Introduction aux pavages: du local au global

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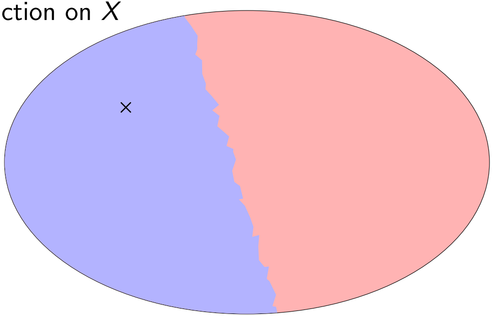
13 June 2019

Local rules

Coding of dynamical systems

Given a dynamical system and a partition, it is possibly to code the trajectory.



F is a $\mathbb{Z}$-action on X

Coding: $\cdots$ 🟦

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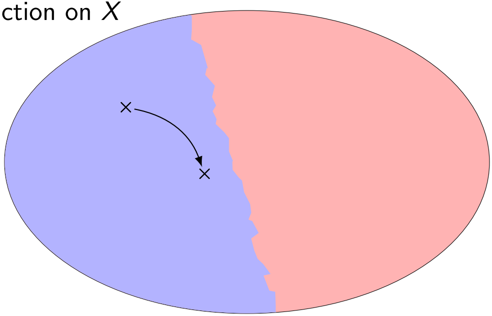

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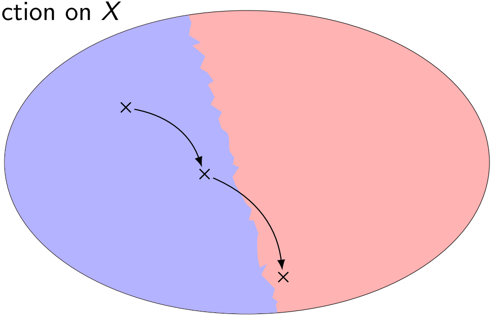

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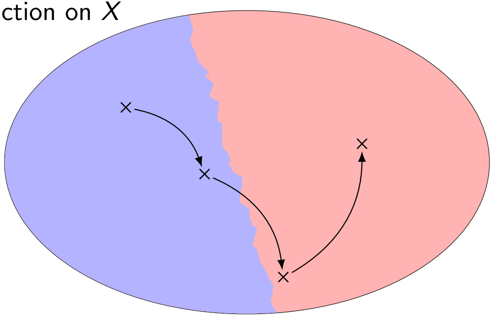

Coding: $\cdots$ 🟦🟦🟥🟥

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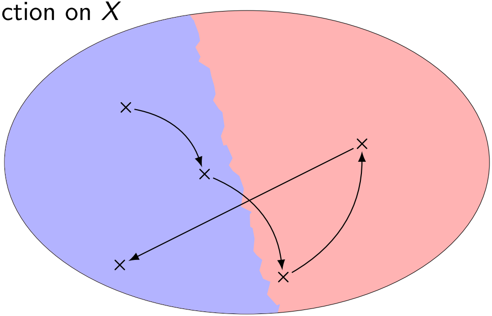

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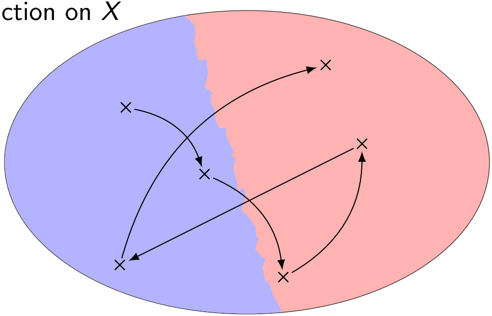


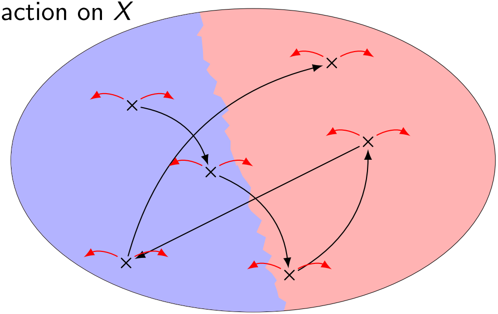
F is a $\mathbb{Z}$-action on X

Coding: $\cdots$ 🟦🟦🟥🟥🟦🟥 $\cdots \in \mathcal{A}^{\mathbb{Z}}$

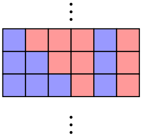
Coding of dynamical systems

Given a dynamical system and a partition, it is possibly to code the trajectory.

F is a $\mathbb{Z}^2$-action on X



Coding: $\cdots$



🟦	🟥	🟥	🟥	🟦	🟥
🟦	🟦	🟥	🟥	🟦	🟥
🟦	🟦	🟦	🟥	🟦	🟥

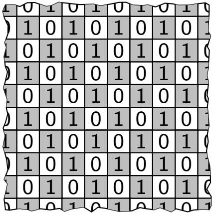
$\cdots \in \mathcal{A}^{\mathbb{Z}^2}$

Configuration and patterns

Let $\mathcal{A}$ be a finite alphabet.

▶$x : \mathbb{Z}^d \to \mathcal{A} \in \mathcal{A}^{\mathbb{Z}^d}$ is a *configuration*.

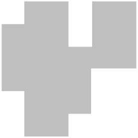
$x =$



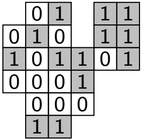
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1

$\in \{0, 1\}^{\mathbb{Z}^d}$

▶Let $\mathbb{U} \subset \mathbb{Z}^d$ be a finite set. A *pattern* is an element $p \in \mathcal{A}^{\mathbb{U}}$.



Support: $\mathbb{U} \subset \mathbb{Z}^2$ finite



	0	1		1	1
0	1	0		1	1
1	0	1	1	0	1
0	0	0	1		
	0	0	0		
	1	1			

$p \notin x$

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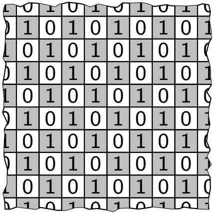

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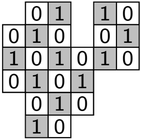
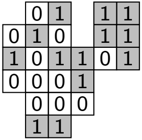


$p \sqsubset x$



$p \not\sqsubset x$

Subshift as dynamical system

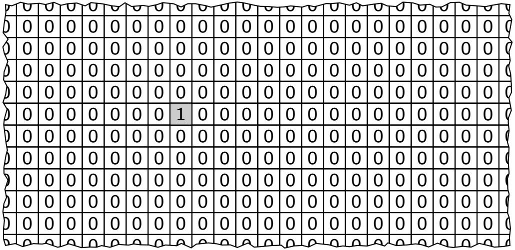
Combinatory definition of subshifts

A subshift $\mathbf{T}$ is defined with a set of forbidden patterns $\mathcal{F}$:

$$\mathbf{T}_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^{\boldsymbol{d}}} \ : \ \text{patterns of } \mathcal{F} \text{ does not appear in } x \right\} \subseteq \mathcal{A}^{\mathbb{Z}^{\boldsymbol{d}}}$$

Exemple:

$$\mathbf{T}_{\leq 1} = \left\{ x \in \{0,1\}^{\mathbb{Z}^{\boldsymbol{d}}} : \text{ there is at most one } g \in \mathbb{Z}^{\boldsymbol{d}} \text{ such that } x_g = 1 \right\}$$



0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Local rules vs colored local rules

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Some classes of subshifts invariant by conjugacy

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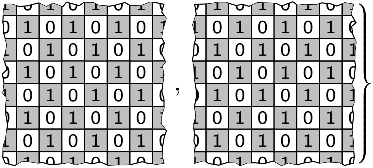
Some classes of subshifts invariant by conjugacy

$\mathbf{T}$ *subshift of finite type* $\Longleftrightarrow$ $\exists\mathcal{F}$ finite set such that $\mathbf{T} = \mathbf{T}_{\mathcal{F}}$

The chessboard: Let $\mathcal{A} = \{0,1\}$ and $\mathcal{F} = \left\{ \begin{array}{|c|}\hline i \\ \hline i \\ \hline \end{array}, \begin{array}{|c|c|}\hline i & i \\ \hline \end{array} : i \in \mathcal{A} \right\}$. Consider



$$\mathbf{T}_{\mathcal{F}} = \left\{ \begin{array}{cccccc} 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1 \end{array}, \begin{array}{cccccc} 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0 \end{array} \right\}$$

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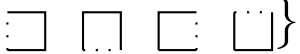
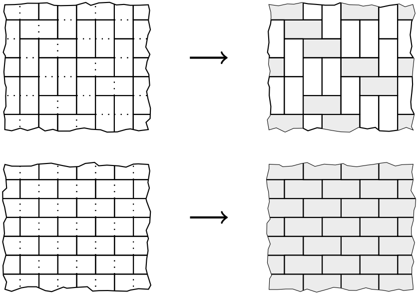
Domino:

$T_D = \{$  $\}$



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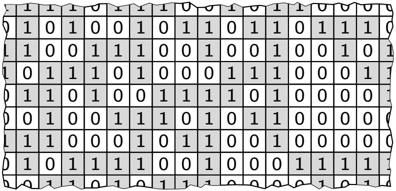
Algebraic subshift of finite type:



1	0	1	0	0	1	0	1	1	0	1	1	0	1	1	1
1	0	0	1	1	1	0	0	1	0	0	1	0	0	1	0
0	1	1	1	0	1	0	0	0	1	1	1	0	0	0	1
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0	1	0	0	1	1	1	0	1	0	1	1	0	0	0	0
1	1	0	0	0	1	0	1	1	0	0	1	0	0	0	0
1	0	1	1	1	1	0	0	1	0	0	0	1	1	1	1

$$\mathbf{T} = \left\{ x \in \{0,1\}^{\mathbb{Z}^{\mathbf{2}}} : \forall (i,j) \in \mathbb{Z}^2, \ x_{(i,j)} + x_{(i+1,j)} + x_{i,j+1} = 0 \mod 2 \right\}$$

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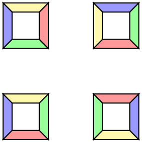
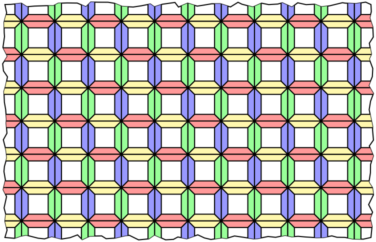
Wang Tiles:





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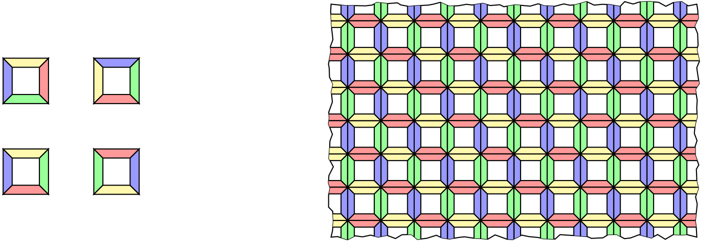

Proposition

If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}^{\boldsymbol{d}}}$ is a SFT then is conjugate to a SFT defined by Wang tiles.

Local rules vs colored local rules

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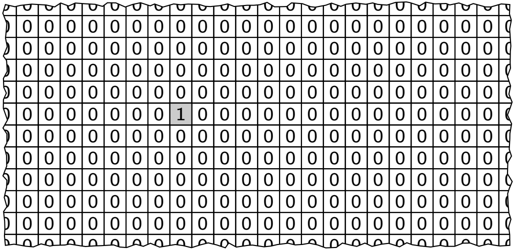

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

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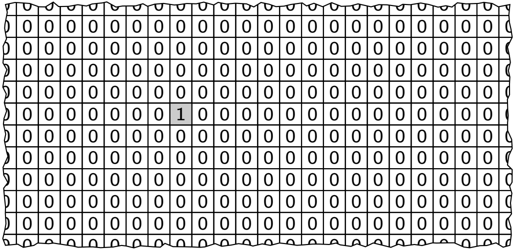
$\mathbf{T}_{\leq 1}$ can be defined by local rules?



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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

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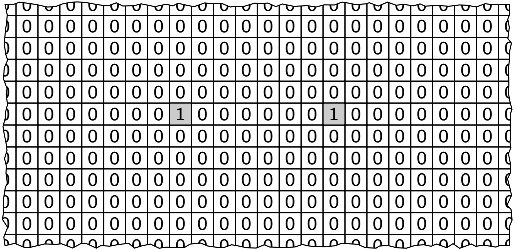
T *subshift of finite type* $\Longleftrightarrow$ $\exists\mathcal{F}$ finite set such that $\mathbf{T} = \mathbf{T}_{\mathcal{F}}$

Consider $\mathbf{T}_{\leq 1} = \left\{ x \in \{0,1\}^{\mathbb{Z}^{\boldsymbol{d}}} : \text{there is at most one } g \in \mathbb{Z}^{d} \text{ such that } x_g = 1 \right\}$.

Assume that $\mathbf{T}_{\leq 1} = \mathbf{T}_{\mathcal{F}}$ with $\mathcal{F} \subset \mathcal{A}^{\mathbf{B}_{\boldsymbol{n}}}$. Then



0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

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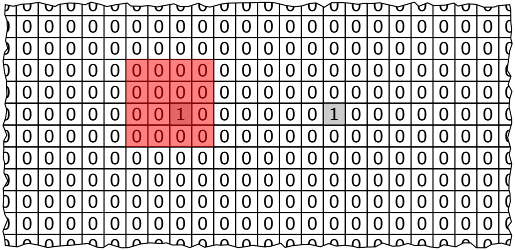


0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Local rules vs colored local rules

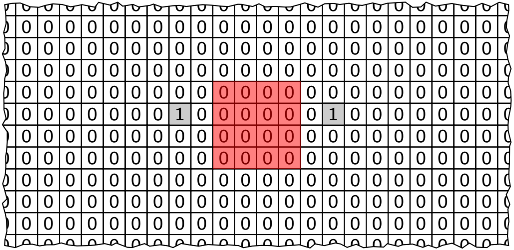
$$\mathbf{T}_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} \ : \ \text{patterns of } \mathcal{F} \text{ does not appear in } x \right\} \subseteq \mathcal{A}^{\mathbb{Z}^d}$$

Some classes of subshifts invariant by conjugacy

T *subshift of finite type* $\iff \exists \mathcal{F}$ finite set such that $\mathbf{T} = \mathbf{T}_{\mathcal{F}}$

Consider $\mathbf{T}_{\leq 1} = \left\{ x \in \{0,1\}^{\mathbb{Z}^d} : \text{there is at most one } g \in \mathbb{Z}^d \text{ such that } x_g = 1 \right\}$.

Assume that $\mathbf{T}_{\leq 1} = \mathbf{T}_{\mathcal{F}}$ with $\mathcal{F} \subset \mathcal{A}^{\mathbf{B}_n}$. Then



0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

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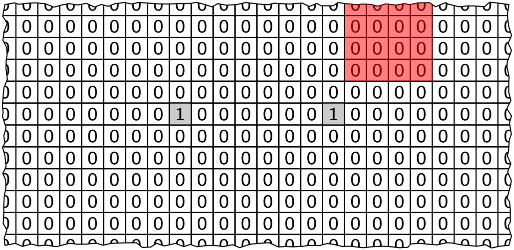

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$\in \mathbf{T}$ contradiction!

Local rules vs colored local rules

$$\mathbf{T}_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^{\boldsymbol{d}}} \ : \ \text{patterns of } \mathcal{F} \text{ does not appear in } x \right\} \subseteq \mathcal{A}^{\mathbb{Z}^{\boldsymbol{d}}}$$

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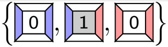
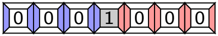
$\mathbf{T}$ *subshift of finite type* $\Longleftrightarrow \exists \mathcal{F}$ finite set such that $\mathbf{T} = \mathbf{T}_{\mathcal{F}}$

$\mathbf{T}$ *subshift sofic* $\Longleftrightarrow \exists \mathcal{F}$ finite set and $\pi : \mathcal{A} \to \mathcal{B}$ such that $\mathbf{T} = \pi(\mathbf{T}_{\mathcal{F}})$

Let $\mathcal{B} = \Big\{$  $\Big\}$ and $\mathcal{F}$ forbid the matching between two different colors.

$\cdots$  $\cdots \in \mathbf{T}_{\mathcal{F}}$ $\xrightarrow{\pi}$ $\cdots \boxed{0}\boxed{0}\boxed{0}\boxed{1}\boxed{0}\boxed{0}\boxed{0} \cdots \in \mathbf{T}$

Local rules vs colored local rules

$$\mathbf{T}_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^{d}} \ : \ \text{patterns of } \mathcal{F} \text{ does not appear in } x \right\} \subseteq \mathcal{A}^{\mathbb{Z}^{d}}$$

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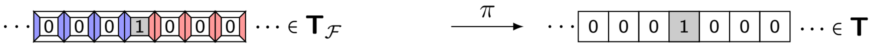
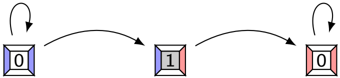


Every configuration can be seen as an infinite path in the following graph:



Local rules vs colored local rules

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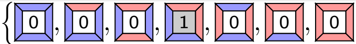
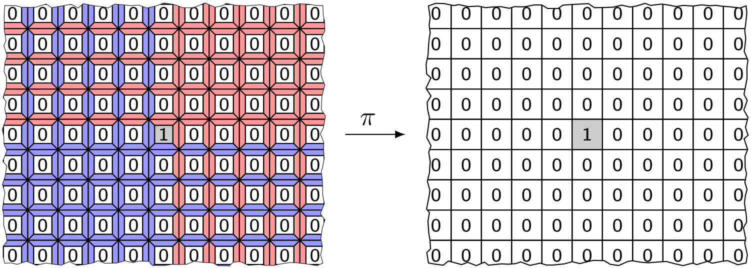
Let $\mathcal{B} = \{$  $\}$ and $\mathcal{F}$ forbid the matching between two different colors.

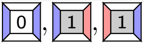


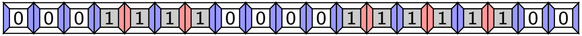
Exemple 2: The even shift

$$X_{\mathtt{even}} = \left\{ x \in \{0,1\}^{\mathbb{Z}^{\boldsymbol{d}}} : \text{Finite connected component have even size} \right\}$$

Proposition

The even shift is sofic.

For $d = 1$, consider $\mathcal{B} = \Big\{$  $\Big\}$, one has

$\pi\Big(\ldots$  $\ldots\Big) \in X_{\mathtt{even}}$

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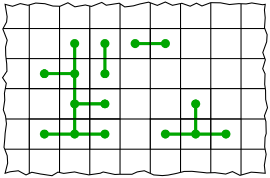
Proposition

The even shift is sofic.

Consider the alphabet $\mathcal{A}_{\text{even}} = \left\{ □ , ╹ , ┴ \right\}$ + rotation

$$\pi(□) = 0 \text{ and } \pi(╹) = \pi(┴) = 1$$



$$\xmapsto[\pi]{}$$

		1	1	1	1			
	1	1	1					
		1	1			1		
	1	1	1		1	1	1	

Green components have even size (handshaking lemma) $\Longrightarrow \pi(X_{\mathcal{A}_{\text{even}}}) \subset X_{\text{even}}$

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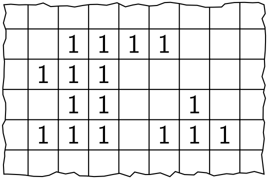

		1	1	1	1			
	1	1	1					
		1	1			1		
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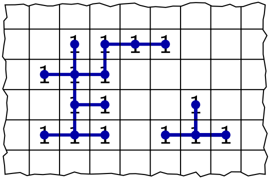
The even shift is sofic.

- Consider $x \in X_{\texttt{even}}$
- There exist trees which cover each connected component



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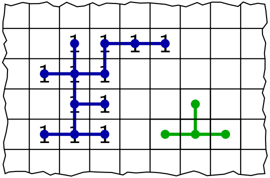

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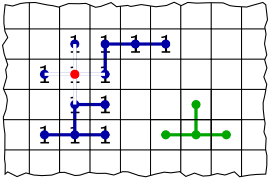
- If all verices of a tree have odd degree, then we done.
- Otherwise, delete a vertex v with even degree
- In $\mathcal{T} \setminus v$, odd number of trees with odd cardinality: connect v with them.



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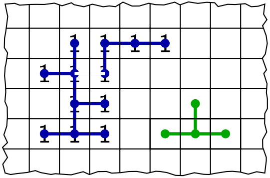

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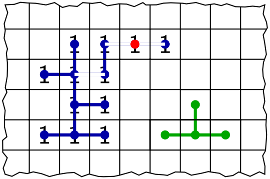

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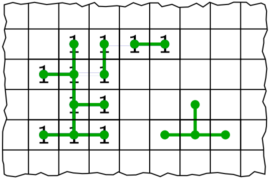


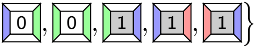
So $\pi(X_{\mathcal{A}_{\mathtt{even}}}) \subset X_{\mathtt{even}}$.

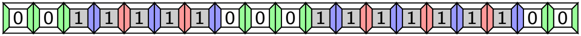
Exemple 3: The odd shift

$$X_{\texttt{Odd}} = \left\{ x \in \{0,1\}^{\mathbb{Z}^{\boldsymbol{d}}} : \text{Finite connected component have even size} \right\}$$

Proposition

The odd shift is sofic if $d = 1$ or 2.

- For $d = 1$, consider $\mathcal{B} = \Bigg\{$  $\Bigg\}$, one has

$$\pi \Big(\dots$$  $$\dots \Big) \in X_{\texttt{even}}$$

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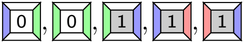
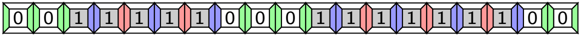
- For $d = 1$, consider $\mathcal{B} = \Big\{$  $\Big\}$, one has

$$\pi\Big(\ldots$$  $$\ldots\Big) \in X_{\mathtt{even}}$$

- The construction is quite hard for $d = 2$

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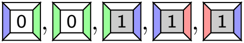
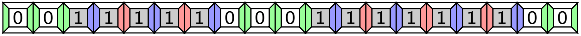
- For $d = 1$, consider $\mathcal{B} = \Big\{$  $\Big\}$, one has

$$\pi\Big(\dots$$  $$\dots\Big) \in X_{\mathtt{even}}$$

- The construction is quite hard for $d = 2$
- It is conjectured that it is not sofic for $d \geq 3$.

Exemple 4: Election of a leader shift

$$X_{\texttt{Leader}} = \left\{ x \in \{0,1,2\}^{\mathbb{Z}^{\boldsymbol{d}}} : \text{Finite connected component have exactly one 2} \right\}$$

Proposition

The election of a leader shift is sofic if $d = 1$ or 2.

Proposition

The election of a leader shift is not sofic for $d = 3$ but it is effective.

