

Introduction aux pavages: du local au global

M. Sablik

IMT, Université Paul Sabatier

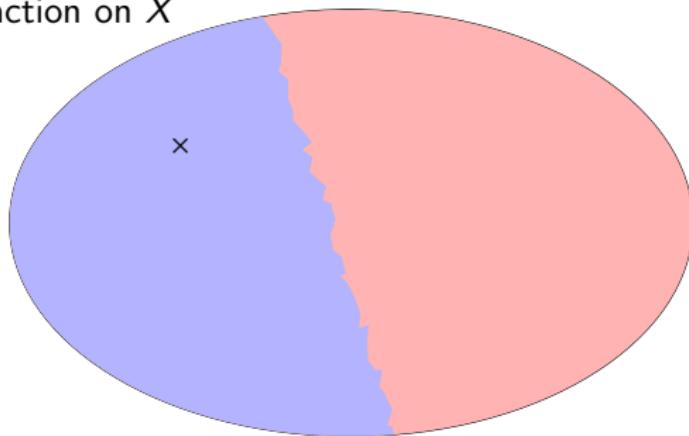
13 June 2019

Local rules

Coding of dynamical systems

Given a dynamical system and a partition, it is possibly to code the trajectory.

F is a \mathbb{Z} -action on X

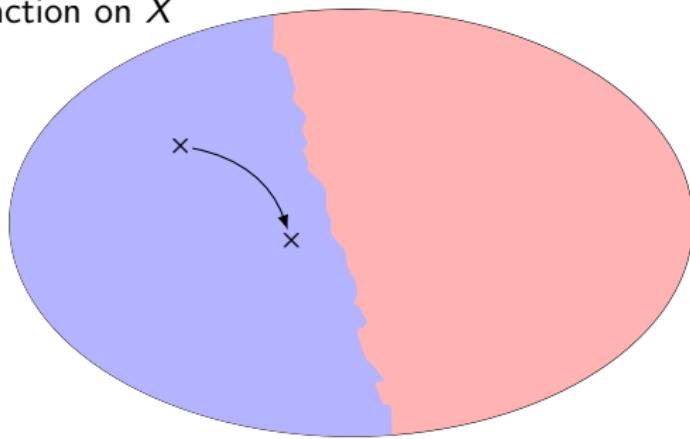


Coding: \dots █

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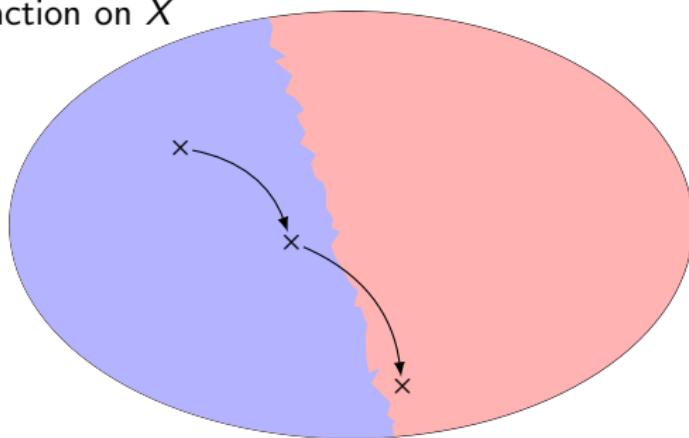
Coding:

... 

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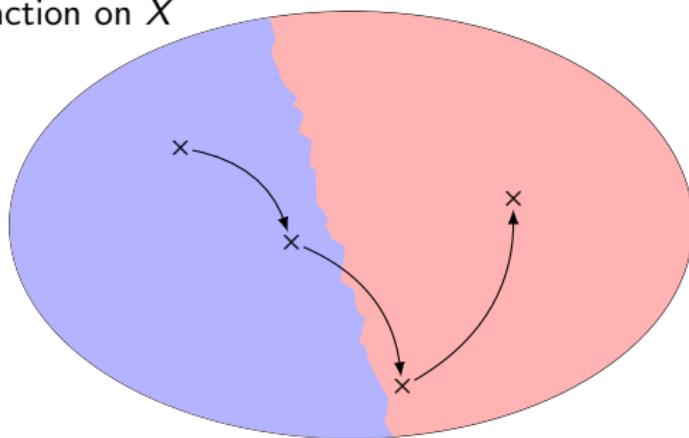
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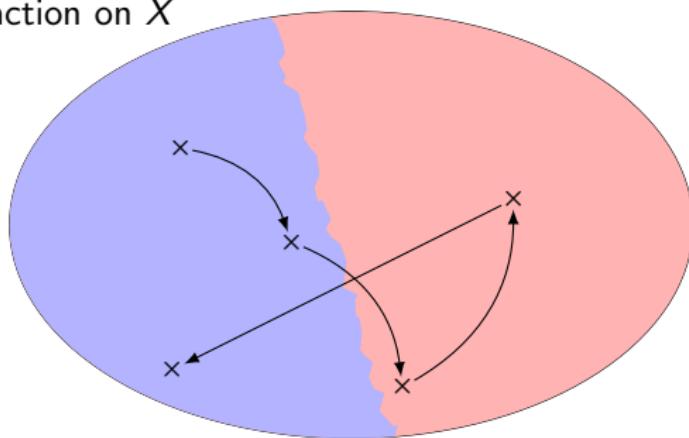
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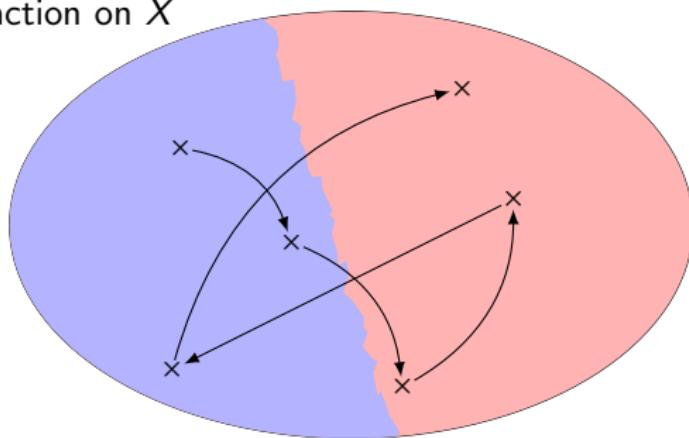
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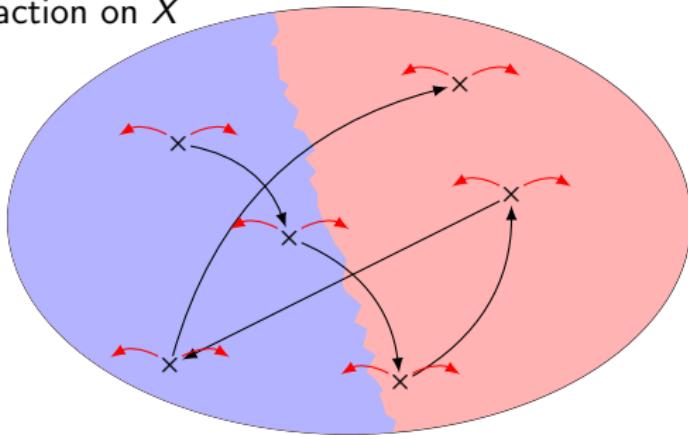
Coding:

$\cdots \square \square \square \square \square \square \cdots \in \mathcal{A}^{\mathbb{Z}}$

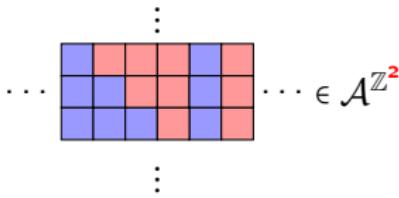
Coding of dynamical systems

Given a dynamical system and a partition, it is possible to code the trajectory.

F is a \mathbb{Z}^2 -action on X



Coding:



Configuration and patterns

Let \mathcal{A} be a **finite** alphabet.

► $x : \mathbb{Z}^d \rightarrow \mathcal{A} \in \mathcal{A}^{\mathbb{Z}^d}$ is a **configuration**.

$$x = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \in \{0, 1\}^{\mathbb{Z}^d}$$

► Let $\mathbb{U} \subset \mathbb{Z}^d$ be a finite set. A **pattern** is an element $p \in \mathcal{A}^{\mathbb{U}}$.



Support: $\mathbb{U} \subset \mathbb{Z}^2$ finite

0	1	1	1
0	1	0	1
1	0	1	1
0	0	0	1
0	0	0	
1	1		

$p \notin x$

Configuration and patterns

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► Let $\mathbb{U} \subset \mathbb{Z}^d$ be a finite set. A **pattern** is an element $p \in \mathcal{A}^{\mathbb{U}}$.

0	1	1	0
0	1	0	0
1	0	1	0
0	1	0	1
0	1	0	
1	0		

$p \sqsubset x$

0	1	1	1
0	1	0	1
1	0	1	1
0	0	0	1
0	0	0	
1	1		

$p \notin x$

Subshift as dynamical system

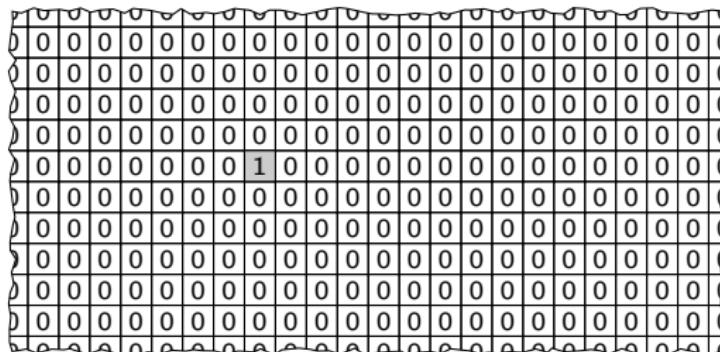
Combinatory definition of subshifts

A subshift \mathbf{T} is defined with a set of forbidden patterns \mathcal{F} :

$$\mathbf{T}_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \text{patterns of } \mathcal{F} \text{ does not appear in } x \right\} \subseteq \mathcal{A}^{\mathbb{Z}^d}$$

Exemple:

$$\mathbf{T}_{\leq 1} = \left\{ x \in \{0, 1\}^{\mathbb{Z}^d} : \text{there is at most one } g \in \mathbb{Z}^d \text{ such that } x_g = 1 \right\}$$



Local rules vs colored local rules

$$\mathbf{T}_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \text{ patterns of } \mathcal{F} \text{ does not appear in } x \right\} \subseteq \mathcal{A}^{\mathbb{Z}^d}$$

Some classes of subshifts invariant by conjugacy

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\mathbf{T} subshift of finite type $\iff \exists \mathcal{F}$ finite set such that $\mathbf{T} = \mathbf{T}_{\mathcal{F}}$

The chessboard: Let $\mathcal{A} = \{0, 1\}$ and $\mathcal{F} = \left\{ \begin{array}{|c|c|} \hline i & \\ \hline i & \\ \hline \end{array}, \begin{array}{|c|c|} \hline i & i \\ \hline i & i \\ \hline \end{array} : i \in \mathcal{A} \right\}$. Consider

$$\mathbf{T}_{\mathcal{F}} = \left\{ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline \end{array} \right\}$$

Local rules vs colored local rules

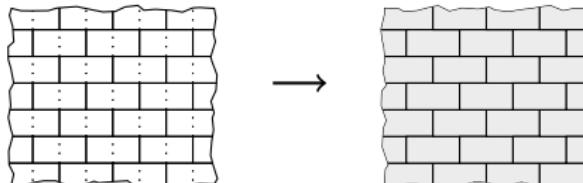
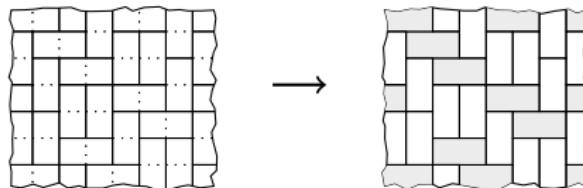
$$\mathbf{T}_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \text{patterns of } \mathcal{F} \text{ does not appear in } x \right\} \subseteq \mathcal{A}^{\mathbb{Z}^d}$$

Some classes of subshifts invariant by conjugacy

\mathbf{T} subshift of finite type $\iff \exists \mathcal{F}$ finite set such that $\mathbf{T} = \mathbf{T}_{\mathcal{F}}$

Domino:

$$T_D = \{ \square \square \square \square \}$$



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Algebraic subshift of finite type:

1	0	1	0	0	1	0	1	1	0	1	1	1	1	0
1	1	0	0	1	1	1	0	0	1	0	0	1	0	1
1	0	1	1	1	0	1	0	0	0	1	1	1	0	0
0	1	1	0	1	0	0	1	1	1	1	0	1	0	0
0	0	1	0	0	1	1	1	1	0	1	0	0	0	0
0	0	1	0	0	1	1	1	0	1	0	1	1	0	0
1	1	1	0	0	0	1	0	1	1	0	0	1	0	0
0	1	0	1	1	1	1	0	0	1	0	0	0	1	1

$$\mathbf{T} = \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \forall (i,j) \in \mathbb{Z}^2, x_{(i,j)} + x_{(i+1,j)} + x_{(i,j+1)} = 0 \pmod{2} \right\}$$

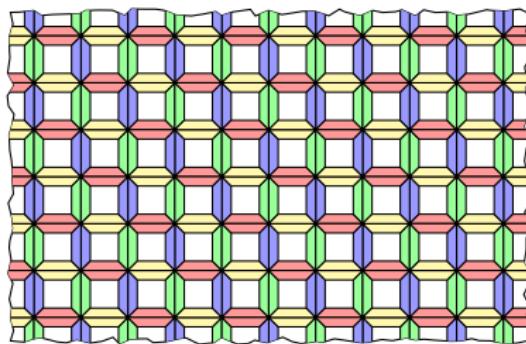
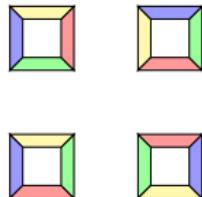
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$$\mathbf{T}_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \text{patterns of } \mathcal{F} \text{ does not appear in } x \right\} \subseteq \mathcal{A}^{\mathbb{Z}^d}$$

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Wang Tiles:



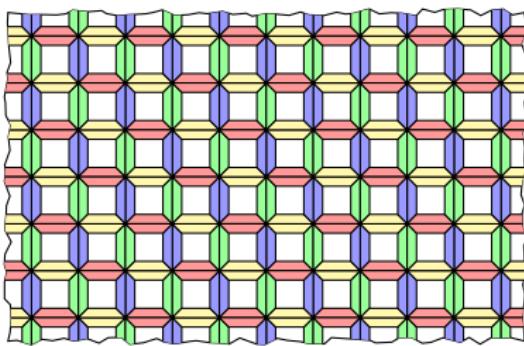
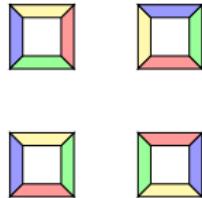
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Wang Tiles:



Proposition

If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}^d}$ is a SFT then is conjugate to a SFT defined by Wang tiles.

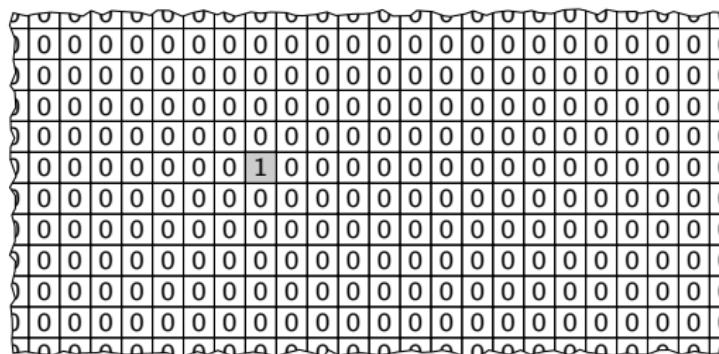
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Consider $\mathbf{T}_{\leq 1} = \left\{ x \in \{0, 1\}^{\mathbb{Z}^d} : \text{there is at most one } g \in \mathbb{Z}^d \text{ such that } x_g = 1 \right\}$.



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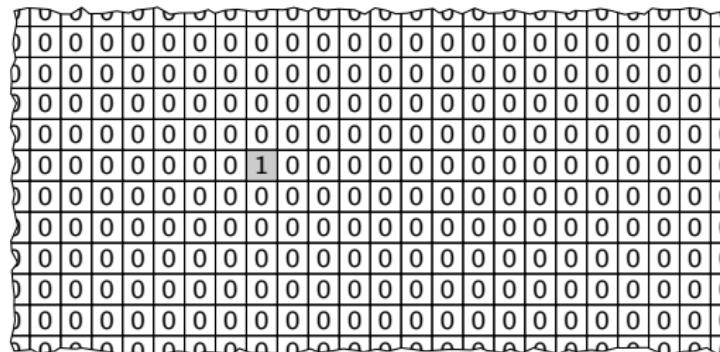
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$\mathbf{T}_{\leq 1}$ can be defined by local rules?



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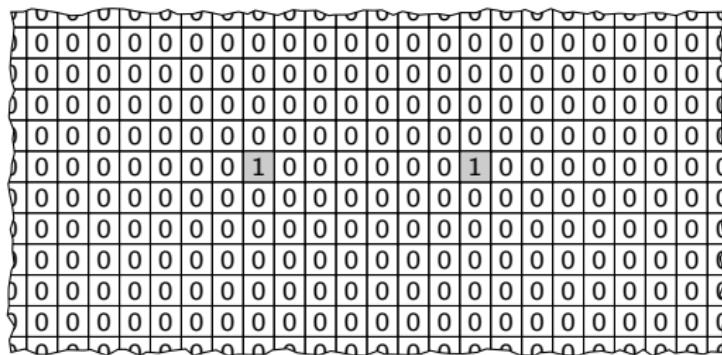
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Assume that $\mathbf{T}_{\leq 1} = \mathbf{T}_{\mathcal{F}}$ with $\mathcal{F} \subset \mathcal{A}^{\mathbb{B}_n}$. Then



Local rules vs colored local rules

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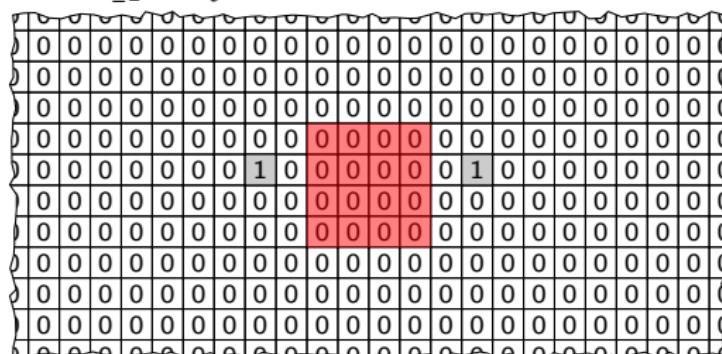
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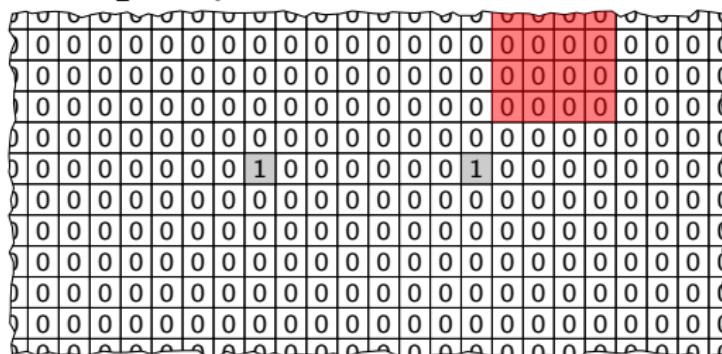
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Assume that $\mathbf{T}_{\leq 1} = \mathbf{T}_{\mathcal{F}}$ with $\mathcal{F} \subset \mathcal{A}^{\mathbb{B}_n}$. Then



$\in \mathbf{T}$ contradiction!

Local rules vs colored local rules

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Some classes of subshifts invariant by conjugacy

\mathbf{T} subshift of finite type $\iff \exists \mathcal{F}$ finite set such that $\mathbf{T} = \mathbf{T}_{\mathcal{F}}$

\mathbf{T} subshift sofic $\iff \exists \mathcal{F}$ finite set and $\pi: \mathcal{A} \rightarrow \mathcal{B}$ such that $\mathbf{T} = \pi(\mathbf{T}_{\mathcal{F}})$

Let $\mathcal{B} = \left\{ \begin{array}{|c|} \hline 0 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 0 \\ \hline \end{array} \right\}$ and \mathcal{F} forbid the matching between two different colors.

$$\cdots \begin{array}{ccccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array} \cdots \in \mathbf{T}_{\mathcal{F}} \quad \xrightarrow{\pi} \quad \cdots \begin{array}{ccccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \cdots \in \mathbf{T}$$

Local rules vs colored local rules

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Some classes of subshifts invariant by conjugacy

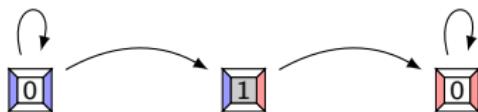
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Let $\mathcal{B} = \{ \boxed{0}, \boxed{1}, \boxed{0} \}$ and \mathcal{F} forbid the matching between two different colors.

$$\cdots \boxed{0} \boxed{0} \boxed{0} \boxed{1} \boxed{0} \boxed{0} \boxed{0} \cdots \in \mathbf{T}_{\mathcal{F}} \xrightarrow{\pi} \cdots 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \cdots \in \mathbf{T}$$

Every configuration can be seen as an infinite path in the following graph:



Local rules vs colored local rules

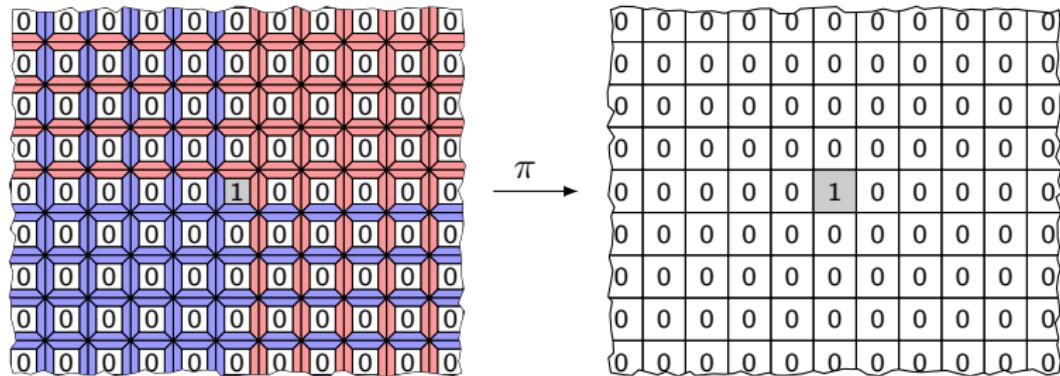
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Exemple 2: The even shift

$$X_{\text{even}} = \left\{ x \in \{0, 1\}^{\mathbb{Z}^d} : \text{Finite connected component have even size} \right\}$$

Proposition

The even shift is sofic.

For $d = 1$, consider $\mathcal{B} = \left\{ \begin{array}{c} \boxed{0} \\ \boxed{1} \\ \boxed{1} \end{array} \right\}$, one has

$$\pi(\dots \boxed{0} \boxed{0} \boxed{0} \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{0} \boxed{0} \boxed{0} \boxed{0} \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{0} \boxed{0} \dots) \in X_{\text{even}}$$

Exemple 2: The even shift

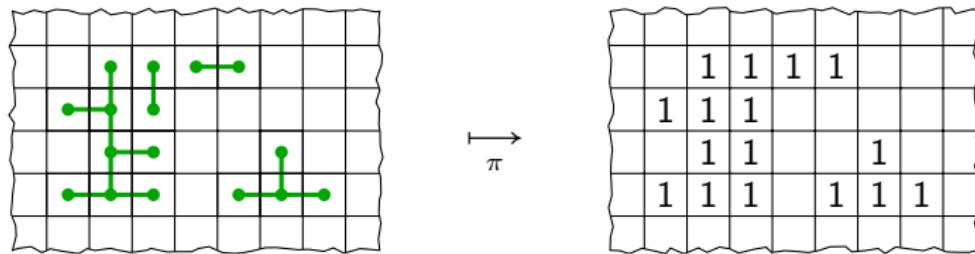
$$X_{\text{even}} = \left\{ x \in \{0, 1\}^{\mathbb{Z}^d} : \text{Finite connected component have even size} \right\}$$

Proposition

The even shift is sofic.

Consider the alphabet $\mathcal{A}_{\text{even}} = \left\{ \square, \boxed{\bullet}, \boxed{\text{---}} \right\} + \text{rotation}$

$$\pi(\square) = 0 \text{ and } \pi(\boxed{\bullet}) = \pi(\boxed{\text{---}}) = 1$$



Green components have even size (handshaking lemma) $\implies \pi(X_{\mathcal{A}_{\text{even}}}) \subset X_{\text{even}}$

Exemple 2: The even shift

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Proposition

The even shift is sofic.

- Consider $x \in X_{\text{even}}$

		1	1	1	1		
1	1	1					
	1	1				1	
1	1	1			1	1	1

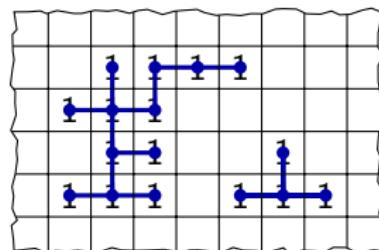
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- Consider $x \in X_{\text{even}}$
- There exist trees which cover each connected component



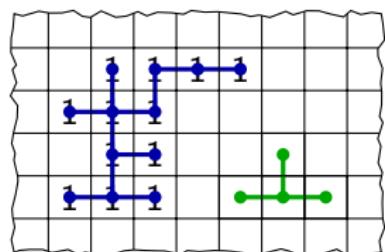
Exemple 2: The even shift

$$X_{\text{even}} = \left\{ x \in \{0, 1\}^{\mathbb{Z}^d} : \text{Finite connected component have even size} \right\}$$

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The even shift is sofic.

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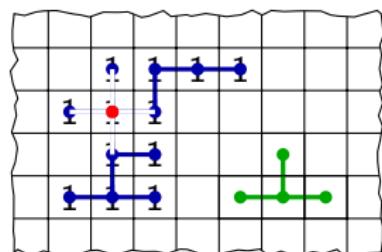
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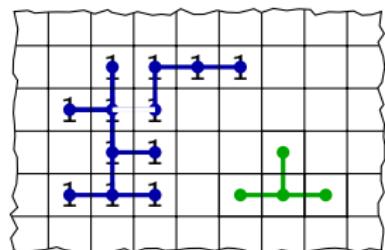
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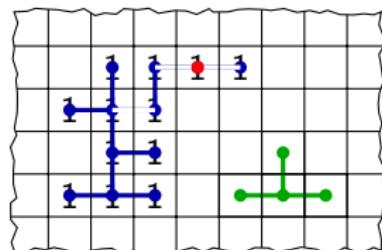
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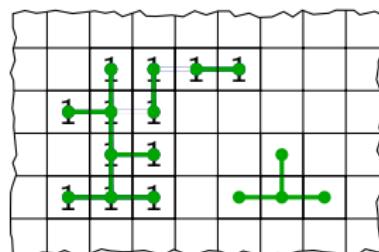
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So $\pi(X_{\mathcal{A}_{\text{even}}}) \subset X_{\text{even}}$.

Exemple 3: The odd shift

$$X_{\text{odd}} = \left\{ x \in \{0, 1\}^{\mathbb{Z}^d} : \text{Finite connected component have even size} \right\}$$

Proposition

The odd shift is sofic if $d = 1$ or 2 .

- For $d = 1$, consider $\mathcal{B} = \left\{ \begin{array}{c} \text{blue box} \\ 0 \end{array}, \begin{array}{c} \text{green box} \\ 0 \end{array}, \begin{array}{c} \text{green box} \\ 1 \end{array}, \begin{array}{c} \text{blue box} \\ 1 \end{array}, \begin{array}{c} \text{red box} \\ 1 \end{array} \end{array} \right\}$, one has

$$\pi \left(\dots \begin{array}{cccccccccccccccccccc} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array} \dots \right) \in X_{\text{even}}$$

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- The construction is quite hard for $d = 2$
- It is conjectured that it is not sofic for $d \geq 3$.

Exemple 4: Election of a leader shift

$$X_{\text{Leader}} = \left\{ x \in \{0, 1, 2\}^{\mathbb{Z}^d} : \text{Finite connected component have exactly one } 2 \right\}$$

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The election of a leader shift is sofic if $d = 1$ or 2 .

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The election of a leader shift is not sofic for $d = 3$ but it is effective.

