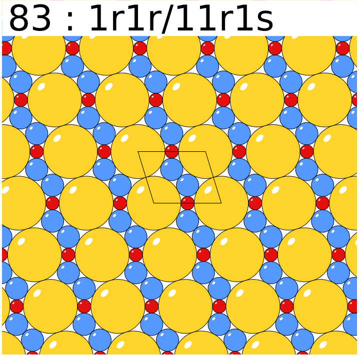

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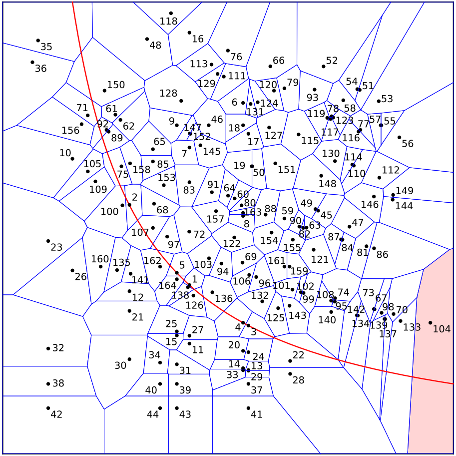

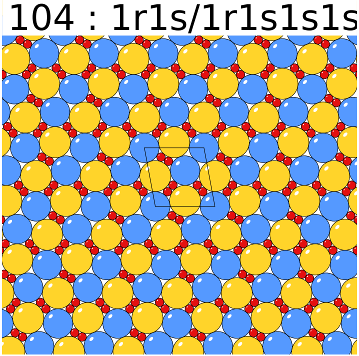

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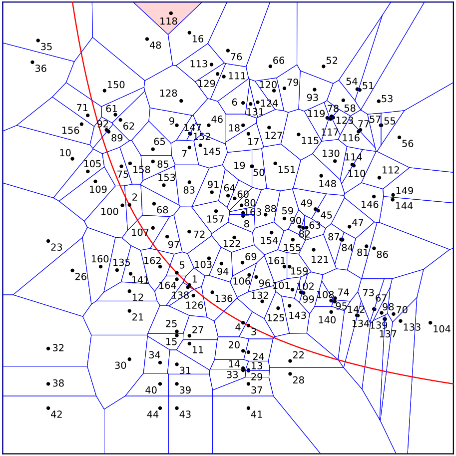

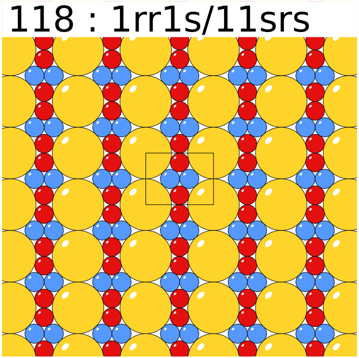

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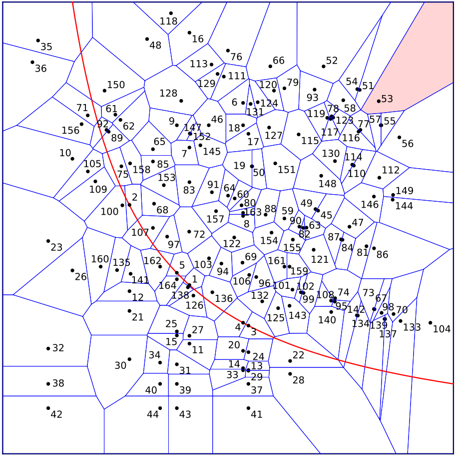

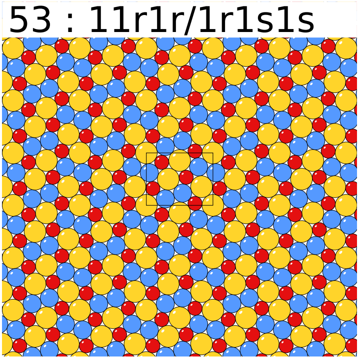

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Let $s < r < 1$ be the three sizes of discs.