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The election of a leader shift is not sofic for $d = 3$ but it is effective.

$$\mathbf{T}_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{G}} \ : \ \text{patterns of } \mathcal{F} \text{ does not appear in } x \right\} \subseteq \mathcal{A}^{\mathbb{G}}$$

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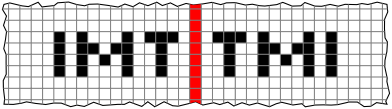
$\mathbf{T}$ *subshift sofic* $\Longleftrightarrow \exists \mathcal{F}$ finite set and $\pi : \mathcal{A} \to \mathcal{B}$ such that $\mathbf{T} = \pi(\mathbf{T}_{\mathcal{F}})$

$\mathbf{T}$ *effective subshift* $\Longleftrightarrow \exists \mathcal{F}$ recursively enumerable set such that $\mathbf{T} = \mathbf{T}_{\mathcal{F}}$

Exemple 5: Miror shift

Let $\mathcal{A} = \{\square, \blacksquare, \text{🟥}\}^{\mathbb{Z}^2}$ and consider the following effective subshift:

$$X_{\texttt{Miror}} = \{\square, \blacksquare\}^{\mathbb{Z}^2} \bigcup \left\{ \qquad \right\}$$



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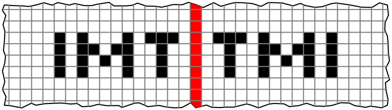
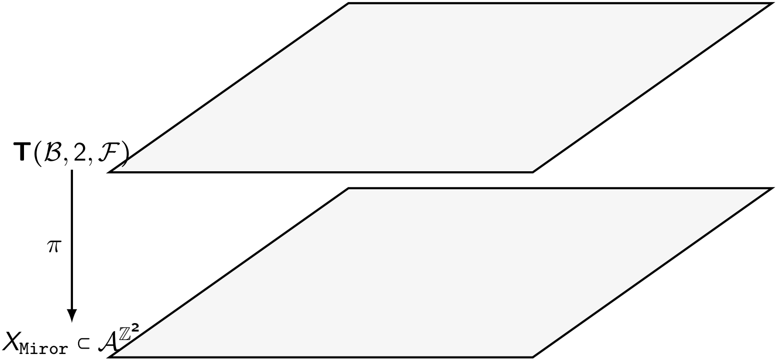


Assume that $X_{\texttt{Miror}}$ is sofic, that is to say $X_{\texttt{Miror}} = \pi(\mathbf{T}(\mathcal{B}, 2, \mathcal{F}))$



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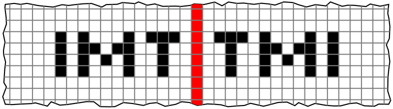

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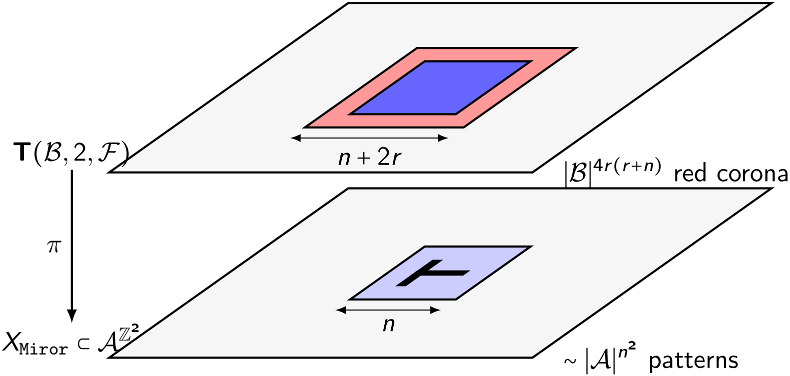

Exemple 5: Miror shift

Let $\mathcal{A} = \{\square, \blacksquare, 🟥\}^{\mathbb{Z}^2}$ and consider the following effective subshift:

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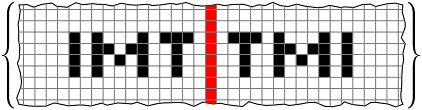

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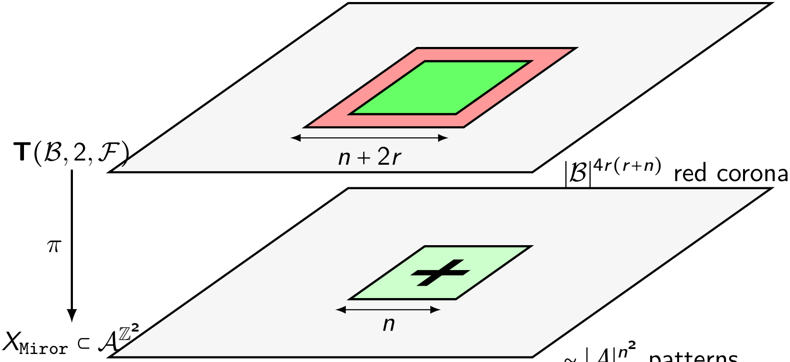

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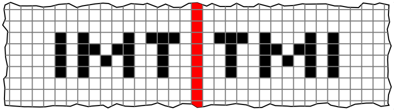

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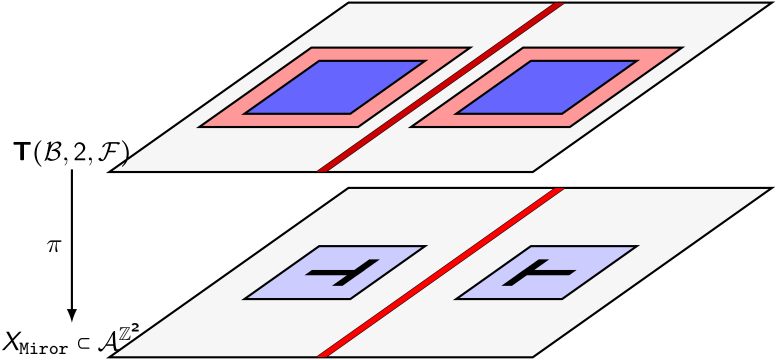

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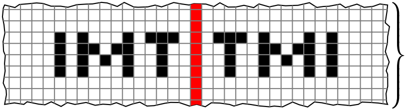

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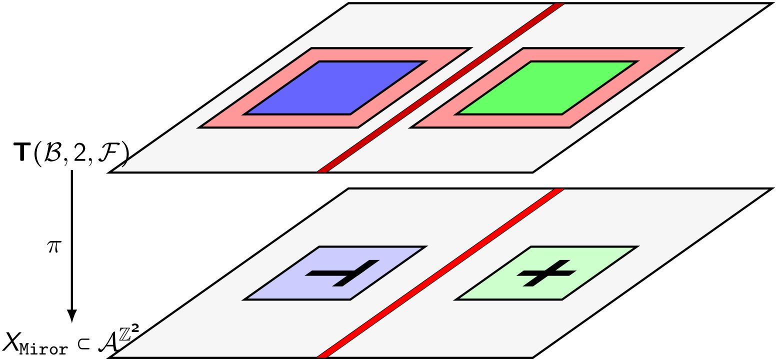
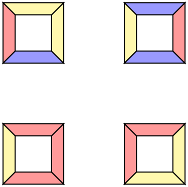


The domino problem

Domino problem for Wang's tilings

Tiles set

An associated tiling



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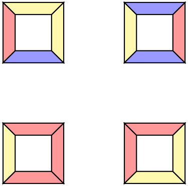

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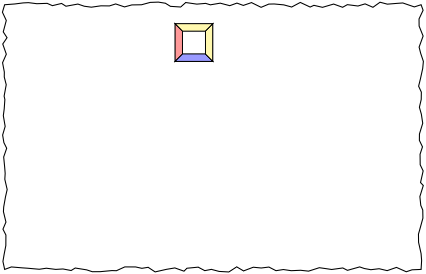

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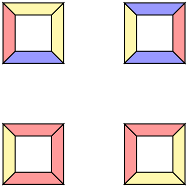

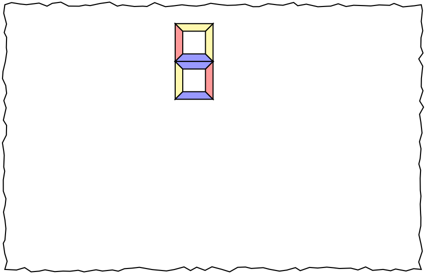

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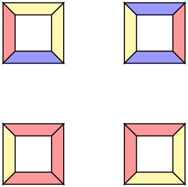

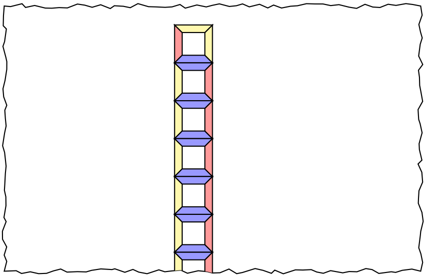

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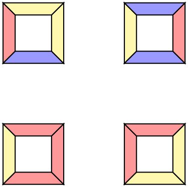

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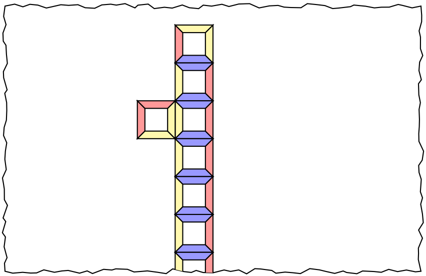

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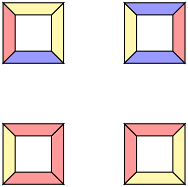

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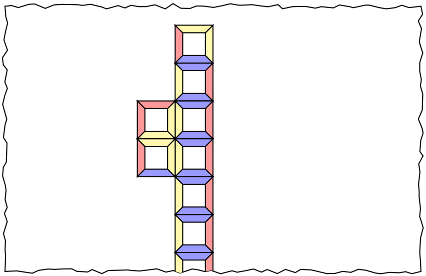

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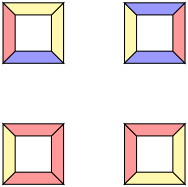

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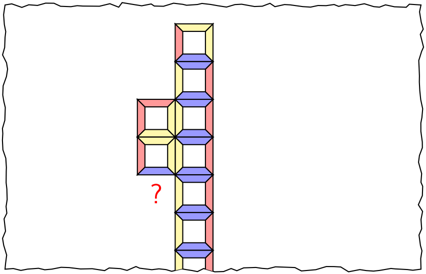

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- The complement of the tiling problem is semi-decidable.

Domino problem for Wang's tilings

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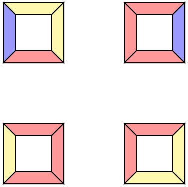

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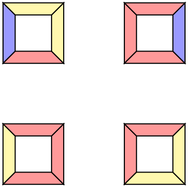

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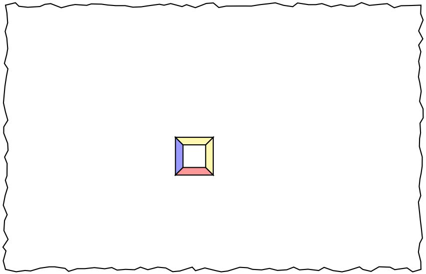

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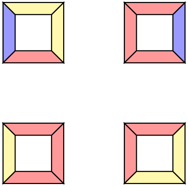

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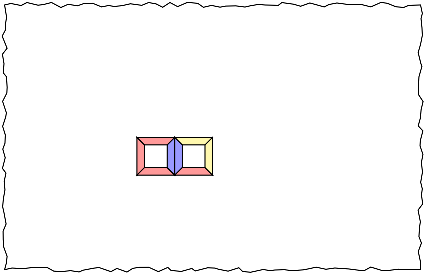

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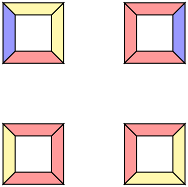

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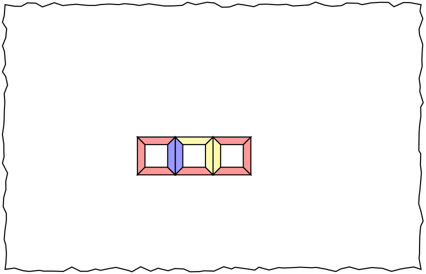

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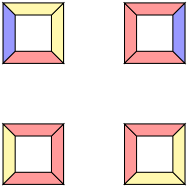

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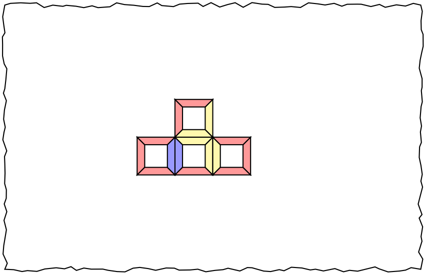

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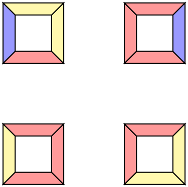

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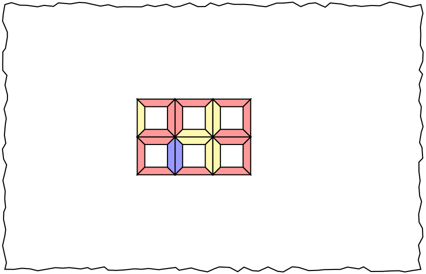

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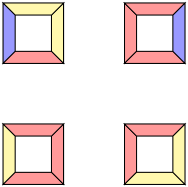

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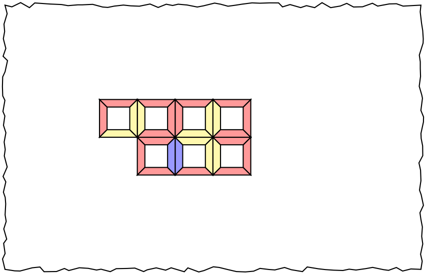

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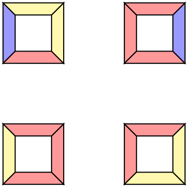

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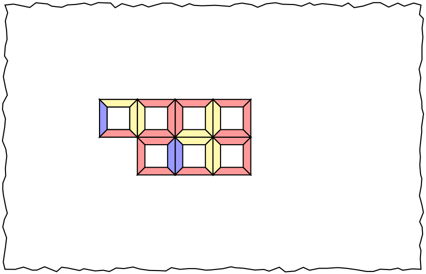

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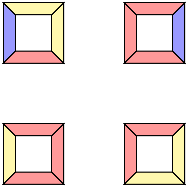

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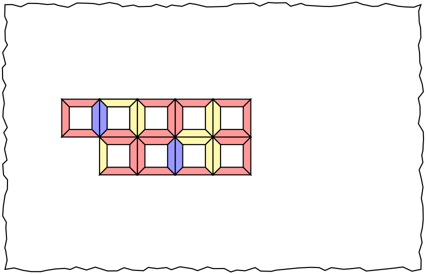

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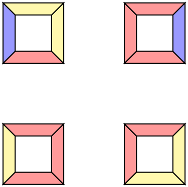

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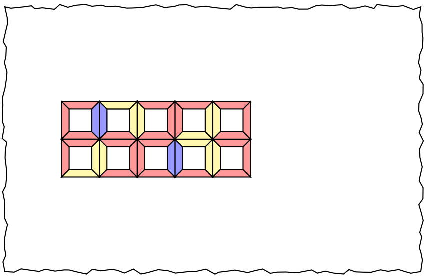

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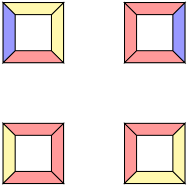

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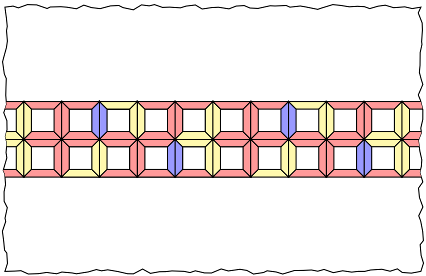

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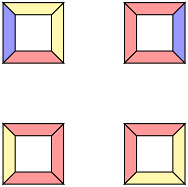

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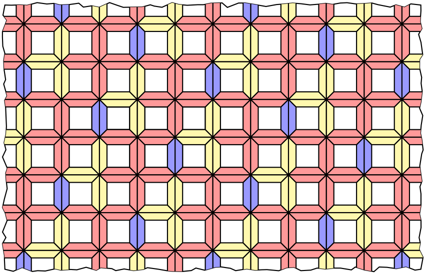


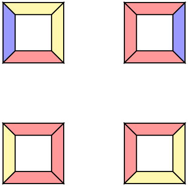
Domino problem

Does a given finite set $\mathcal{P}$ of Wang prototiles admit a tiling ?

- The complement of the tiling problem is semi-decidable.
- If there exists a periodic tiling, finding “algorithmically" a configuration is easy!

Domino problem for Wang's tilings

Tiles set



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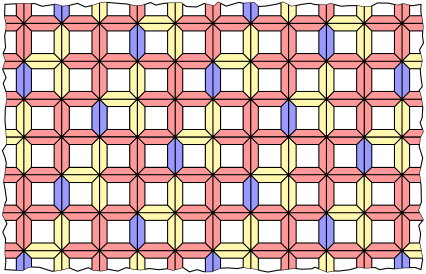


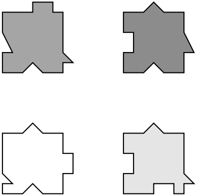
Theorem *(Berger-66, Robinson-71, Mozes-89, Kari-96...)*

- There exists tiles sets which produce only aperiodic tilings.
- The domino problem is undecidable.

...

Domino problem for Wang's tilings

Tiles set



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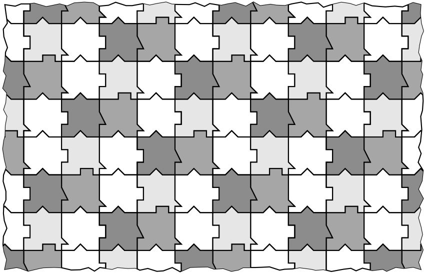

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Domino problem in dimension 1

Tile set:

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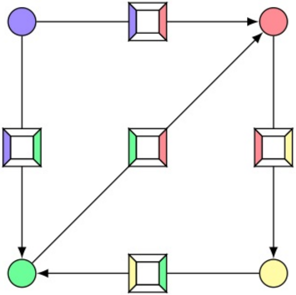
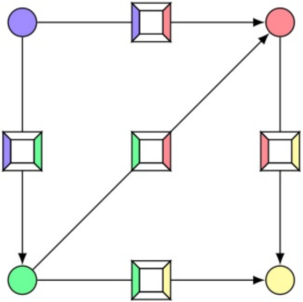
Every configuration can be seen as an infinite path in the following graphs:





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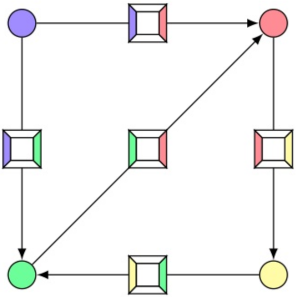
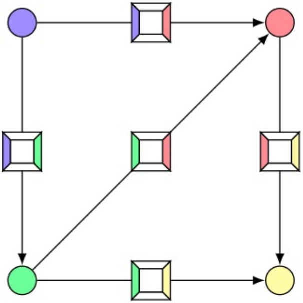


Tile the line

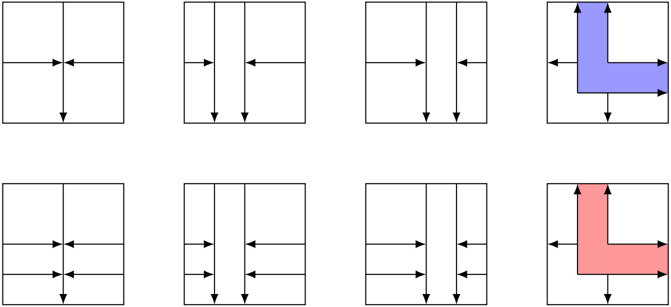


Do not tile the line

My first aperiodic tiling: Robinson's tiling

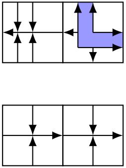
Alphabet of the tiling of Robinson

There exists different means to define the Robinson tiling. Consider $\mathcal{R}obi$ the next set of tiles modulo the rotation

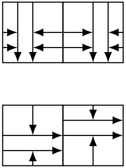


Local rules

Incoming and outgoing arrows must be respected:

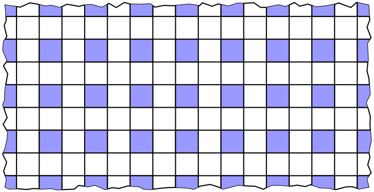


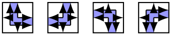
Allowed



Not allowed

We add the forbidden patterns $\mathcal{F}$ which impose the alternating of the colors:

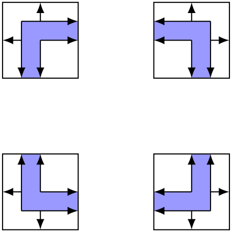


où 🟦 $\in \{$  $\}$

Denote $\mathbf{T}_{\mathcal{R}obi}$ the SFT described by these rules.

Existence of the tiling (level 1)

The Robinson tiling is based on a hierarchical structure. Nine of these tiles can be assembled to form a *super-tile of level 1*:



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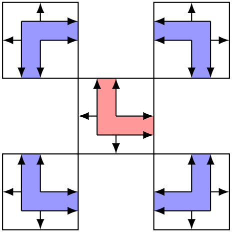

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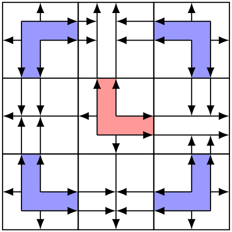
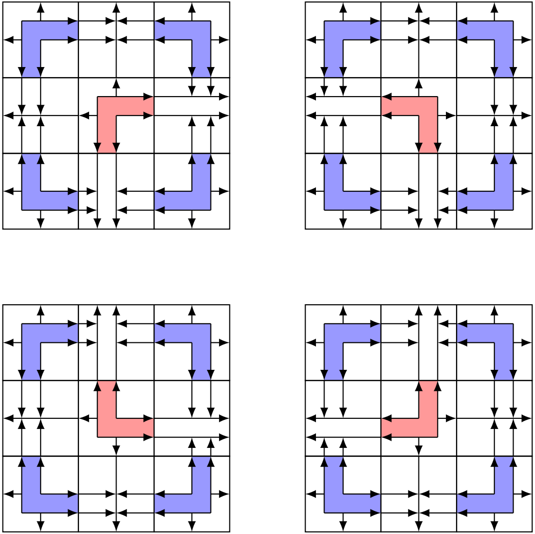


Existence of the tiling (level 2)

Super-tiles of level 1 can be assembled to form *super-tiles of level 2*:



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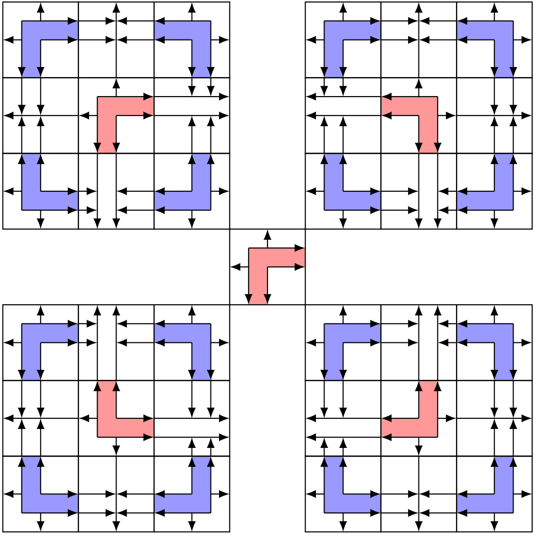

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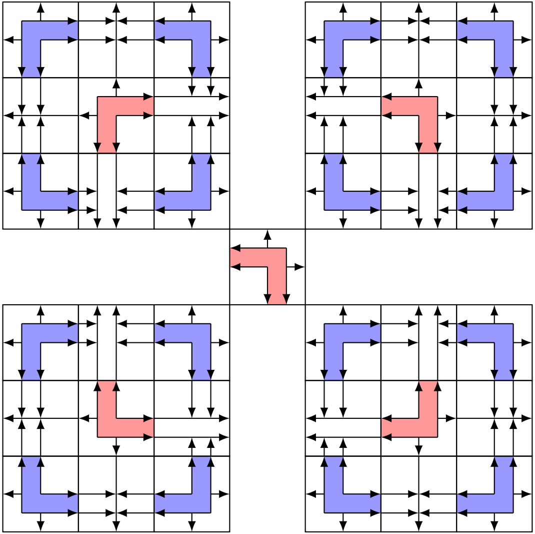

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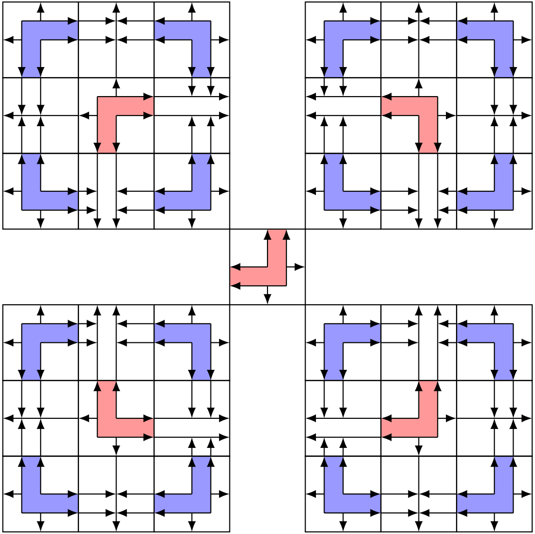