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$$\mathbf{T}_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{G}} : \text{patterns of } \mathcal{F} \text{ does not appear in } x \right\} \subseteq \mathcal{A}^{\mathbb{G}}$$

Some classes of subshifts invariant by conjugacy

\mathbf{T} subshift of finite type $\iff \exists \mathcal{F}$ finite set such that $\mathbf{T} = \mathbf{T}_{\mathcal{F}}$

\mathbf{T} subshift sofic $\iff \exists \mathcal{F}$ finite set and $\pi: \mathcal{A} \rightarrow \mathcal{B}$ such that $\mathbf{T} = \pi(\mathbf{T}_{\mathcal{F}})$

\mathbf{T} effective subshift $\iff \exists \mathcal{F}$ recursively enumerable set such that $\mathbf{T} = \mathbf{T}_{\mathcal{F}}$

Exemple 5: Miror shift

Let $\mathcal{A} = \{\square, \blacksquare, \blacksquare\}^{\mathbb{Z}^2}$ and consider the following effective subshift:

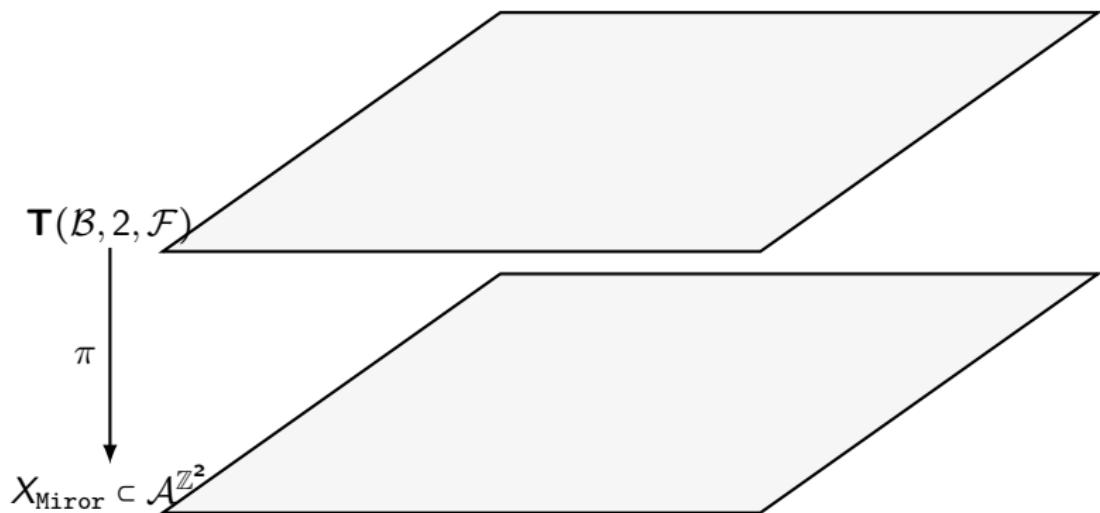
$$X_{\text{Mirror}} = \{\square, \blacksquare\}^{\mathbb{Z}^2} \cup \left\{ \begin{array}{|c|c|c|c|c|c|c|c|} \hline & \blacksquare \\ \hline & \blacksquare \\ \hline & \blacksquare \\ \hline & \blacksquare \\ \hline & \blacksquare \\ \hline & \blacksquare \\ \hline \end{array} \right\}$$

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Assume that X_{Mirror} is sofic, that is to say $X_{\text{Mirror}} = \pi(\mathbf{T}(\mathcal{B}, 2, \mathcal{F}))$

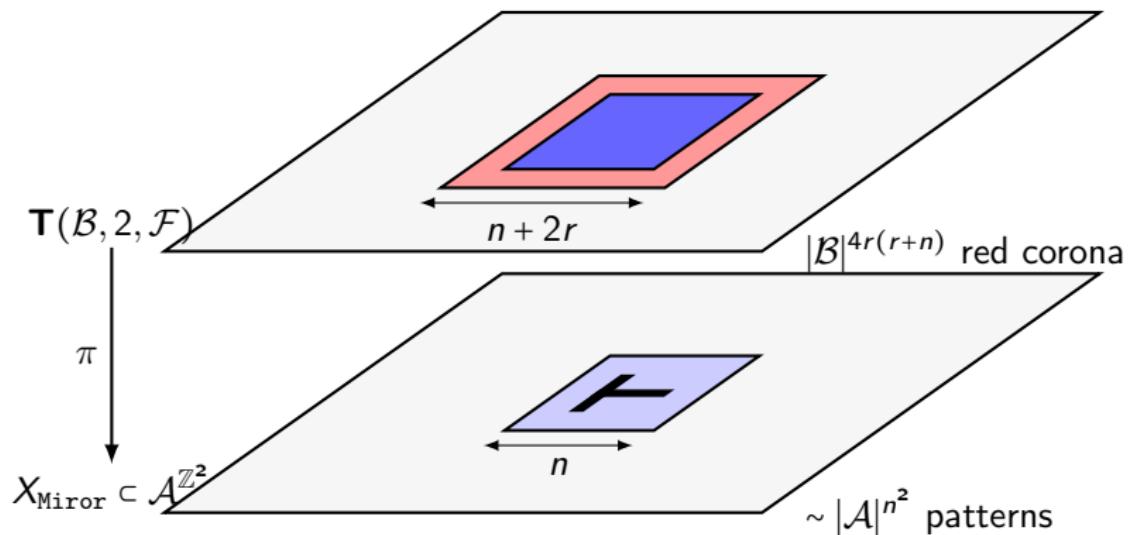


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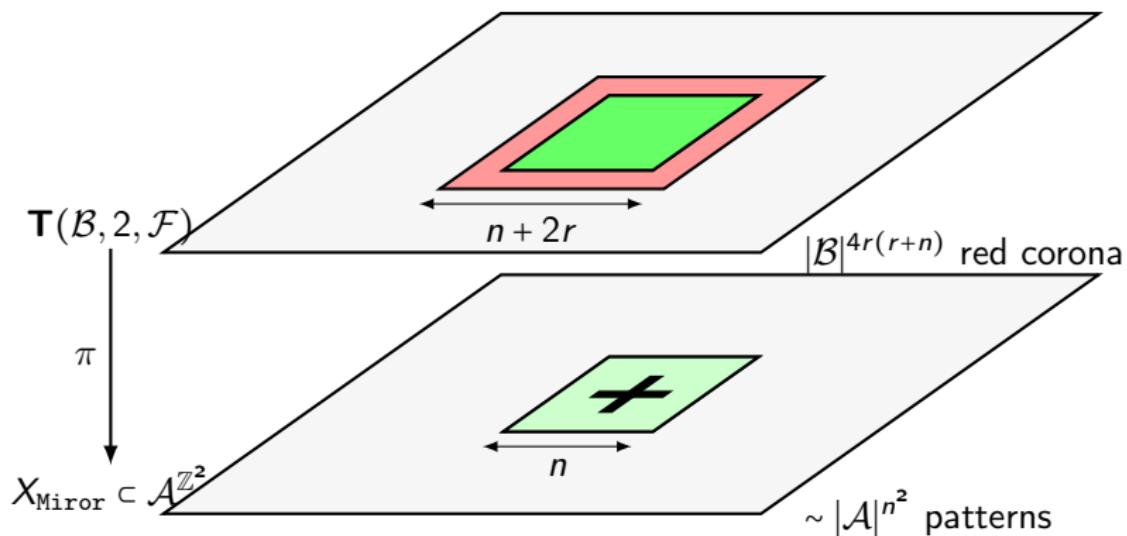


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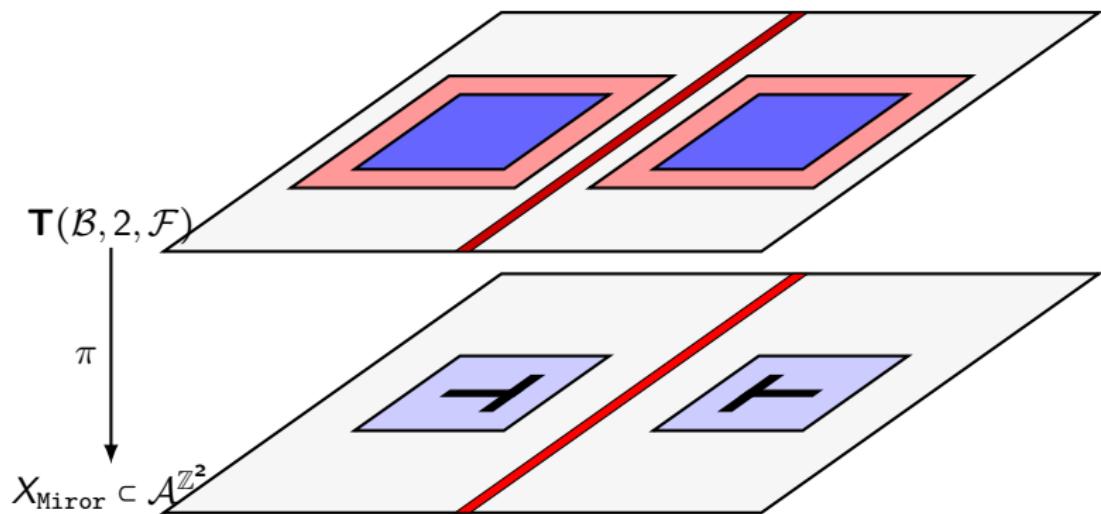


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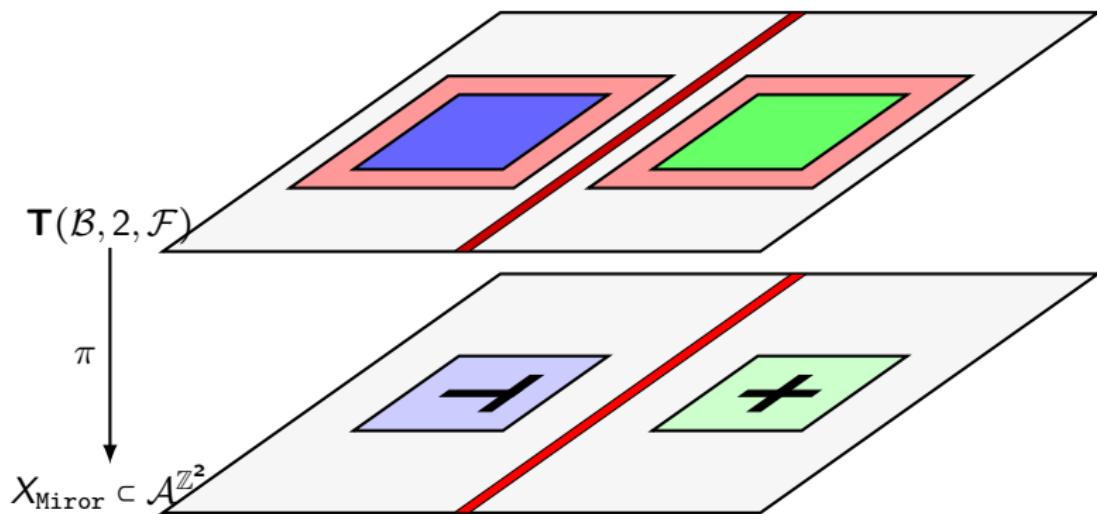


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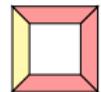


The domino problem

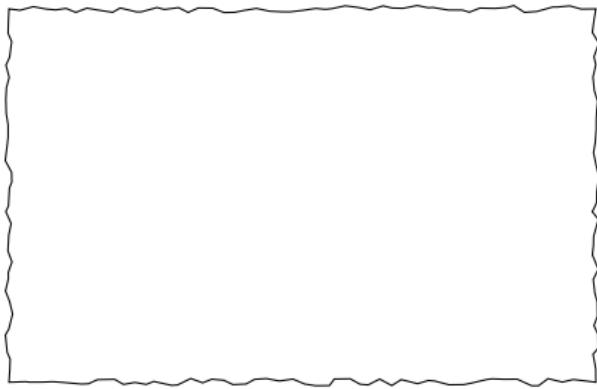
Domino problem for Wang's tilings



Tiles set



An associated tiling



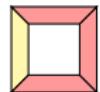
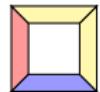
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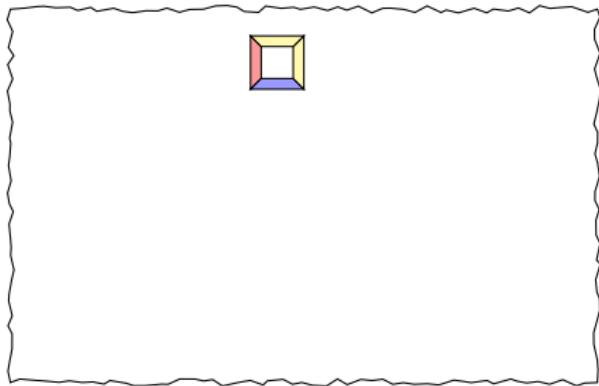
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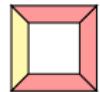
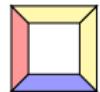
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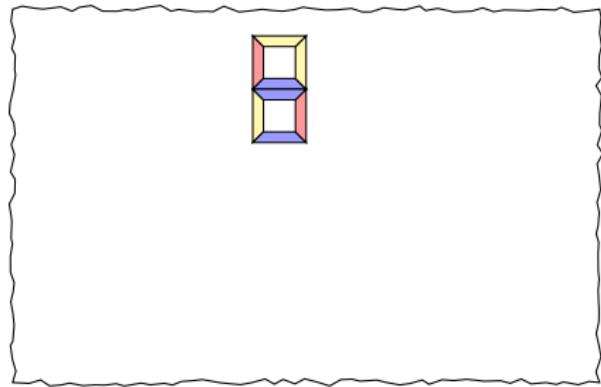
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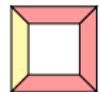
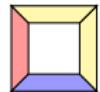
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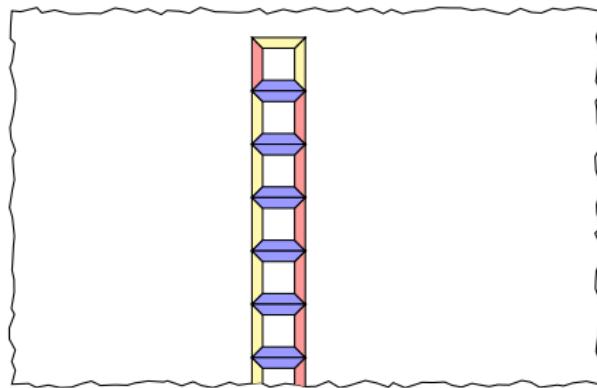
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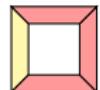
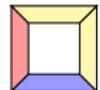
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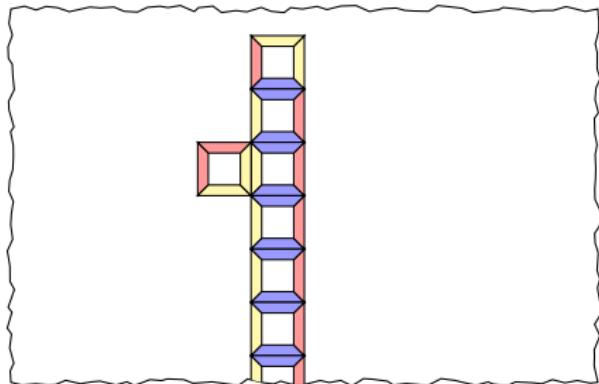
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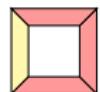
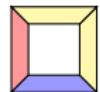
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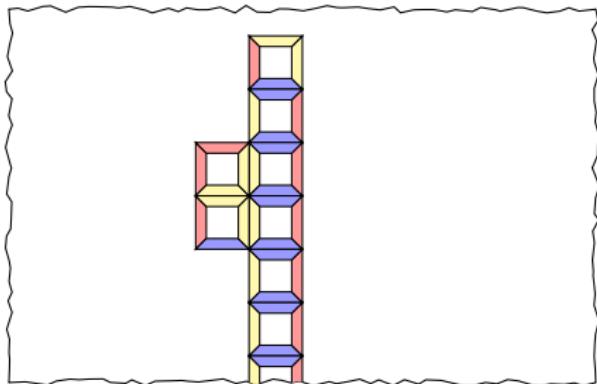
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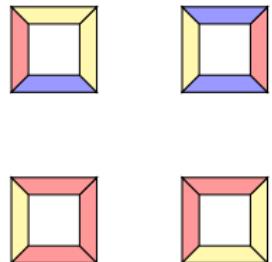
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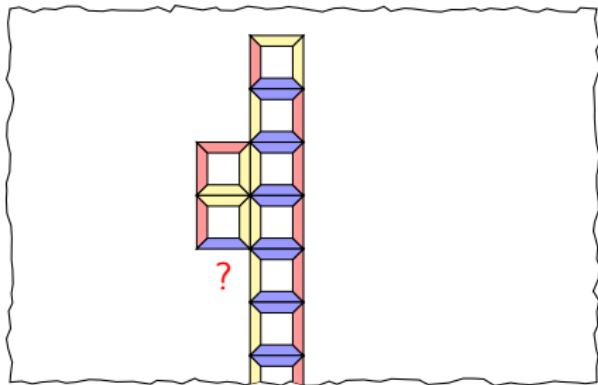
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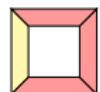
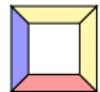
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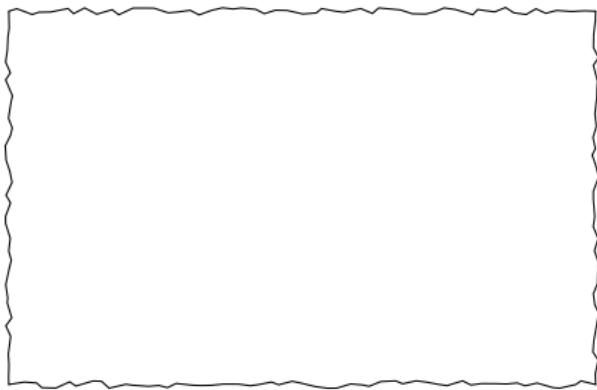
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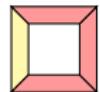
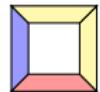
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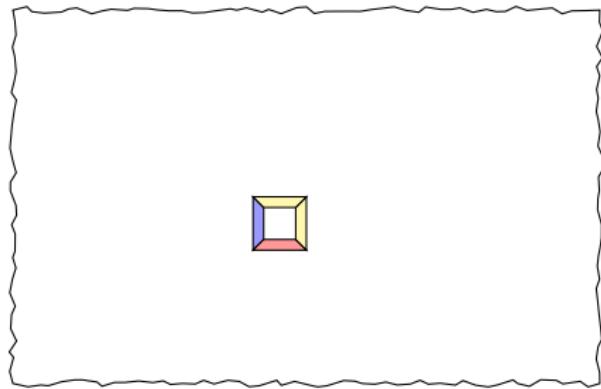
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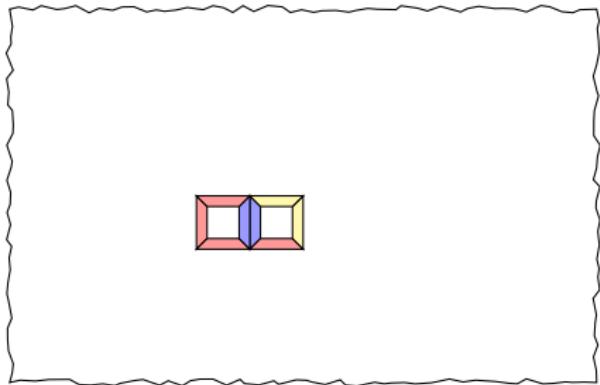
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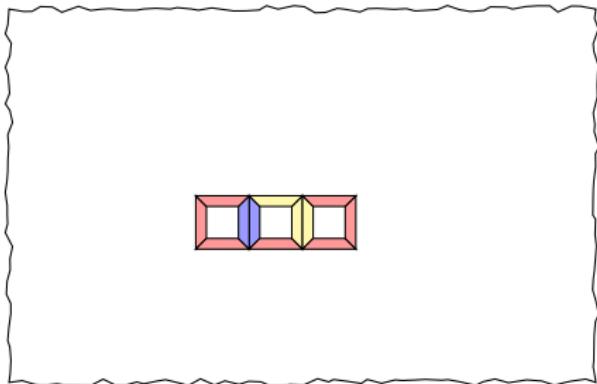
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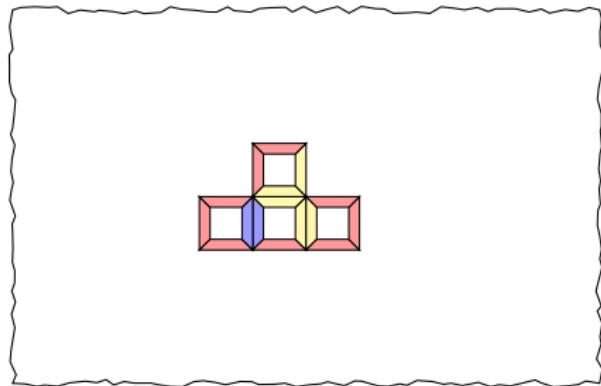
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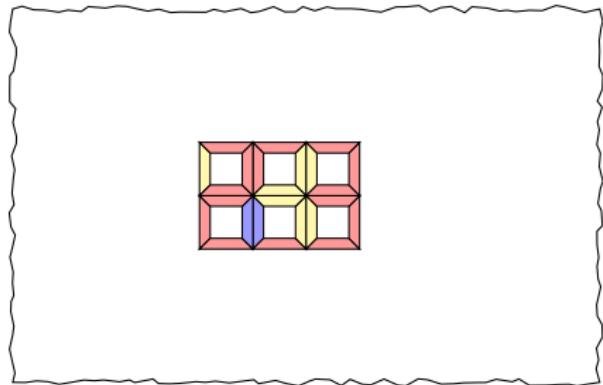
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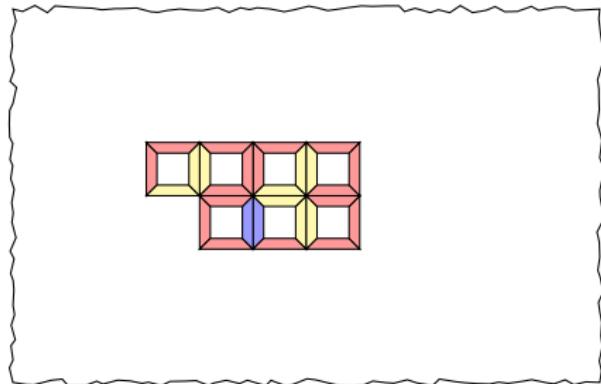
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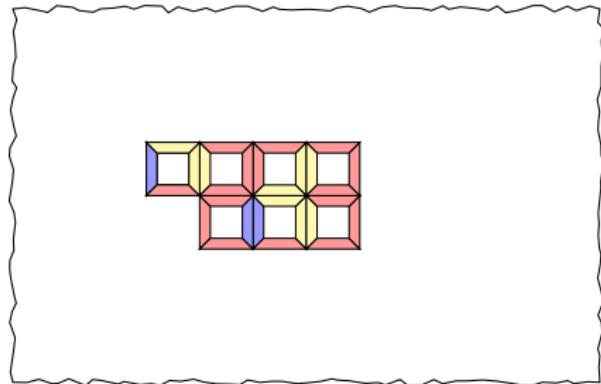
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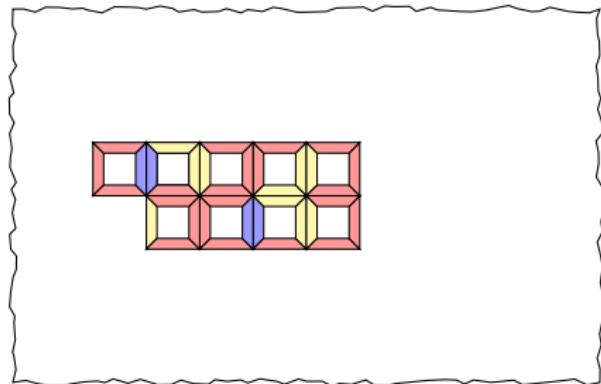
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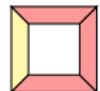
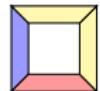
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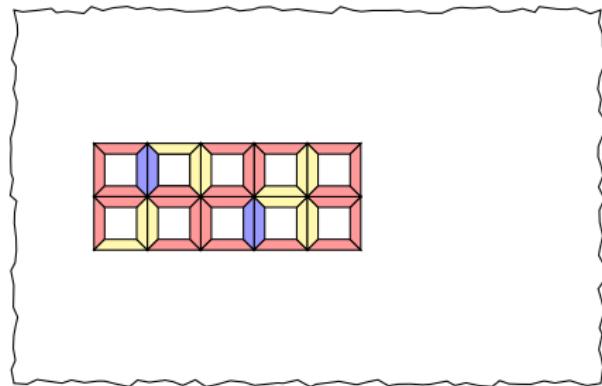
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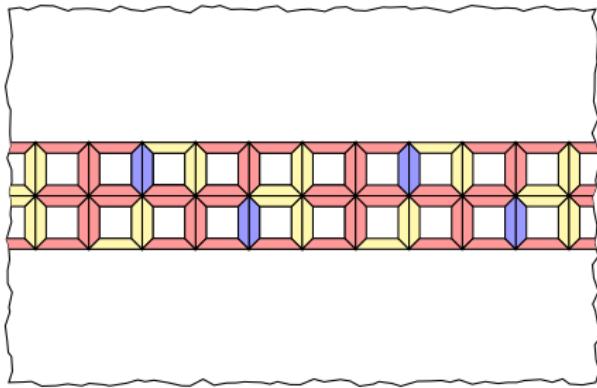
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An associated tiling



Domino problem

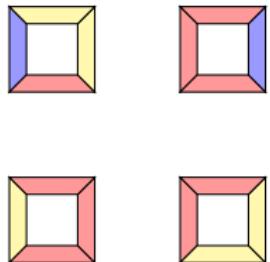
Does a given finite set \mathcal{P} of Wang prototiles admit a tiling ?

- The complement of the tiling problem is semi-decidable.

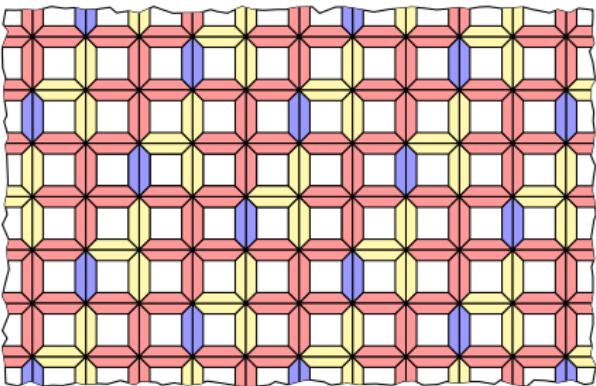
Domino problem for Wang's tilings



Tiles set



An associated tiling



Domino problem

Does a given finite set \mathcal{P} of Wang prototiles admit a tiling ?

- The complement of the tiling problem is semi-decidable.
- If there exists a periodic tiling, finding “algorithmically” a configuration is easy!

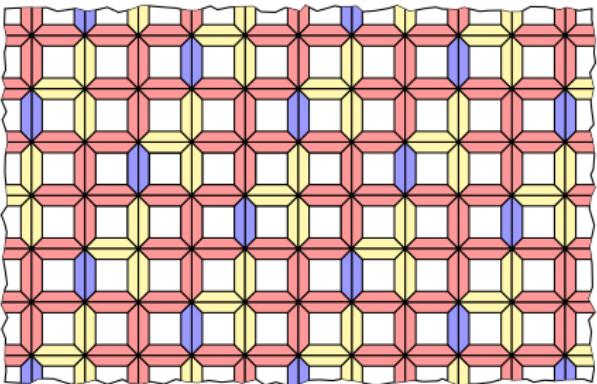
Domino problem for Wang's tilings



Tiles set



An associated tiling



Theorem (*Berger-66, Robinson-71, Mozes-89, Kari-96...*)

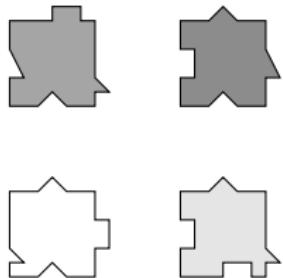
- There exists tile sets which produce only aperiodic tilings.
- The domino problem is undecidable.



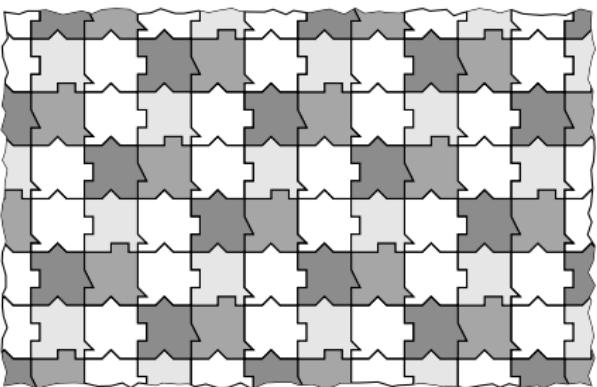
Domino problem for Wang's tilings



Tiles set



An associated tiling



Theorem (*Berger-66, Robinson-71, Mozes-89, Kari-96...*)

- There exists tile sets which produce only aperiodic tilings.
- The domino problem is undecidable.



...

Domino problem in dimension 1

Tile set:



Tile set:



Domino problem in dimension 1

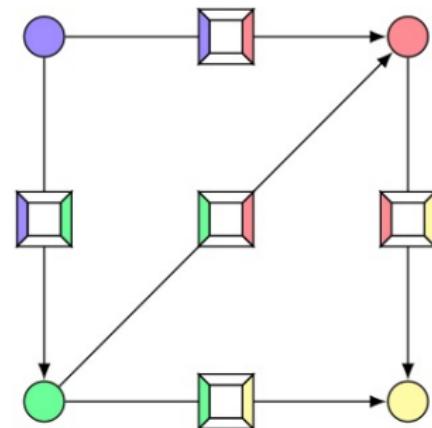
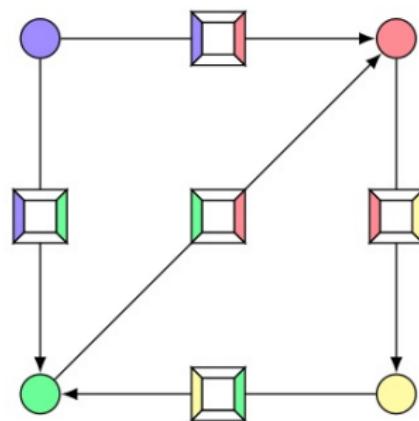
Tile set:



Tile set:



Every configuration can be seen as an infinite path in the following graphs:



Domino problem in dimension 1

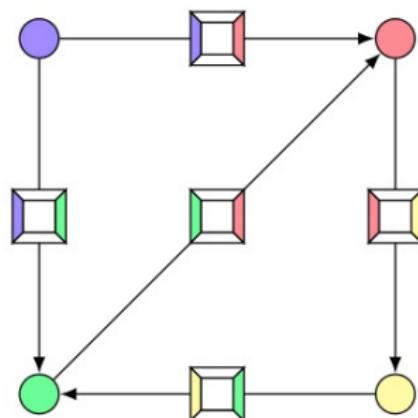
Tile set:



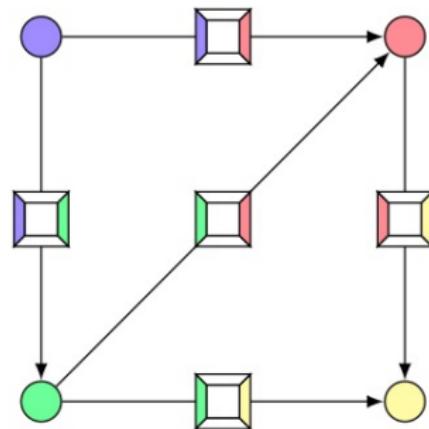
Tile set:



Every configuration can be seen as an infinite path in the following graphs:



Tile the line



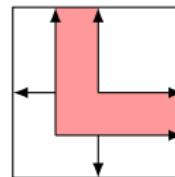
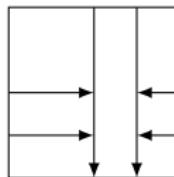
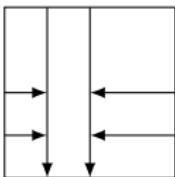
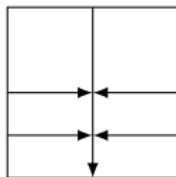
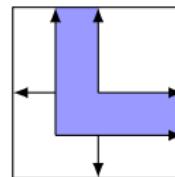
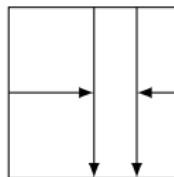
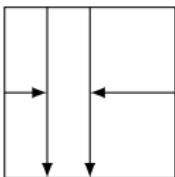
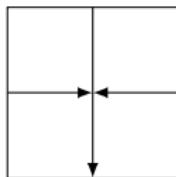
Do not tile the line

My first aperiodic tiling: Robinson's tiling

Alphabet of the tiling of Robinson

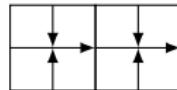
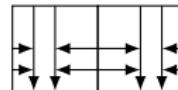
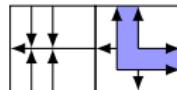


There exists different means to define the Robinson tiling. Consider $\mathcal{R}obi$ the next set of tiles modulo the rotation

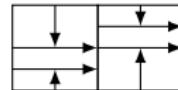


Local rules

Incoming and outgoing arrows must be respected:

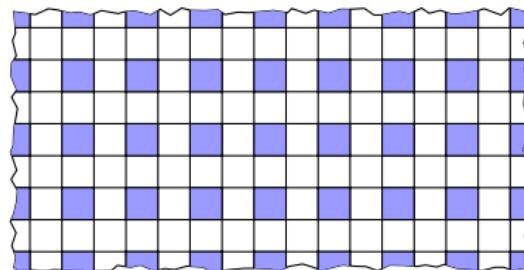


Allowed



Not allowed

We add the forbidden patterns \mathcal{F} which impose the alternating of the colors:

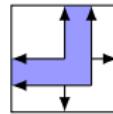
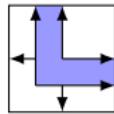
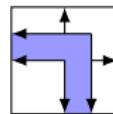
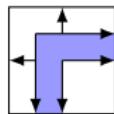


$$\text{ou } \square \in \{ \text{[diagonal]} \text{, [cross]} \text{, [X]} \text{, [diamond]} \}$$

Denote T_{Robi} the SFT described by these rules.