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An *x-corona* is a sequence of sizes of discs around a disc of size x.

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Each pair of s- and r-coronas yields a polynomial system in r and s.

Strategy

For each pair of s- and r-coronas, solve the polynomial system, then find all the possible coronas and finally find the packings.

Compact packing with spheres in $\mathbb{R}^3$

With two spheres, it is boring:

Theorem (Fernique, 2019)

The compact packings by two sizes of spheres are exactly those obtained by filling the octahedral holes of a close-packing.

Compact packing with spheres in $\mathbb{R}^3$

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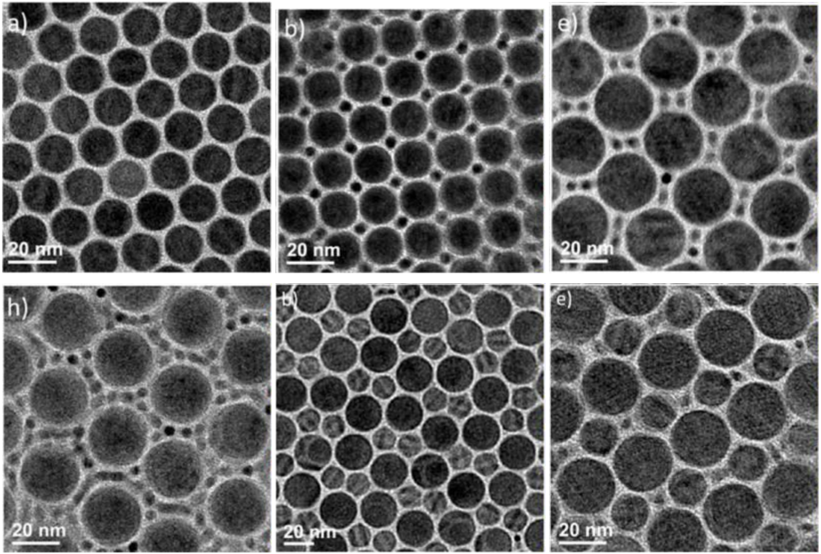
With three spheres, it is still boring:

Theorem (Fernique, expected 2019)

The compact packing by three sizes of spheres are exactly those obtained by filling one of the two types of tetrahedral holes of a compact packing by two sizes of spheres.

Back to material science



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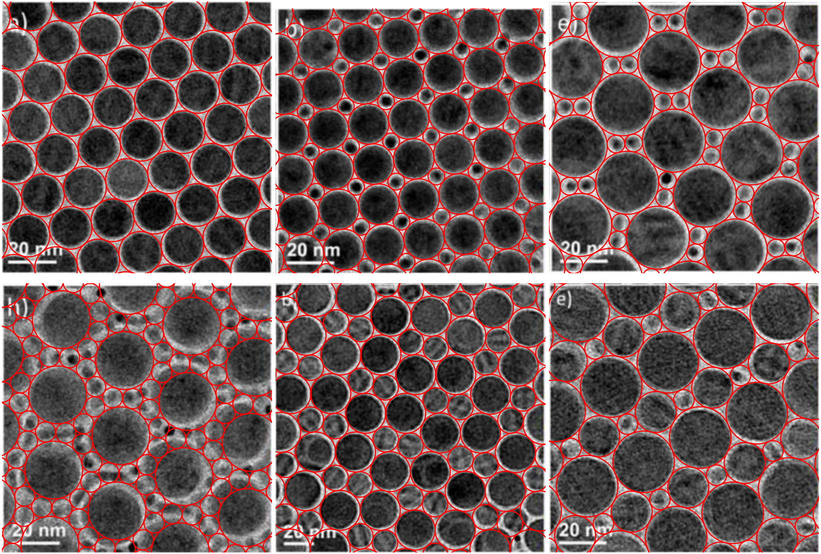