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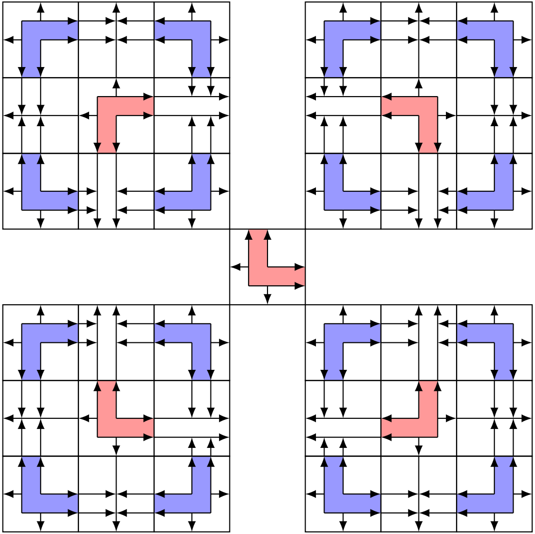

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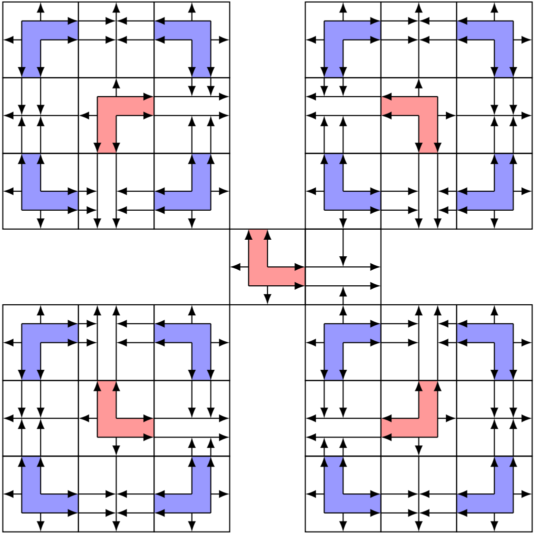

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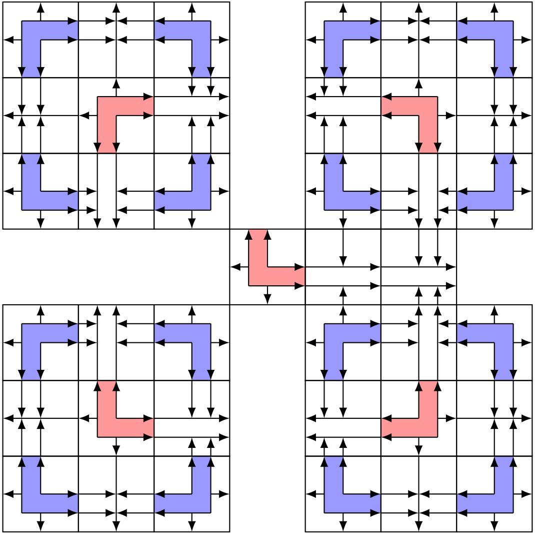

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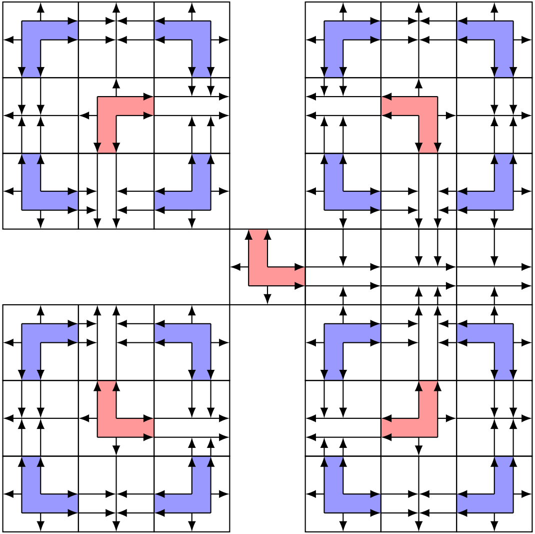

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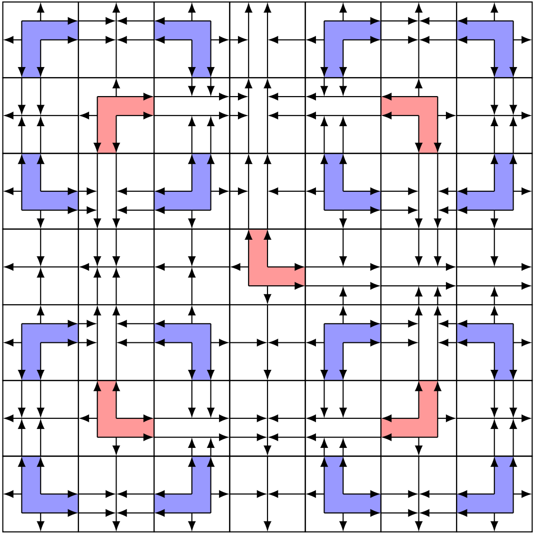
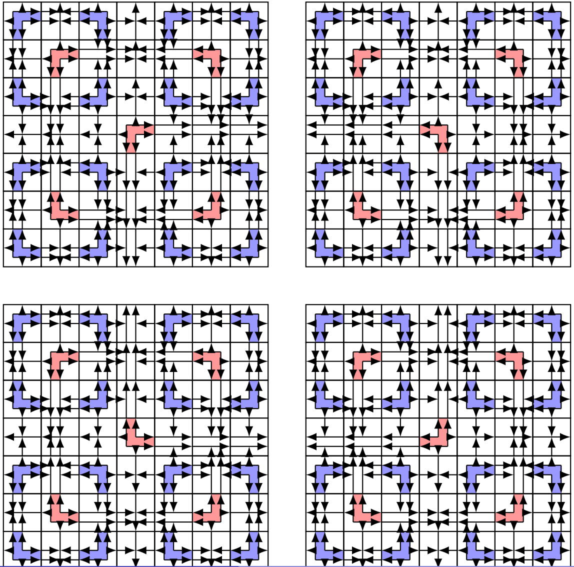


Existence of the tiling

Iterating the operation, super-tiles of level n can be formed and by compacity we conclude that $\mathbf{T}_{\mathcal{R}obi} \neq \emptyset$.



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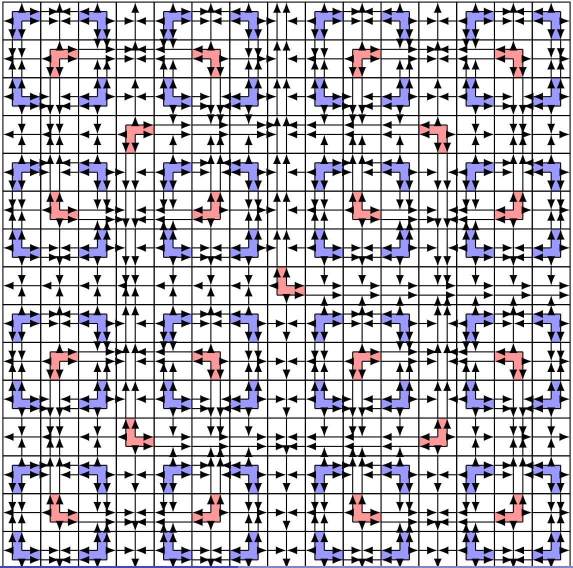
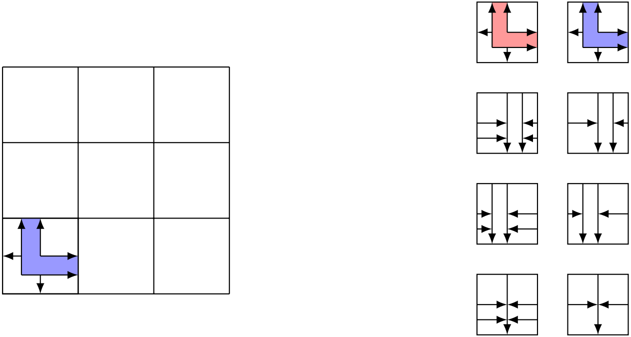


Force the presence of super-tiles (of level 1)



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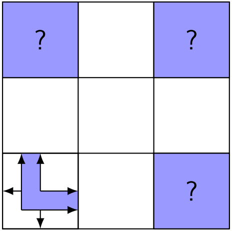
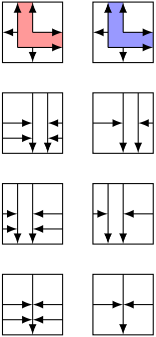
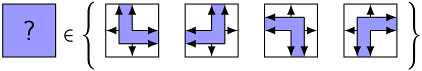




With 

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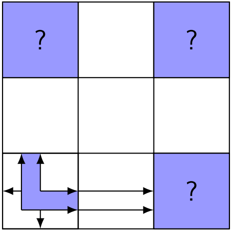

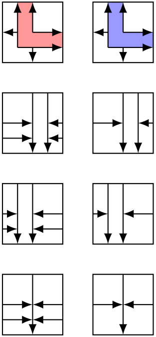

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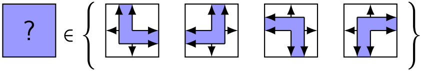

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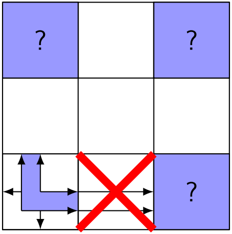

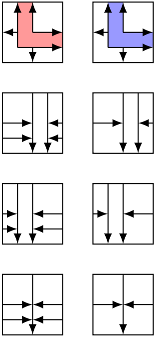

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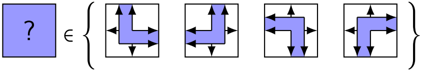

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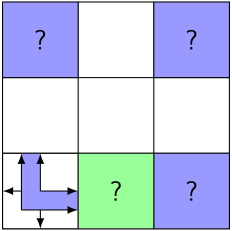
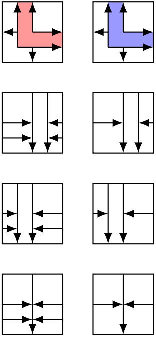
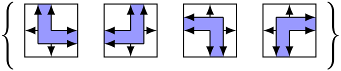




With ? ∈



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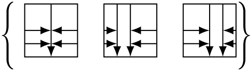

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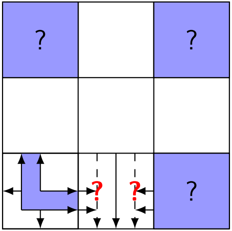

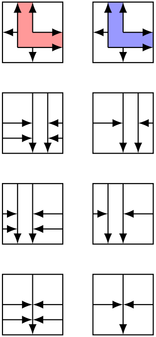

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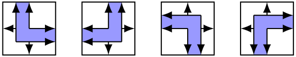
∈ {  }

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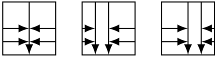
∈ {  }

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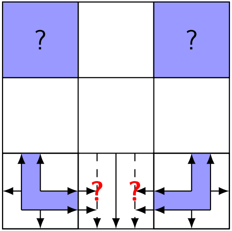
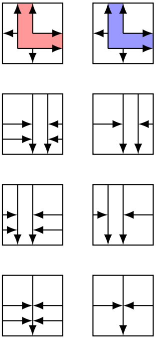
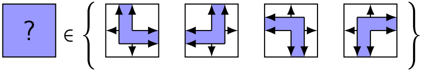
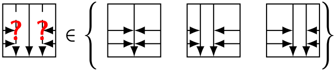




With 

With 

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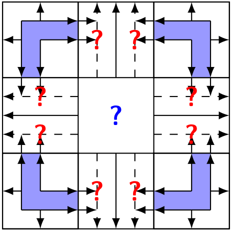
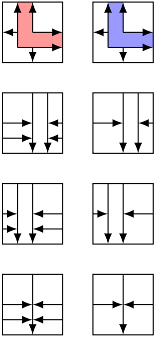
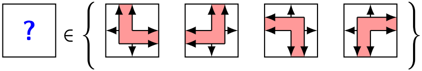
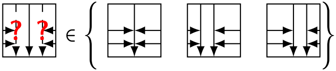




With 

With 

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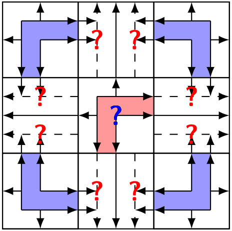
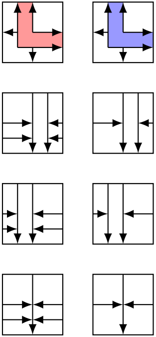
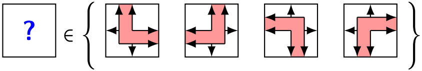
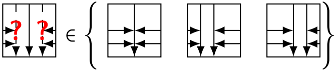




With 

With 

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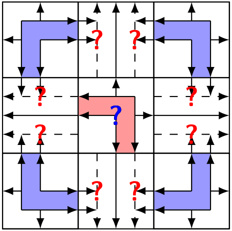
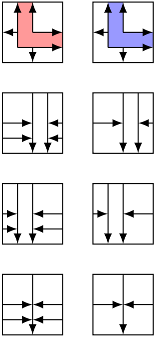
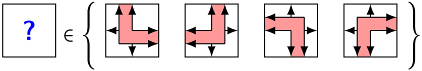
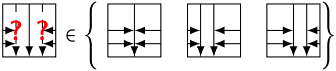




With 

With 

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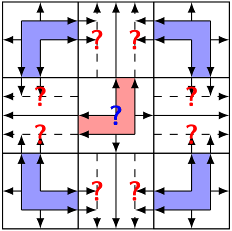
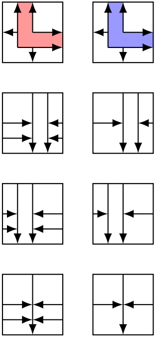
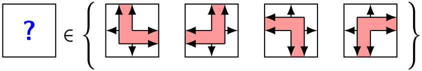
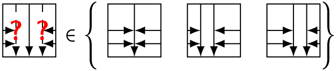




With 

With 

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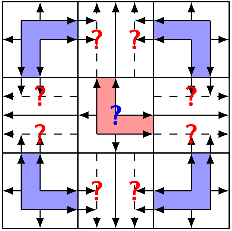
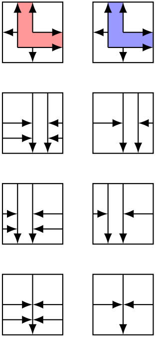
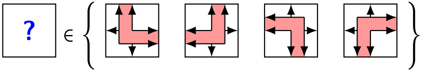
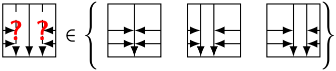




With 

With 

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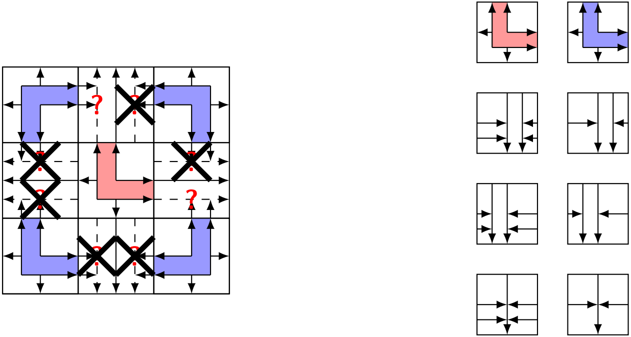
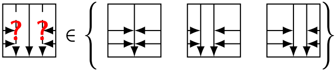


With 

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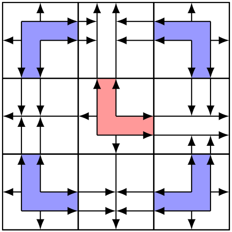

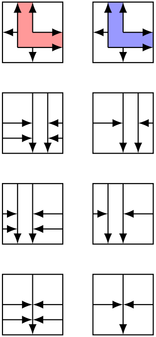

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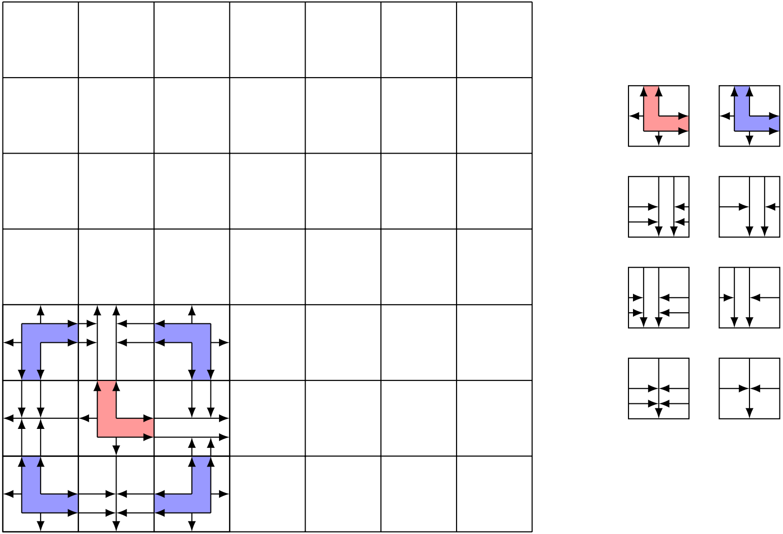

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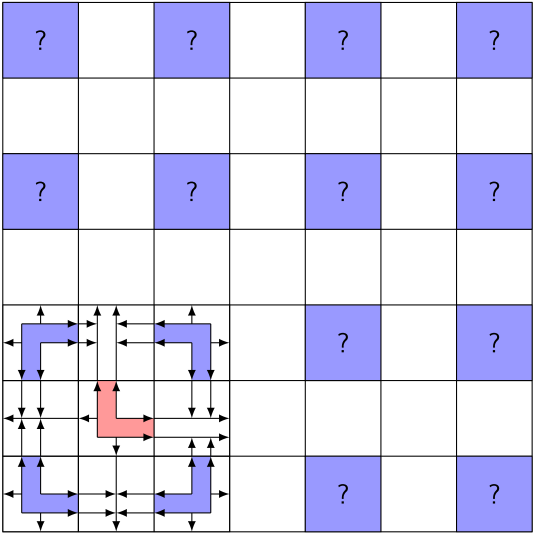

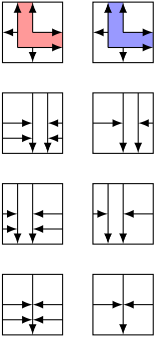

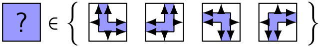

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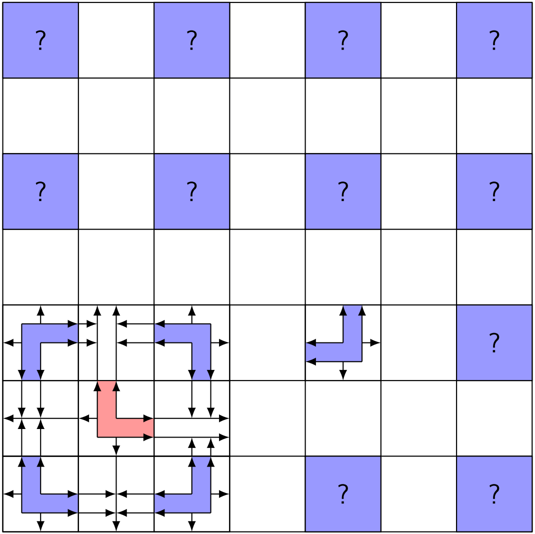

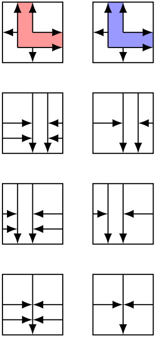

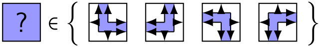

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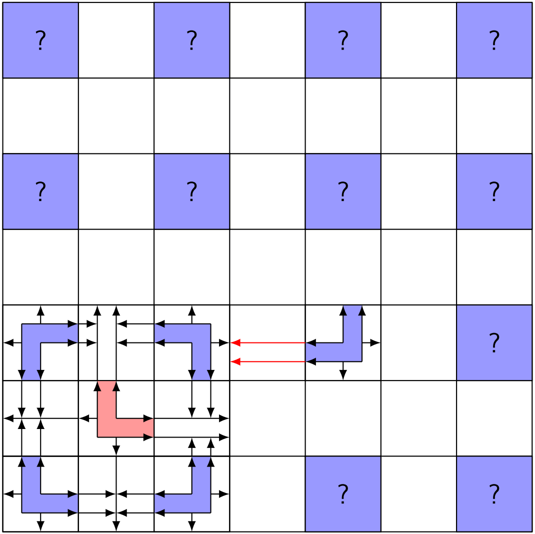

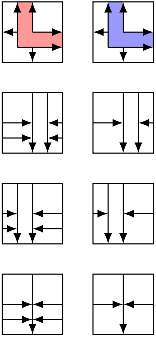

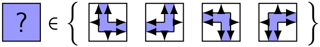

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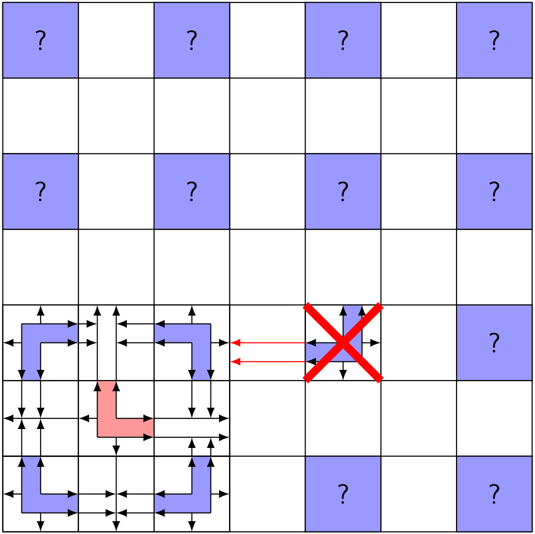

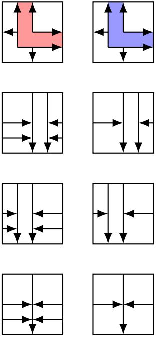

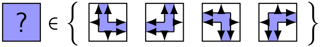

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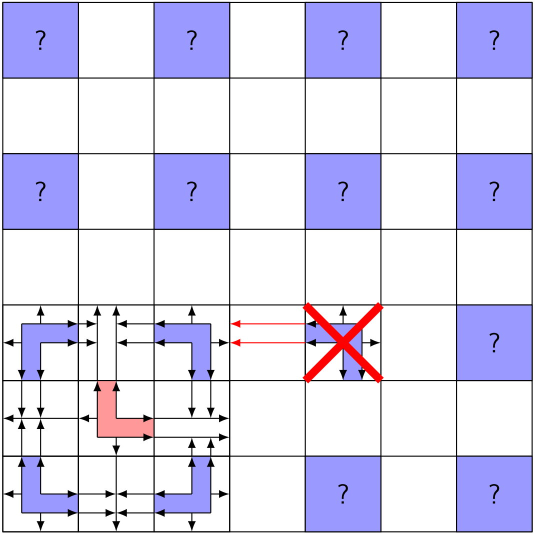