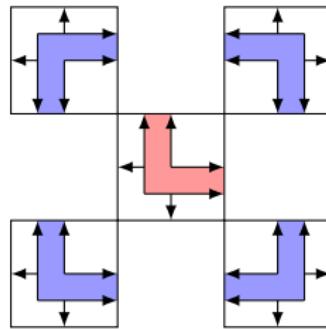
Existence of the tiling (level 1)

The Robinson tiling is based on a hierarchical structure. Nine of these tiles can be assembled to form a *super-tile of level 1*:



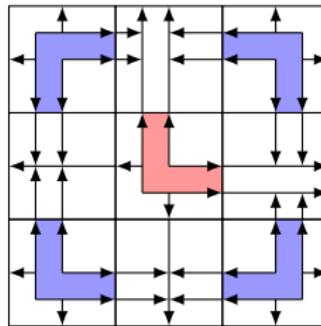
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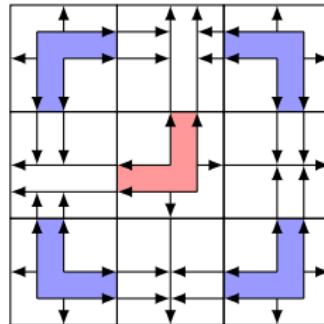
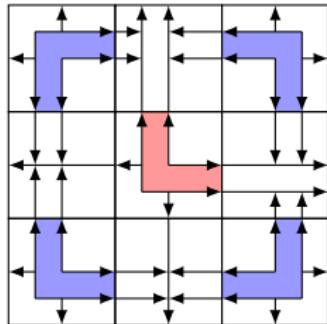
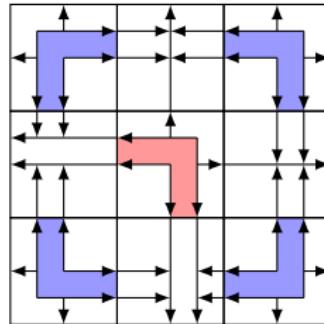
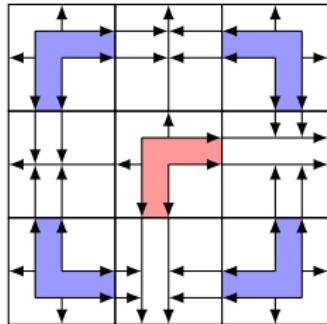
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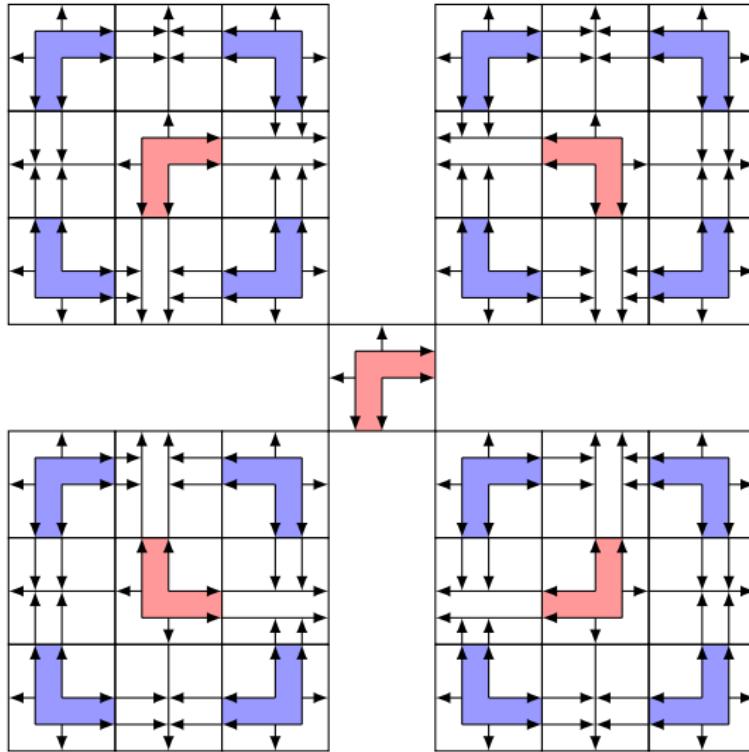
Existence of the tiling (level 2)

Super-tiles of level 1 can be assembled to form *super-tiles of level 2*:



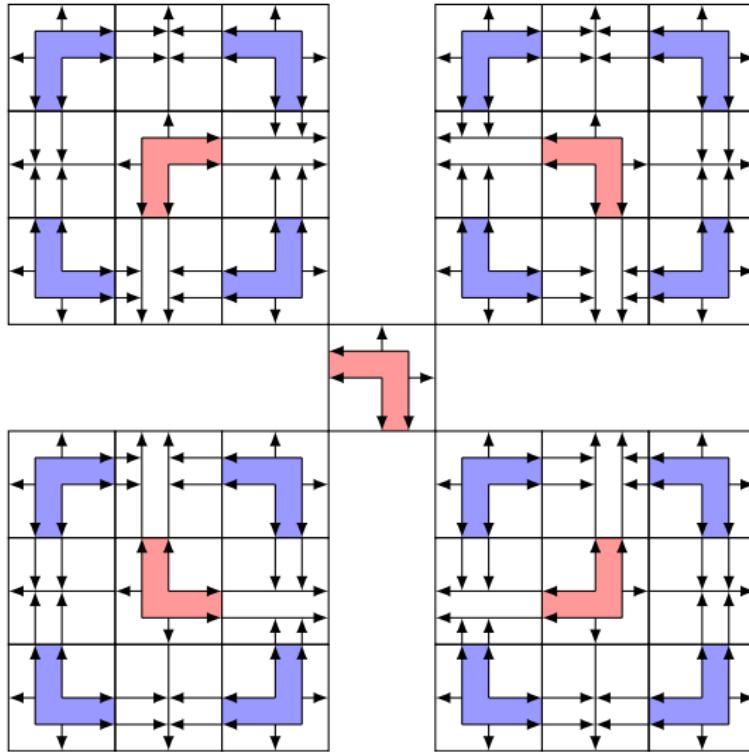
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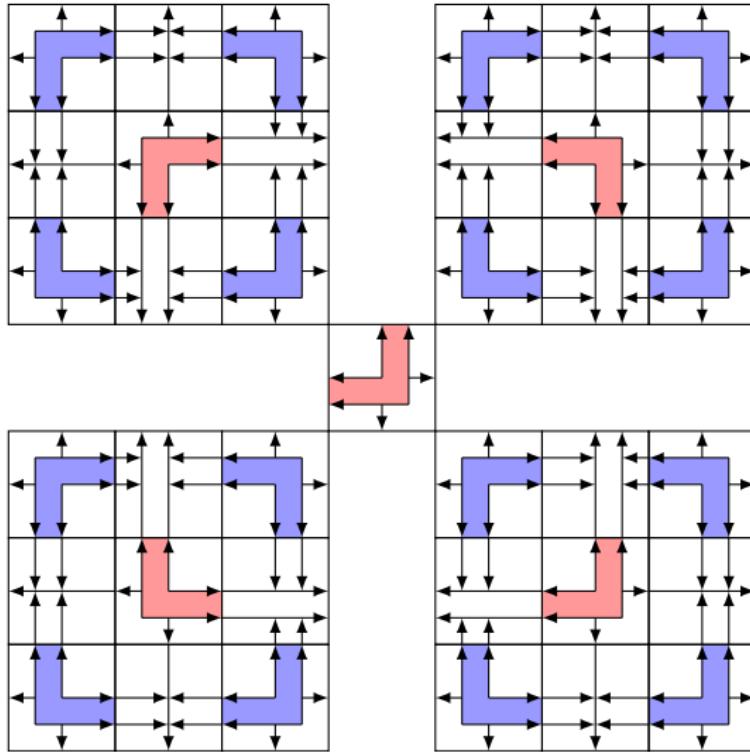
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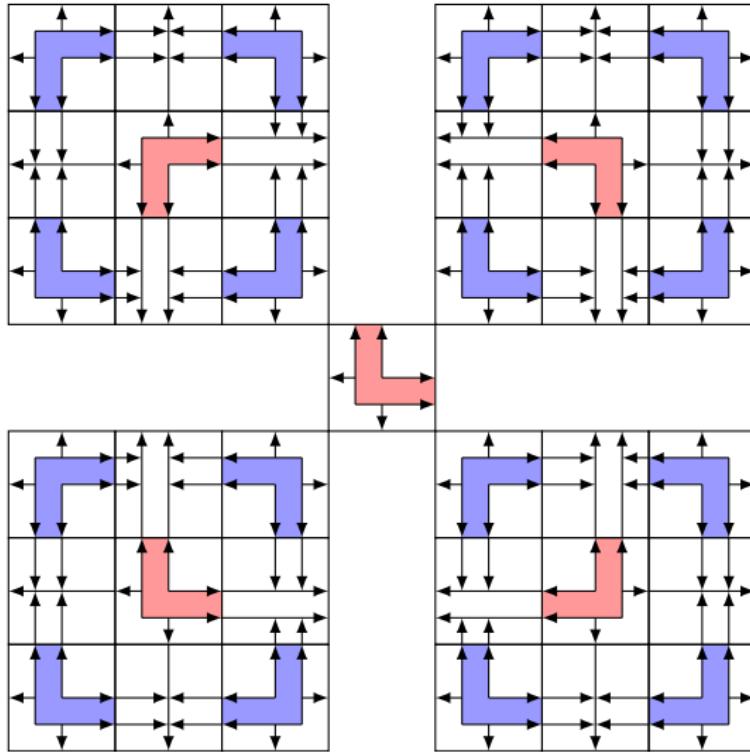
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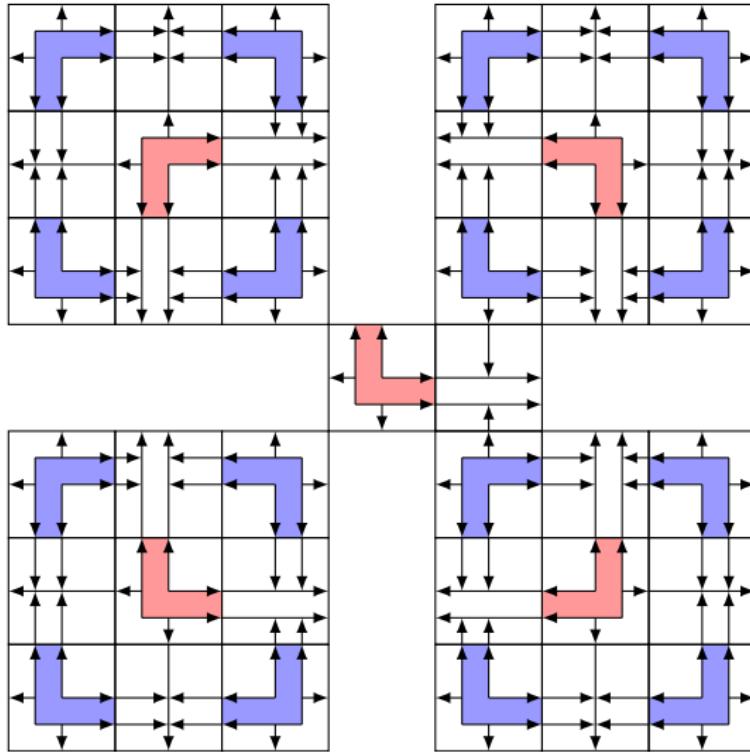
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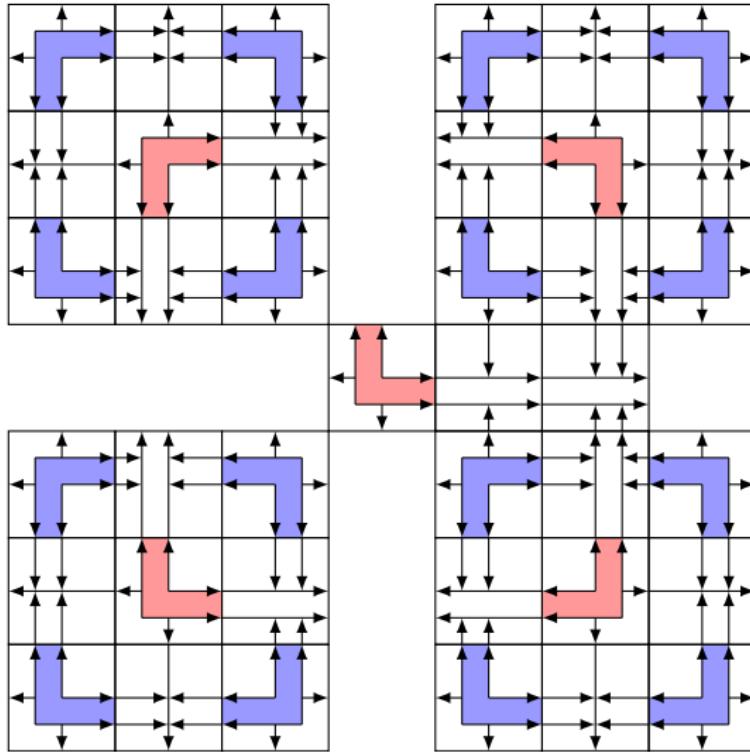
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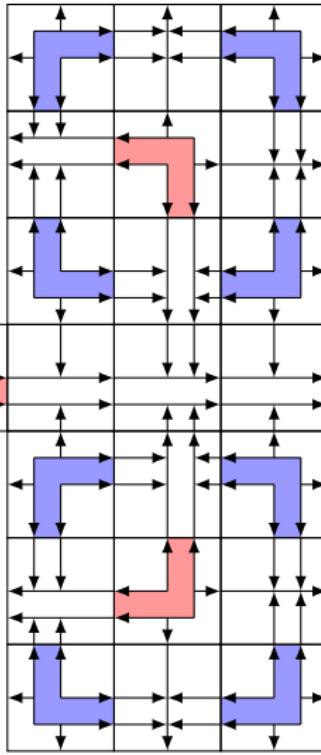
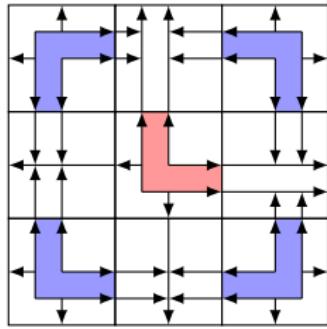
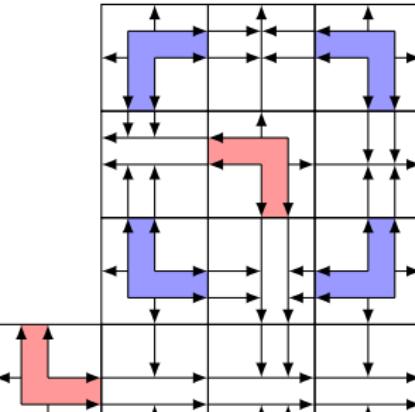
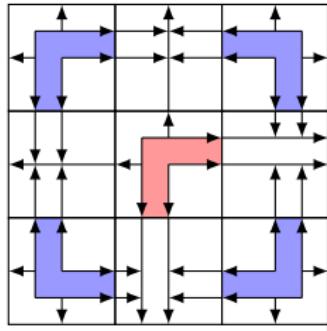
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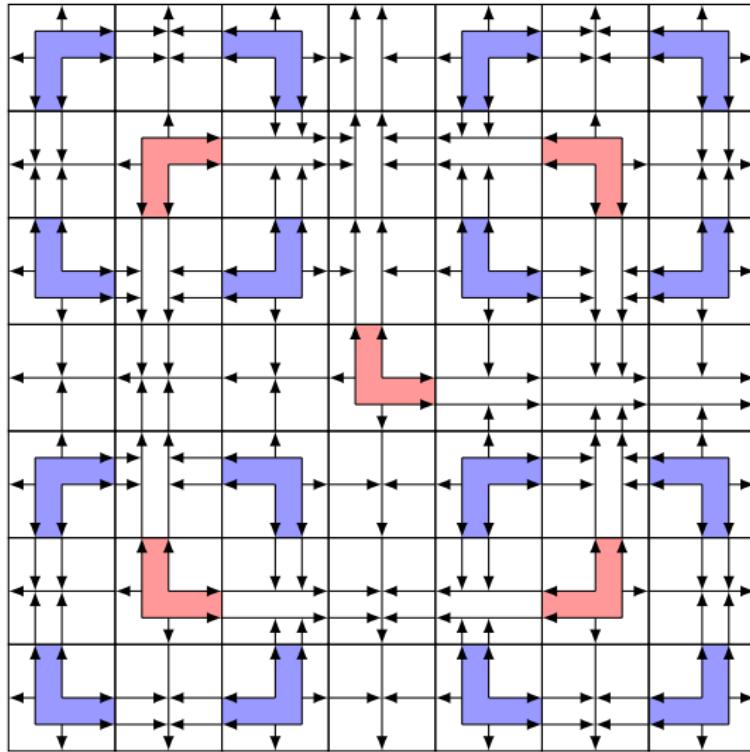
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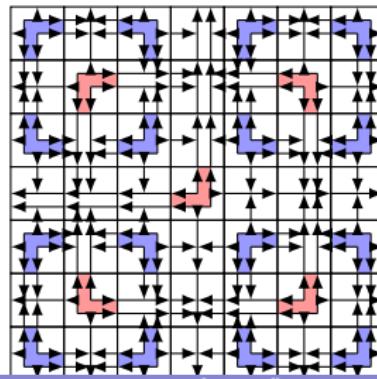
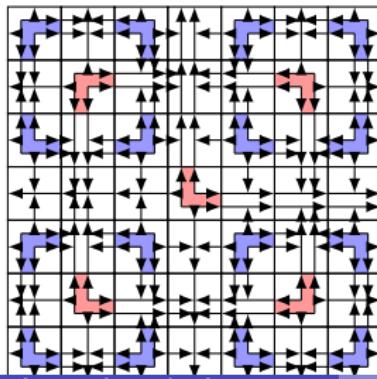
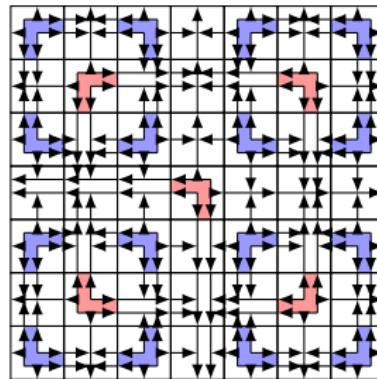
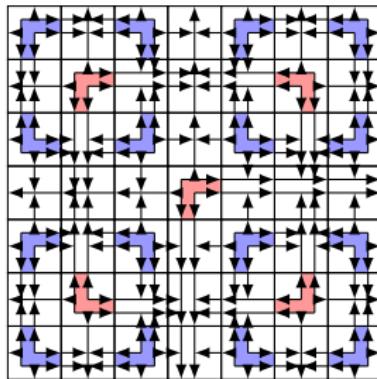
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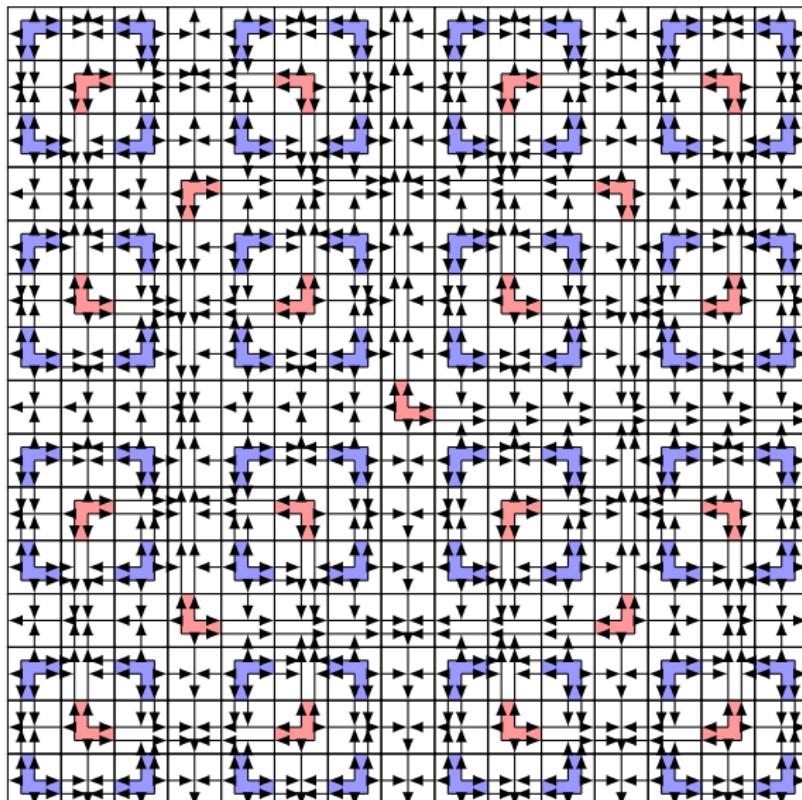
Existence of the tiling

Iterating the operation, super-tiles of level n can be formed and by compacity we conclude that $T_{Robi} \neq \emptyset$.

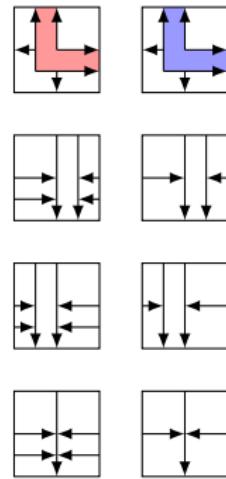
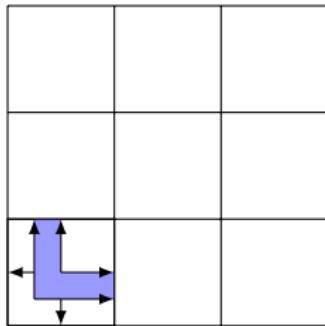


Existence of the tiling

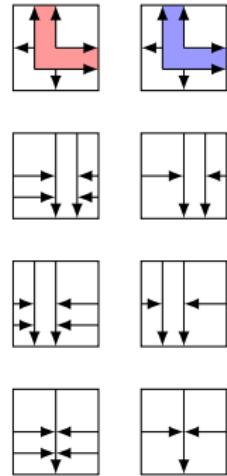
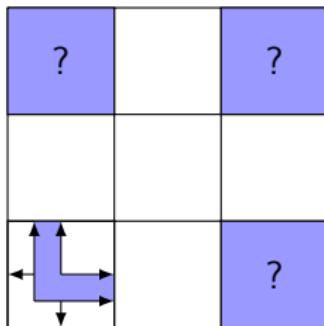
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Force the presence of super-tiles (of level 1)

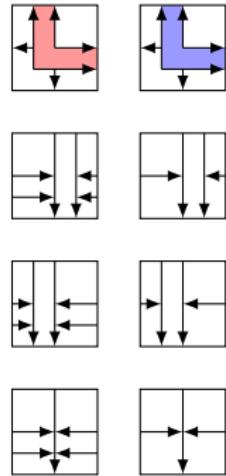
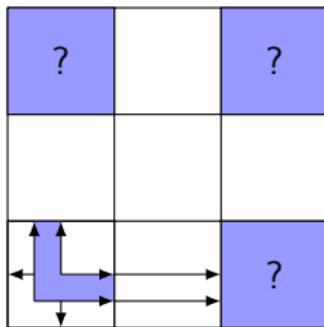


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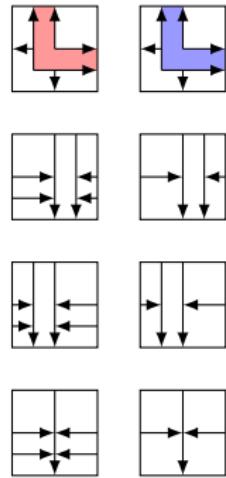
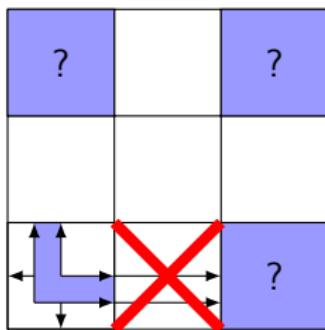
With $\boxed{?} \in \left\{ \begin{array}{c} \text{[L-shaped tile]} \\ \text{[J-shaped tile]} \\ \text{[T-shaped tile]} \\ \text{[F-shaped tile]} \end{array} \right\}$

Force the presence of super-tiles (of level 1)



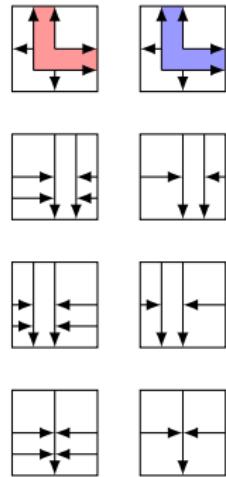
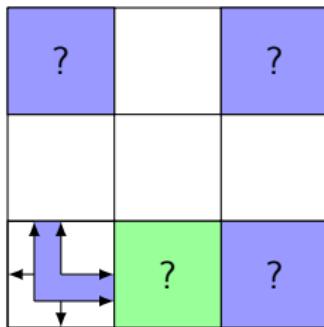
With $\boxed{?} \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{Empty square} \\ \text{Square with horizontal arrows} \\ \text{Square with vertical arrows} \\ \text{Square with diagonal arrows} \end{array} \right\}$

Force the presence of super-tiles (of level 1)



With $\boxed{?} \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{2x2 square with horizontal arrows} \\ \text{2x2 square with vertical arrows} \\ \text{2x2 square with diagonal arrows} \end{array} \right\}$

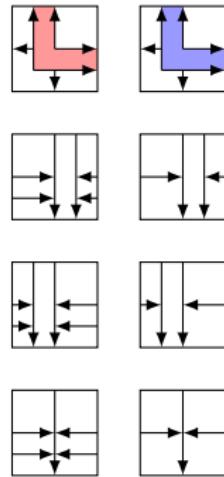
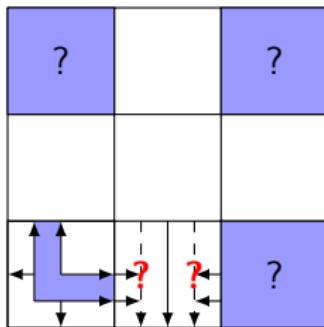
Force the presence of super-tiles (of level 1)



With $\in \left\{ \begin{array}{c} \text{blue L} \\ \text{blue J} \\ \text{blue T} \\ \text{blue F} \end{array} \right\}$

With $\in \left\{ \begin{array}{c} \text{2x1 vertical} \\ \text{2x1 vertical} \\ \text{2x1 vertical} \end{array} \right\}$

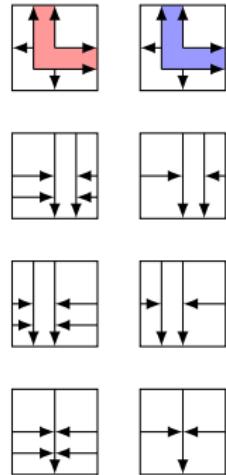
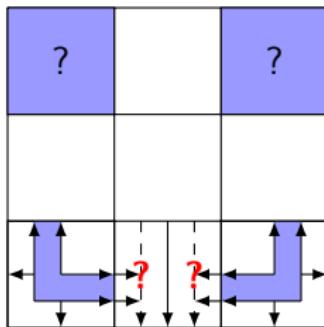
Force the presence of super-tiles (of level 1)



With $\in \left\{ \begin{array}{c} \text{red L-shaped tile} \\ \text{blue L-shaped tile} \\ \text{blue vertical bar tile} \\ \text{blue T-shaped tile} \\ \text{blue F-shaped tile} \end{array} \right\}$

With $\in \left\{ \begin{array}{c} \text{blue vertical bar tile} \\ \text{blue vertical bar tile} \\ \text{blue vertical bar tile} \end{array} \right\}$

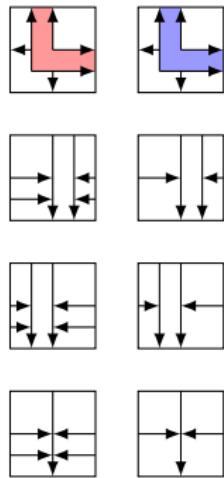
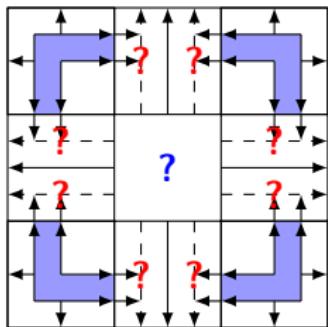
Force the presence of super-tiles (of level 1)



With $\in \left\{ \begin{array}{c} \text{red L} \\ \text{blue L} \\ \text{blue T} \\ \text{blue F} \end{array} \right\}$

With $\in \left\{ \begin{array}{c} \text{red 3x2} \\ \text{blue 3x2} \\ \text{blue 3x2} \end{array} \right\}$

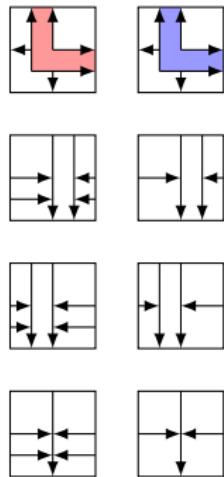
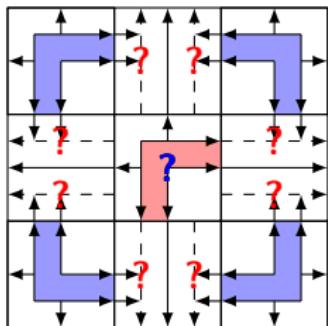
Force the presence of super-tiles (of level 1)



With $\boxed{?} \in \left\{ \begin{array}{c} \text{red L} \\ \text{red T} \\ \text{blue T} \\ \text{red F} \end{array} \right\}$

With $\boxed{\text{??}} \in \left\{ \begin{array}{c} \text{red 1x2} \\ \text{blue 1x2} \\ \text{red 2x1} \\ \text{blue 2x1} \end{array} \right\}$

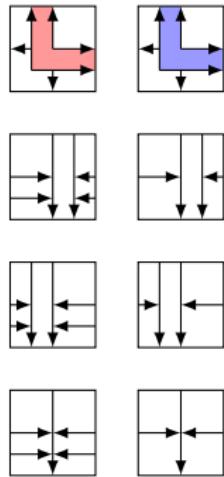
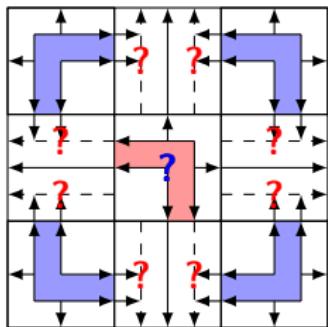
Force the presence of super-tiles (of level 1)



With $\boxed{?} \in \left\{ \begin{array}{c} \text{red L-shaped tile} \\ \text{blue L-shaped tile} \end{array} \right\}$

With $\boxed{\text{? ?}} \in \left\{ \begin{array}{c} \text{red L-shaped tile} \\ \text{blue L-shaped tile} \end{array} \right\}$

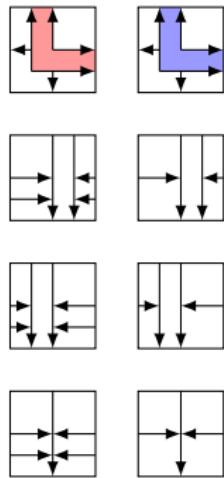
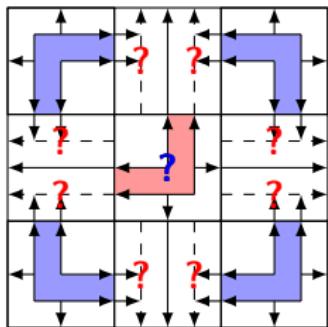
Force the presence of super-tiles (of level 1)



With $\boxed{?} \in \left\{ \begin{array}{c} \text{Red L-shaped tile} \\ \text{Red T-shaped tile} \\ \text{Blue L-shaped tile} \\ \text{Blue T-shaped tile} \end{array} \right\}$

With $\boxed{\text{??}} \in \left\{ \begin{array}{c} \text{Red 2x2 tile} \\ \text{Blue 2x2 tile} \\ \text{Blue 3x2 tile} \end{array} \right\}$

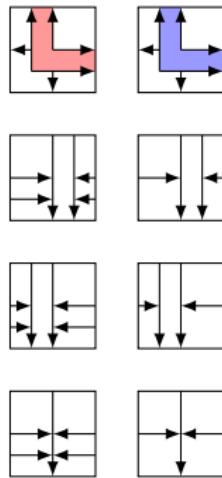
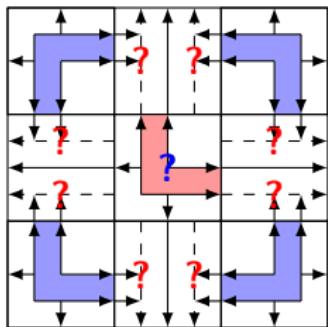
Force the presence of super-tiles (of level 1)



With $\boxed{?} \in \left\{ \begin{array}{c} \text{red L-shape} \\ \text{red T-shape} \\ \text{red T-shape} \\ \text{red L-shape} \end{array} \right\}$

With $\boxed{\text{? ?}} \in \left\{ \begin{array}{c} \text{blue 2x2 square} \\ \text{blue 2x2 square} \\ \text{blue 2x2 square} \end{array} \right\}$