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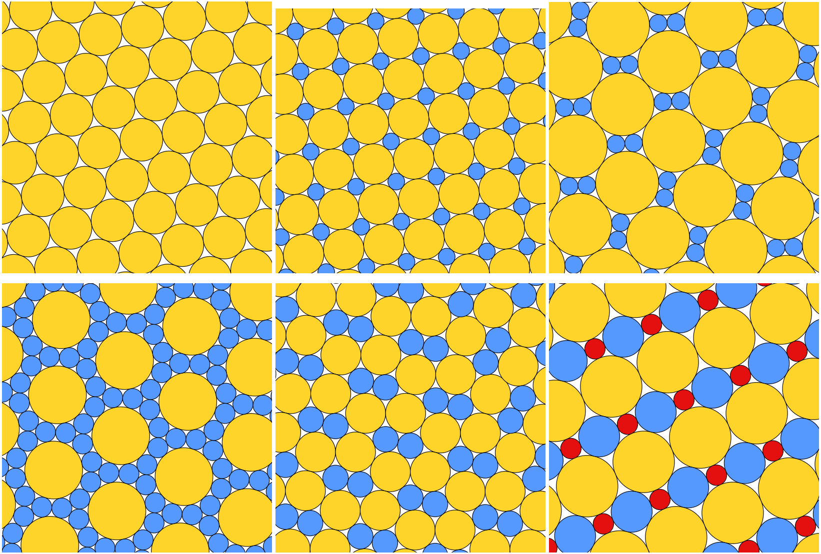
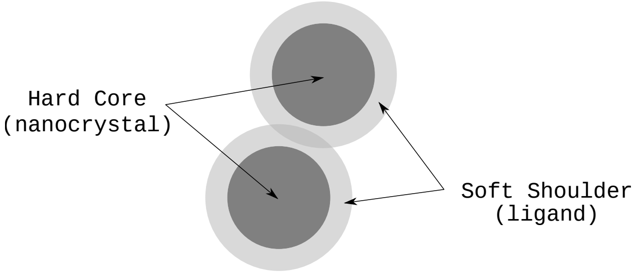
Back to material science



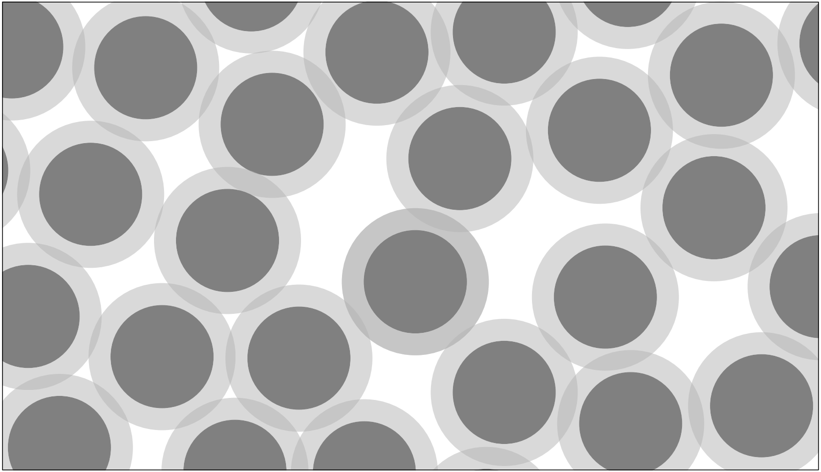
T. Paik, B. Diroll, C. Kagan, Ch. Murray J. Am. Chem. Soc., 2015, 137
Binary and ternary superlattices self-assembled from colloidal nanodisks and nanorods