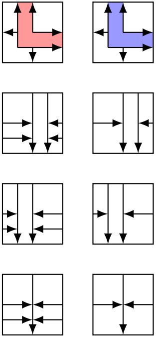

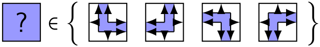

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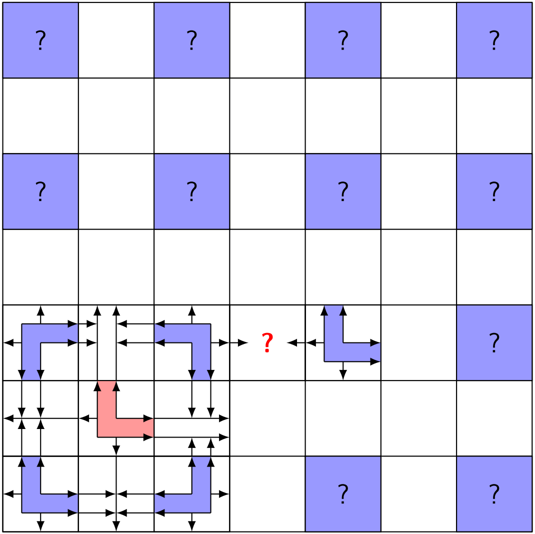

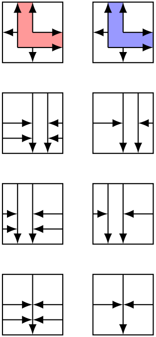

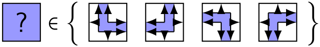

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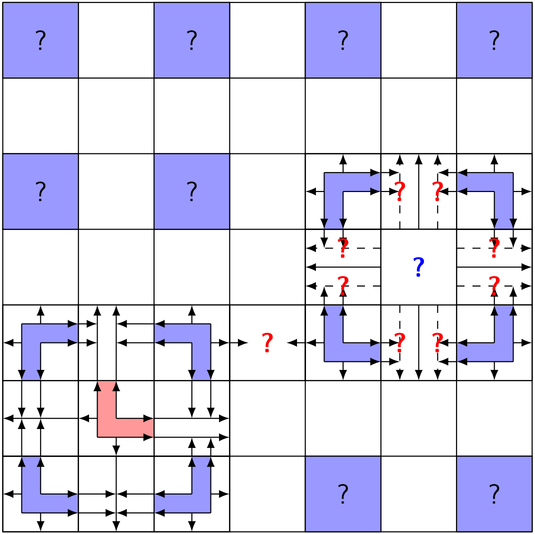

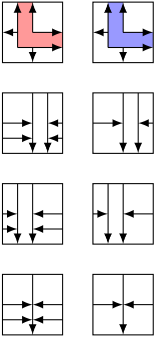

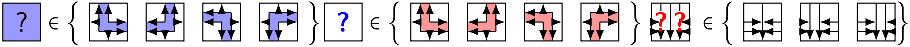

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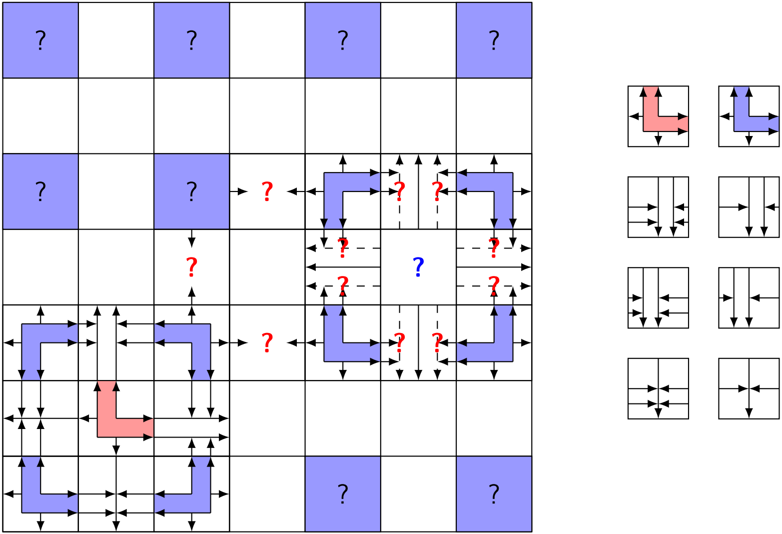

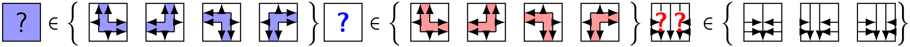

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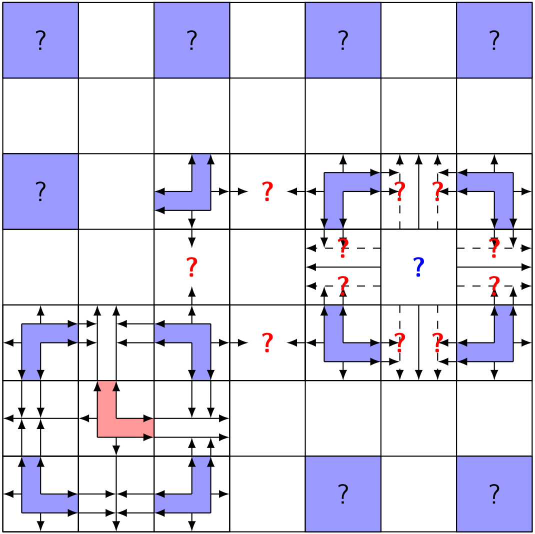

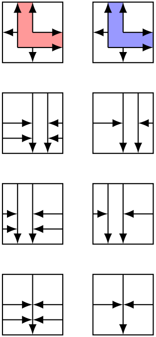

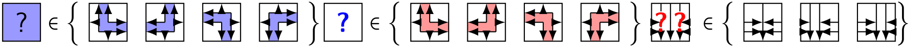

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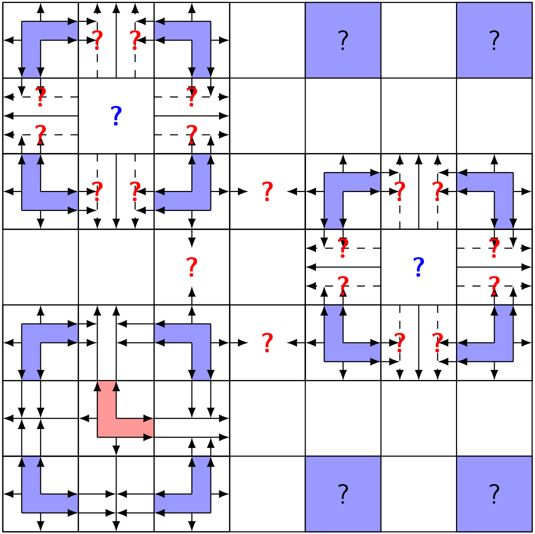

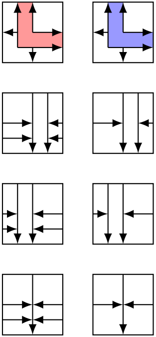

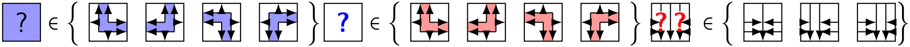

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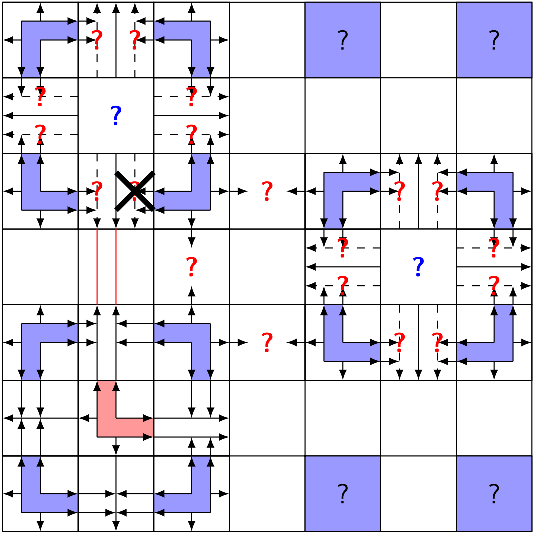

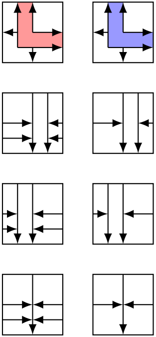

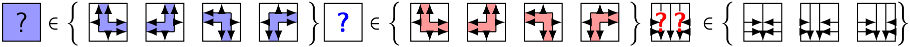

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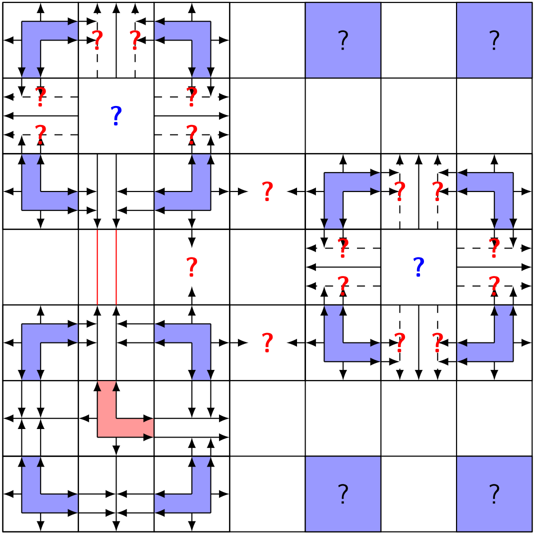

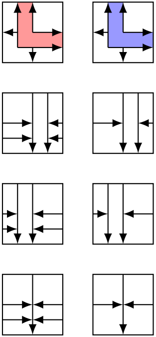

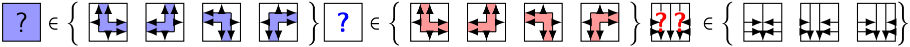

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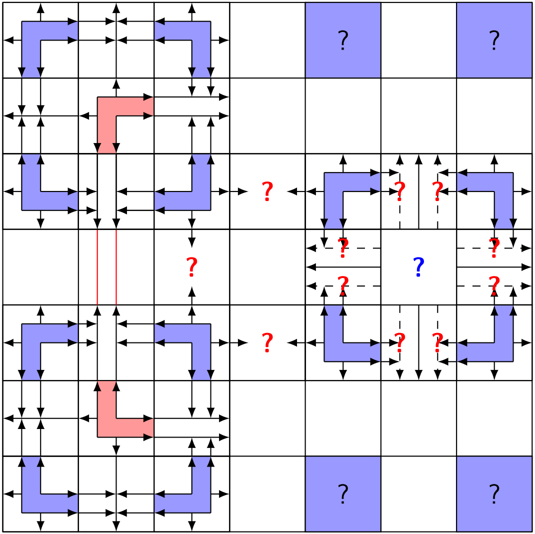

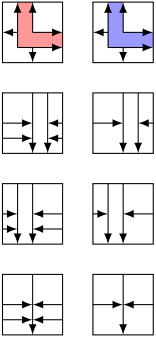

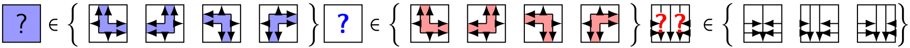

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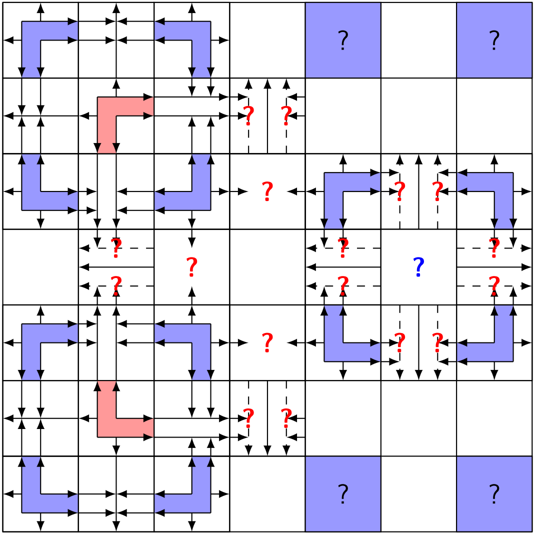

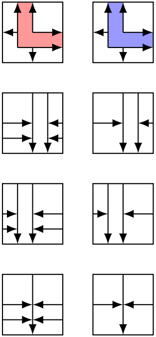

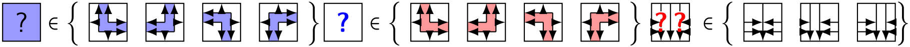

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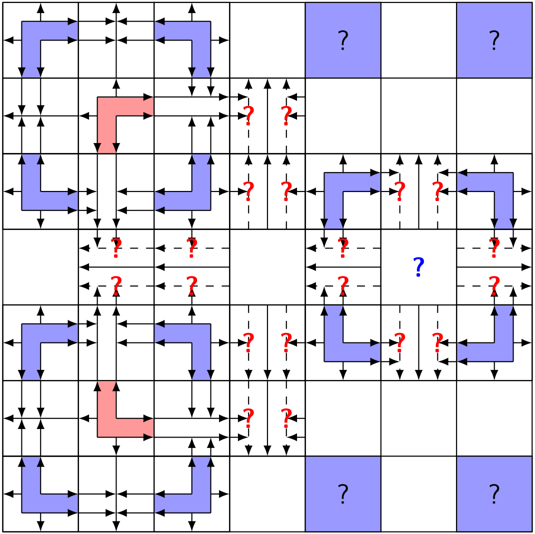

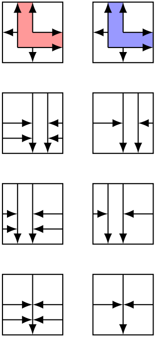

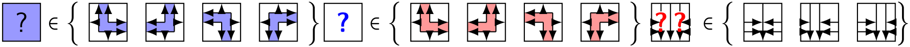

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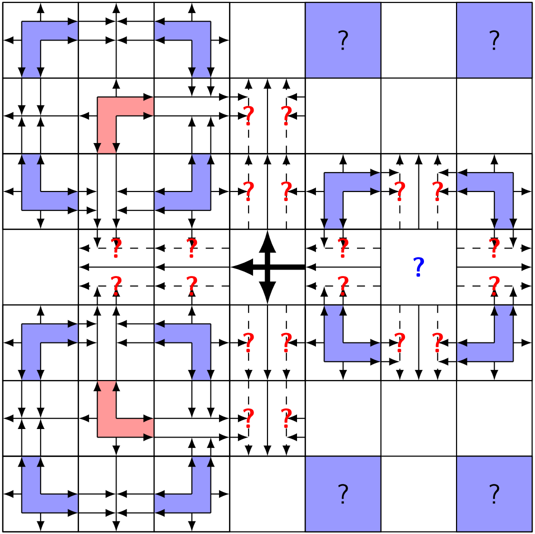

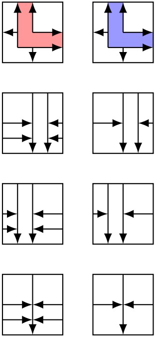

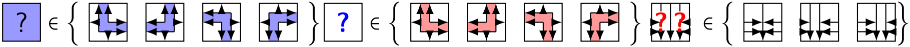

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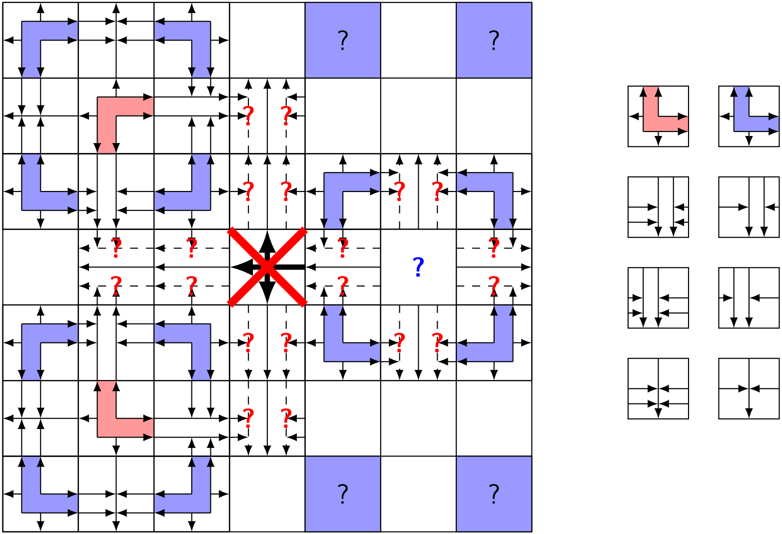

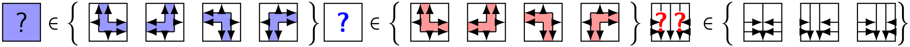

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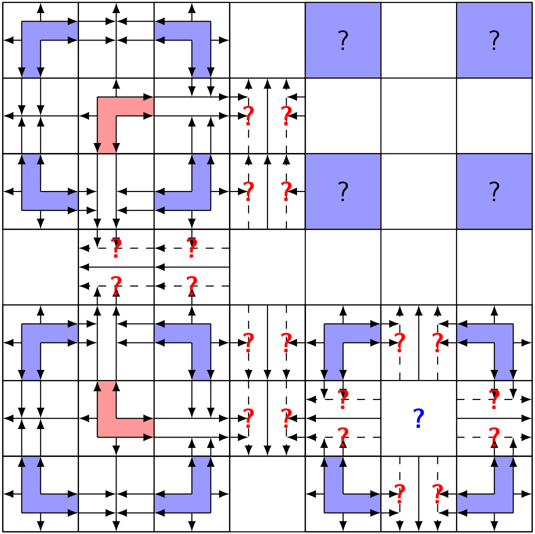

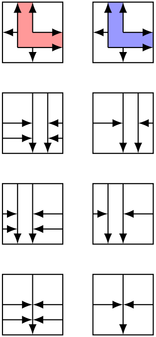

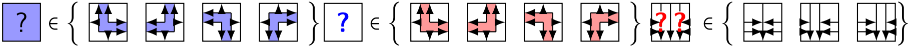

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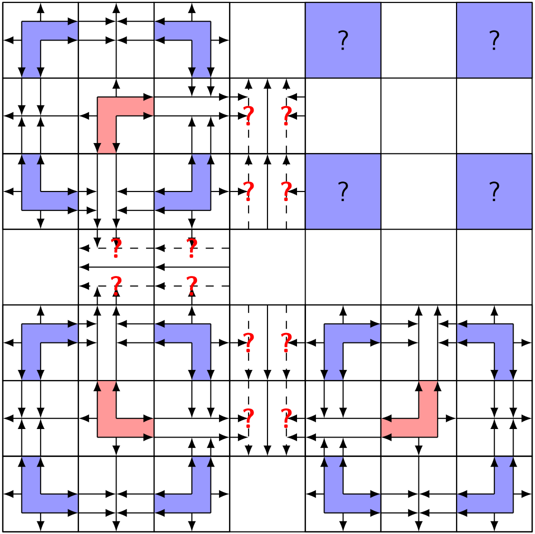

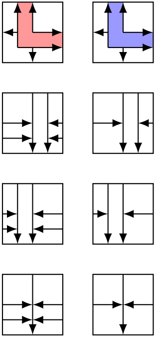

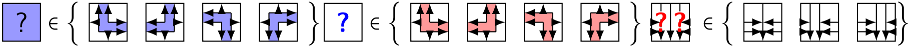

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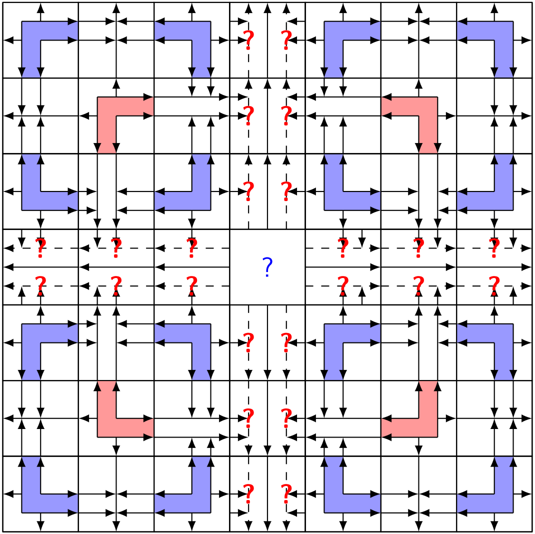

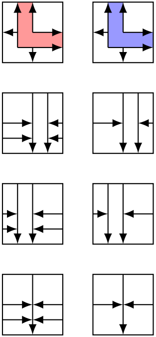

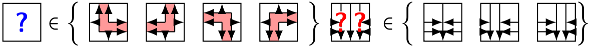

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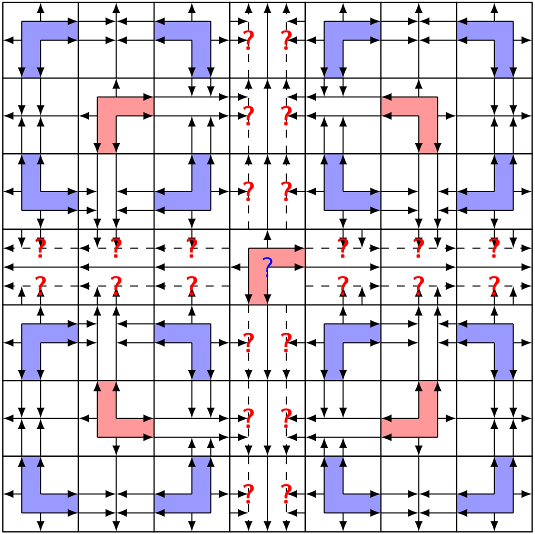

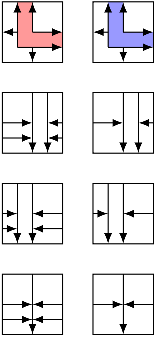

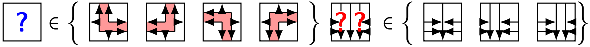

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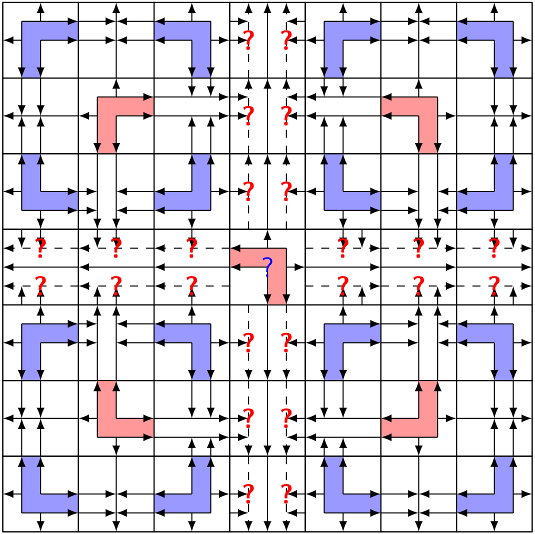

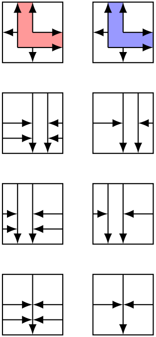

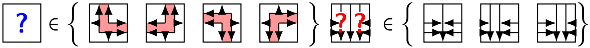

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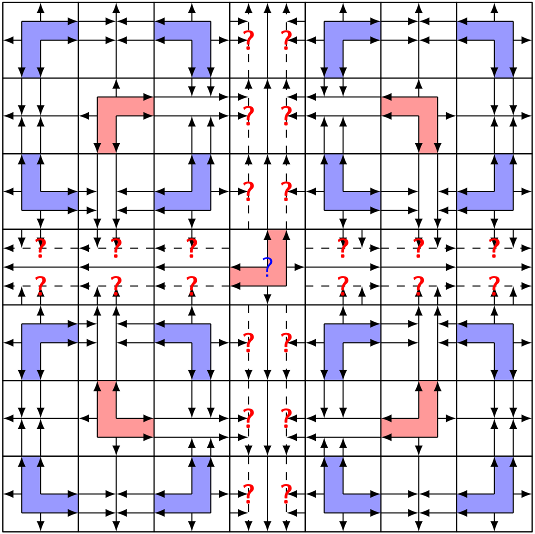