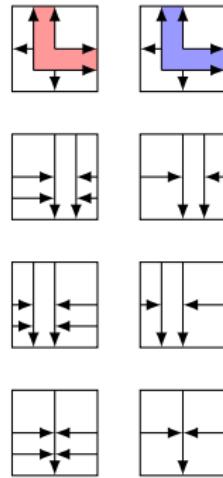
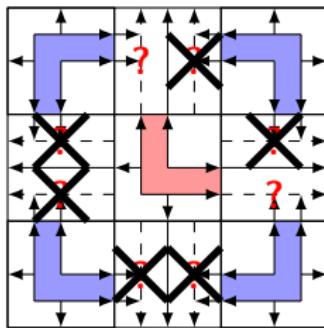
Force the presence of super-tiles (of level 1)



With $\boxed{?} \in \left\{ \begin{array}{c} \text{red L-shaped tile} \\ \text{white tile with horizontal arrows} \\ \text{white tile with vertical arrows} \end{array} \right\}$

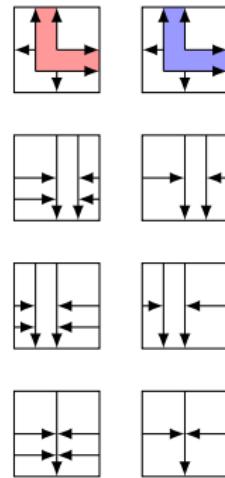
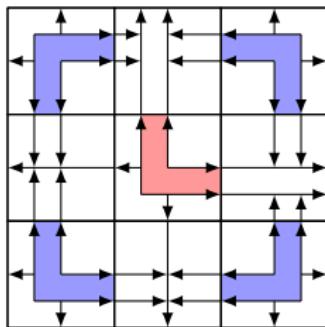
With $\boxed{\text{? ?}} \in \left\{ \begin{array}{c} \text{white tile with horizontal arrows} \\ \text{white tile with vertical arrows} \end{array} \right\}$

Force the presence of super-tiles (of level 1)

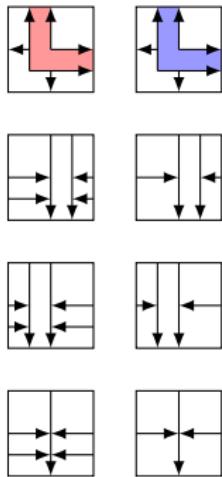
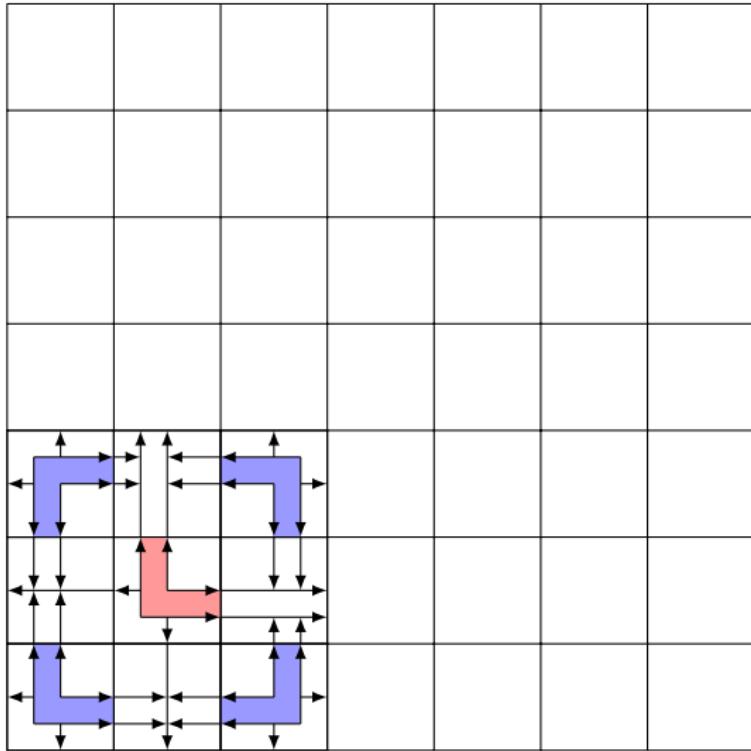


With  $\in \left\{ \begin{array}{c} \text{[grid with arrows pointing right]} \\ \text{[grid with arrows pointing down]} \\ \text{[grid with arrows pointing left]} \end{array} \right\}$

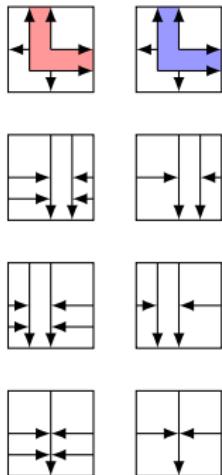
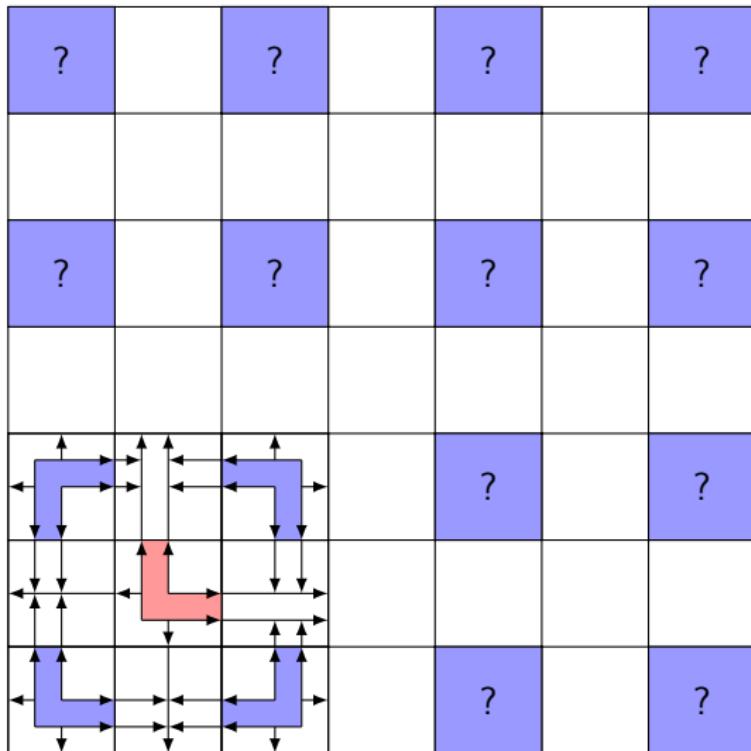
Force the presence of super-tiles (of level 1)



Force the presence of super-tiles (of level 2)

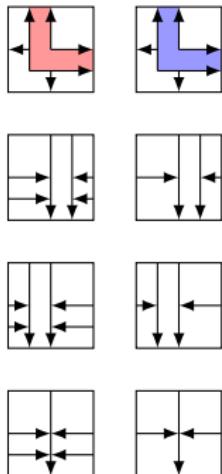
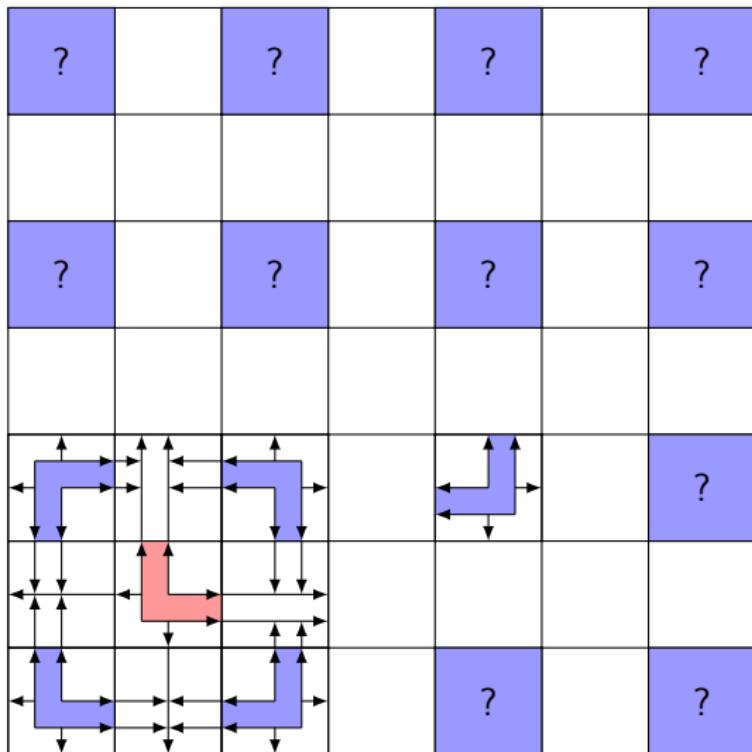


Force the presence of super-tiles (of level 2)



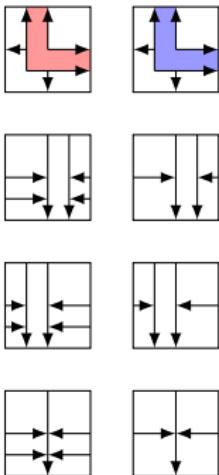
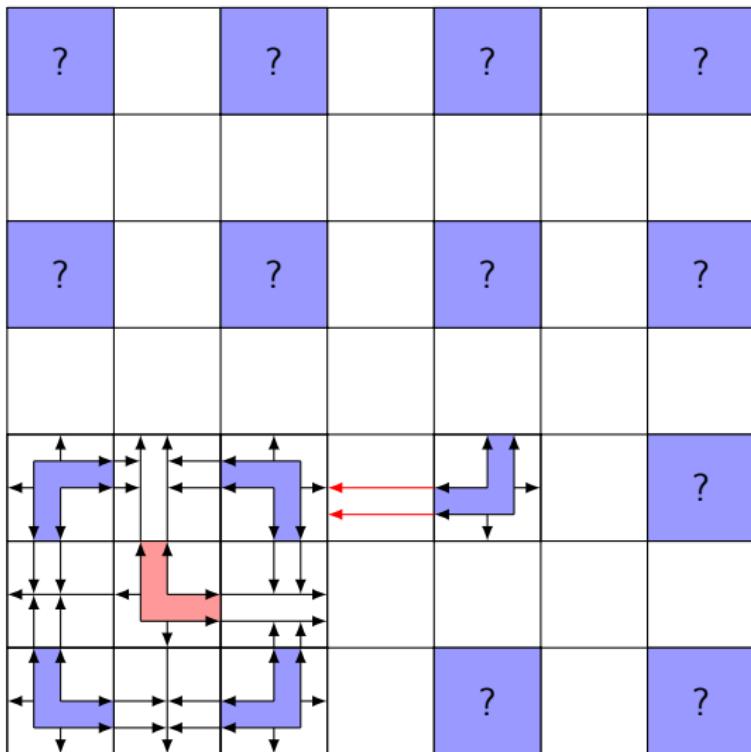
$$\boxed{?} \in \left\{ \begin{array}{c} \text{Red L-shaped tile} \\ \text{Blue L-shaped tile} \\ \text{Purple L-shaped tile} \\ \text{Blue L-shaped tile (rotated)} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



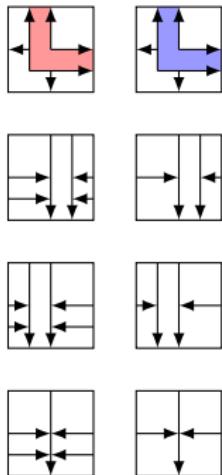
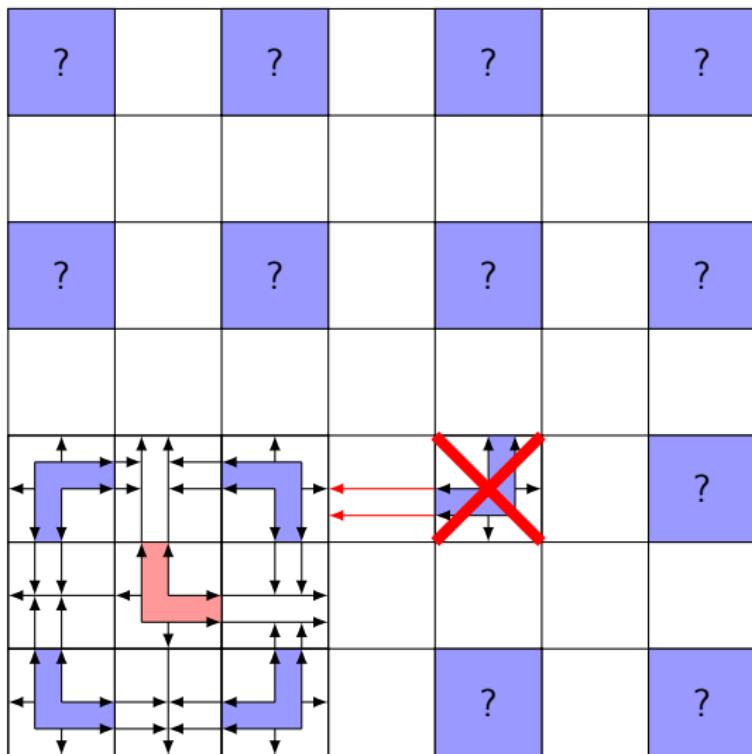
$$\boxed{?} \in \left\{ \begin{array}{c} \text{[red L]} \\ \text{[blue L]} \\ \text{[blue T]} \\ \text{[blue F]} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



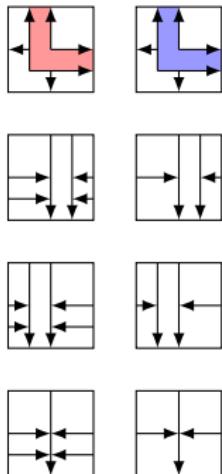
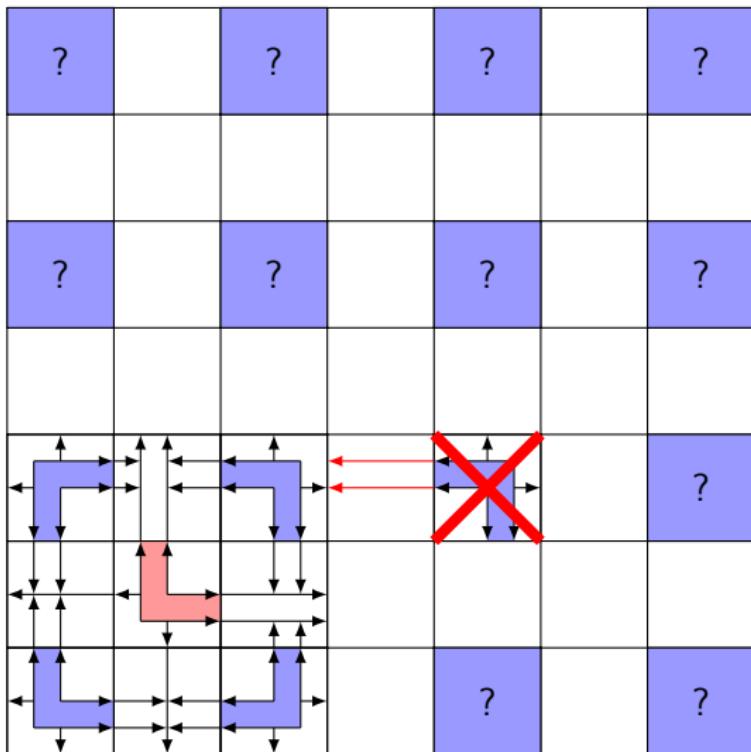
$$\boxed{?} \in \left\{ \begin{array}{c} \text{red L} \\ \text{blue L} \\ \text{blue T} \\ \text{blue F} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



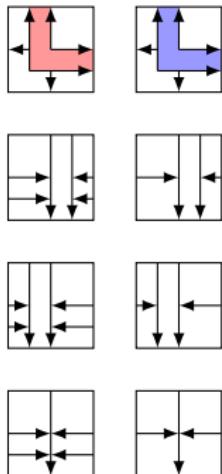
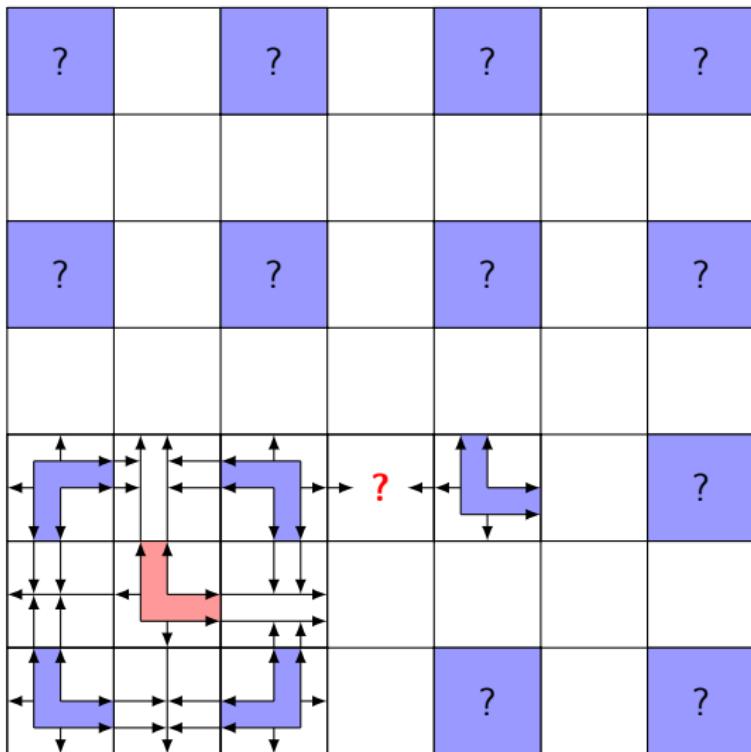
$$\boxed{?} \in \left\{ \begin{array}{c} \text{Red L-shape} \\ \text{Blue L-shape (up, left, down)} \\ \text{Blue L-shape (up, right, down)} \\ \text{Blue L-shape (up, left, right)} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



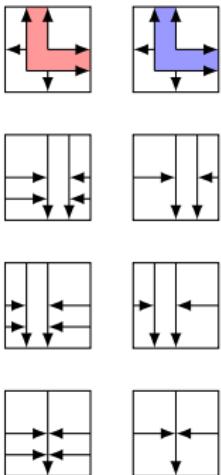
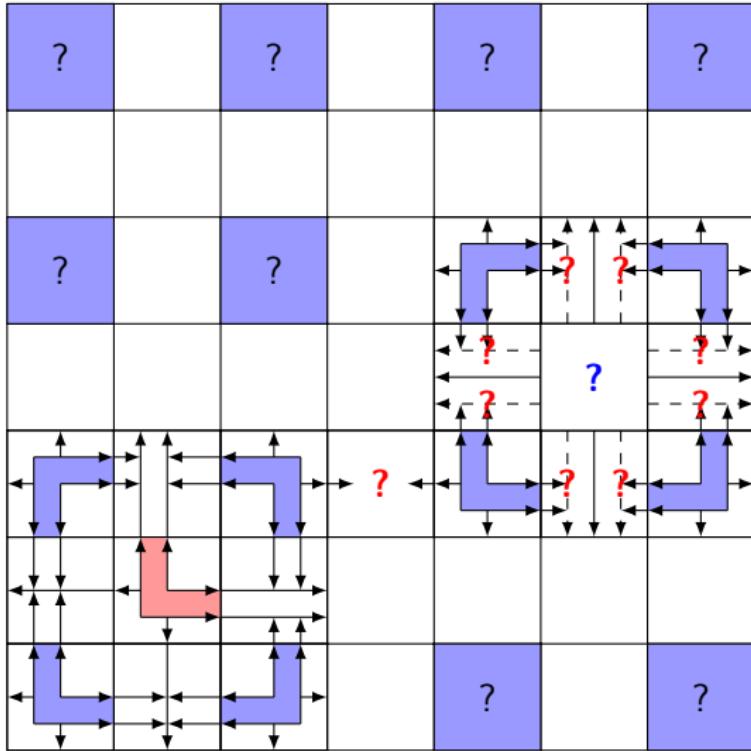
$$? \in \left\{ \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



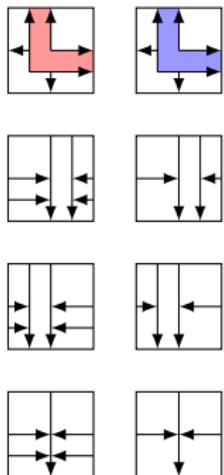
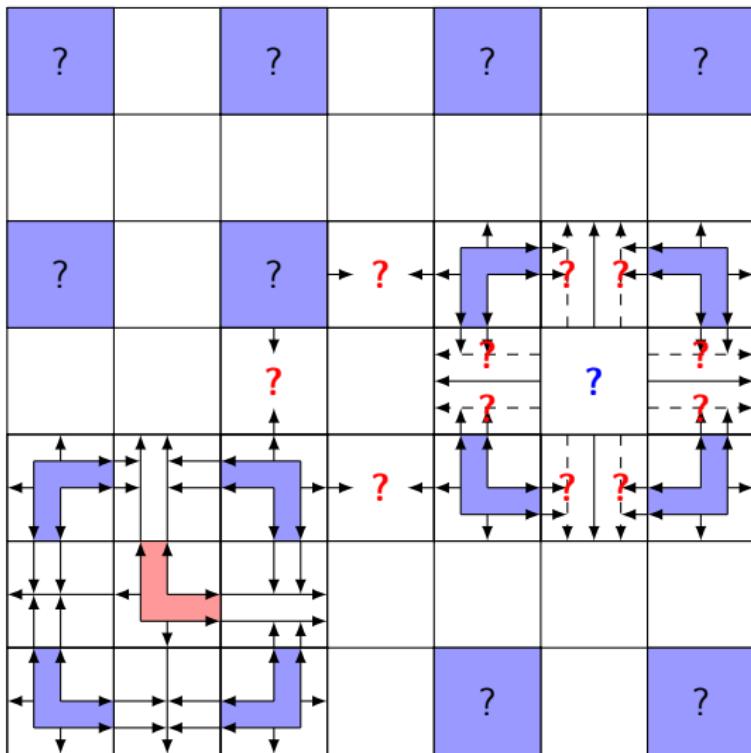
$$? \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



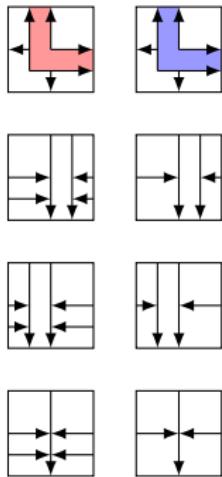
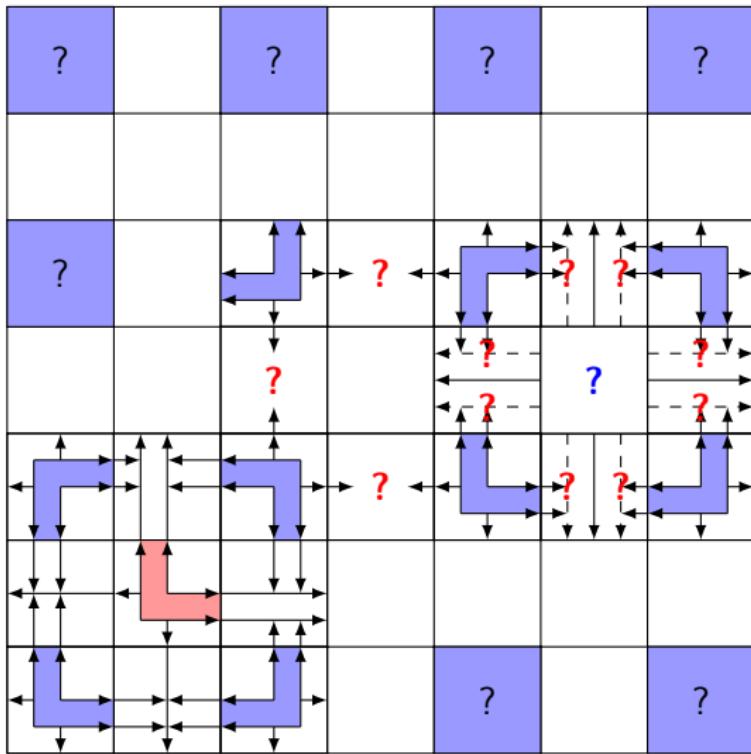
$$\boxed{?} \in \left\{ \begin{array}{c} \text{red L} \\ \text{blue L} \\ \text{blue T} \\ \text{blue F} \end{array} \right\} \boxed{?} \in \left\{ \begin{array}{c} \text{red L} \\ \text{red T} \\ \text{red F} \end{array} \right\} \boxed{??} \in \left\{ \begin{array}{c} \text{blue L} \\ \text{blue T} \\ \text{blue F} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



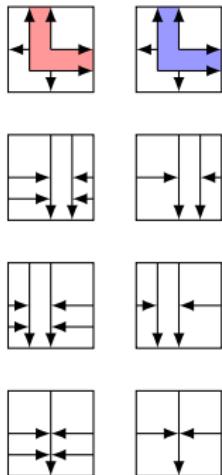
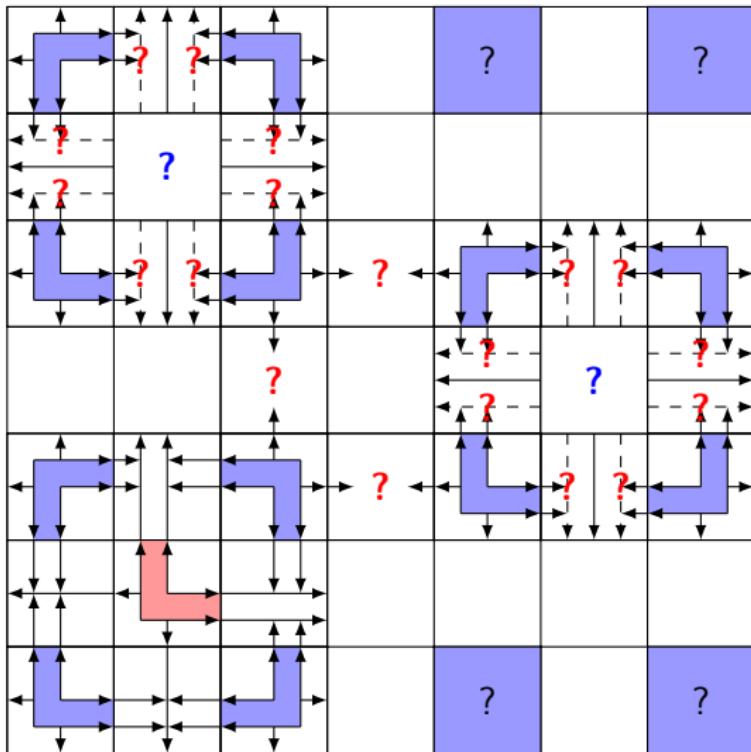
$$? \in \left\{ \begin{array}{c} \text{L-shaped tile 1} \\ \text{L-shaped tile 2} \\ \text{L-shaped tile 3} \\ \text{L-shaped tile 4} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{L-shaped tile 5} \\ \text{L-shaped tile 6} \\ \text{L-shaped tile 7} \\ \text{L-shaped tile 8} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{L-shaped tile 9} \\ \text{L-shaped tile 10} \\ \text{L-shaped tile 11} \\ \text{L-shaped tile 12} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



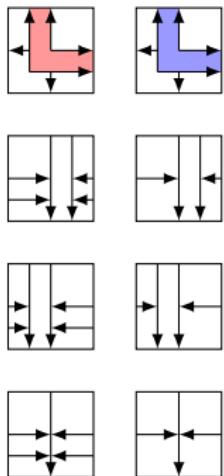
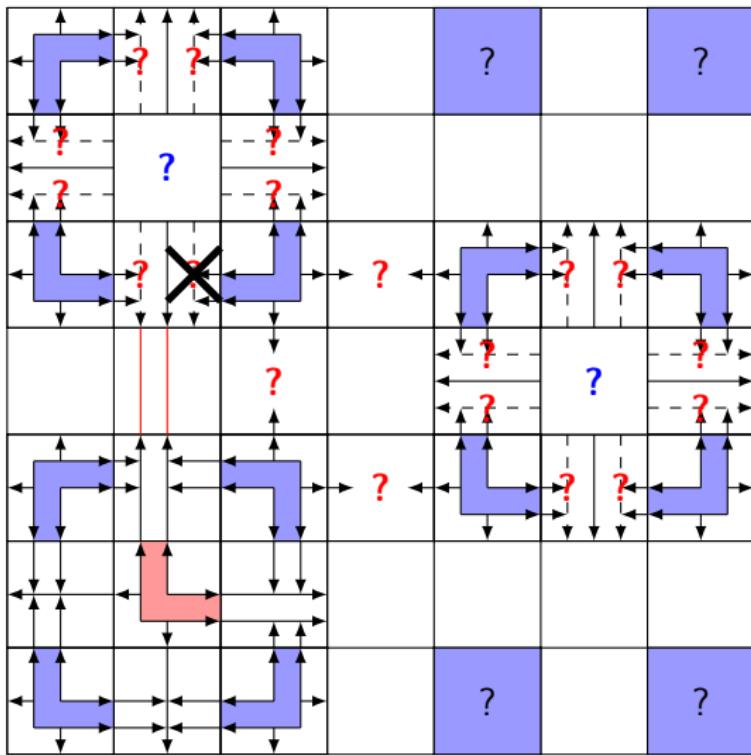
$$\boxed{?} \in \left\{ \begin{array}{c} \text{L-shaped tile 1} \\ \text{L-shaped tile 2} \\ \text{L-shaped tile 3} \\ \text{L-shaped tile 4} \end{array} \right\} \boxed{?} \in \left\{ \begin{array}{c} \text{L-shaped tile 1} \\ \text{L-shaped tile 2} \\ \text{L-shaped tile 3} \\ \text{L-shaped tile 4} \end{array} \right\} \boxed{??} \in \left\{ \begin{array}{c} \text{L-shaped tile 1} \\ \text{L-shaped tile 2} \\ \text{L-shaped tile 3} \\ \text{L-shaped tile 4} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



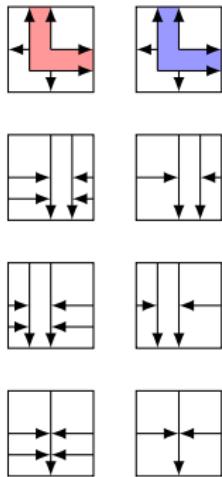
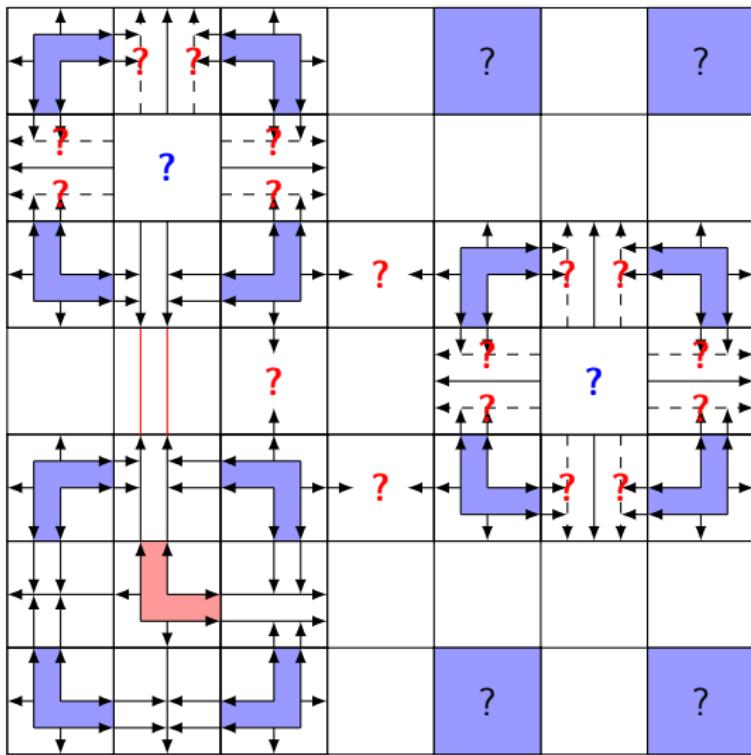
$$\boxed{?} \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\} \boxed{?} \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\} \boxed{??} \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