Quasicompact packings



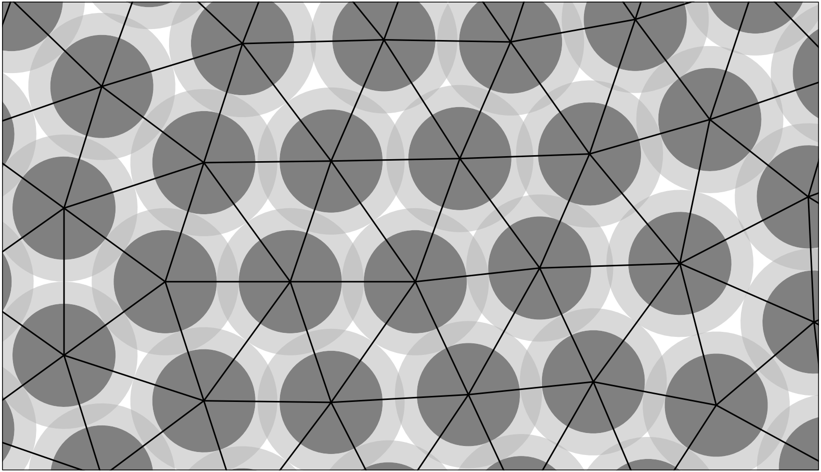
More realistic model: Hard Core + Soft Shoulder (HCSS).

Quasicompact packings



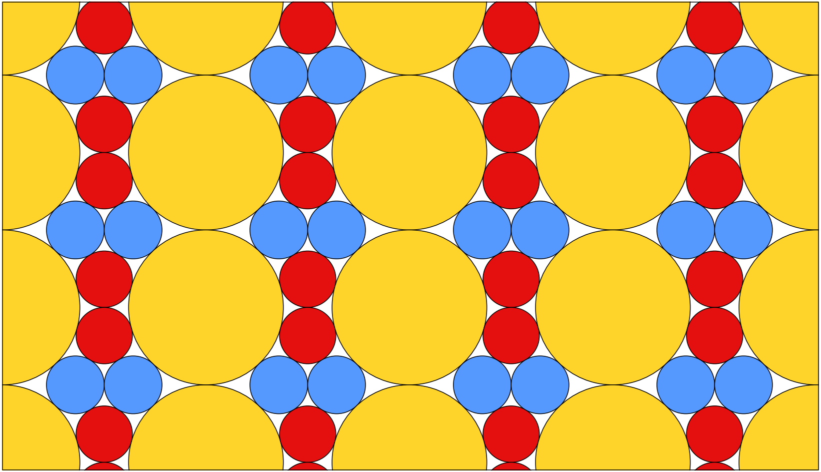
Which quantity has to be minimized or maximized (formally)?

Quasicompact packings



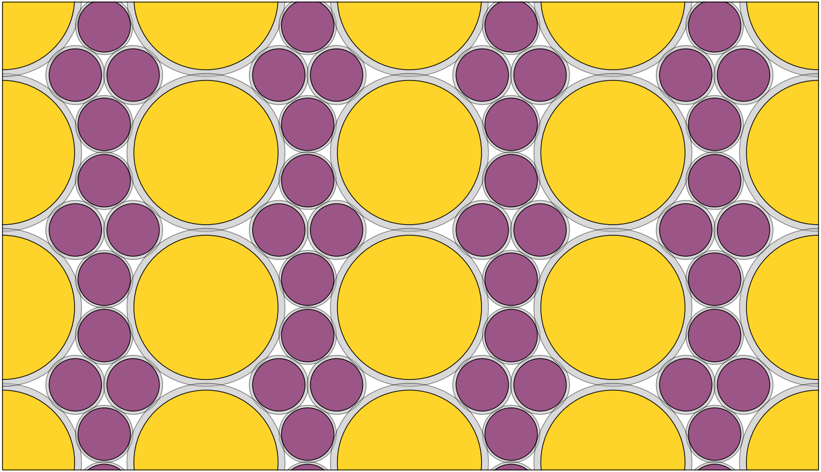
Compact packings naturally extend to *quasicompact* packings.

Quasicompact packings



Some compact packings with three sizes of discs. . .

Quasicompact packings



. . . can be seen as a quasicompact packings with two sizes of discs.

ASAP

Project 80-PRIME CNRS 2019-2021

- INS2I: LIPN (Thomas Fernique)
- INC/INP: LPCNO (Simon Tricard)

Goals:

- experimental nanosynthesis of "compact supercrystals";
- new theoretical questions raised by experiences.