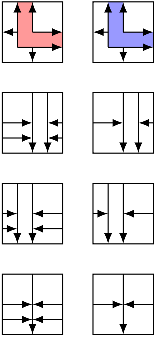

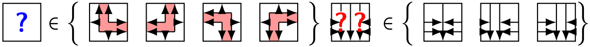

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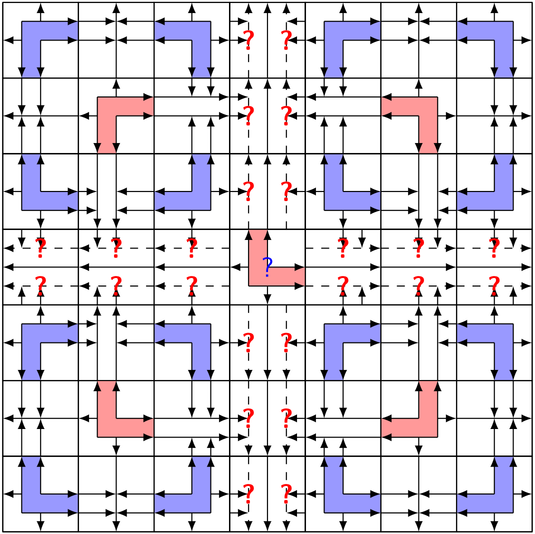

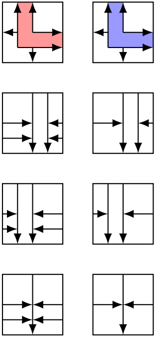

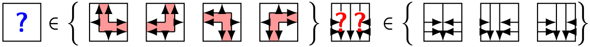

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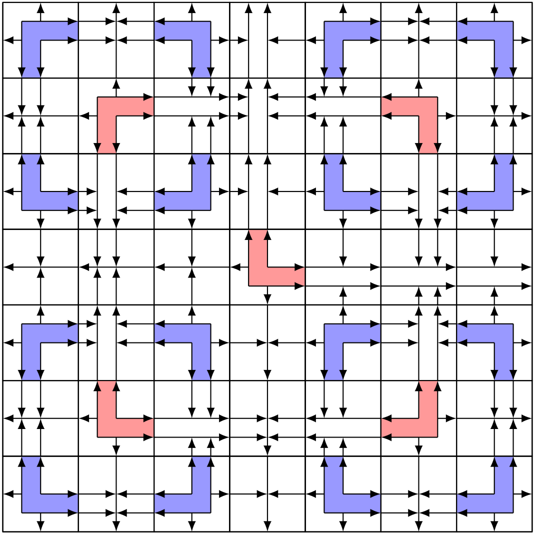
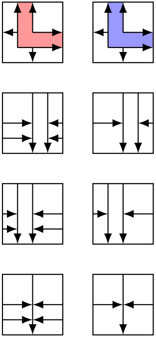
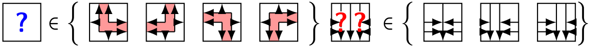
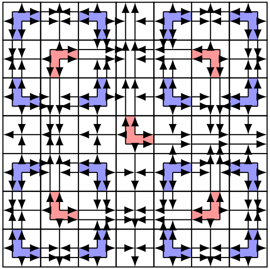






Force the presence of higher levels super-tiles



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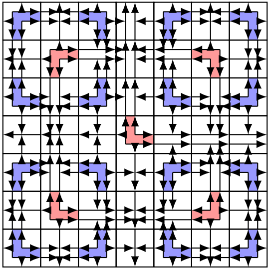

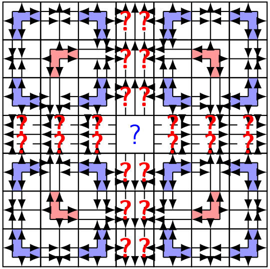

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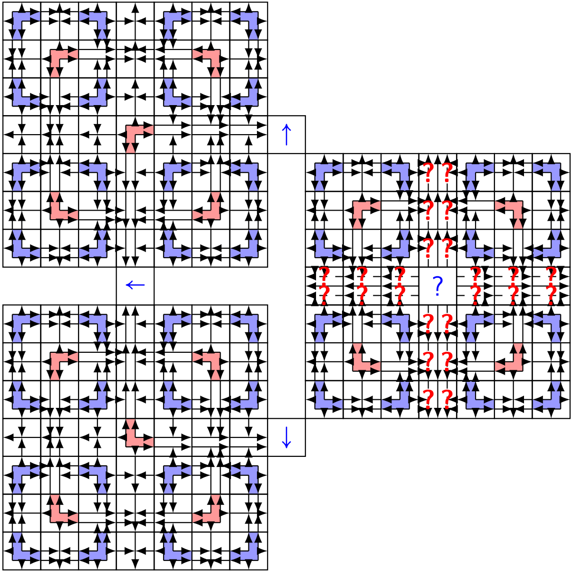

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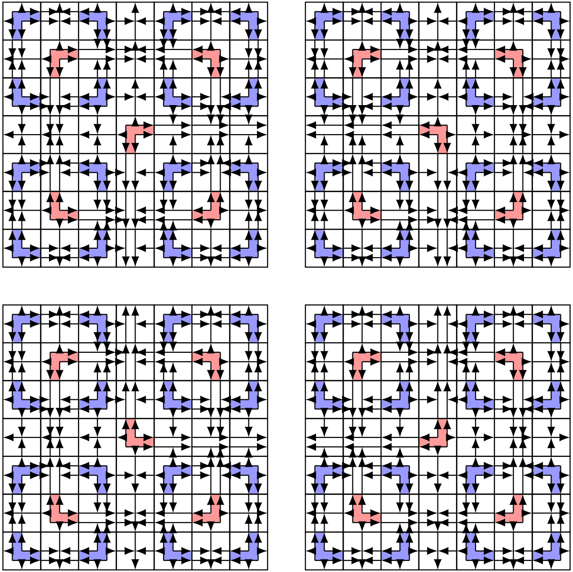
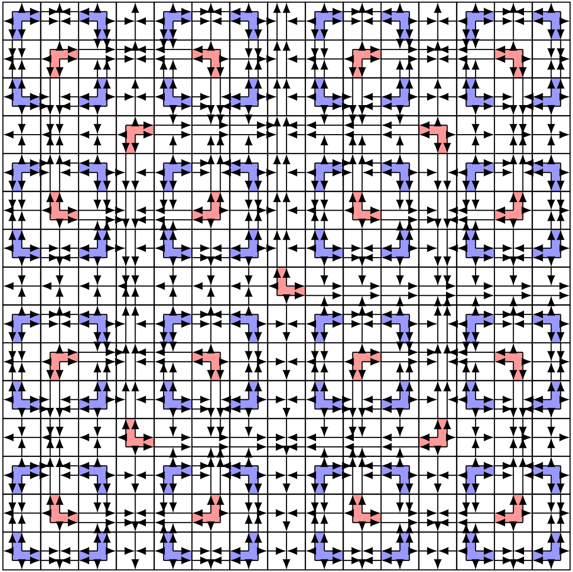


Geometric vision: Hierarchical structure



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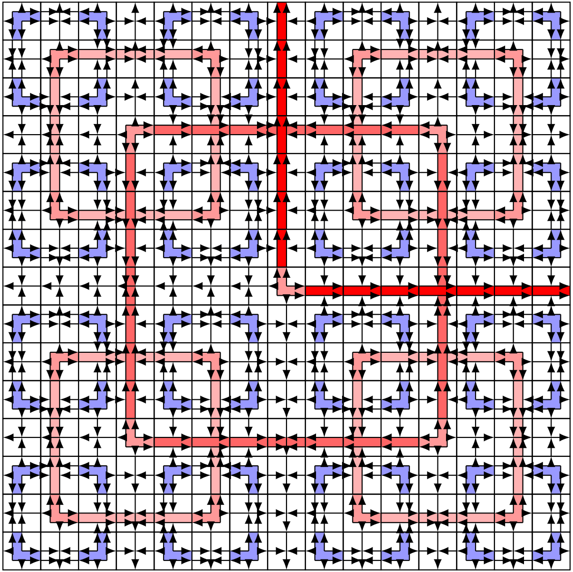
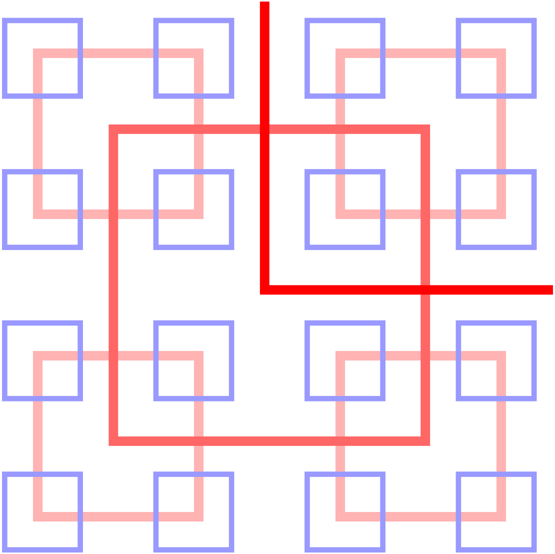
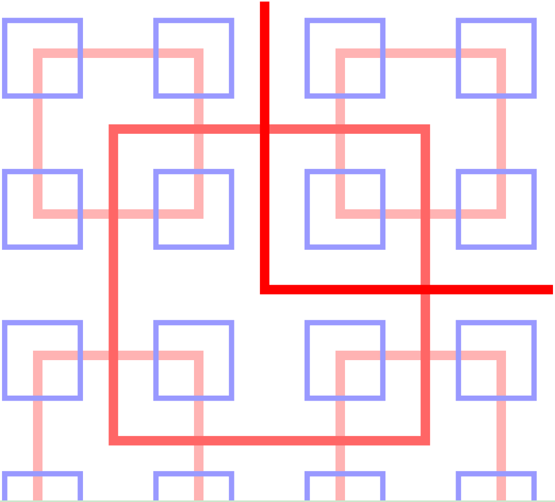


Geometric vision: Hierarchical structure



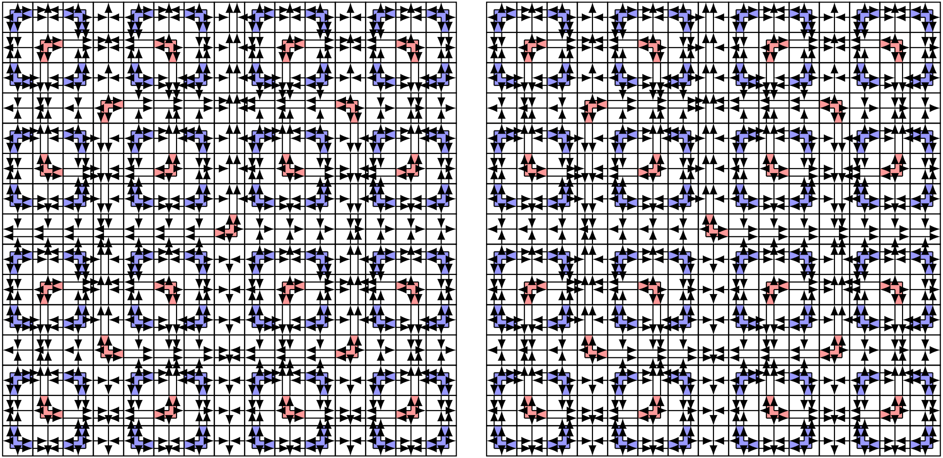
Geometric vision: Hierarchical structure



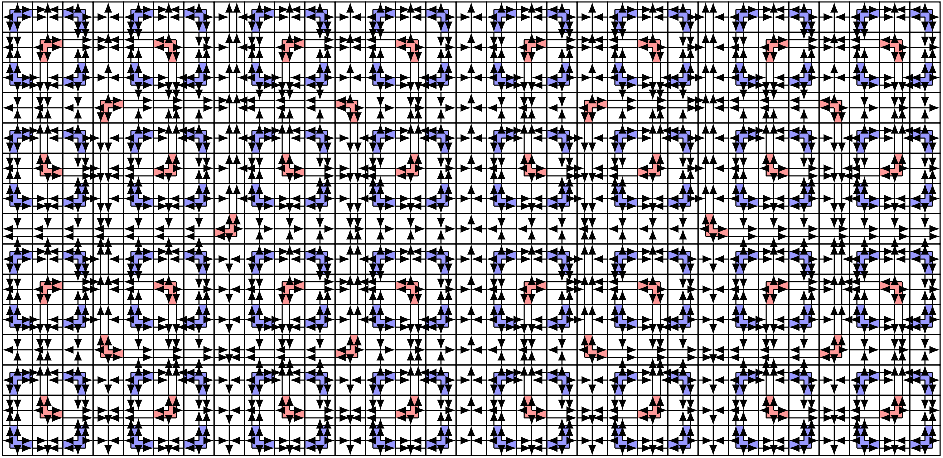
Theorem *(Robinson 1971)*

The SFT $\mathbf{T}_{\mathcal{R}obi}$ is not empty and all configurations are aperiodics.

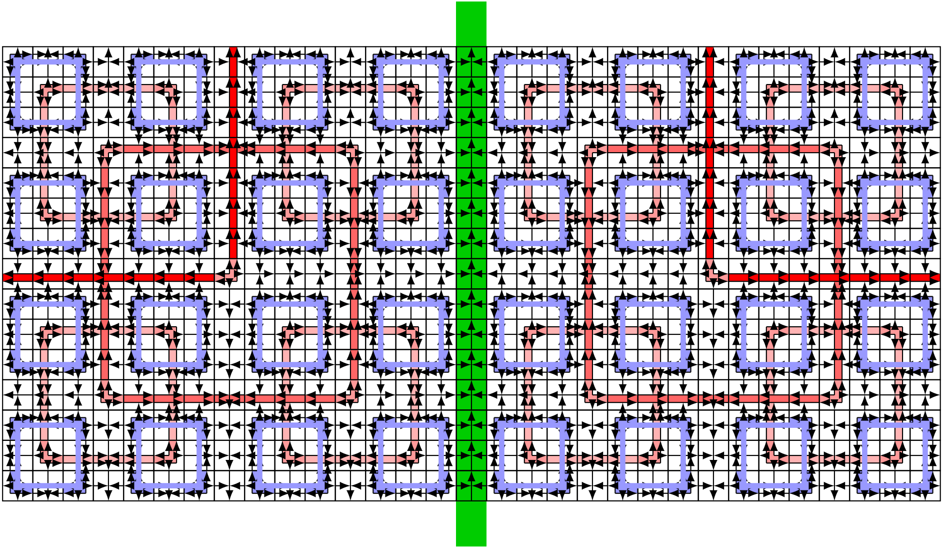
Geometric Vision: Fractured lines



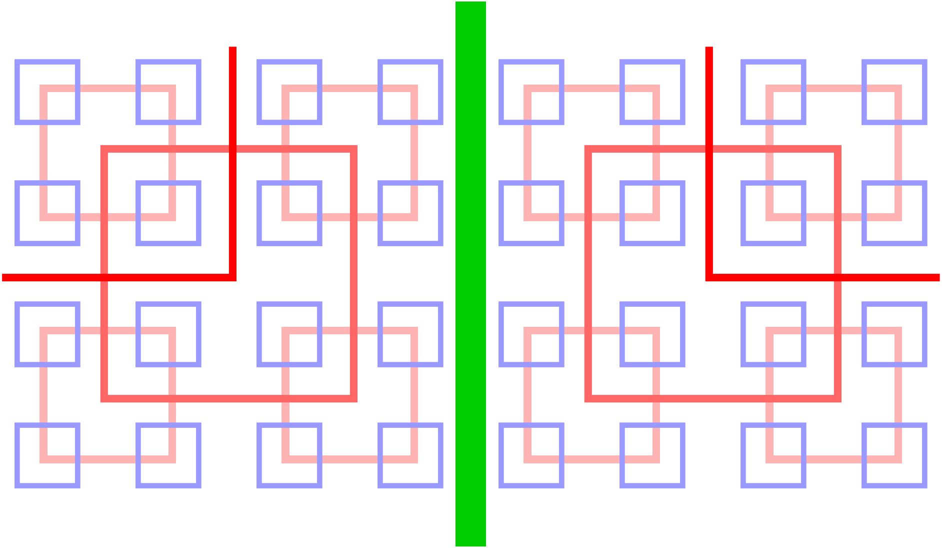
Geometric Vision: Fractured lines



Geometric Vision: Fractured lines



Geometric Vision: Fractured lines



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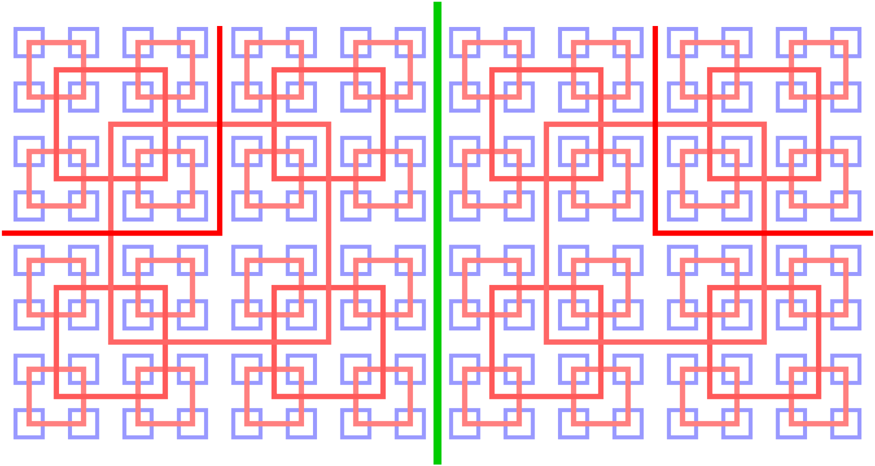

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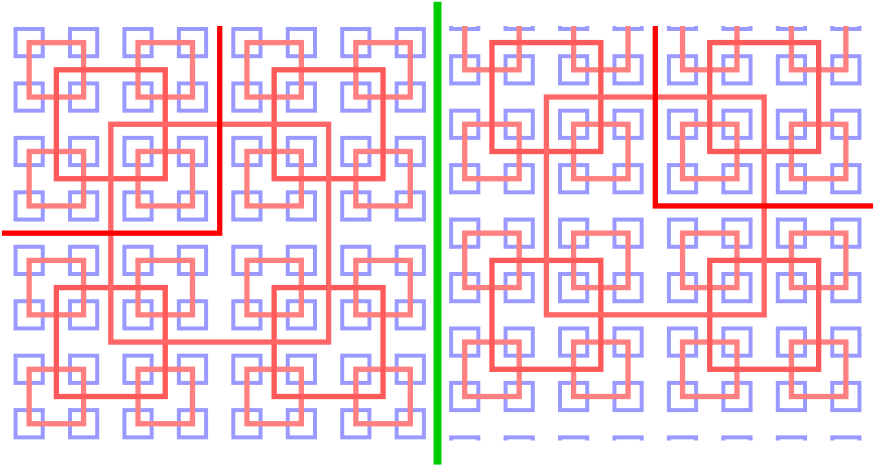

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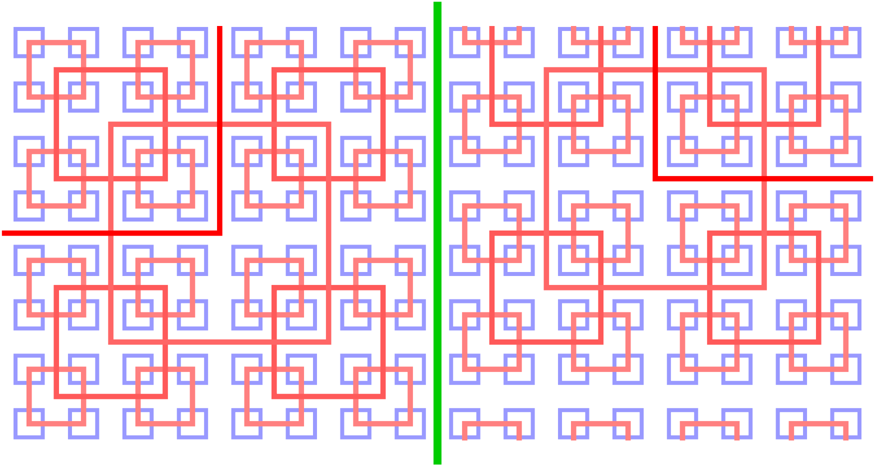
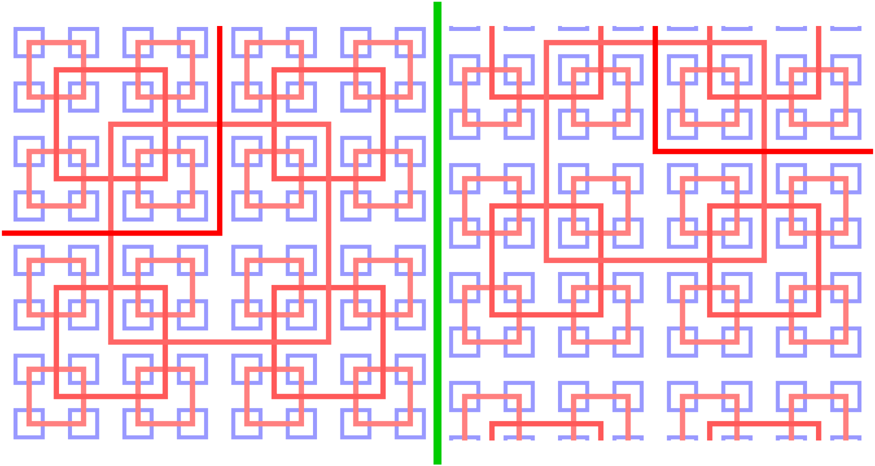


Geometric Vision: Fractured lines



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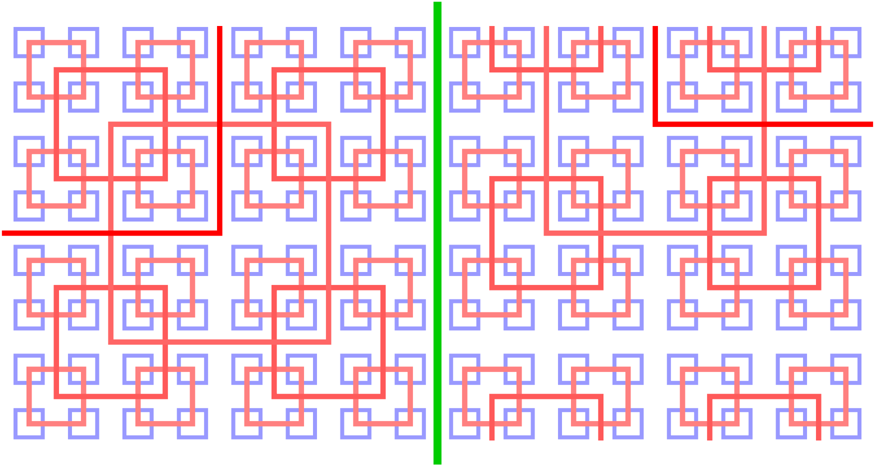
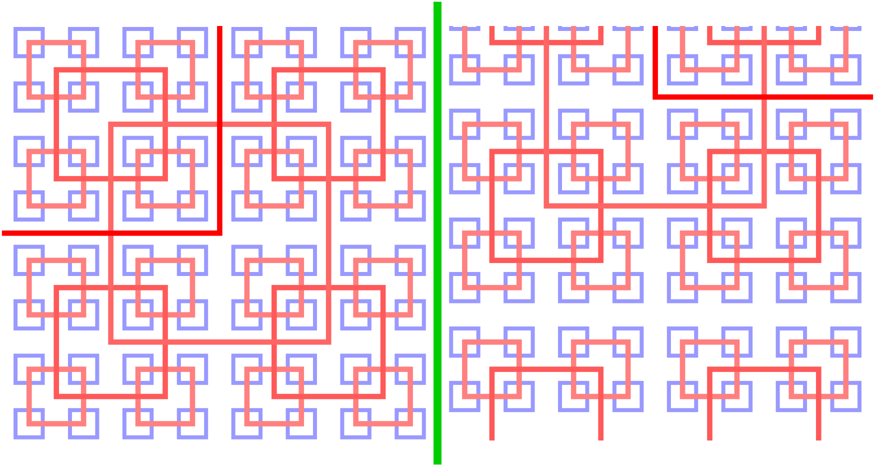


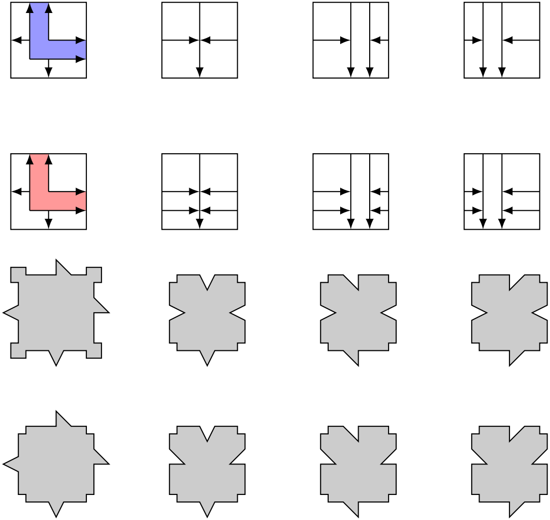
Geometric Vision: Fractured lines



Proposition

$\mu(\{x \in \mathbf{T}_{\mathcal{R}obin} \text{ with fracture lines}\}) = 0$ for all probability measure μ σ-invariante.