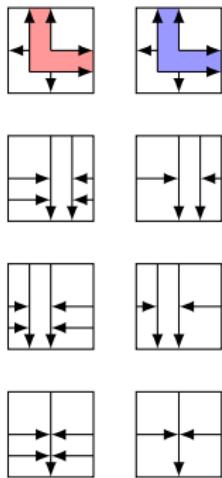
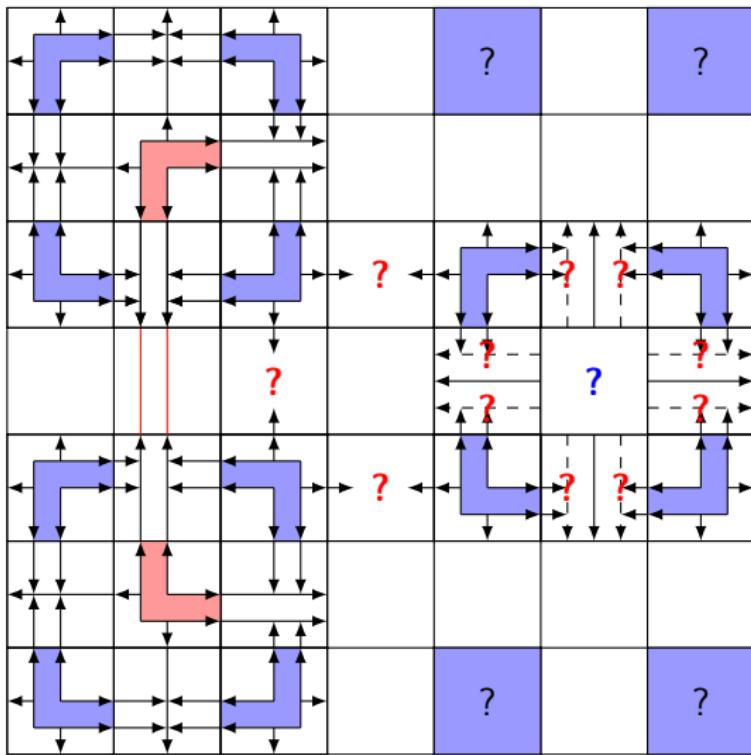
$$? \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{2x2 block} \\ \text{2x2 block} \\ \text{2x2 block} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



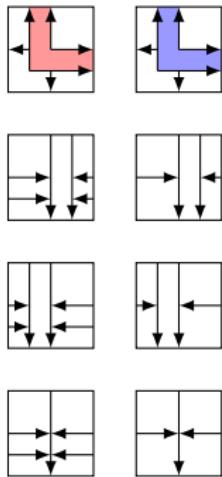
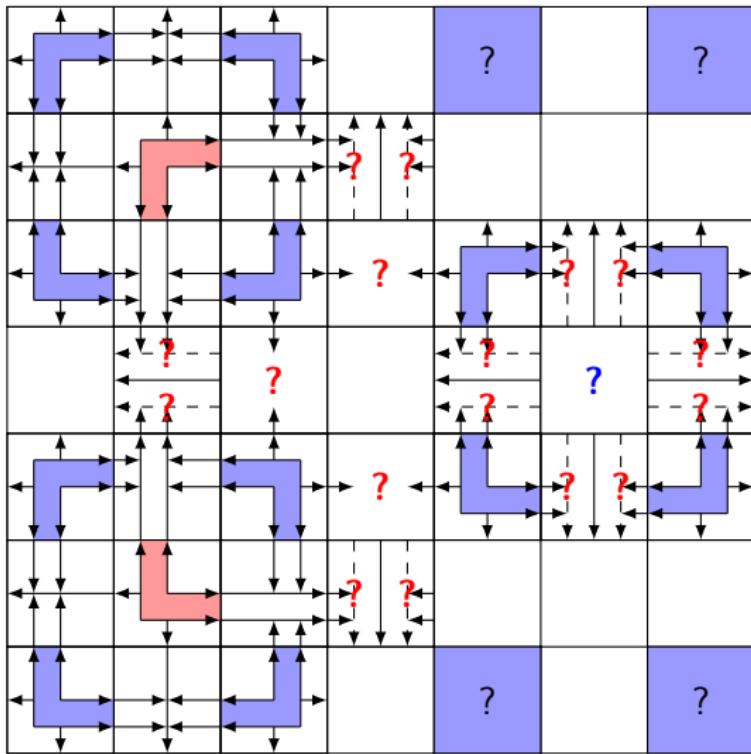
$$? \in \left\{ \begin{array}{c} \text{blue L-shaped tile} \\ \text{red L-shaped tile} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{blue L-shaped tile} \\ \text{red L-shaped tile} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{blue L-shaped tile} \\ \text{red L-shaped tile} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



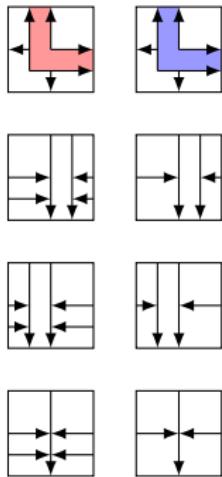
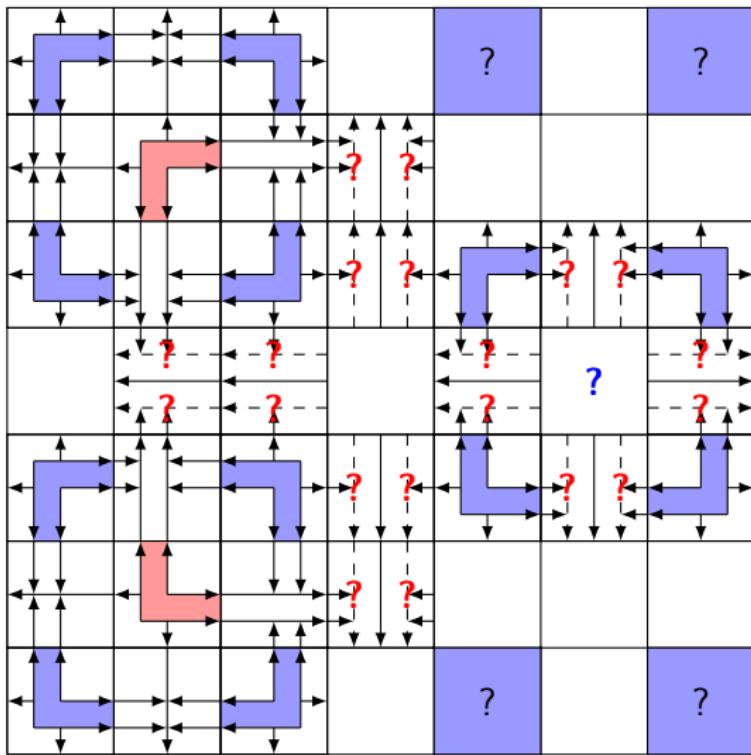
$$? \in \left\{ \begin{array}{c} \text{Blue L-shaped tile} \\ \text{Red L-shaped tile} \end{array} \right\} \quad ? \in \left\{ \begin{array}{c} \text{Blue L-shaped tile} \\ \text{Red L-shaped tile} \end{array} \right\} \quad ?? \in \left\{ \begin{array}{c} \text{Blue L-shaped tile} \\ \text{Red L-shaped tile} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



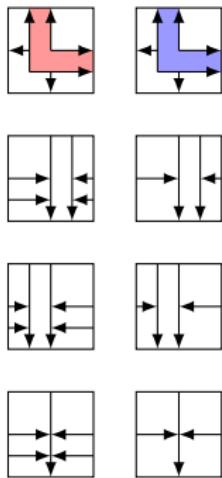
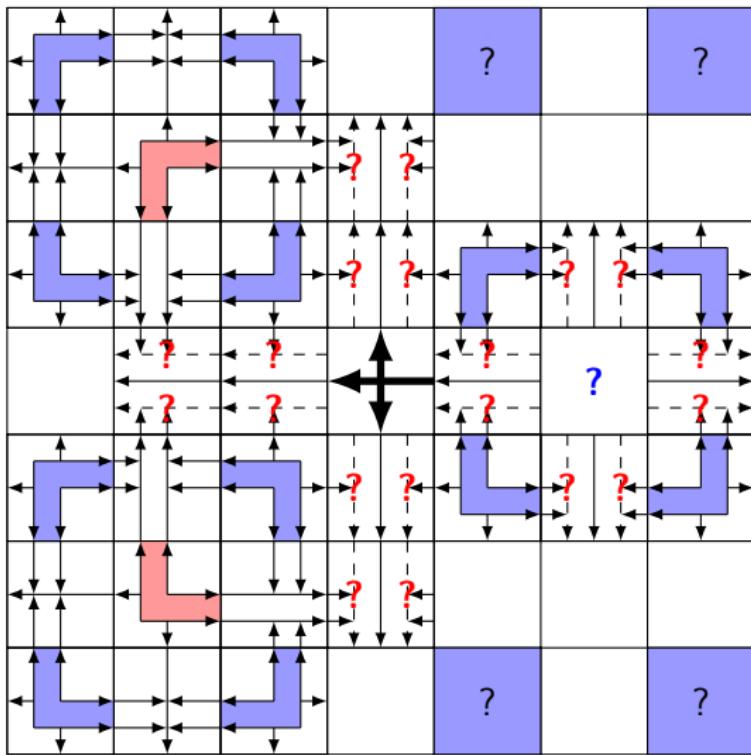
$$? \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{Movement path} \\ \text{Movement path} \\ \text{Movement path} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



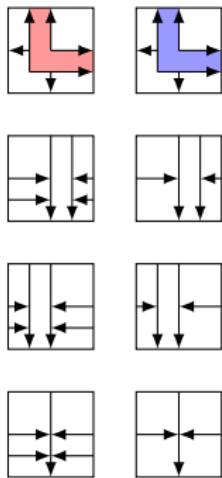
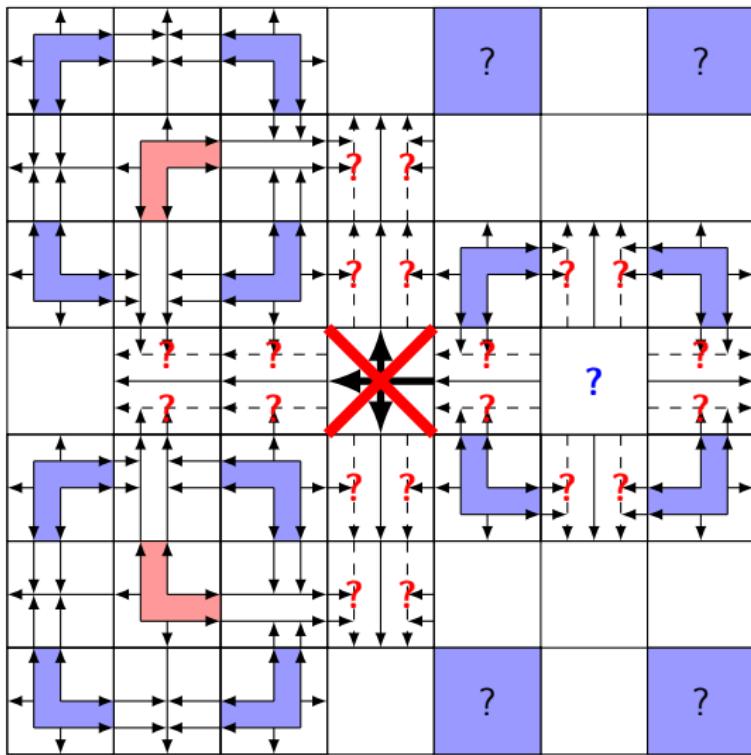
$$? \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



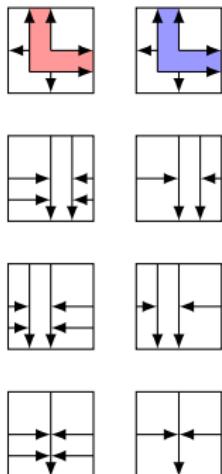
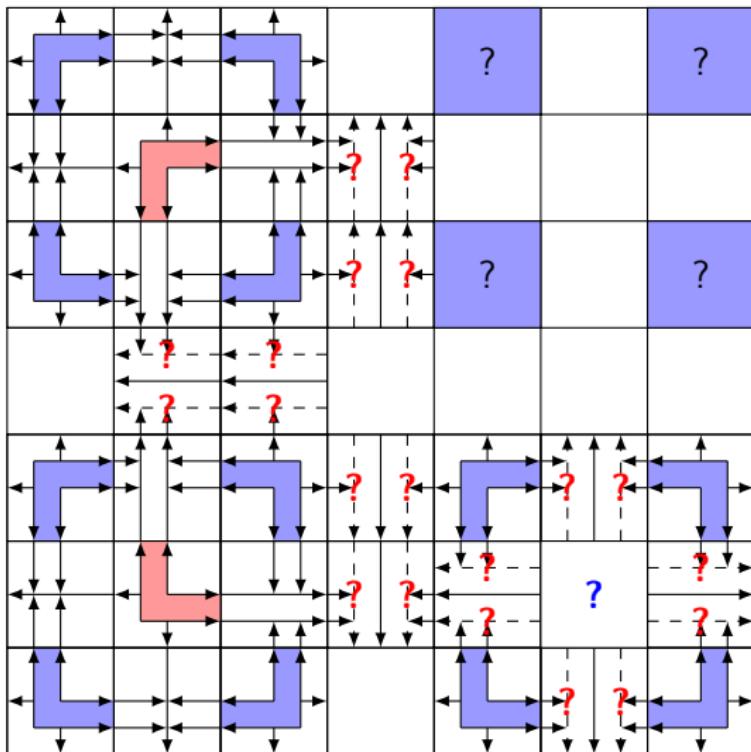
$$? \in \left\{ \begin{array}{c} \text{blue L} \\ \text{red L} \\ \text{blue T} \\ \text{red T} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{blue L} \\ \text{red L} \\ \text{blue T} \\ \text{red T} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{blue L} \\ \text{red L} \\ \text{blue T} \\ \text{red T} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



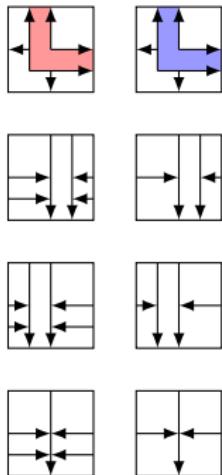
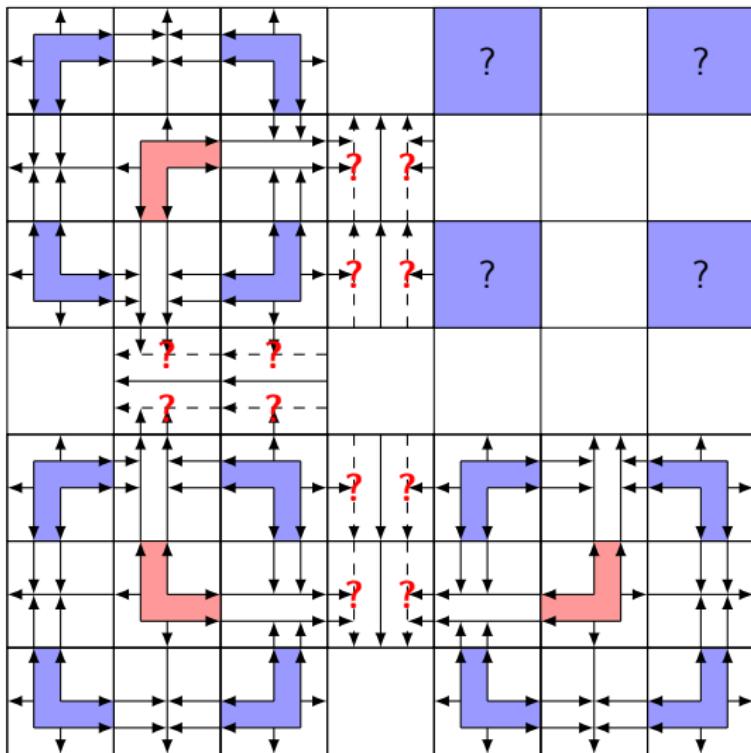
$$? \in \left\{ \begin{array}{c} \text{blue L} \\ \text{red L} \\ \text{blue T} \\ \text{red T} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{blue L} \\ \text{red L} \\ \text{blue T} \\ \text{red T} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{blue T} \\ \text{red T} \\ \text{blue T} \\ \text{red T} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



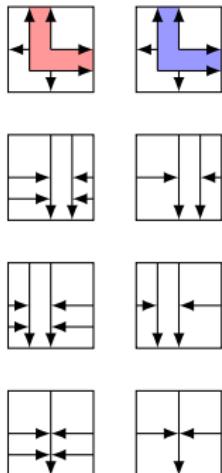
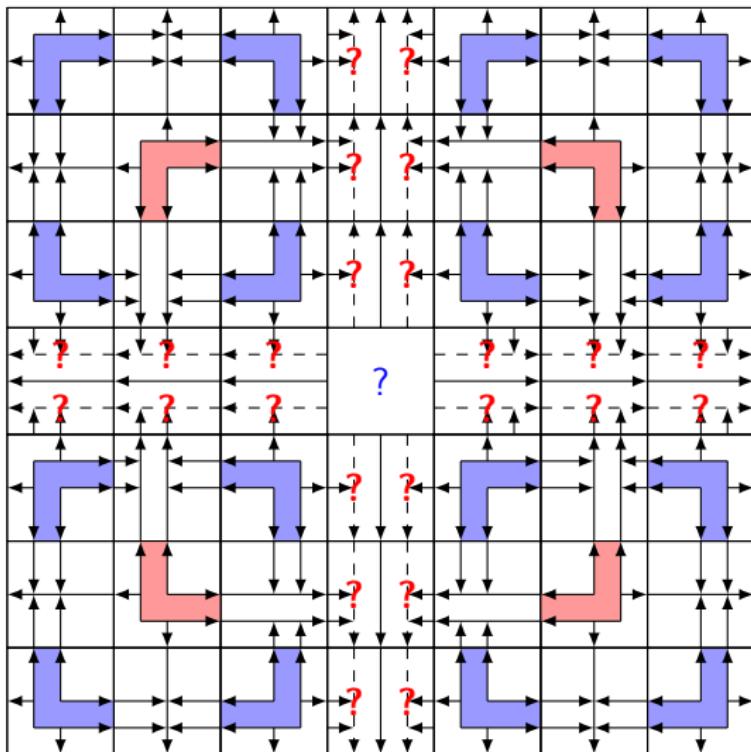
$$? \in \left\{ \begin{array}{c} \text{blue L-tile} \\ \text{red L-tile} \\ \text{empty square} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{blue L-tile} \\ \text{red L-tile} \\ \text{empty square} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{blue L-tile} \\ \text{red L-tile} \\ \text{empty square} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



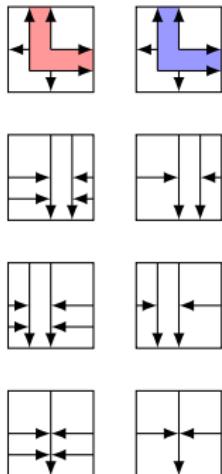
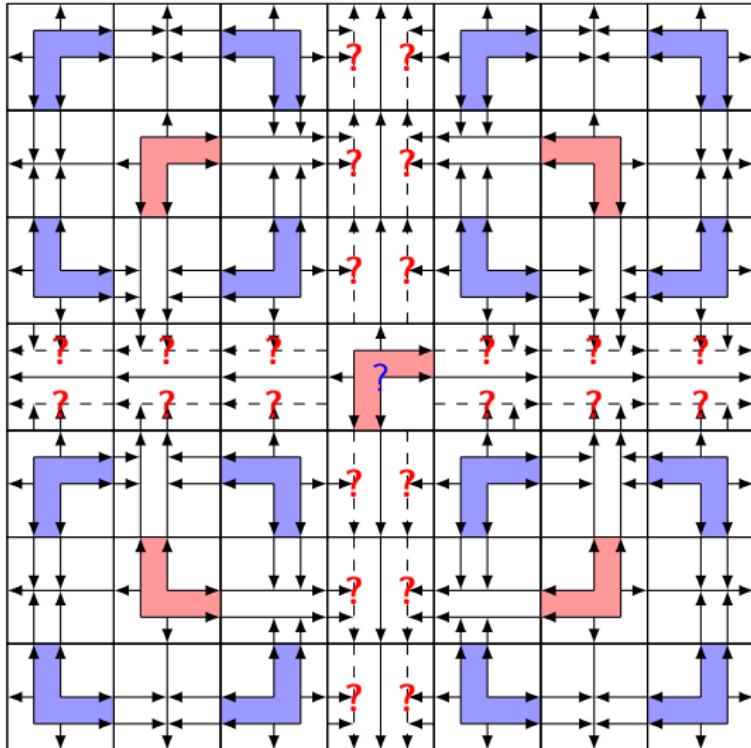
$$? \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



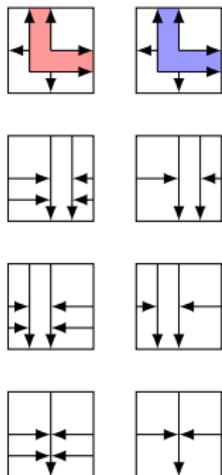
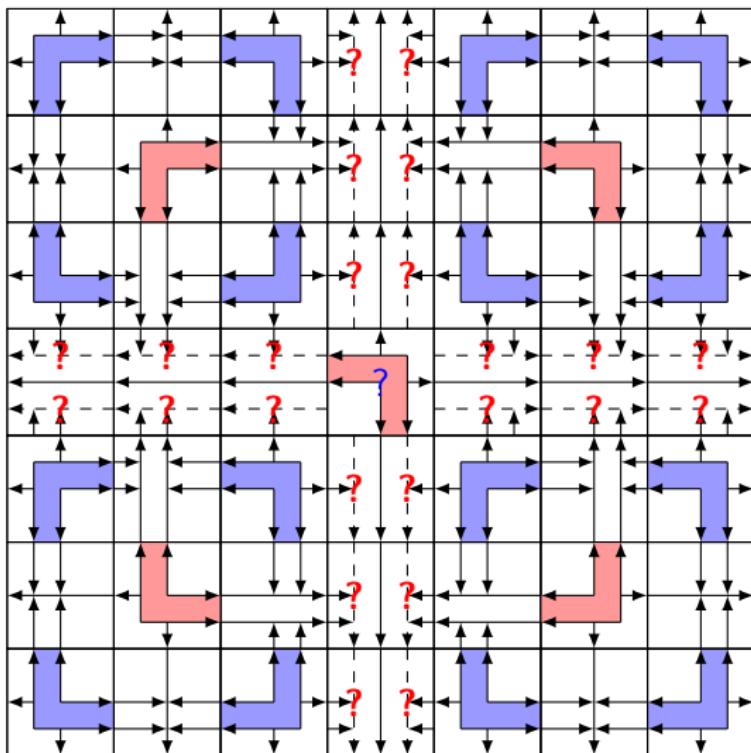
$$\boxed{?} \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\} \quad \boxed{??} \in \left\{ \begin{array}{c} \text{2x2 tile} \\ \text{2x2 tile} \\ \text{2x2 tile} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



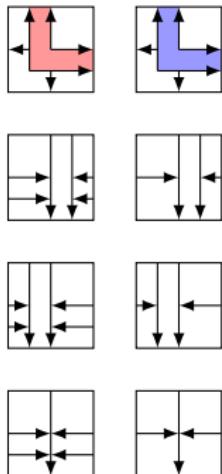
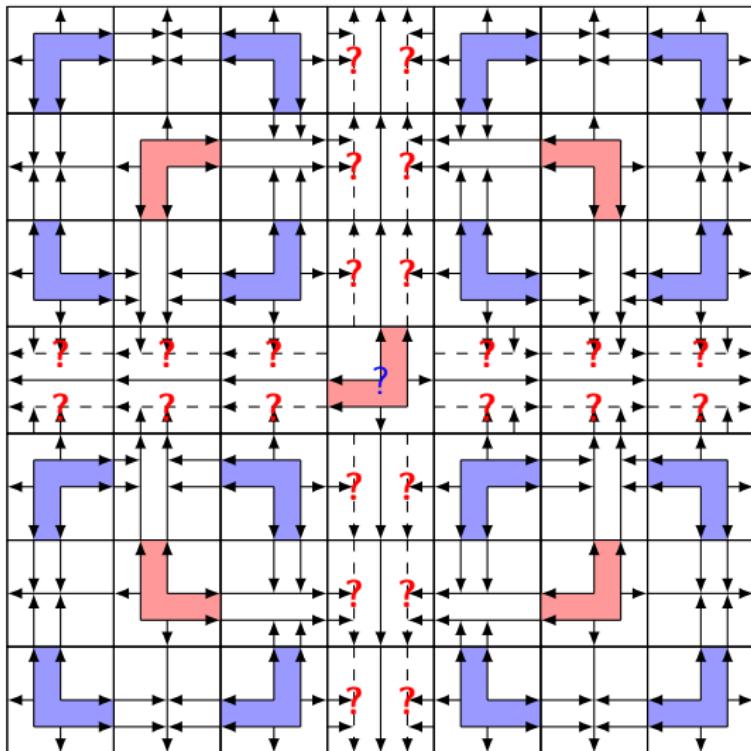
$$\boxed{?} \in \left\{ \begin{array}{c} \text{red L} \\ \text{blue L} \\ \text{red T} \\ \text{blue T} \end{array} \right\} \quad \boxed{??} \in \left\{ \begin{array}{c} \text{vertical bars} \\ \text{horizontal bars} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



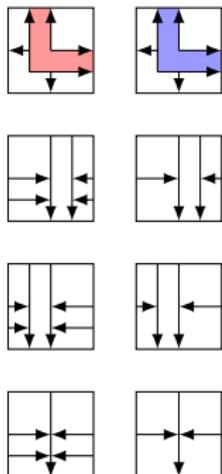
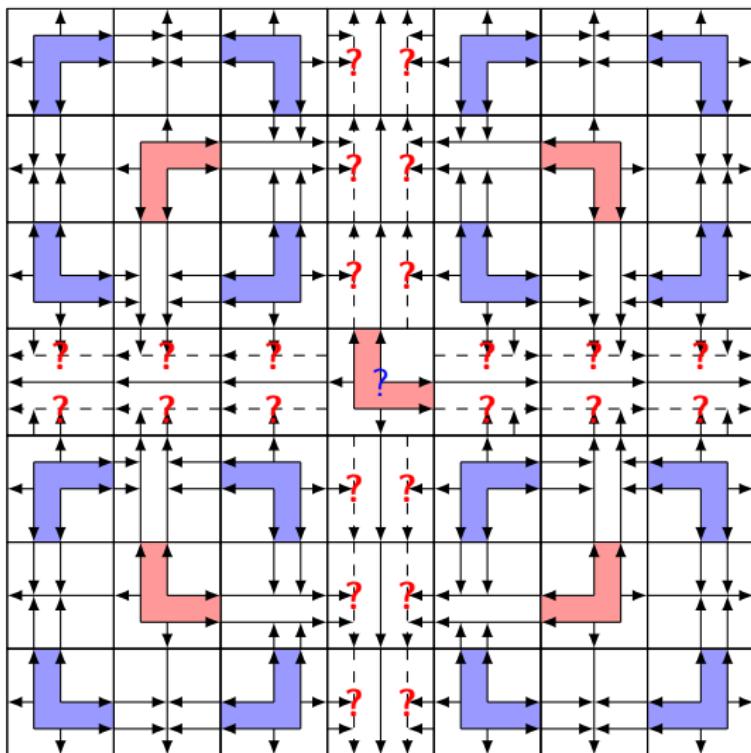
$$\boxed{?} \in \left\{ \begin{array}{c} \text{Red L} \\ \text{Blue L} \\ \text{Red T} \\ \text{Blue T} \end{array} \right\} \quad \boxed{??} \in \left\{ \begin{array}{c} \text{Red 2x2} \\ \text{Blue 2x2} \\ \text{Red 2x2} \\ \text{Blue 2x2} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



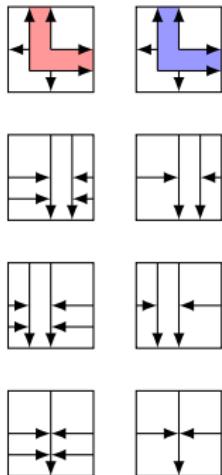
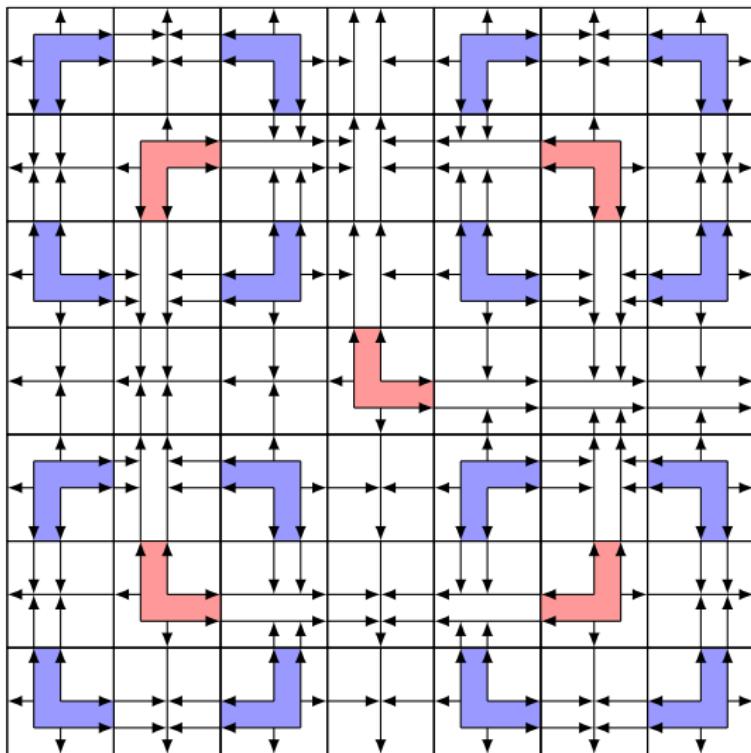
$$\boxed{?} \in \left\{ \begin{array}{c} \text{blue L} \\ \text{red L} \\ \text{red T} \\ \text{red F} \end{array} \right\} \quad \boxed{??} \in \left\{ \begin{array}{c} \text{blue 2x2} \\ \text{red 2x2} \\ \text{red 3x2} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



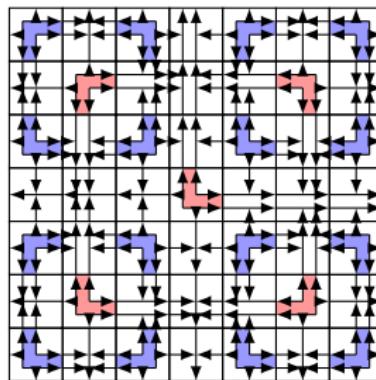
$$\boxed{?} \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\} \quad \boxed{??} \in \left\{ \begin{array}{c} \text{2x2 square tile} \\ \text{2x2 square tile} \\ \text{2x2 square tile} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)

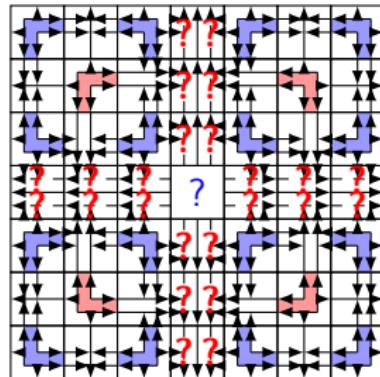
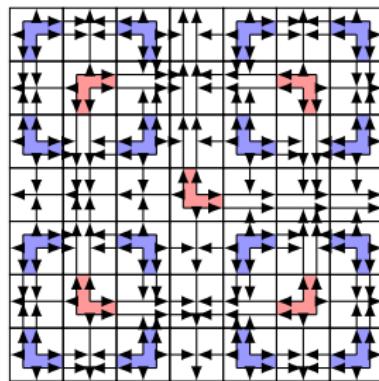


$$\boxed{?} \in \left\{ \begin{array}{c} \text{red L} \\ \text{red T} \\ \text{blue L} \\ \text{blue T} \end{array} \right\} \quad \boxed{??} \in \left\{ \begin{array}{c} \text{red 2x2} \\ \text{blue 2x2} \\ \text{black 2x2} \end{array} \right\}$$