Robinson's tiles for children



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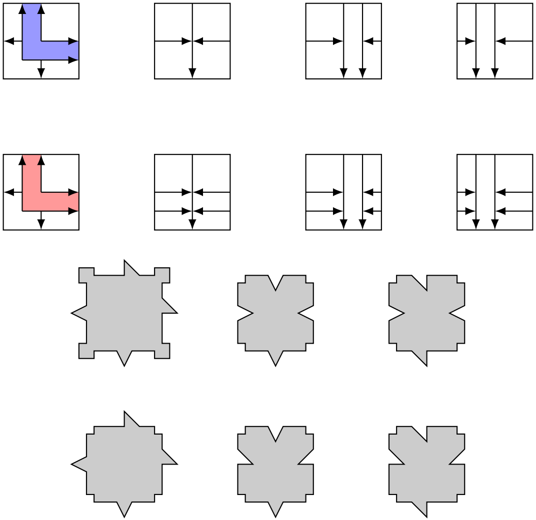

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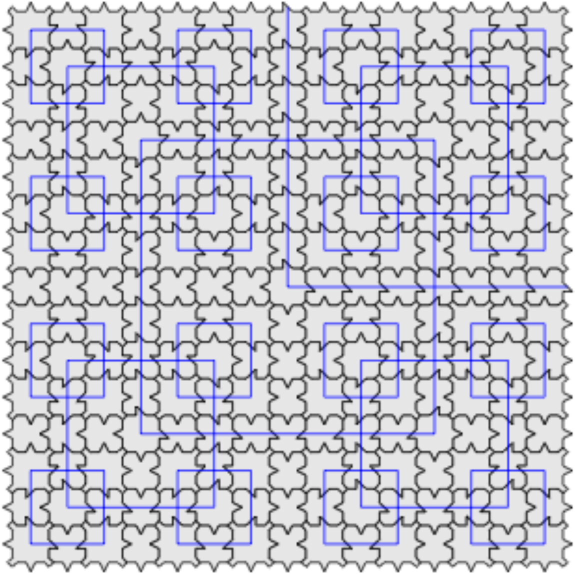
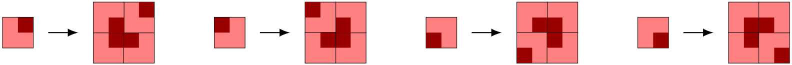


Multidimensional substitutions

Rectangular substitution

Let $\mathcal{A} = \Big\{$  $\Big\}$. Consider the next substitution:

$s:$



Rectangular substitution

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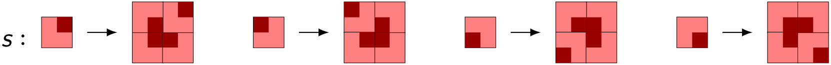
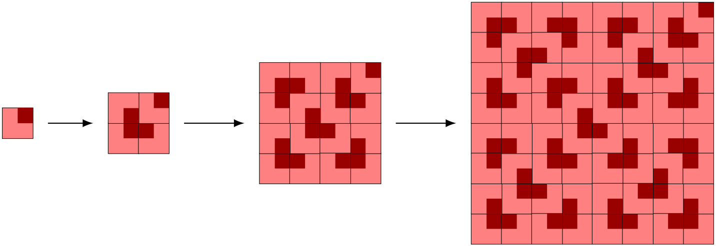


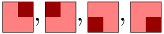
After iteration, we obtain:

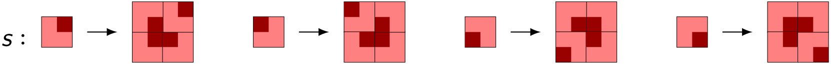


Define *the substitutif subshift*:

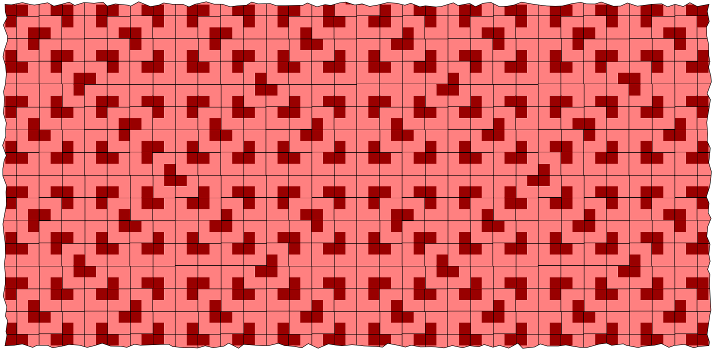
$$\mathbf{T}_s = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \forall p \sqsubset x \qquad \exists a \in \mathcal{A},\ n \in \mathbb{N},\ tel\ que\ p \sqsubset s^n(a) \right\}$$

Rectangular substitution

Let $\mathcal{A} = \Big\{$  $\Big\}$. Consider the next substitution:



After iteration, we obtain:



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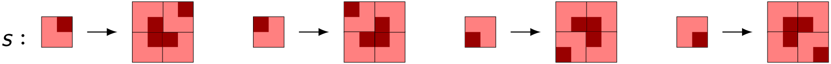
Rectangular substitution

Let $\mathcal{A} = \{$▝, ▘, ▖, ▗$\}$. Consider the next substitution:



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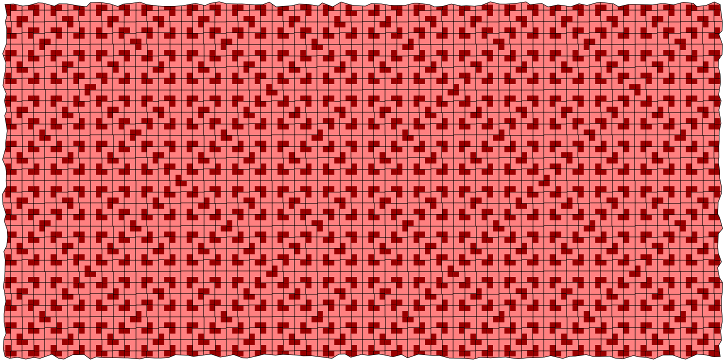

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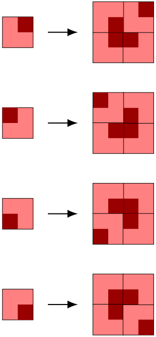
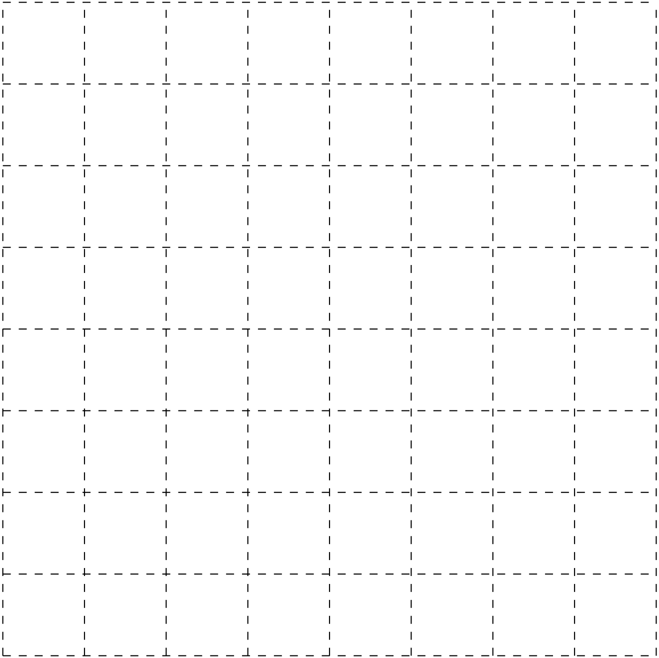
How forces the substitutive tiling with local rules?





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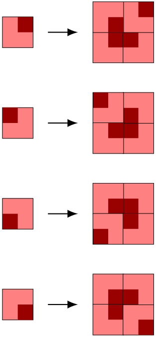

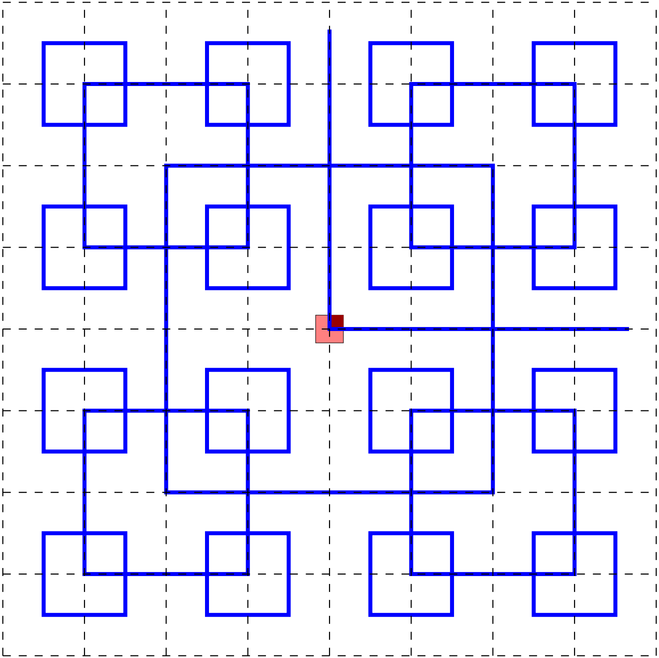

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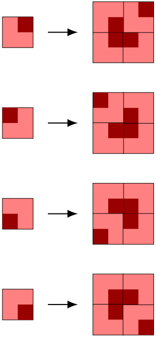

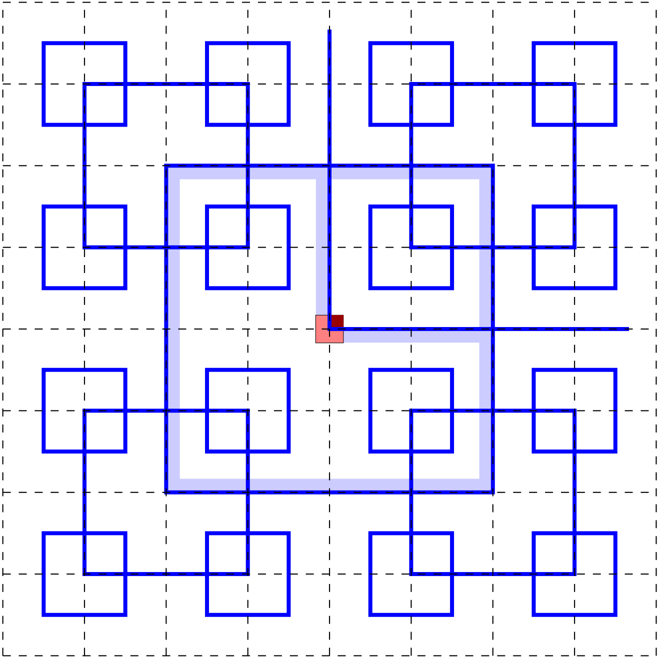

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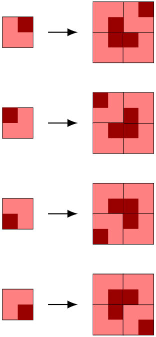

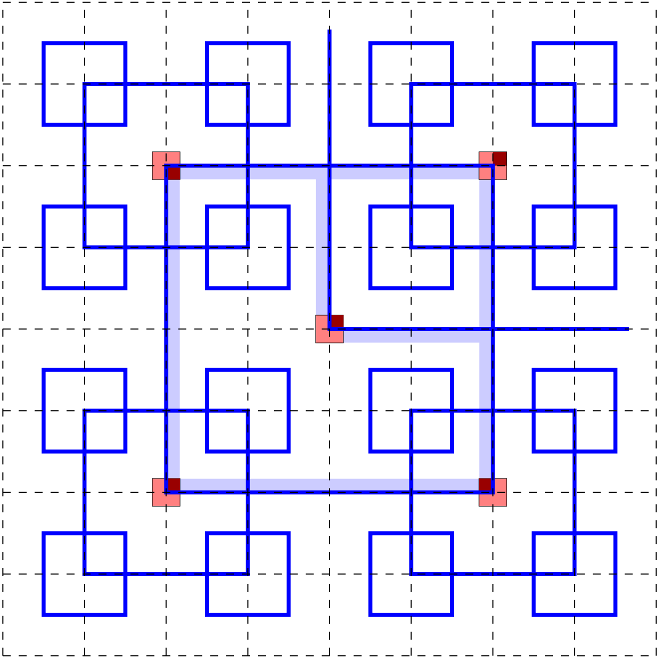

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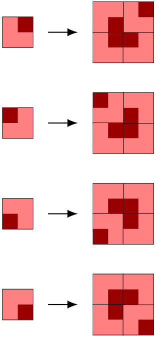

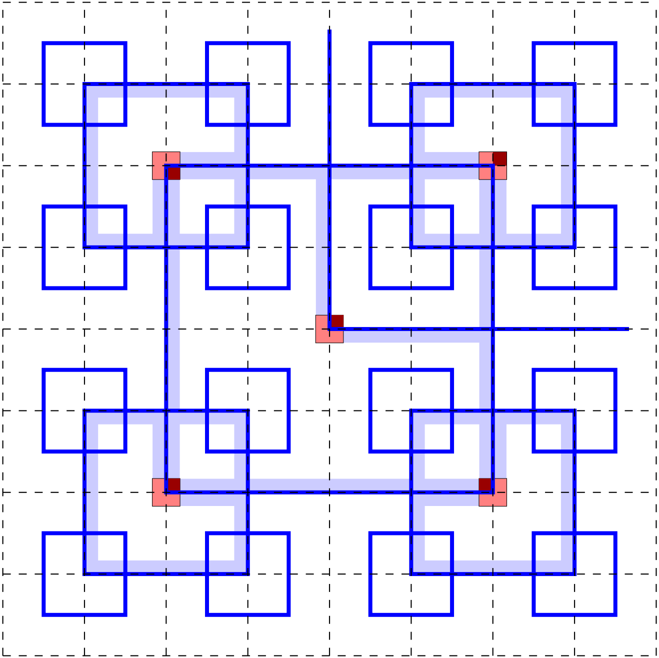

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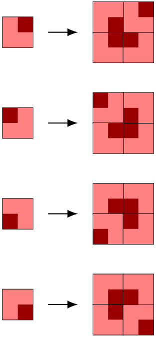

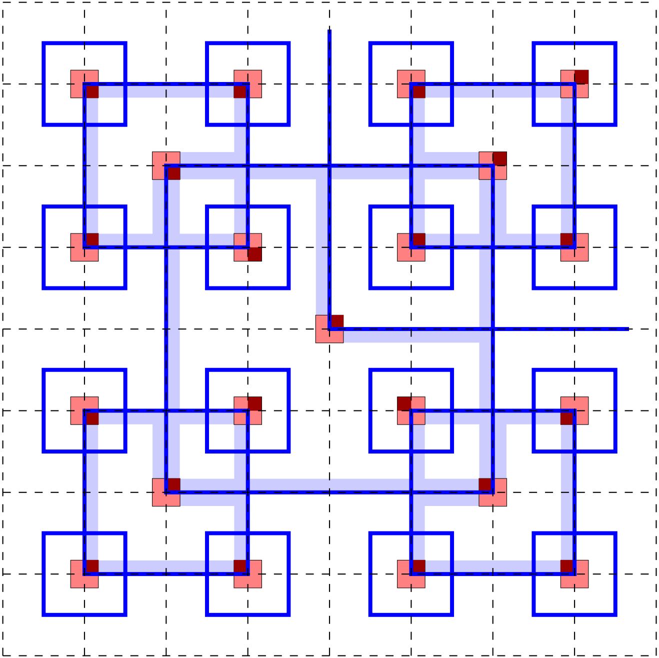

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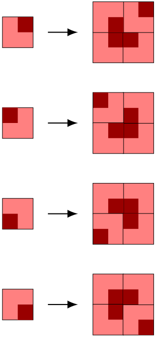

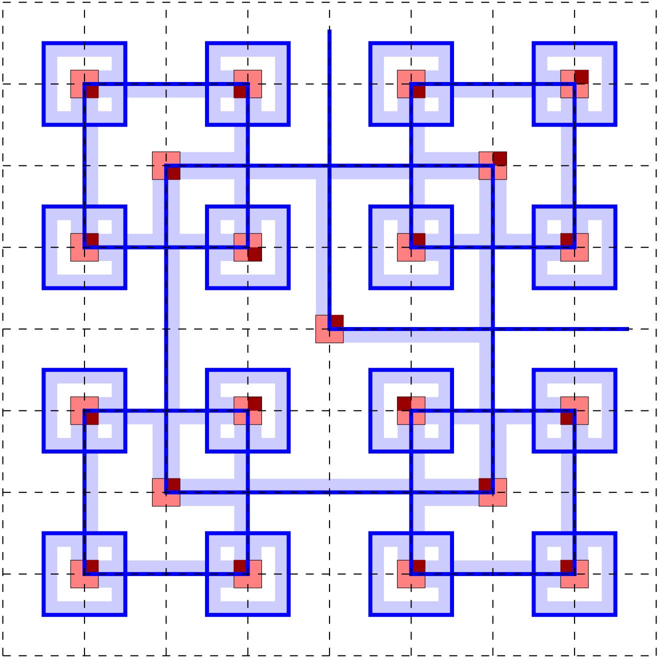

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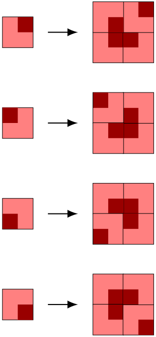

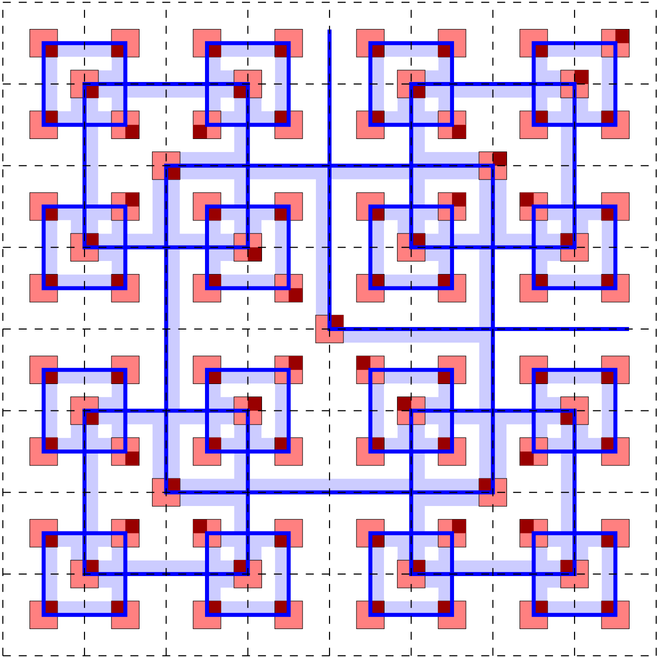

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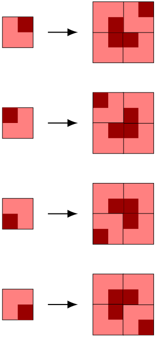

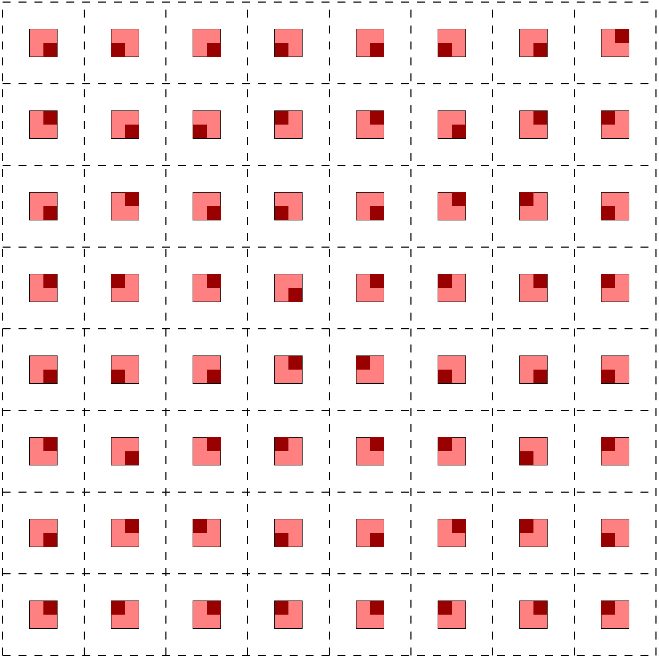

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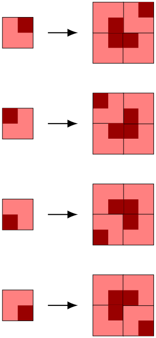
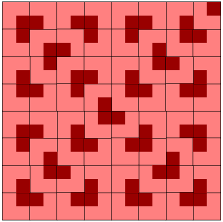
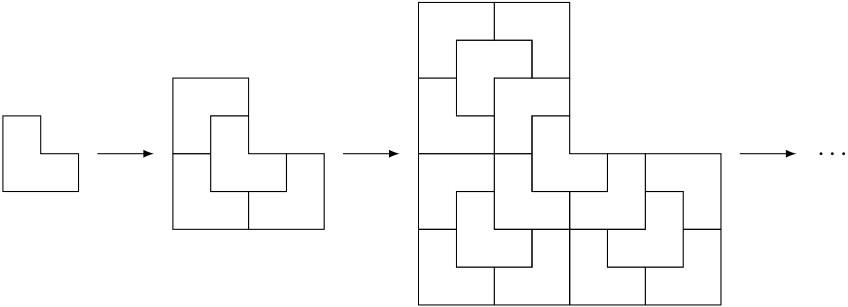




Substitution of polygons



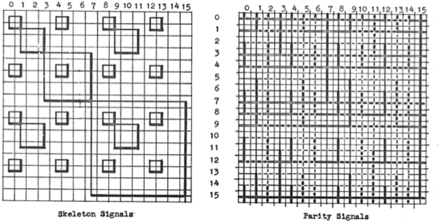
Theorem *(Goodman-Strauss 1998)*

The tiling space defined by substitution on polygon can be defined by local rules.

Little historic of aperiodic tilings and perspectives

Little historic of aperiodic tilings

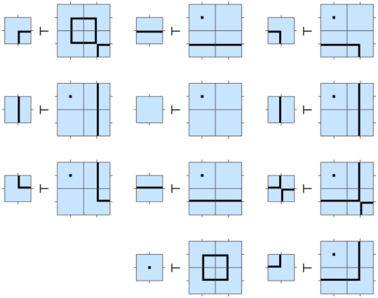
Auteurs	Number of Wang	Nombre de tuiles à translation près	Nombres de tuiles à isométrie près
R. Breger 1966	20 426		
			



Skeleton Signals

Parity Signals

Figure 24 Part of the Solution of Q



Little historic of aperiodic tilings

Auteurs	Number of Wang	Nombre de tuiles à translation près	Nombres de tuiles à isométrie près
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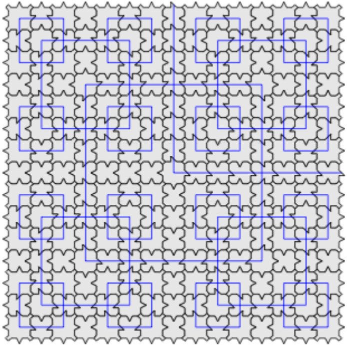

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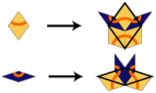

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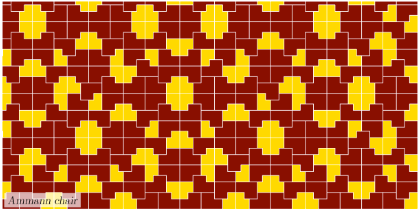

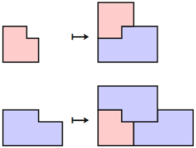

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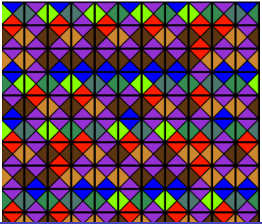
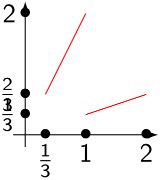
J. Kari 1996	14		
K. Culick 1996	13		





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J. Kari 1996	14		
K. Culick 1996	13		
M. Rao-E. Jeandel 2017	11		

Problematic

There exists other type of aperiodic tilings?

Decision problem

Decision problem

Domino problem in the SFT setting:

Given $\mathcal{A}$ a finite alphabet and $\mathcal{F}$ a finite set of 2-dimensional patterns.

$$\mathbf{T}_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \forall p \in \mathcal{F},\ p \not\sqsubseteq x \right\} \neq \varnothing ?$$

Decision problem

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There is an program which halts if and only if $\mathbf{T}_{\mathcal{F}} = \varnothing$

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In dimension 1, it is decidable!

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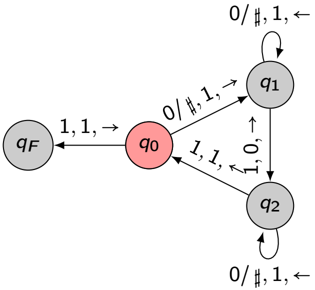
In dimension 1, it is decidable!

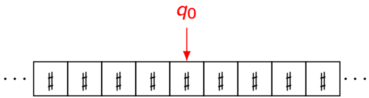
Theorem *(Berger 1966, Robinson 1971)*

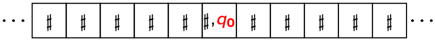
The domino problem is undecidable in dimension $d \geq 2$.

Computation of Turing machine

An example of model of computation:







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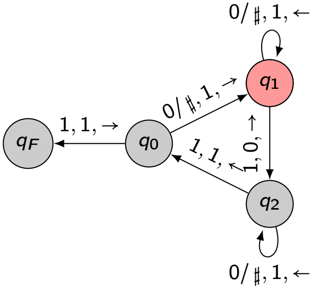
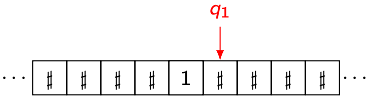


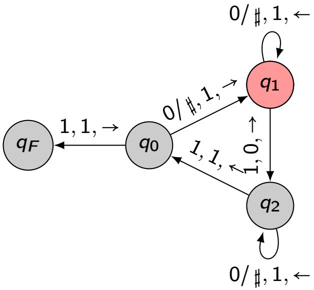


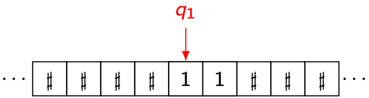
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$

$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$\sharp, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$

Computation of Turing machine

An example of model of computation:

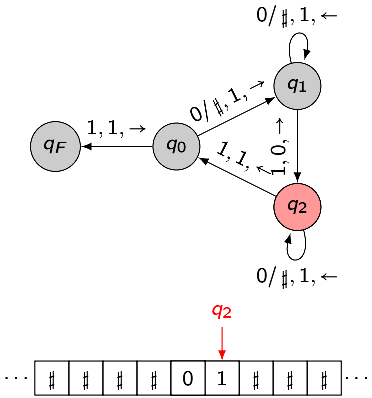




…	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	…

Computation of Turing machine

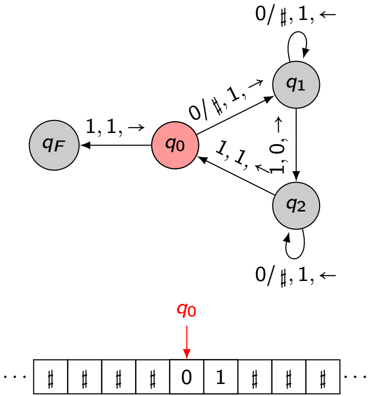
An example of model of computation:



···	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	···

Computation of Turing machine

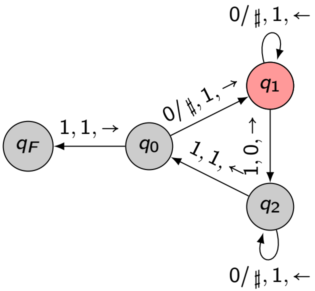
An example of model of computation:

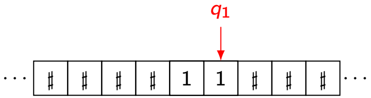


⋯	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	⋯

Computation of Turing machine

An example of model of computation:

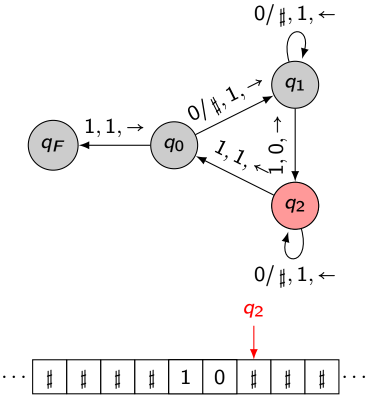




…	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	…

Computation of Turing machine

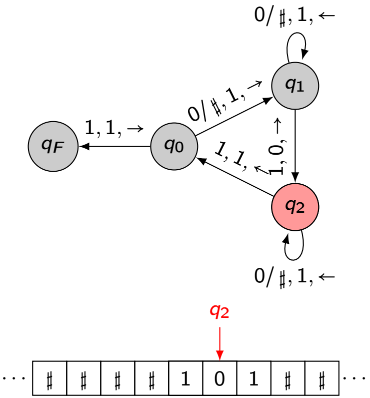
An example of model of computation:



…	♯	♯	♯	♯	♯	1	0	♯, q_2	♯	♯	♯	…
…	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	…

Computation of Turing machine

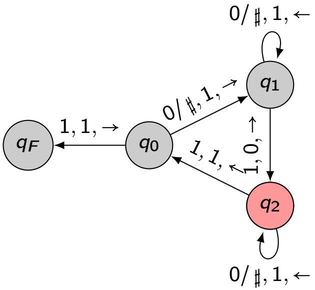
An example of model of computation:

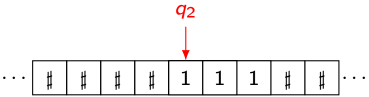


⋯	♯	♯	♯	♯	♯	1	0, q_2	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0	♯, q_2	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	⋯

Computation of Turing machine

An example of model of computation:

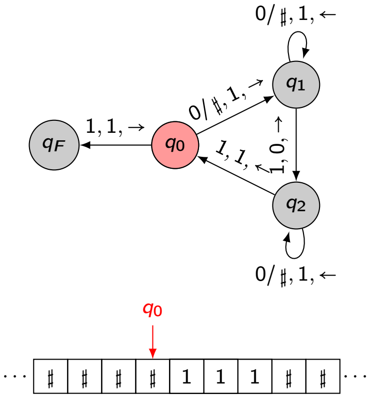




⋯	♯	♯	♯	♯	♯	$1, q_2$	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	$0, q_2$	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0	$\sharp, q_2$	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	$1, q_1$	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	$0, q_0$	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0	$1, q_2$	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	$1, q_1$	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	$\sharp, q_1$	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	$\sharp, q_0$	♯	♯	♯	♯	♯	⋯

Computation of Turing machine

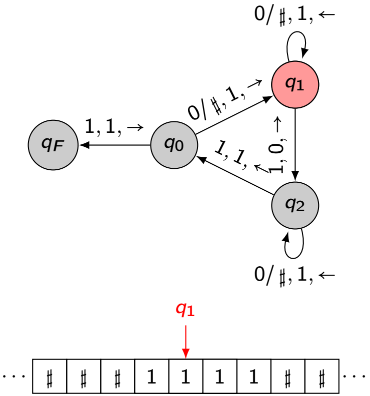
An example of model of computation:



$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	1	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_2$	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$0, q_2$	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	0	$\sharp, q_2$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$1, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$0, q_0$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0	$1, q_2$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_1$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$\sharp, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$

Computation of Turing machine

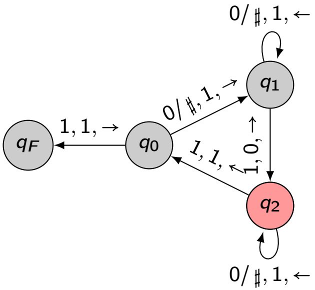
An example of model of computation:

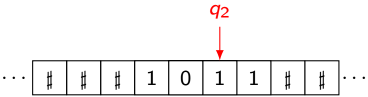


	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	♯	♯	♯	♯	1	1, q_1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯, q_0	1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_2	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0, q_2	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0	♯, q_2	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	⋯

Computation of Turing machine

An example of model of computation:



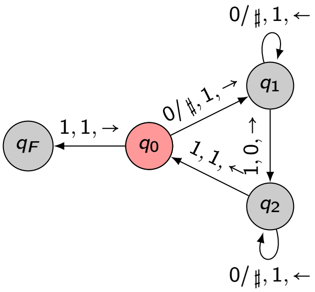


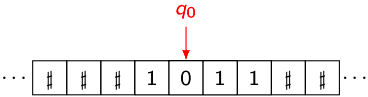
⋯	♯	♯	♯	1	0	1	1	♯	♯	⋯		

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	♯	♯	♯	♯	1	1, q_1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯, q_0	1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_2	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0, q_2	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0	♯, q_2	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	⋯

Computation of Turing machine

An example of model of computation:

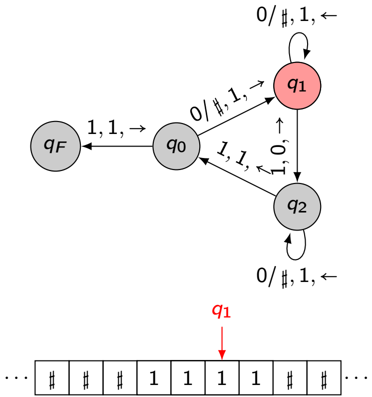




	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	♯	♯	♯	♯	1	1, q_1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯, q_0	1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_2	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0, q_2	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0	♯, q_2	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	⋯

Computation of Turing machine

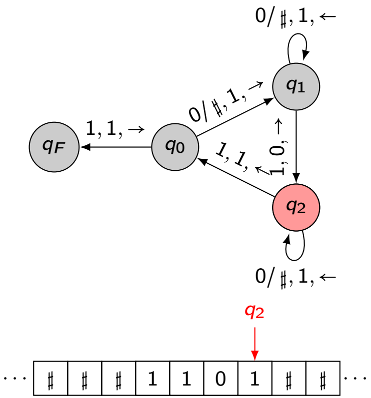
An example of model of computation:



	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
···	♯	♯	♯	♯	1	1, q_1	1	1	♯	♯	♯	···
···	♯	♯	♯	♯	♯, q_0	1	1	1	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1, q_2	1	1	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1	0, q_2	1	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1	0	♯, q_2	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	···

Computation of Turing machine

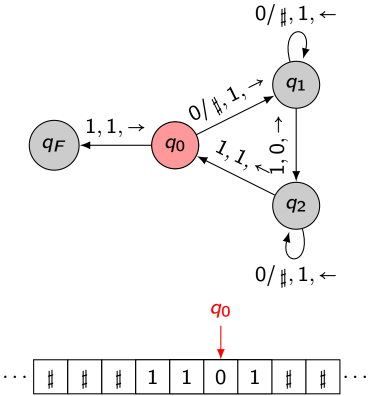
An example of model of computation:



	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
···	♯	♯	♯	♯	1	1, q_1	1	1	♯	♯	♯	···
···	♯	♯	♯	♯	♯, q_0	1	1	1	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1, q_2	1	1	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1	0, q_2	1	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1	0	♯, q_2	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	···

Computation of Turing machine

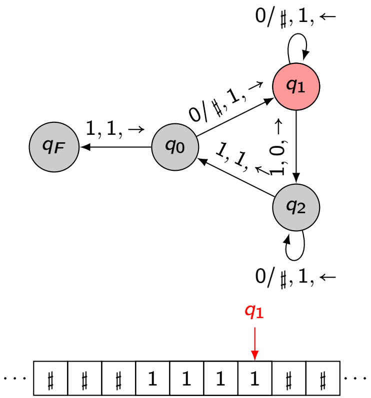
An example of model of computation:



	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	♯	♯	♯	♯	1	1, q_1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯, q_0	1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_2	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0, q_2	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0	♯, q_2	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	⋯

Computation of Turing machine

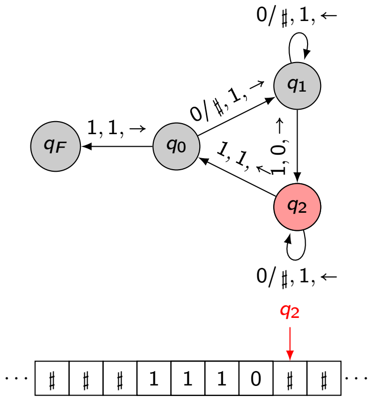
An example of model of computation:



	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	♯	♯	♯	♯	1	1, q_1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯, q_0	1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_2	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0, q_2	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0	♯, q_2	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	⋯

Computation of Turing machine

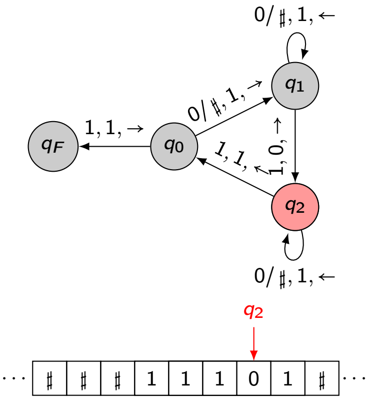
An example of model of computation:



	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
…	♯	♯	♯	♯	1	$1,q_1$	1	1	♯	♯	♯	…
…	♯	♯	♯	♯	$♯,q_0$	1	1	1	♯	♯	♯	…
…	♯	♯	♯	♯	♯	$1,q_2$	1	1	♯	♯	♯	…
…	♯	♯	♯	♯	♯	1	$0,q_2$	1	♯	♯	♯	…
…	♯	♯	♯	♯	♯	1	0	$♯,q_2$	♯	♯	♯	…
…	♯	♯	♯	♯	♯	1	$1,q_1$	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	$0,q_0$	1	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	0	$1,q_2$	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	$1,q_1$	1	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	1	$♯,q_1$	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	$♯,q_0$	♯	♯	♯	♯	♯	…

Computation of Turing machine

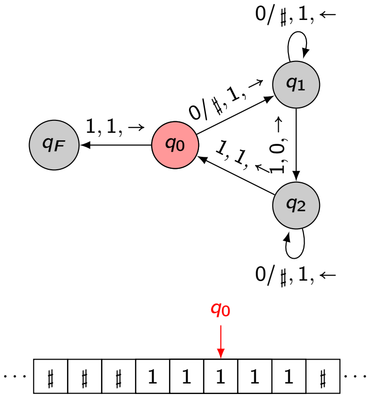
An example of model of computation:



	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	♯	♯	♯	♯	1	1, q_1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯, q_0	1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_2	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0, q_2	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0	♯, q_2	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	⋯

Computation of Turing machine

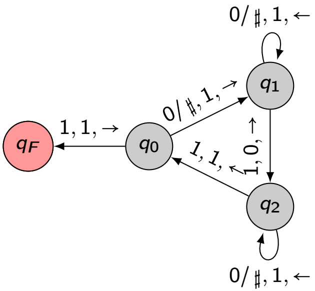
An example of model of computation:

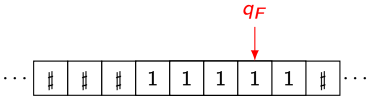


	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$1, q_1$	1	1	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	1	1	1	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_2$	1	1	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$0, q_2$	1	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	0	$\sharp, q_2$	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$1, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$0, q_0$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0	$1, q_2$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_1$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$\sharp, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯

Computation of Turing machine

An example of model of computation:

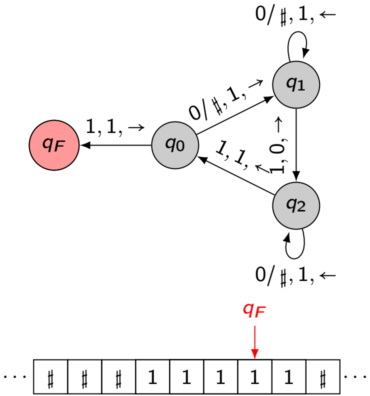




	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
···	♯	♯	♯	♯	1	1, q_1	1	1	♯	♯	♯	···
···	♯	♯	♯	♯	♯, q_0	1	1	1	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1, q_2	1	1	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1	0, q_2	1	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1	0	♯, q_2	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	···

Computation of Turing machine

An example of model of computation:

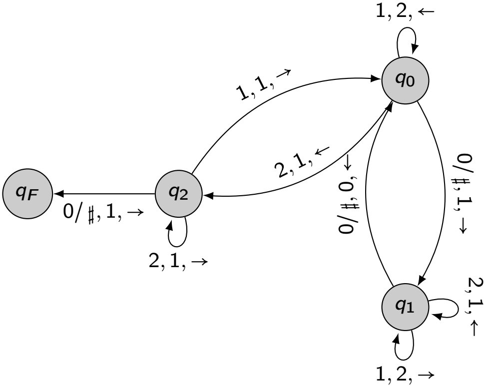


	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	♯	♯	♯	♯	1	$1, q_1$	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	$\sharp, q_0$	1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	$1, q_2$	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	$0, q_2$	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0	$\sharp, q_2$	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	$1, q_1$	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	$0, q_0$	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0	$1, q_2$	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	$1, q_1$	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	$\sharp, q_1$	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	$\sharp, q_0$	♯	♯	♯	♯	♯	⋯

Theorem *(Turing 1937)*

The halting problem is undecidable (but it is semi-decidable).

Combinatory monster



Starting from an empty tape, this Turing machine write 374×10^6 letters in 119×10^{15} steep of computation.

From the behavior of a Turing machine to SFT

Completion problem

Given a SFT $\mathbf{T}$ and a pattern p, it is possible to find $x \in \mathbf{T}$ such that $p \sqsubset x$?

From the behavior of a Turing machine to SFT

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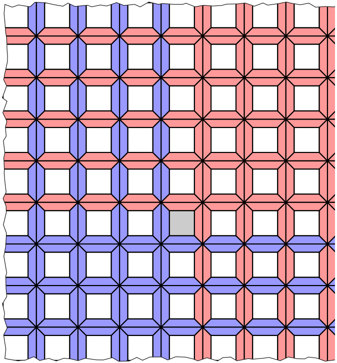
Consider

$$\mathbf{T}(\mathcal{F}_{\leq 1}) \subset \mathcal{A}_{\leq 1}$$



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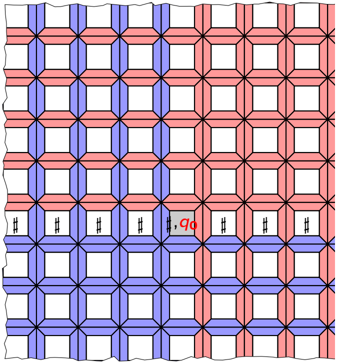
Consider

$$\mathbf{T}(\mathcal{F}_{\leq 1} \cup \mathcal{F}_{\mathtt{Ini}}) \subset \mathcal{A}_{\leq 1} \times \mathcal{A}_{\mathcal{M}}$$



From the behavior of a Turing machine to SFT

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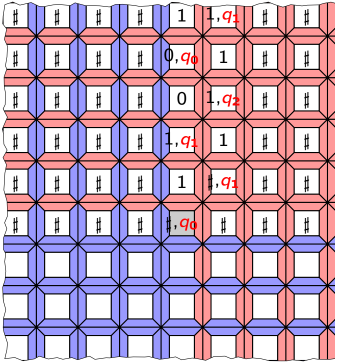

From the behavior of a Turing machine to SFT

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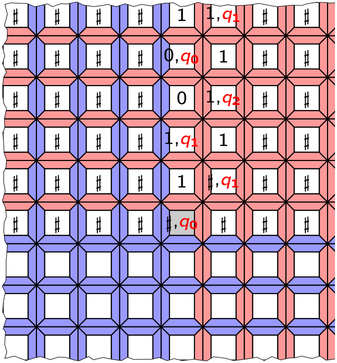
$$\mathbf{T}(\mathcal{F}_{\leq 1} \cup \mathcal{F}_{\mathtt{Ini}} \cup \mathcal{F}_{\mathcal{M}} \cup \{q_F\}) \subset \mathcal{A}_{\leq 1} \times \mathcal{A}_{\mathcal{M}}$$

can be completed in **T** $\Longleftrightarrow \mathcal{M}$ does not halt



From the behavior of a Turing machine to SFT

Completion problem

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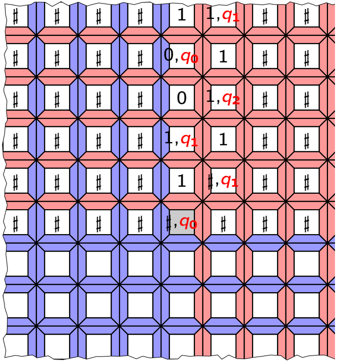


Consider

$$\mathbf{T}(\mathcal{F}_{\leq 1} \cup \mathcal{F}_{\texttt{Ini}} \cup \mathcal{F}_{\mathcal{M}} \cup \{q_F\}) \subset \mathcal{A}_{\leq 1} \times \mathcal{A}_{\mathcal{M}}$$

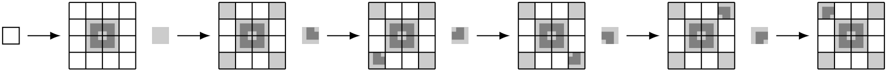
can be completed in **T** $\iff \mathcal{M}$ does not halt

Theorem *(Wang 1961)*

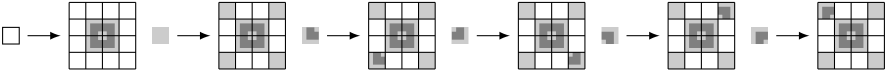
The completion problem is undecidable.

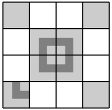
There is no links with the domino problem: by compacity there is no subshift such that a tile appear exactly one times.