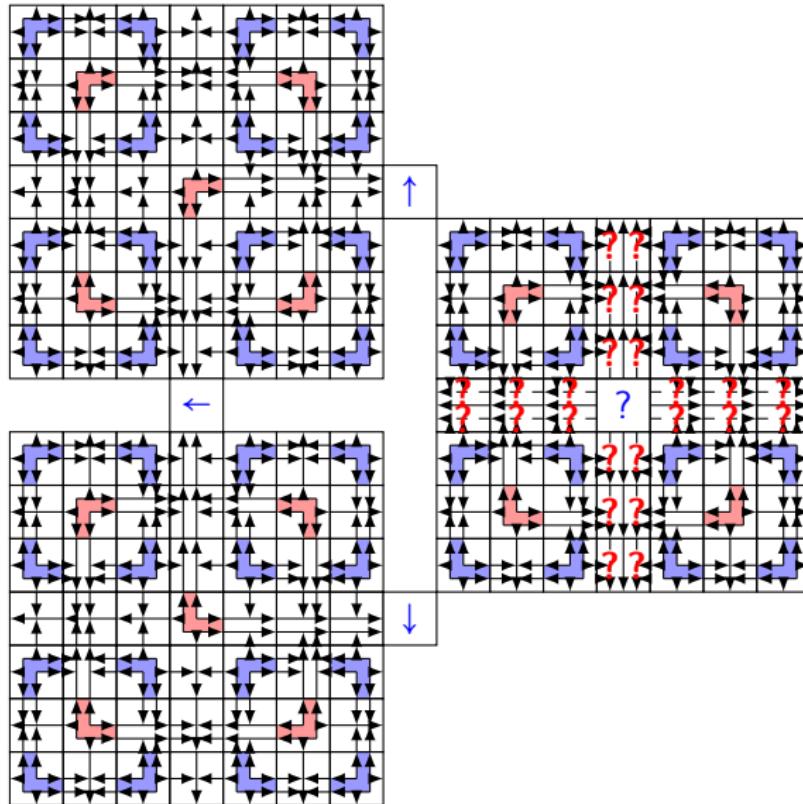
Force the presence of higher levels super-tiles



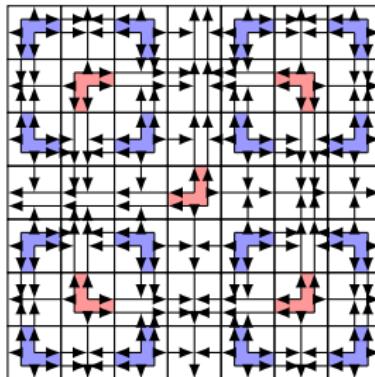
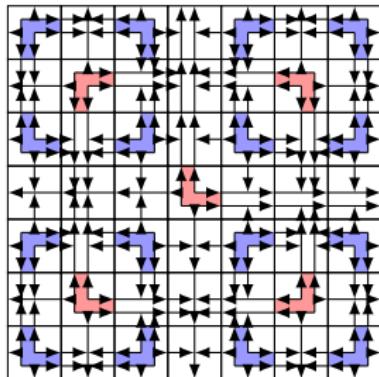
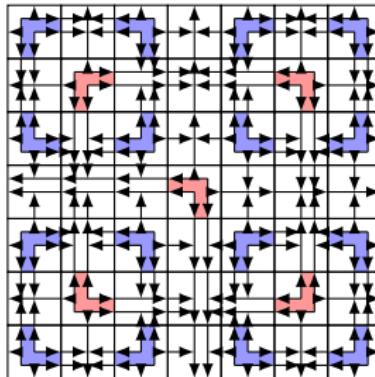
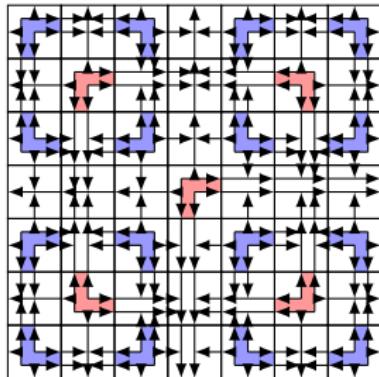
Force the presence of higher levels super-tiles



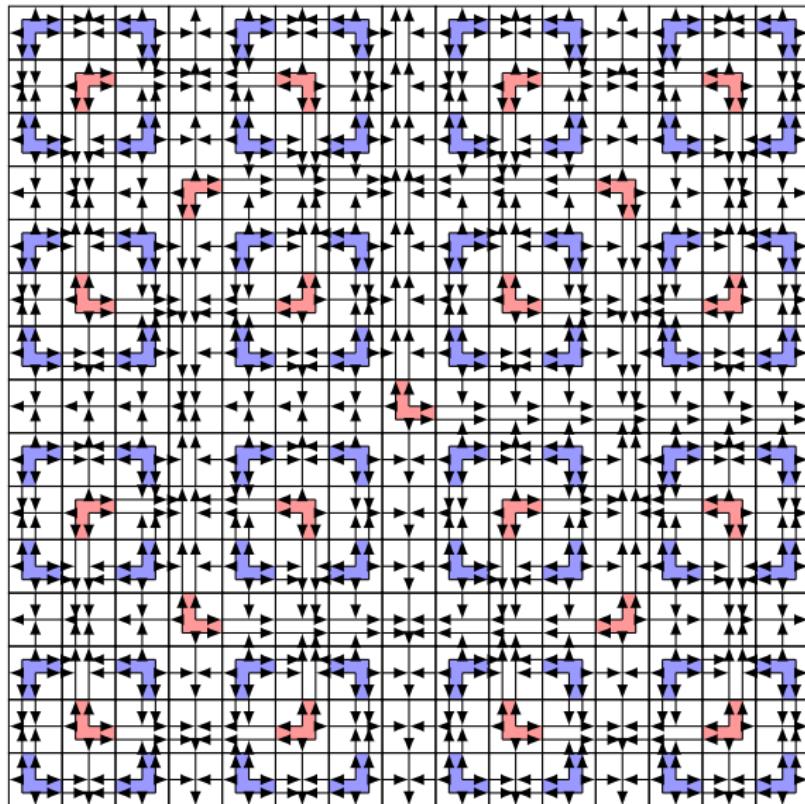
Force the presence of higher levels super-tiles



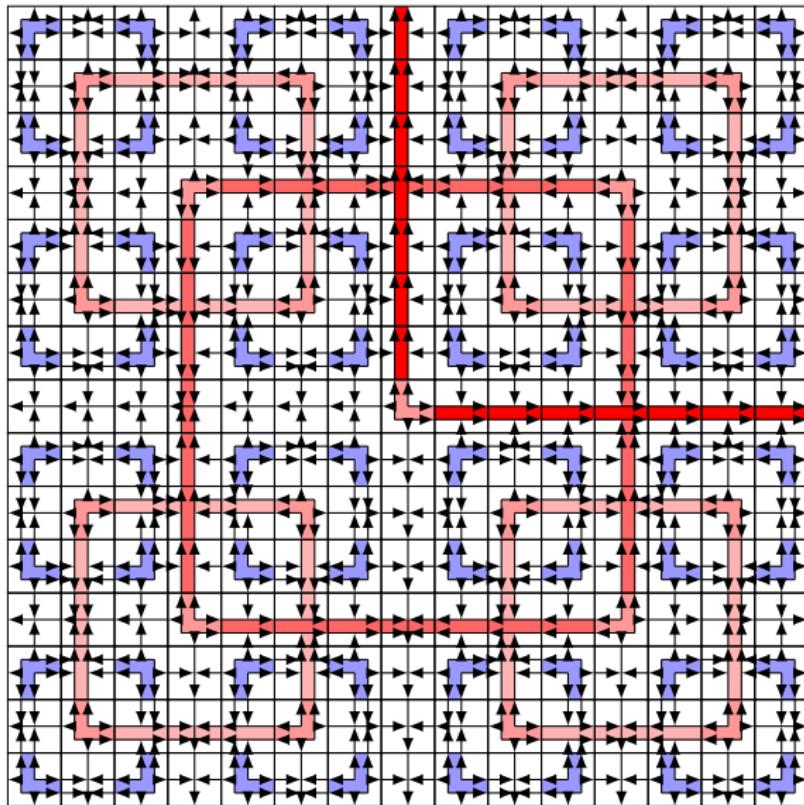
Force the presence of higher levels super-tiles



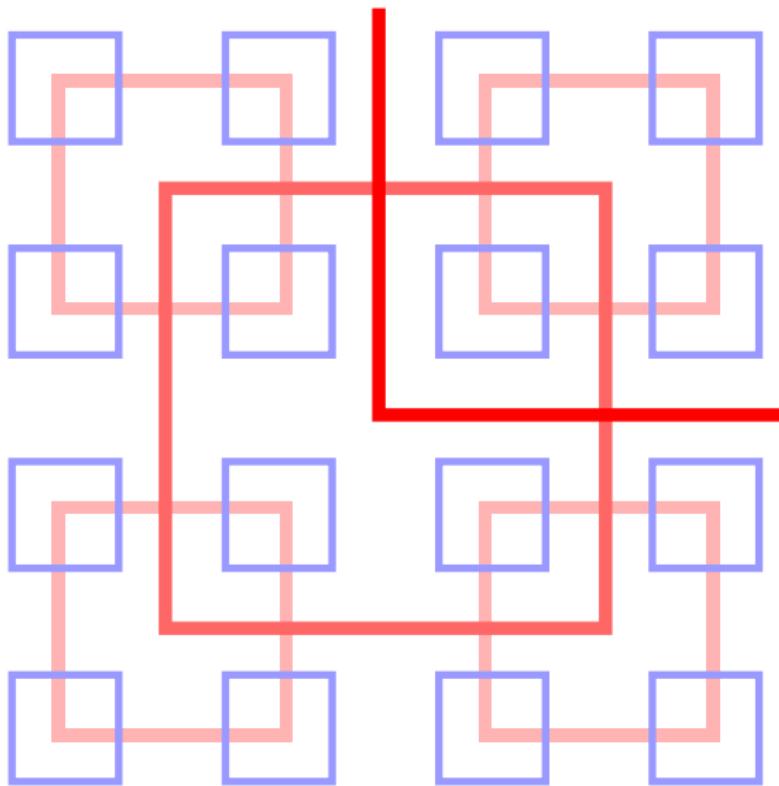
Geometric vision: Hierarchical structure



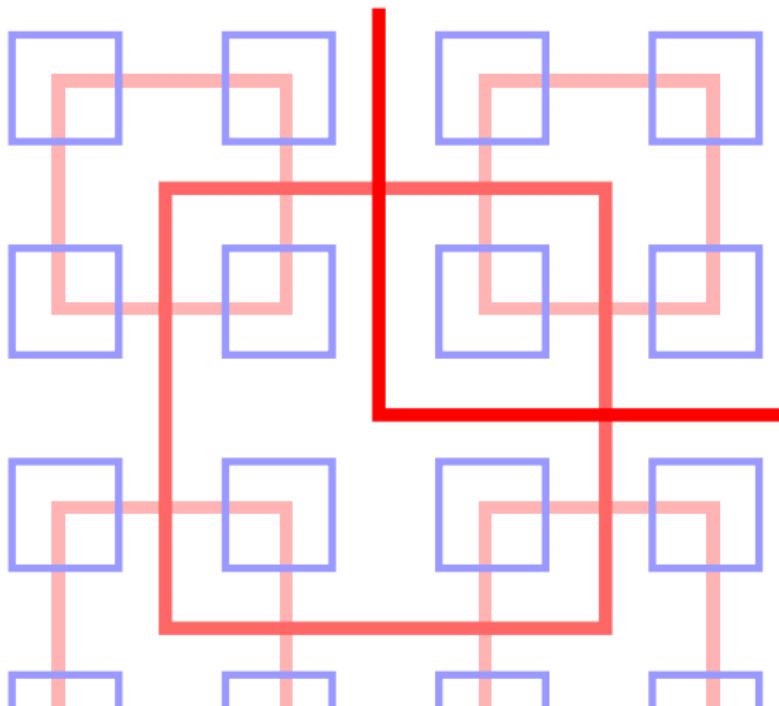
Geometric vision: Hierarchical structure



Geometric vision: Hierarchical structure



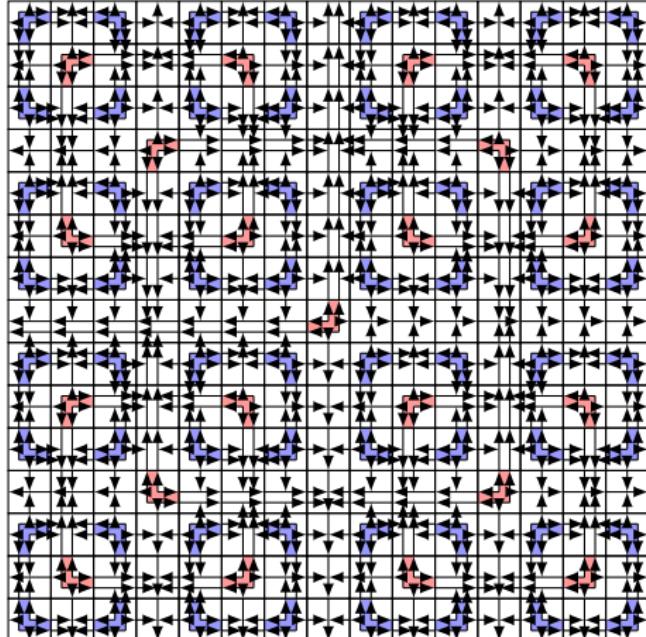
Geometric vision: Hierarchical structure



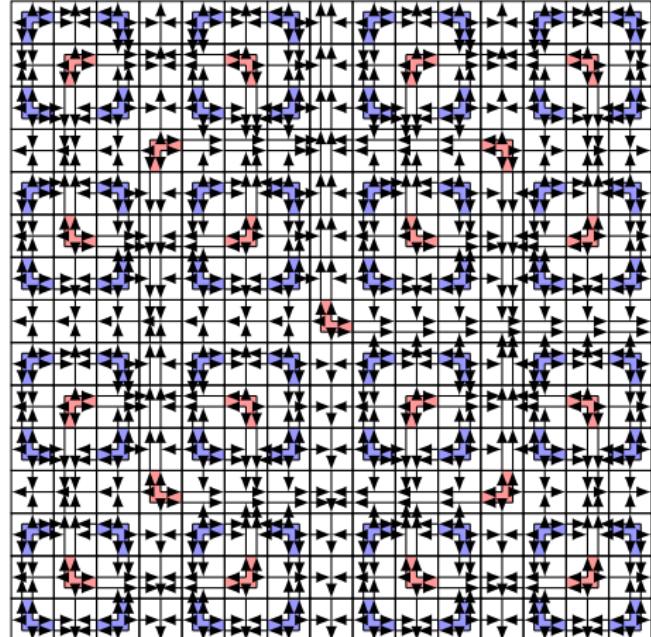
Theorem (*Robinson 1971*)

The SFT $\mathbf{T}_{\mathcal{R}obi}$ is not empty and all configurations are aperiodic.