Computation zone

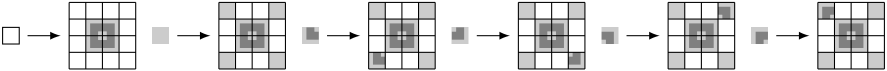


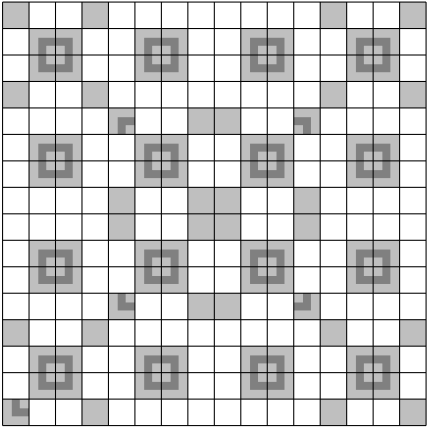
Computation zone



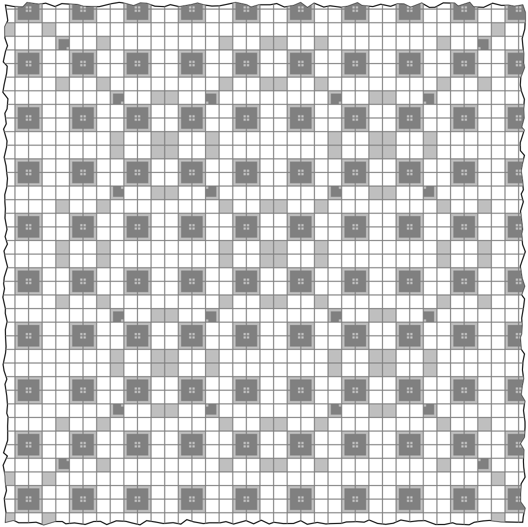


Computation zone

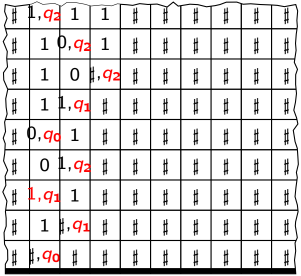




Computation zone

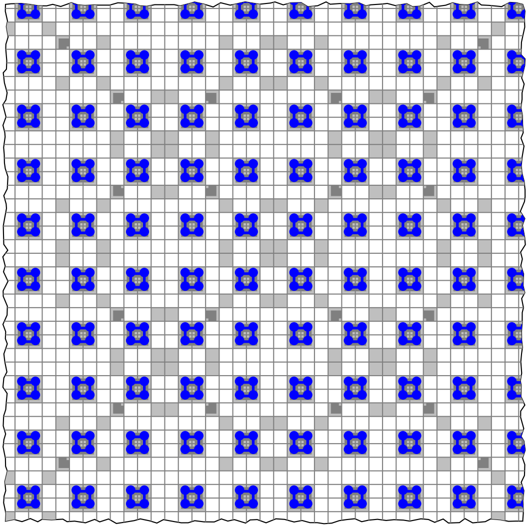


Space time diagram of a Turing machine:



♯	$1,q_2$	1	1	♯	♯	♯	♯	♯	♯
♯	1	$0,q_2$	1	♯	♯	♯	♯	♯	♯
♯	1	0	$♯,q_2$	♯	♯	♯	♯	♯	♯
♯	1	$1,q_1$	♯	♯	♯	♯	♯	♯	♯
♯	$0,q_0$	1	♯	♯	♯	♯	♯	♯	♯
♯	0	$1,q_2$	♯	♯	♯	♯	♯	♯	♯
♯	$1,q_1$	1	♯	♯	♯	♯	♯	♯	♯
♯	1	$♯,q_1$	♯	♯	♯	♯	♯	♯	♯
♯	$♯,q_0$	♯	♯	♯	♯	♯	♯	♯	♯

Computation zone



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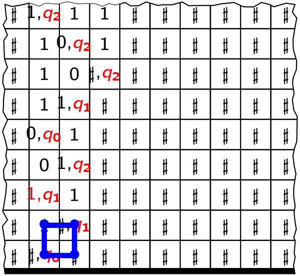
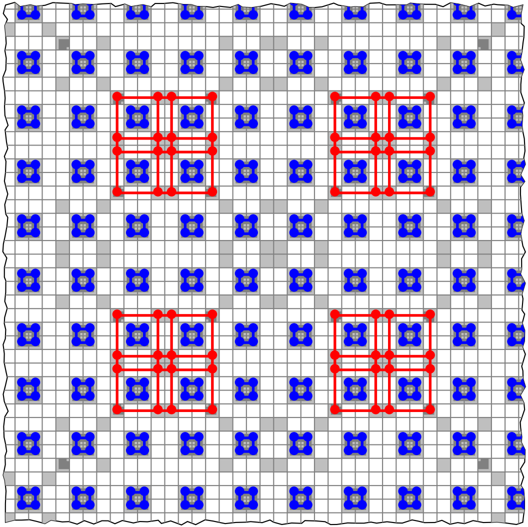
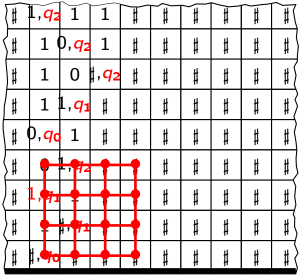


♯	1,q_2	1	1	♯	♯	♯	♯	♯	♯
♯	1	0,q_2	1	♯	♯	♯	♯	♯	♯
♯	1	0	♯,q_2	♯	♯	♯	♯	♯	♯
♯	1	1,q_1	♯	♯	♯	♯	♯	♯	♯
♯	0,q_0	1	♯	♯	♯	♯	♯	♯	♯
♯	0	1,q_2	♯	♯	♯	♯	♯	♯	♯
♯	1,q_1	1	♯	♯	♯	♯	♯	♯	♯
♯	[illegible]	♯,q_1	♯	♯	♯	♯	♯	♯	♯
♯	♯,q_0	♯	♯	♯	♯	♯	♯	♯	♯

Computation zone

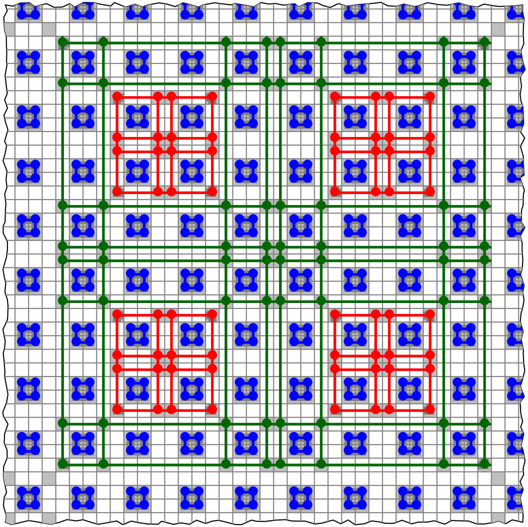


Space time diagram of a Turing machine:

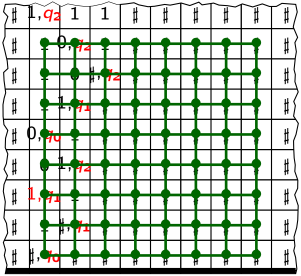


♯	$1,q_2$	1	1	♯	♯	♯	♯	♯	♯
♯	1	$0,q_2$	1	♯	♯	♯	♯	♯	♯
♯	1	0	$\sharp,q_2$	♯	♯	♯	♯	♯	♯
♯	1	$1,q_1$	♯	♯	♯	♯	♯	♯	♯
♯	$0,q_0$	1	♯	♯	♯	♯	♯	♯	♯
♯	0	$1,q_2$	♯	♯	♯	♯	♯	♯	♯
♯	$1,q_1$	1	♯	♯	♯	♯	♯	♯	♯
♯	1	$\sharp,q_1$	♯	♯	♯	♯	♯	♯	♯
♯	$\sharp,q_0$	♯	♯	♯	♯	♯	♯	♯	♯

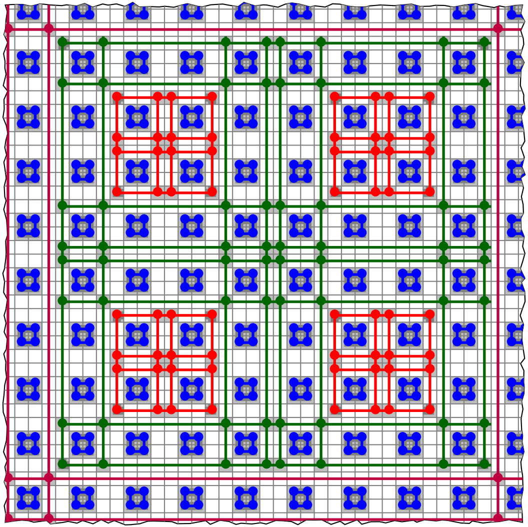
Computation zone



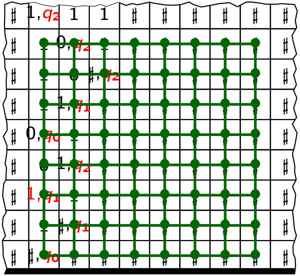
Space time diagram of a Turing machine:



Computation zone



Space time diagram of a Turing machine:



Theorem: Undecidability of the domino problem

The tile set without q_F tiles the plane. $\iff$ The Turing machine does not halt.

Links between effective subshift and local rules

Realization of effective subshifts by sofic

$$\mathbf{T}_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^{\boldsymbol{d}}} \ : \ \text{patterns of } \mathcal{F} \text{ does not appear in } x \right\} \subseteq \mathcal{A}^{\mathbb{Z}^{\boldsymbol{d}}}$$

Some classes of subshifts invariant by conjugacy

T *subshift of finite type* $\iff \exists \mathcal{F}$ finite set such that $\mathbf{T} = \mathbf{T}_{\mathcal{F}}$

T *subshift sofic* $\iff \exists \mathcal{F}$ finite set and $\pi : \mathcal{A} \to \mathcal{B}$ such that $\mathbf{T} = \pi(\mathbf{T}_{\mathcal{F}})$

T *effective subshift* $\iff \exists \mathcal{F}$ recursively enumerable such that $\mathbf{T} = \mathbf{T}_{\mathcal{F}}$

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Some classes of subshifts invariant by conjugacy

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$\mathbf{T}$ *subshift sofic* $\Longleftrightarrow$ $\exists \mathcal{F}$ finite set and $\pi : \mathcal{A} \to \mathcal{B}$ such that $\mathbf{T} = \pi(\mathbf{T}_{\mathcal{F}})$

$\mathbf{T}$ *effective subshift* $\Longleftrightarrow$ $\exists \mathcal{F}$ recursively enumerable such that $\mathbf{T} = \mathbf{T}_{\mathcal{F}}$

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

Let $\Sigma \subset \mathcal{A}^{\mathbb{Z}^{\boldsymbol{d}}}$ be an effective subshift, then the following subshift is sofic:

$$\Sigma^{\uparrow} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^{\boldsymbol{d+1}}} : \exists y \in \Sigma, \ \forall i \in \mathbb{Z} x_{(\mathbb{Z},i)} = y_{\mathbb{Z}} \right\}.$$

An example

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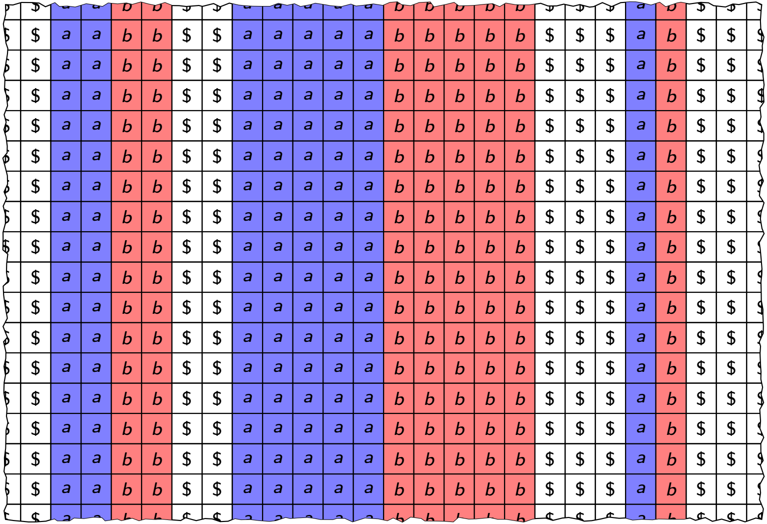

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$	a	a	b	b	$	$	a	a	a	a	a	b	b	b	b	b	$	$	$	a	b	$	$
$	a	a	b	b	$	$	a	a	a	a	a	b	b	b	b	b	$	$	$	a	b	$	$
$	a	a	b	b	$	$	a	a	a	a	a	b	b	b	b	b	$	$	$	a	b	$	$
$	a	a	b	b	$	$	a	a	a	a	a	b	b	b	b	b	$	$	$	a	b	$	$
$	a	a	b	b	$	$	a	a	a	a	a	b	b	b	b	b	$	$	$	a	b	$	$
$	a	a	b	b	$	$	a	a	a	a	a	b	b	b	b	b	$	$	$	a	b	$	$
$	a	a	b	b	$	$	a	a	a	a	a	b	b	b	b	b	$	$	$	a	b	$	$
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$	a	a	b	b	$	$	a	a	a	a	a	b	b	b	b	b	$	$	$	a	b	$	$
$	a	a	b	b	$	$	a	a	a	a	a	b	b	b	b	b	$	$	$	a	b	$	$
$	a	a	b	b	$	$	a	a	a	a	a	b	b	b	b	b	$	$	$	a	b	$	$
$	a	a	b	b	$	$	a	a	a	a	a	b	b	b	b	b	$	$	$	a	b	$	$
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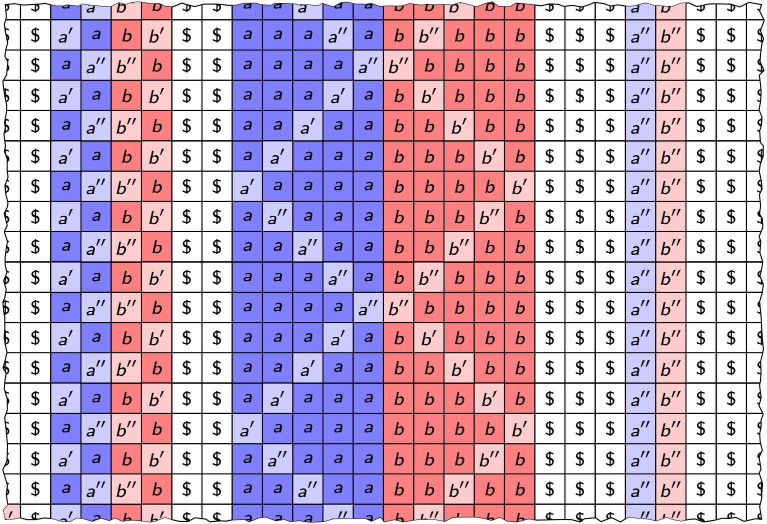

$\$$	a'	a	b	b'	$\$$	$\$$	a	a	a	a''	a	b	b''	b	b	b	$\$$	$\$$	$\$$	a''	b''	$\$$	$\$$
$\$$	a	a''	b''	b	$\$$	$\$$	a	a	a	a	a''	b''	b	b	b	b	$\$$	$\$$	$\$$	a''	b''	$\$$	$\$$
$\$$	a'	a	b	b'	$\$$	$\$$	a	a	a	a'	a	b	b'	b	b	b	$\$$	$\$$	$\$$	a''	b''	$\$$	$\$$
$\$$	a	a''	b''	b	$\$$	$\$$	a	a	a'	a	a	b	b	b'	b	b	$\$$	$\$$	$\$$	a''	b''	$\$$	$\$$
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$\$$	a	a''	b''	b	$\$$	$\$$	a'	a	a	a	a	b	b	b	b	b'	$\$$	$\$$	$\$$	a''	b''	$\$$	$\$$
$\$$	a'	a	b	b'	$\$$	$\$$	a	a''	a	a	a	b	b	b	b''	b	$\$$	$\$$	$\$$	a''	b''	$\$$	$\$$
$\$$	a	a''	b''	b	$\$$	$\$$	a	a	a''	a	a	b	b	b''	b	b	$\$$	$\$$	$\$$	a''	b''	$\$$	$\$$
$\$$	a'	a	b	b'	$\$$	$\$$	a	a	a	a''	a	b	b''	b	b	b	$\$$	$\$$	$\$$	a''	b''	$\$$	$\$$
$\$$	a	a''	b''	b	$\$$	$\$$	a	a	a	a	a''	b''	b	b	b	b	$\$$	$\$$	$\$$	a''	b''	$\$$	$\$$
$\$$	a'	a	b	b'	$\$$	$\$$	a	a	a	a'	a	b	b'	b	b	b	$\$$	$\$$	$\$$	a''	b''	$\$$	$\$$
$\$$	a	a''	b''	b	$\$$	$\$$	a	a	a'	a	a	b	b	b'	b	b	$\$$	$\$$	$\$$	a''	b''	$\$$	$\$$
$\$$	a'	a	b	b'	$\$$	$\$$	a	a'	a	a	a	b	b	b	b'	b	$\$$	$\$$	$\$$	a''	b''	$\$$	$\$$
$\$$	a	a''	b''	b	$\$$	$\$$	a'	a	a	a	a	b	b	b	b	b'	$\$$	$\$$	$\$$	a''	b''	$\$$	$\$$
$\$$	a'	a	b	b'	$\$$	$\$$	a	a''	a	a	a	b	b	b	b''	b	$\$$	$\$$	$\$$	a''	b''	$\$$	$\$$
$\$$	a	a''	b''	b	$\$$	$\$$	a	a	a''	a	a	b	b	b''	b	b	$\$$	$\$$	$\$$	a''	b''	$\$$	$\$$

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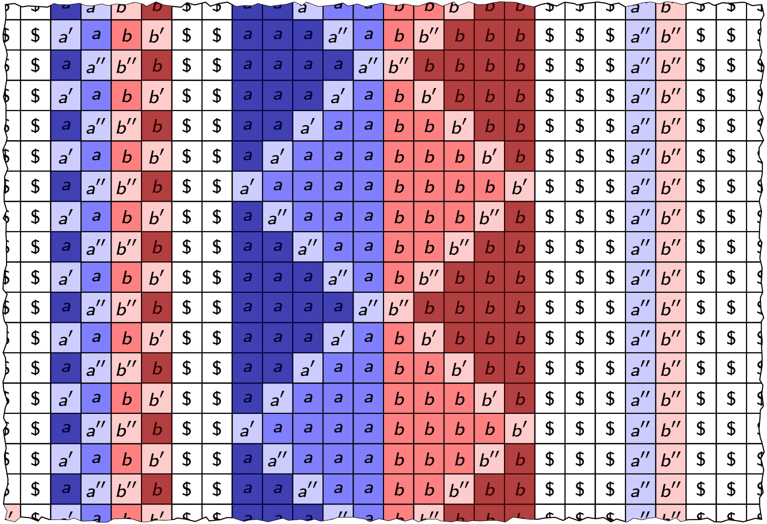

\$	a'	a	b	b'	\$	\$	a	a	a	a''	a	b	b''	b	b	b	\$	\$	\$	a''	b''	\$	\$
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\$	a	a''	b''	b	\$	\$	a	a	a''	a	a	b	b	b''	b	b	\$	\$	\$	a''	b''	\$	\$

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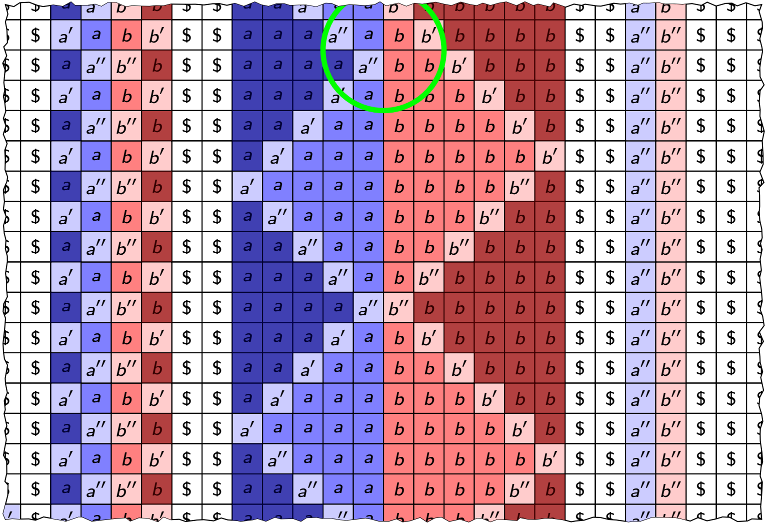

\$	a'	a	b	b'	\$	\$	a	a	a	a''	a	b	b'	b	b	b	b	\$	\$	a''	b''	\$	\$
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\$	a'	a	b	b'	\$	\$	a	a'	a	a	a	b	b	b	b	b	b'	\$	\$	a''	b''	\$	\$
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\$	a	a''	b''	b	\$	\$	a	a	a''	a	a	b	b	b''	b	b	b	\$	\$	a''	b''	\$	\$
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\$	a	a''	b''	b	\$	\$	a'	a	a	a	a	b	b	b	b	b'	b	\$	\$	a''	b''	\$	\$
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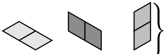
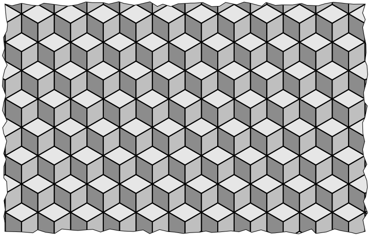
\$	a	a''	b''	b	\$	\$	a	a	a''	a	a	b	b	b	b	b''	b	\$	\$	a''	b''	\$	\$

Which vector spaces admit local rules?

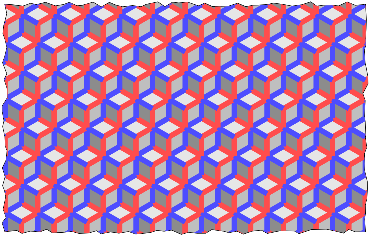
Undecorated local rules

$\mathcal{F} = \Big\{$  $\Big\}$



Decorated local rules

$\mathcal{F} = \{$  $\}$



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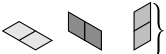
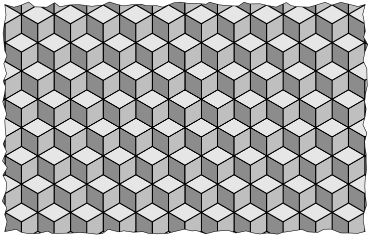
$\mathcal{F} = \Big\{$  $\Big\}$



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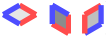
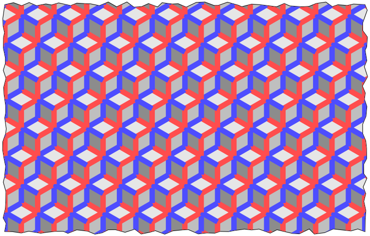
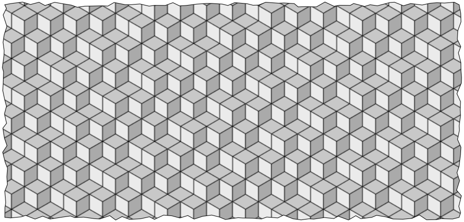
$\mathcal{F} = \{$  $\}$

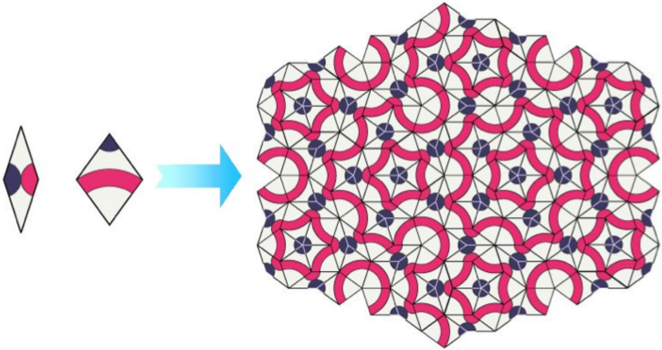


There is local rules if the normal vector is not rational?



Which vector space admits local rules?

- **Some suffisant conditions suffisantes of algebraic nature:**
  Penrose-74, Burkov-88, Levitov-88, Socolar-89, Le-92...



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Theorem of realisation *(Fernique-Sablik-12)*

The tiling associated to $V \subset \mathbb{R}^n$ admits local rules iff V is computable.

How to realize such structure?

Structural problems

- Alphabet of huge cardinal
- It can be difficult to complete a patterns.
- We need a hierarchical structure

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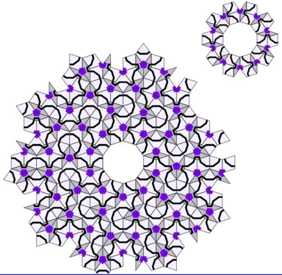

Socolar 1991.

Which rules admits this type of assembly?

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Formation of aperiodic structures

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