Geometric Vision: Fractured lines

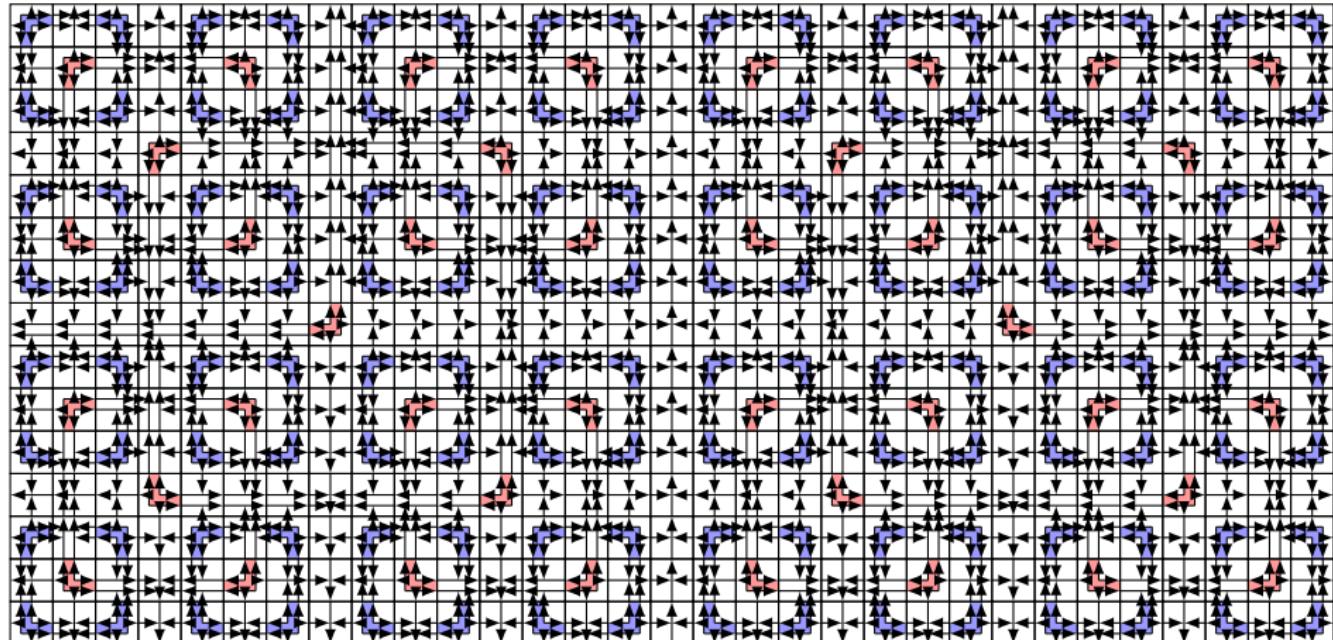


My first aperiodic tiling: Robinson's tiling

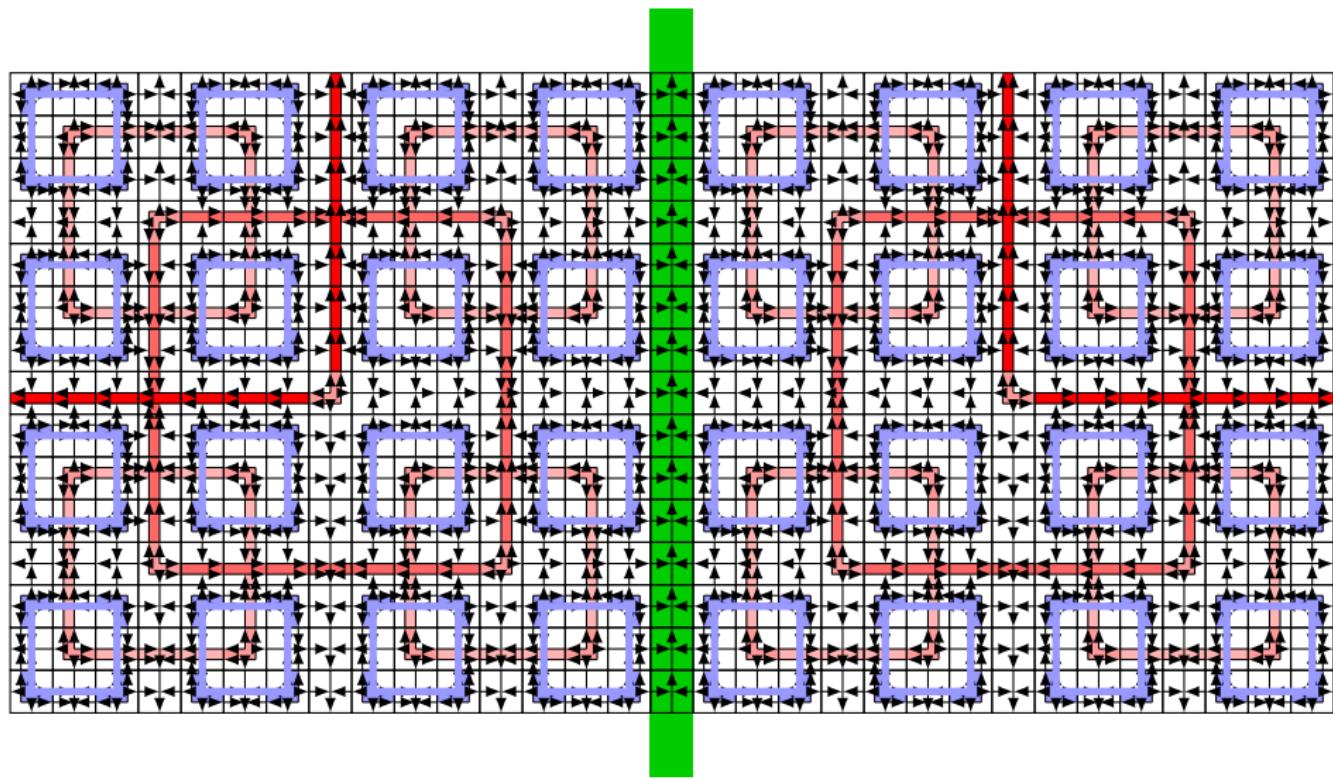


Fractured lines

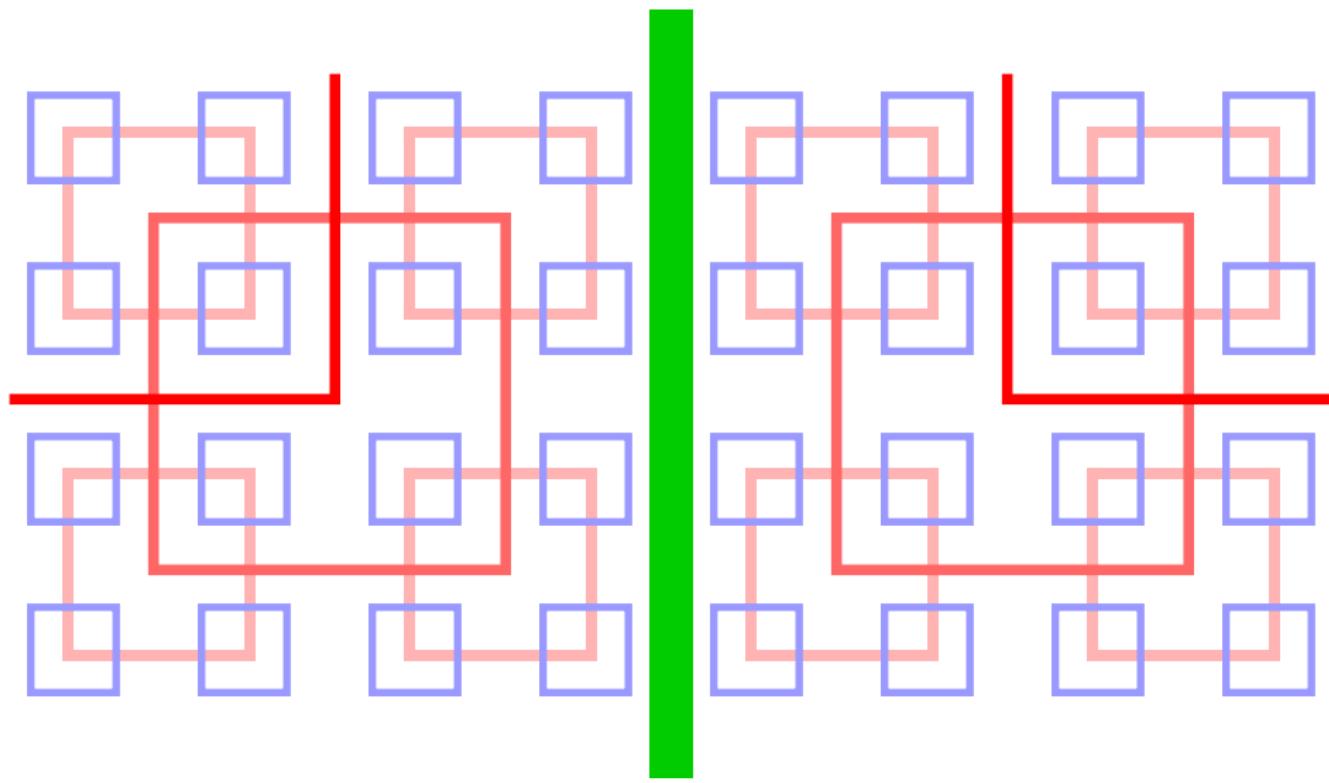
Geometric Vision: Fractured lines



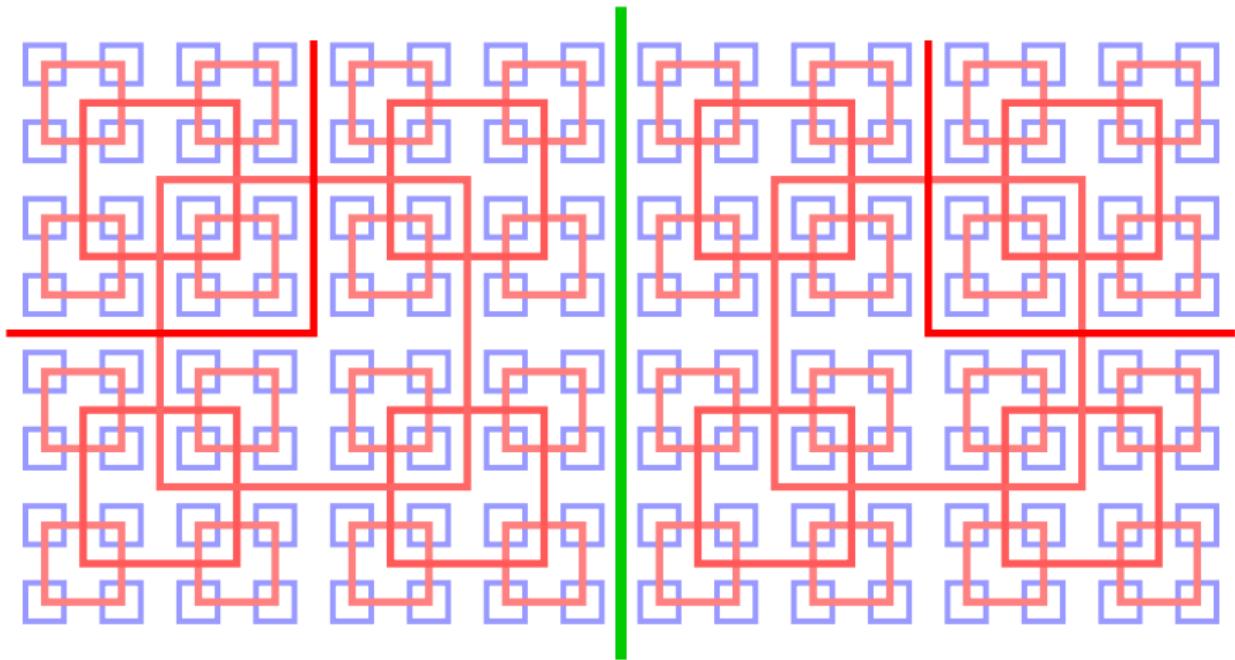
Geometric Vision: Fractured lines



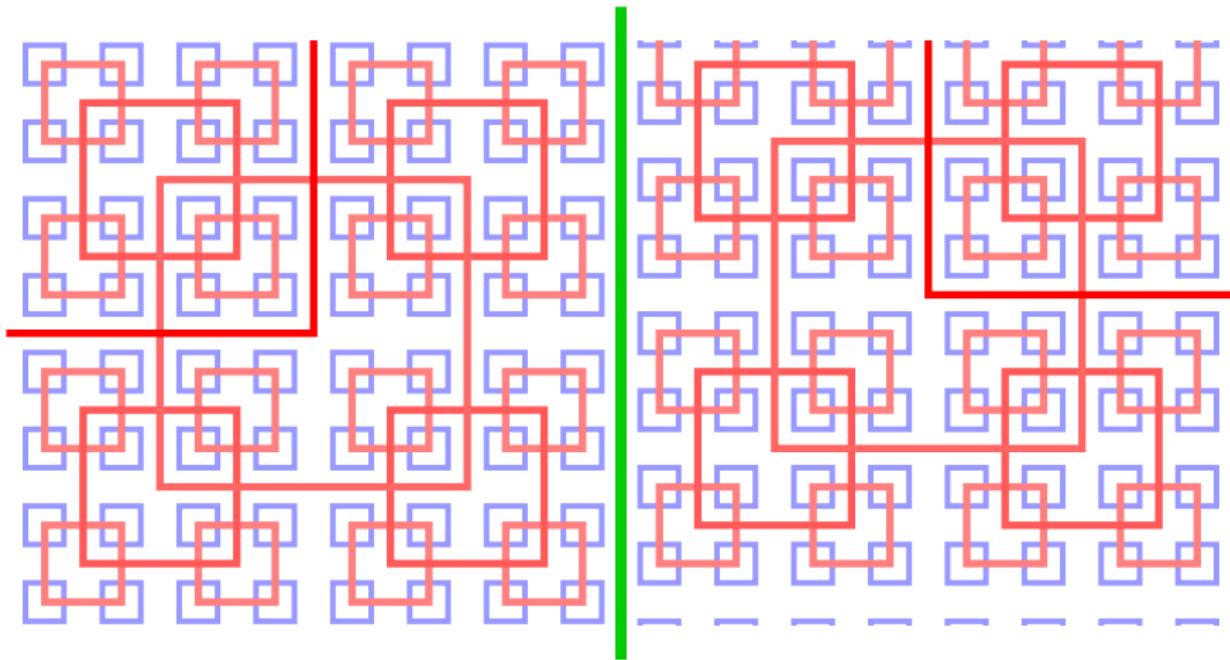
Geometric Vision: Fractured lines



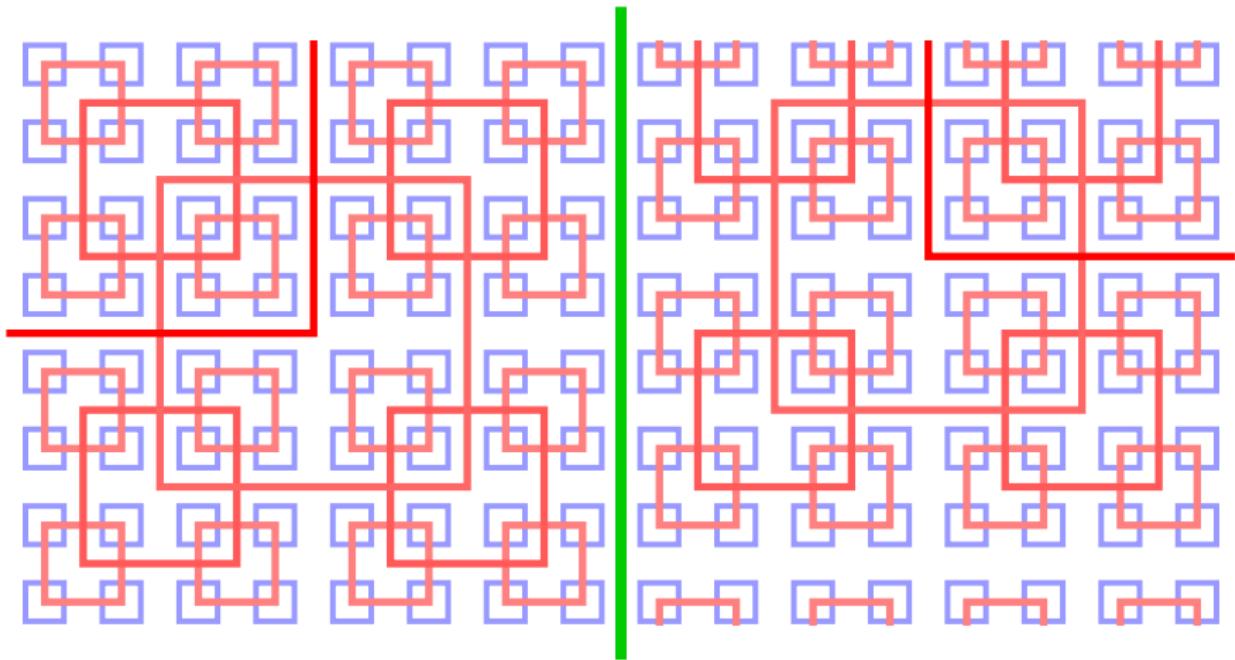
Geometric Vision: Fractured lines



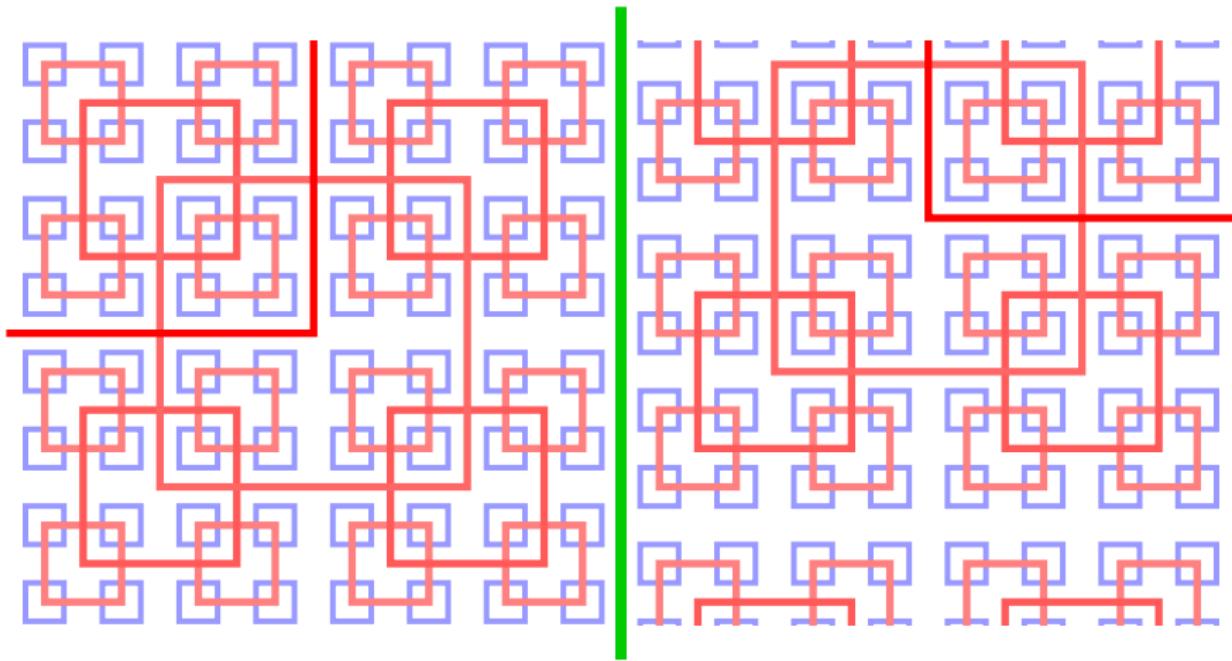
Geometric Vision: Fractured lines



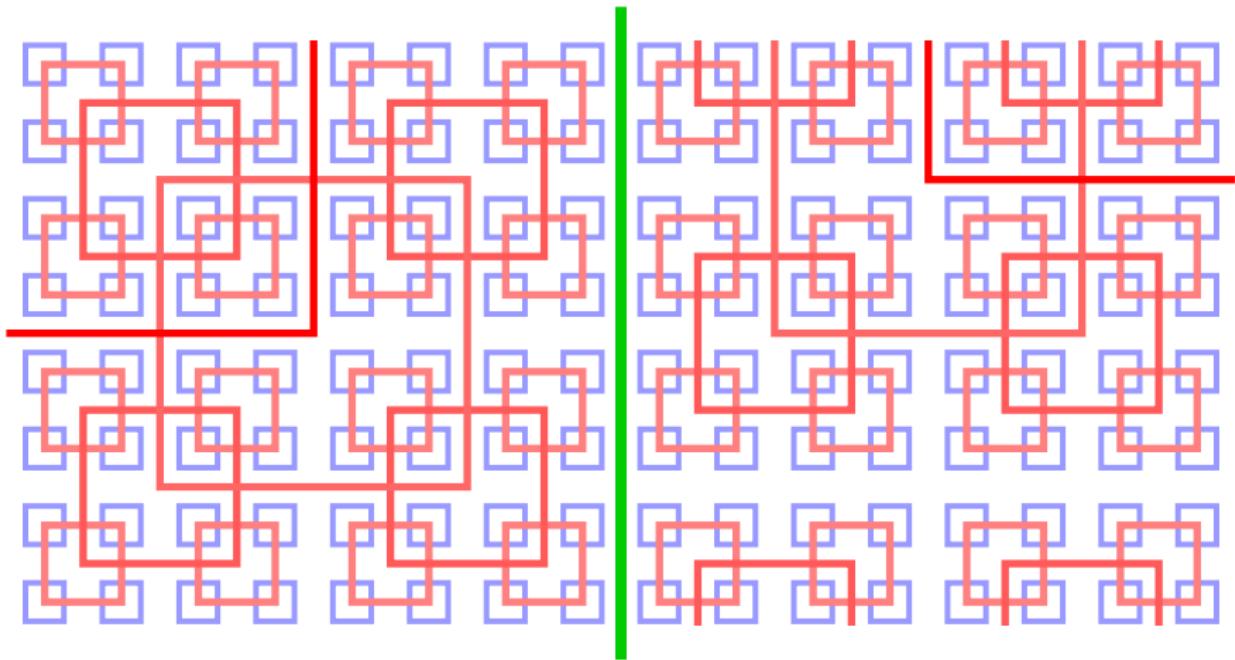
Geometric Vision: Fractured lines



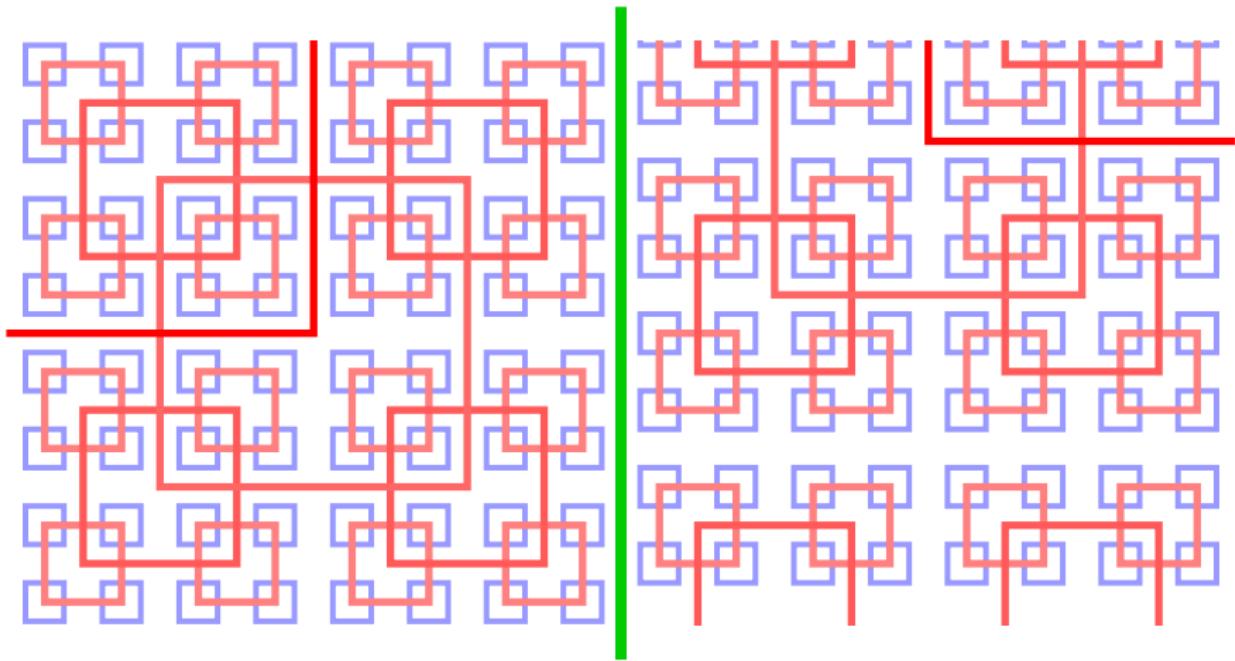
Geometric Vision: Fractured lines



Geometric Vision: Fractured lines



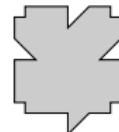
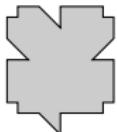
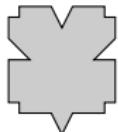
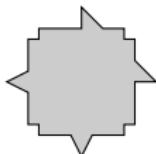
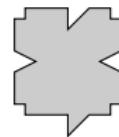
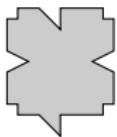
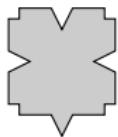
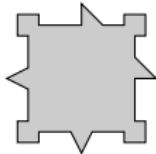
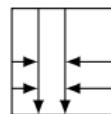
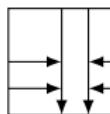
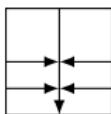
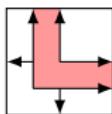
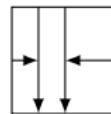
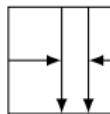
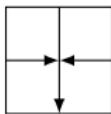
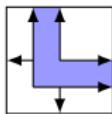
Geometric Vision: Fractured lines



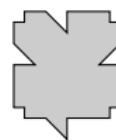
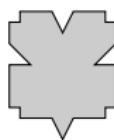
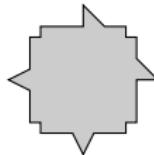
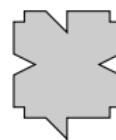
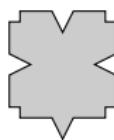
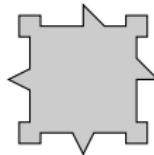
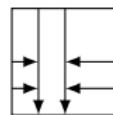
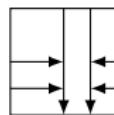
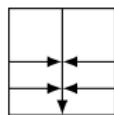
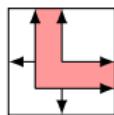
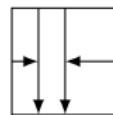
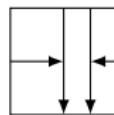
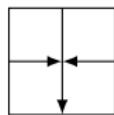
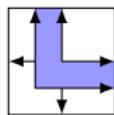
Proposition

$\mu(\{x \in \mathbf{T}_{\text{Robin}} \text{ with fracture lines}\}) = 0$ for all probability measure μ σ -invariante.

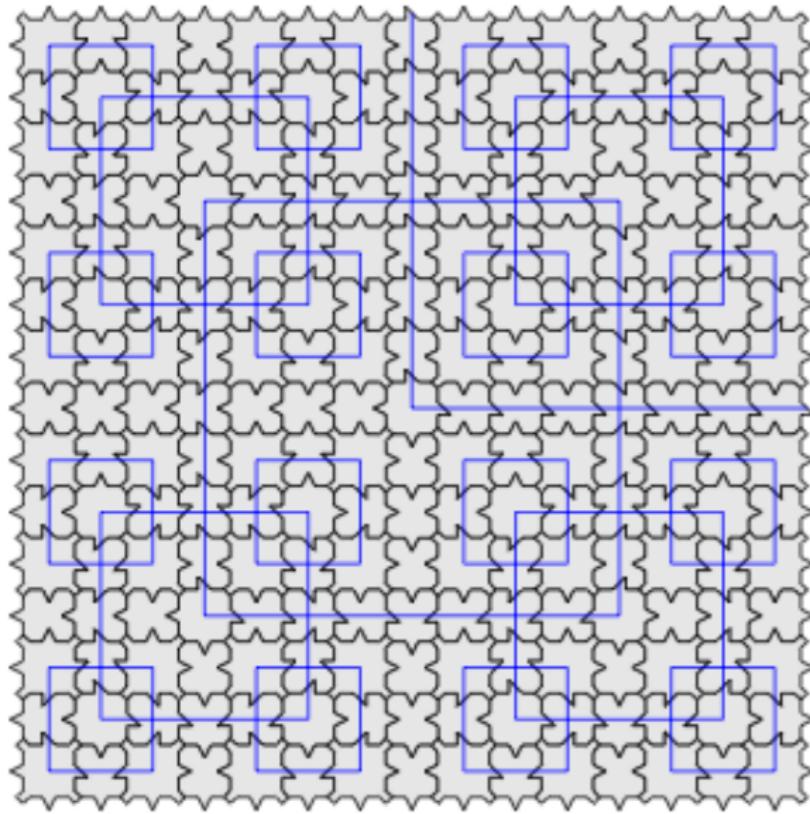
Robinson's tiles for children



Robinson's tiles for children



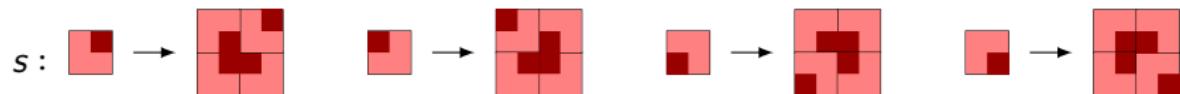
Robinson's tiles for children



Multidimensional substitutions

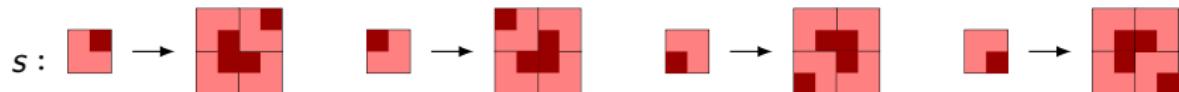
Rectangular substitution

Let $\mathcal{A} = \left\{ \begin{matrix} \text{red square} \\ \text{red L-shape} \\ \text{red T-shape} \\ \text{red cross} \end{matrix} \right\}$. Consider the next substitution:

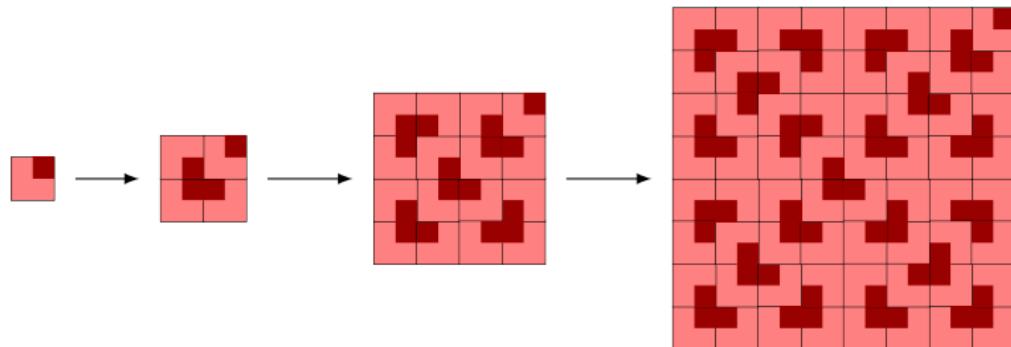


Rectangular substitution

Let $\mathcal{A} = \{\text{red square}, \text{red L-shape}, \text{red T-shape}, \text{red cross}\}$. Consider the next substitution:



After iteration, we obtain:

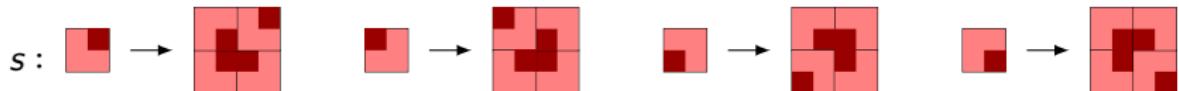


Define *the substitutive subshift*:

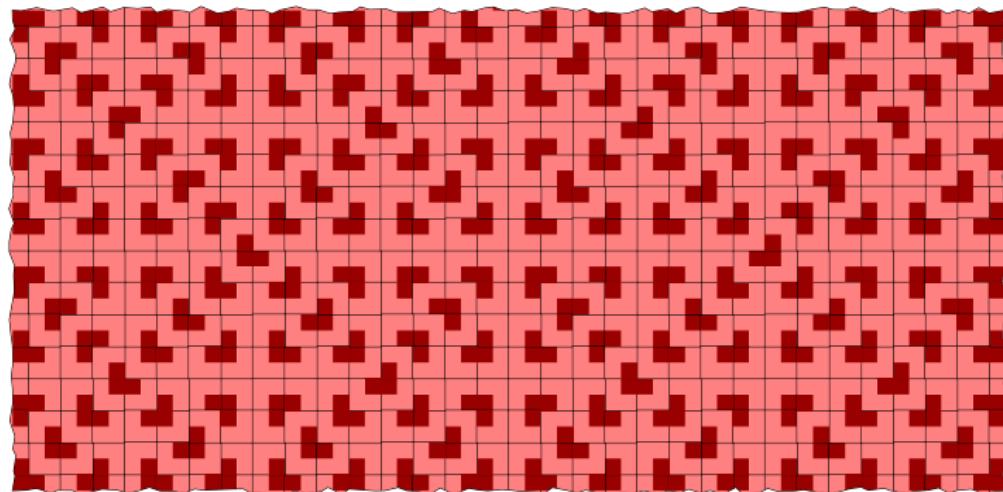
$$\mathbf{T}_s = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \forall p \sqsubset x \quad \exists a \in \mathcal{A}, \ n \in \mathbb{N}, \text{ tel que } p \sqsubset s^n(a) \right\}$$

Rectangular substitution

Let $\mathcal{A} = \{ \text{red square}, \text{dark red square}, \text{pink square}, \text{dark pink square} \}$. Consider the next substitution:



After iteration, we obtain:



Define *the substitutive subshift*:

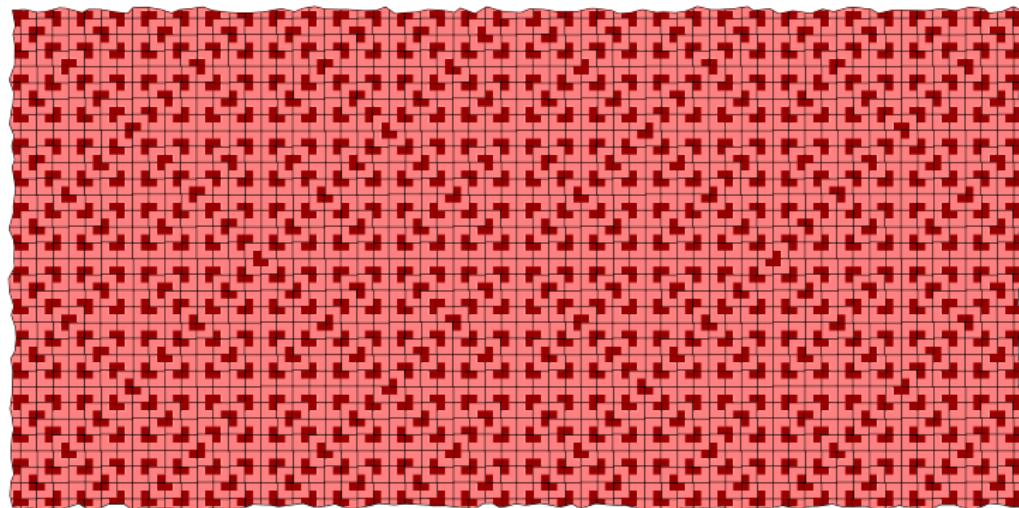
$$\mathbf{T}_s = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \forall p \sqsubset x \quad \exists a \in \mathcal{A}, \ n \in \mathbb{N}, \text{ tel que } p \sqsubset s^n(a) \right\}$$

Rectangular substitution

Let $\mathcal{A} = \left\{ \begin{matrix} & \\ & \end{matrix}, \begin{matrix} & \\ & \end{matrix}, \begin{matrix} & \\ & \end{matrix}, \begin{matrix} & \\ & \end{matrix} \right\}$. Consider the next substitution:

$$s : \begin{matrix} & \\ & \end{matrix} \rightarrow \begin{matrix} & & \\ & & \\ & & \end{matrix} \quad \begin{matrix} & \\ & \end{matrix} \rightarrow \begin{matrix} & & \\ & & \\ & & \end{matrix} \quad \begin{matrix} & \\ & \end{matrix} \rightarrow \begin{matrix} & & \\ & & \\ & & \end{matrix} \quad \begin{matrix} & \\ & \end{matrix} \rightarrow \begin{matrix} & & \\ & & \\ & & \end{matrix}$$

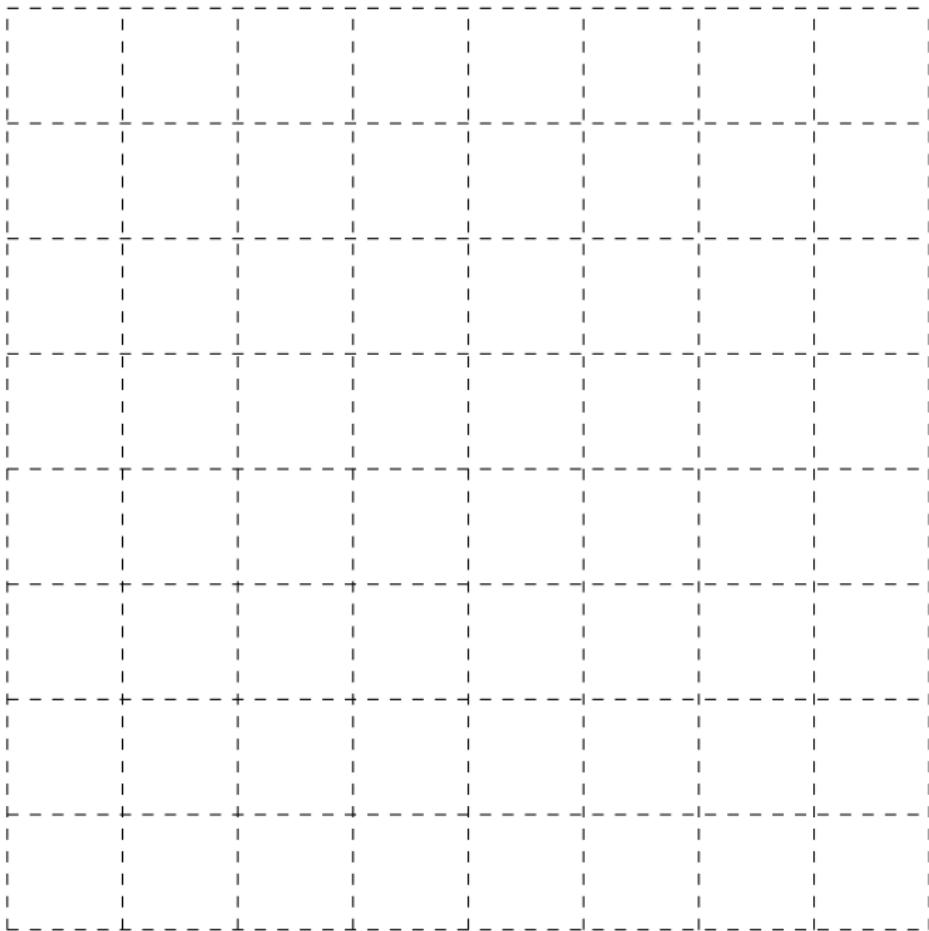
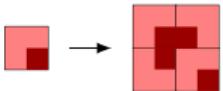
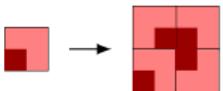
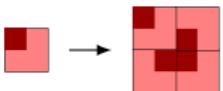
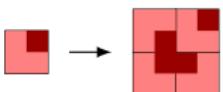
After iteration, we obtain:



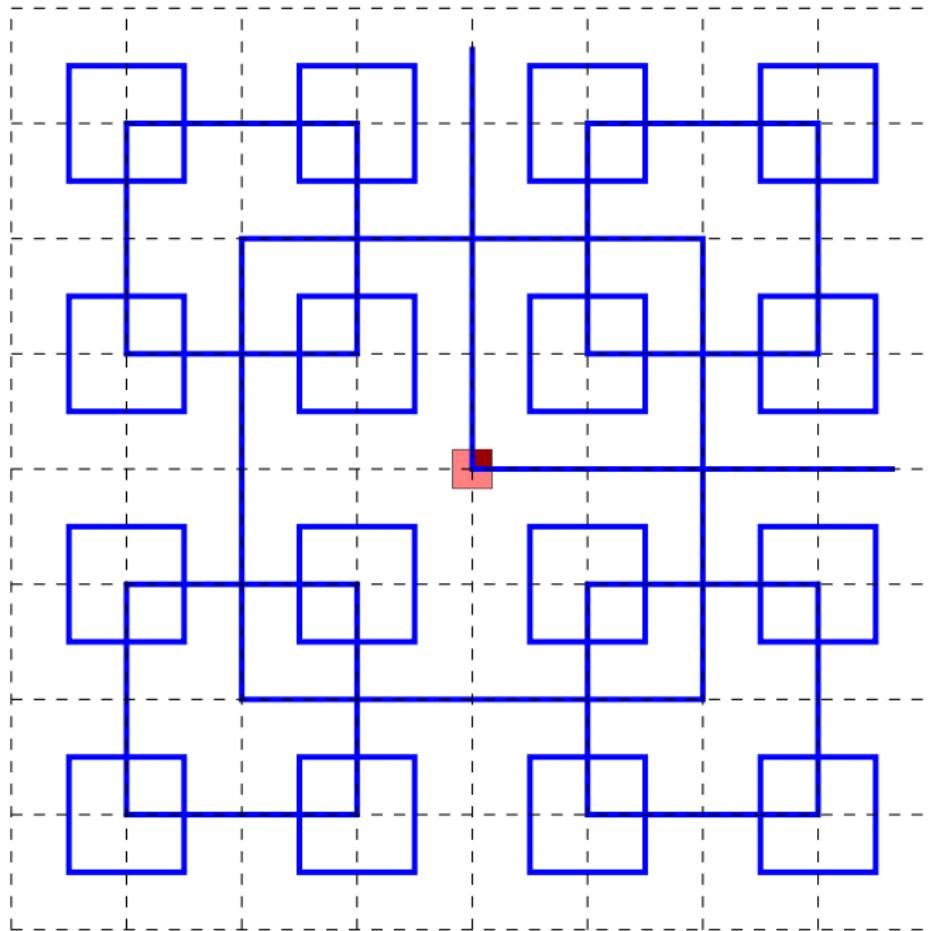
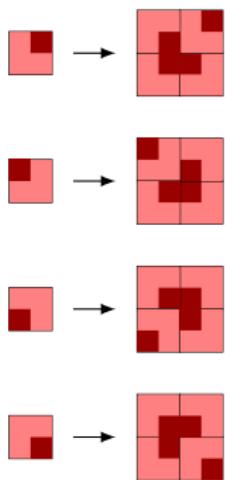
Define *the substitutif subshift*:

$$\mathbf{T}_s = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \forall p \sqsubset x \quad \exists a \in \mathcal{A}, \ n \in \mathbb{N}, \text{ tel que } p \sqsubset s^n(a) \right\}$$

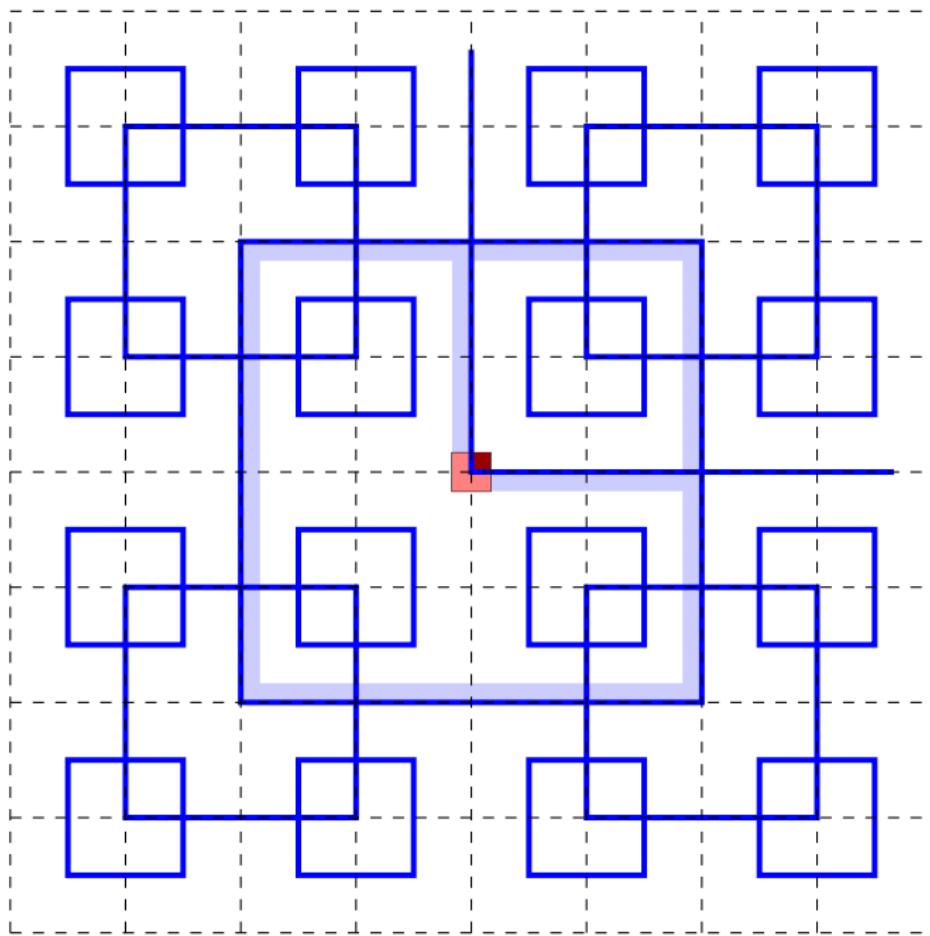
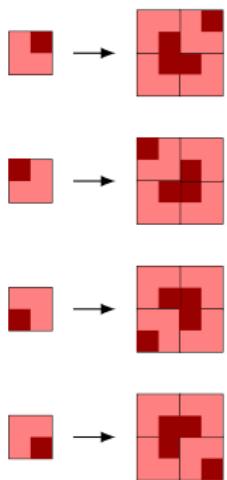
How forces the
substitutive tiling
with local rules?



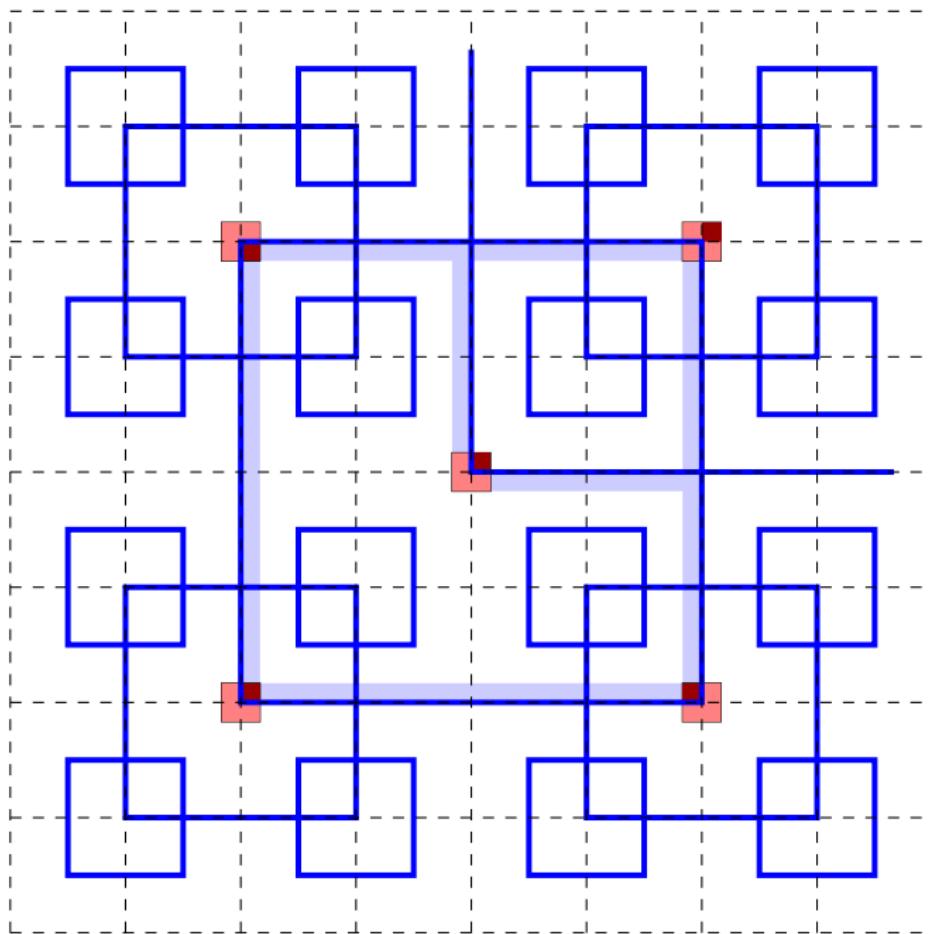
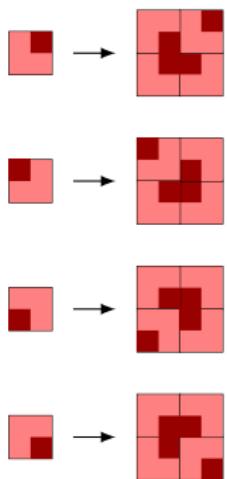
How forces the
substitutive tiling
with local rules?



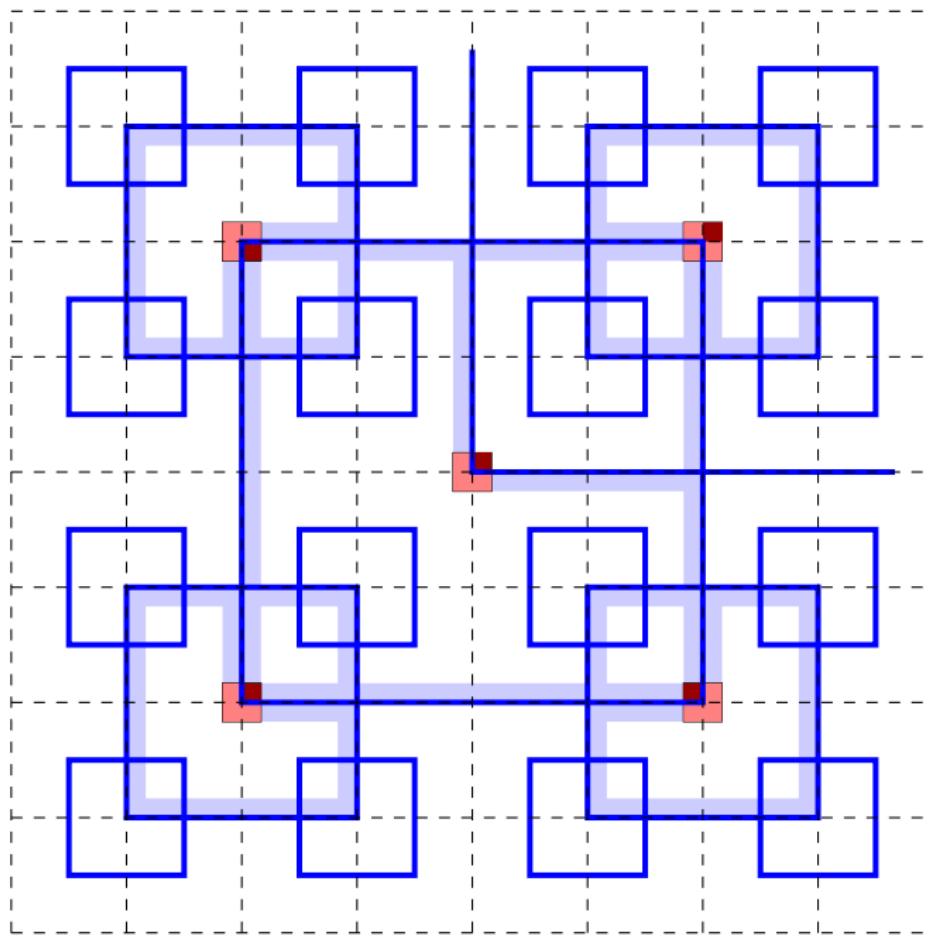
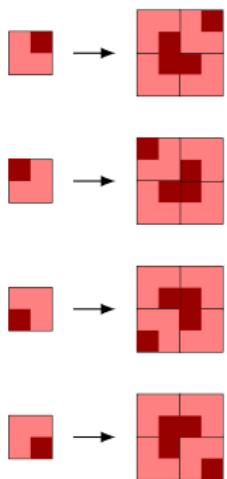
How forces the
substitutive tiling
with local rules?



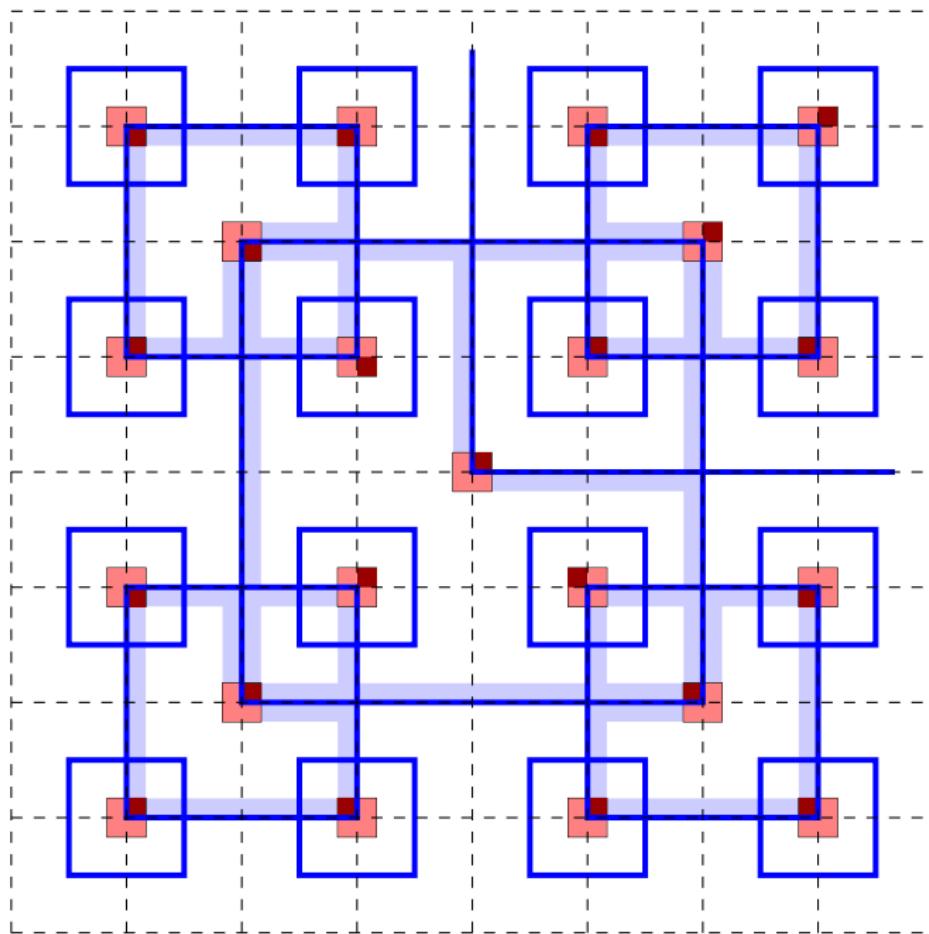
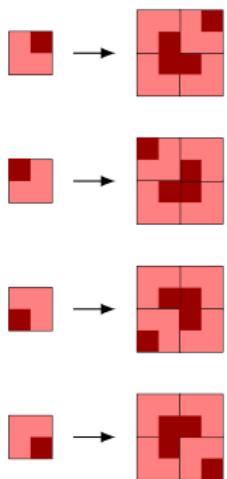
How forces the
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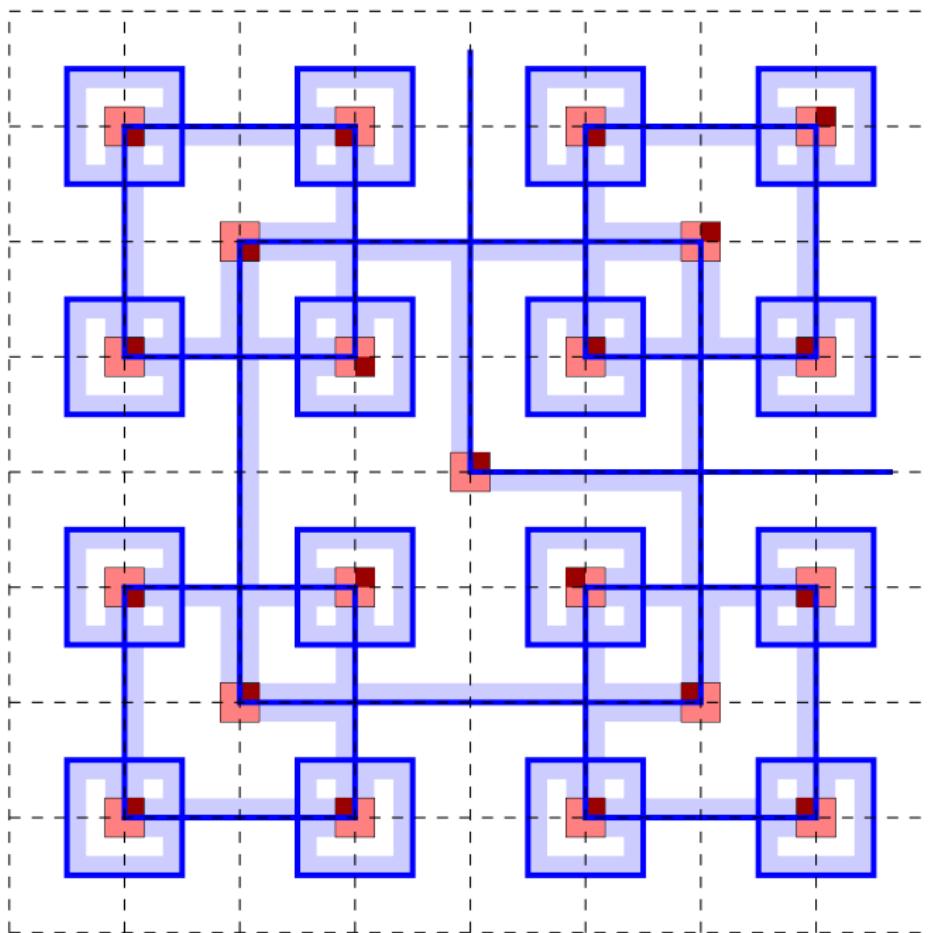
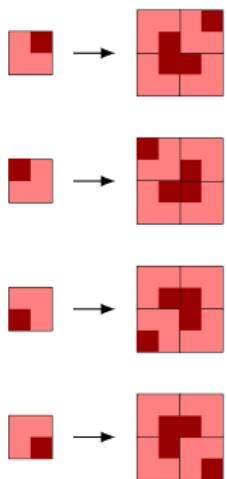
How forces the
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with local rules?



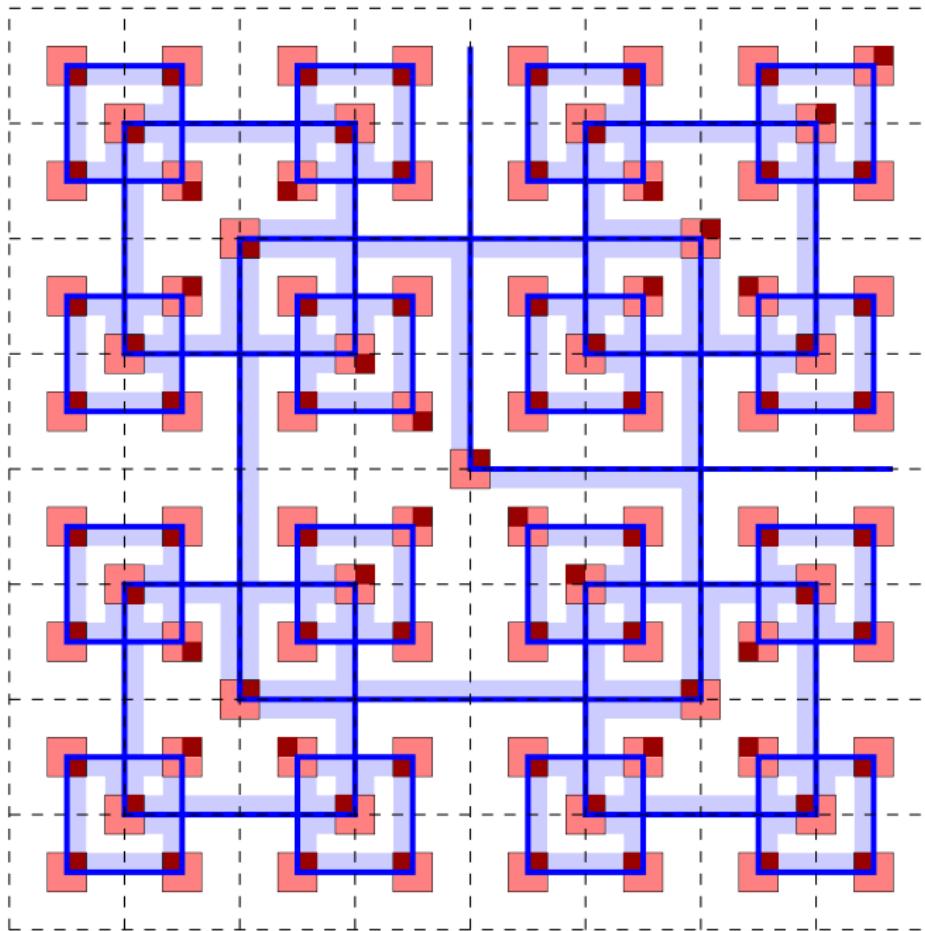
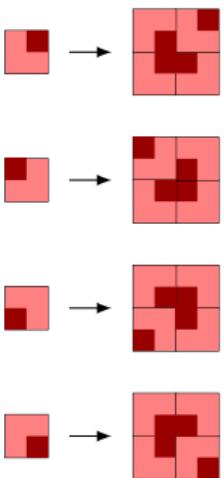
How forces the
substitutive tiling
with local rules?



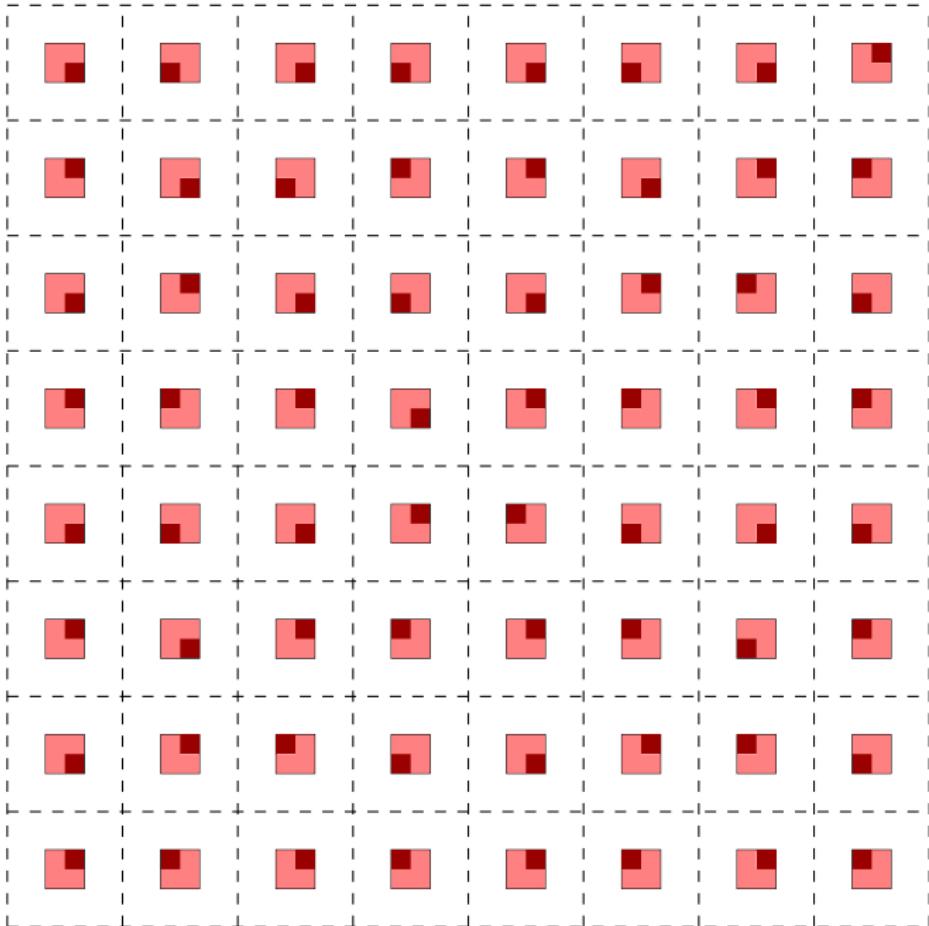
How forces the
substitutive tiling
with local rules?



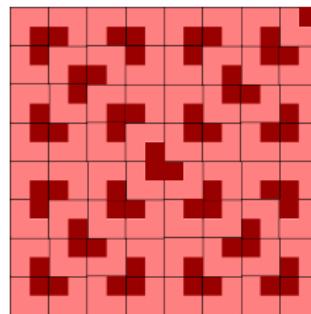
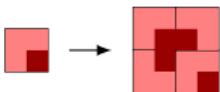
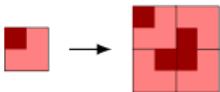
How forces the
substitutive tiling
with local rules?



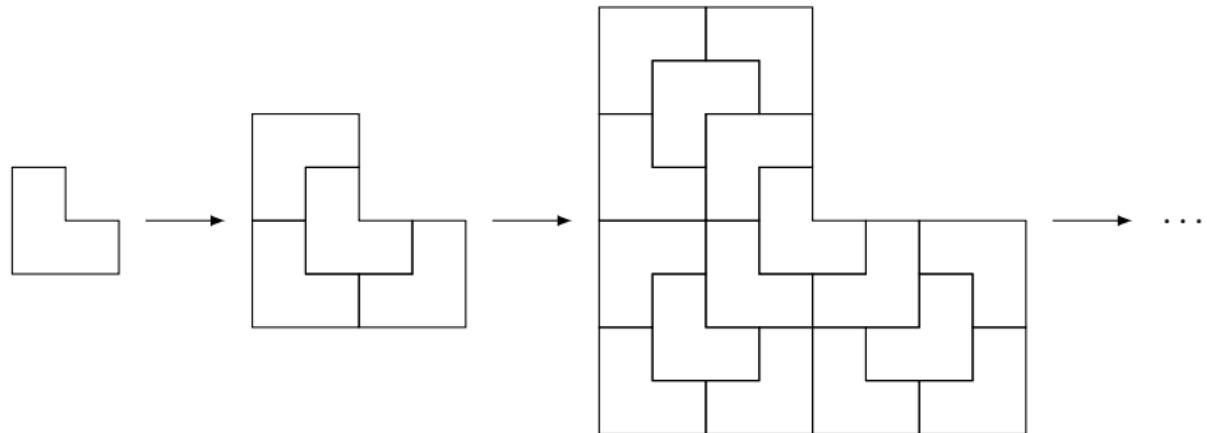
How forces the
substitutive tiling
with local rules?



How forces the
substitutive tiling
with local rules?



Substitution of polygons



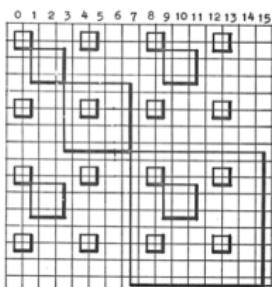
Theorem (*Goodman-Strauss 1998*)

The tiling space defined by substitution on polygon can be defined by local rules.

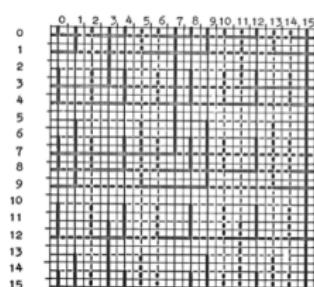
Little historic of aperiodic tilings and perspectives

Little historic of aperiodic tilings

Auteurs	Number of Wang	Nombre de tuiles à translation près	Nombres de tuiles à isométrie près
R. Breger 1966	20 426		

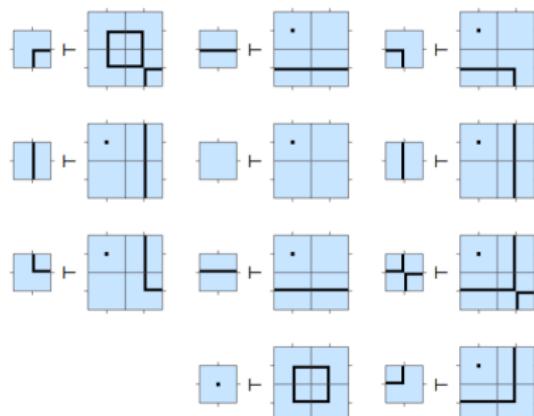


Skeleton Signals



Parity Signals

Figure 24 Part of the Solution of Q



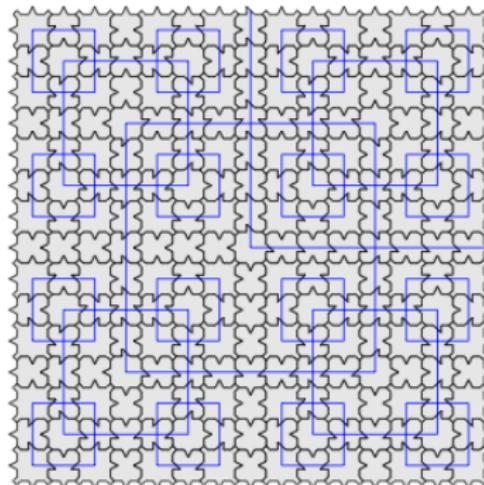
Little historic of aperiodic tilings

Auteurs	Number Wang	of	Nombre de tuiles à translation près	Nombres de tuiles à isométrie près
<i>R. Breger 1966</i>	20 426			
<i>R. Breger 1966</i>	104			
<i>D. E. Knuth 1966</i>	92			



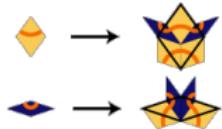
Little historic of aperiodic tilings

Auteurs	Number of Wang	Nombre de tuiles à translation près	Nombres de tuiles à isométrie près
<i>R. Breger 1966</i>	20 426		
<i>R. Breger 1966</i>	104		
<i>D. E. Knuth 1966</i>	92		
<i>R. M. Robinson 1971</i>	56	32	6



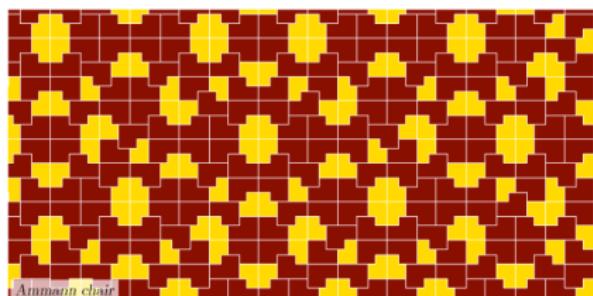
Little historic of aperiodic tilings

Auteurs	Number Wang	of	Nombre de tuiles à translation près	Nombres de tuiles à isométrie près
<i>R. Breger 1966</i>	20 426			
<i>R. Breger 1966</i>	104			
<i>D. E. Knuth 1966</i>	92			
<i>R. M. Robinson 1971</i>	56		32	6
<i>R. Penrose 1978</i>			20	2

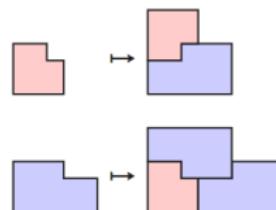


Little historic of aperiodic tilings

Auteurs	Number of Wang	Nombre de tuiles à translation près	Nombres de tuiles à isométrie près
R. Breger 1966	20 426		
R. Breger 1966	104		
D. E. Knuth 1966	92		
R. M. Robinson 1971	56	32	6
R. Penrose 1978		20	2
R. Ammann 1978		16	2

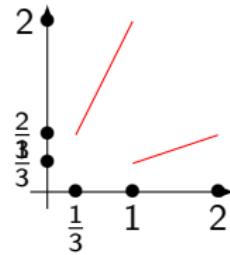
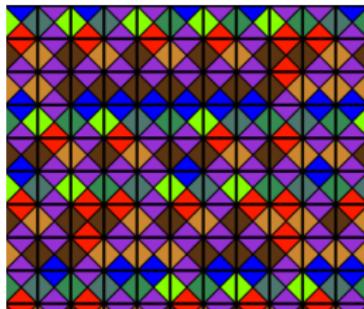


Autumn chair



Little historic of aperiodic tilings

Auteurs	Number of Wang	Nombre de tuiles à translation près	Nombres de tuiles à isométrie près
R. Breger 1966	20 426		
R. Breger 1966	104		
D. E. Knuth 1966	92		
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K. Culick 1996	13		



Little historic of aperiodic tilings

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R. Penrose 1978		20	2
R. Ammann 1978		16	2
J. Kari 1996	14		
K. Culick 1996	13		
M. Rao-E. Jeandel 2017	11		

Problematic

There exists other type of aperiodic tilings?

Decision problem

Decision problem

Domino problem in the SFT setting:

Given \mathcal{A} a finite alphabet and \mathcal{F} a finite set of 2-dimensional patterns.

$$\mathbf{T}_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \forall p \in \mathcal{F}, \ p \notin x \right\} \neq \emptyset?$$

Decision problem

Domino problem in the SFT setting:

Given \mathcal{A} a finite alphabet and \mathcal{F} a finite set of 2-dimensional patterns.

$$\mathbf{T}_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \forall p \in \mathcal{F}, p \notin x \right\} \neq \emptyset?$$

There is a program which **halts** if and only if $\mathbf{T}_{\mathcal{F}} = \emptyset$

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In dimension 1, it is decidable!

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There is a program which **halts** if and only if $\mathbf{T}_{\mathcal{F}} = \emptyset$

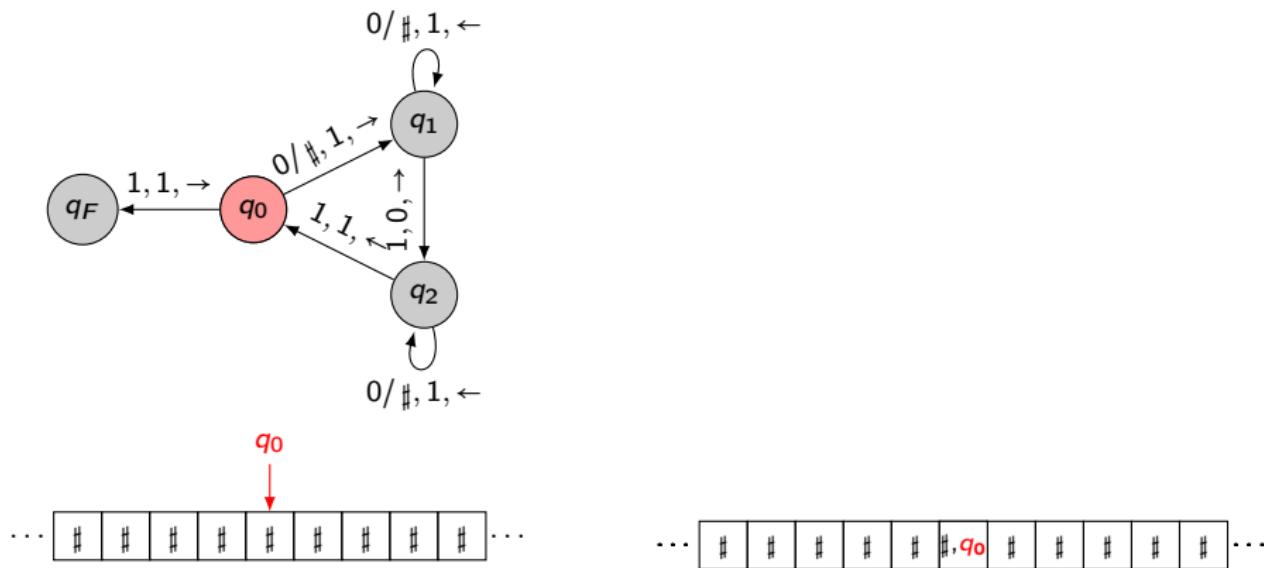
In dimension 1, it is decidable!

Theorem (Berger 1966, Robinson 1971)

The domino problem is undecidable in dimension $d \geq 2$.

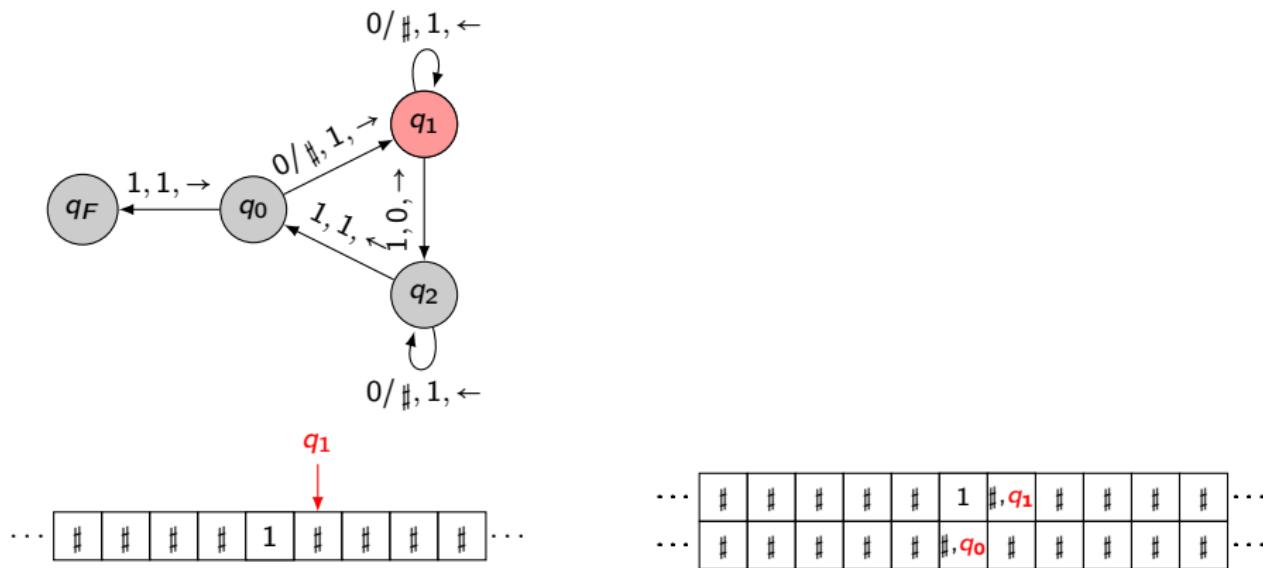
Computation of Turing machine

An example of model of computation:



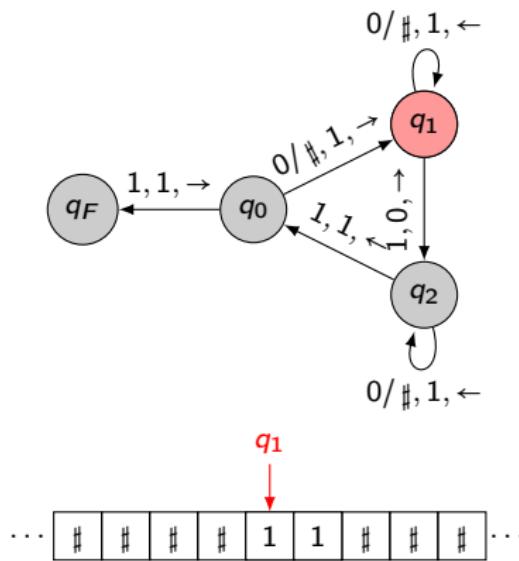
Computation of Turing machine

An example of model of computation:



Computation of Turing machine

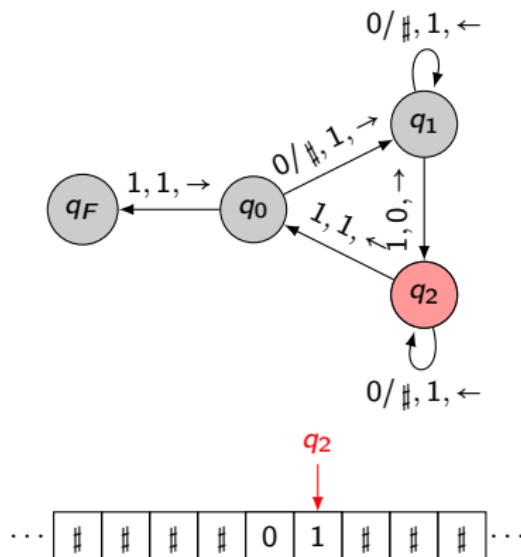
An example of model of computation:



...	#	#	#	#	#	#	1, <i>q₁</i>	1	#	#	#	#	...
...	#	#	#	#	#	#	1	#, <i>q₀</i>	#	#	#	#	...
...	#	#	#	#	#	#	#, <i>q₀</i>	#	#	#	#	#	...

Computation of Turing machine

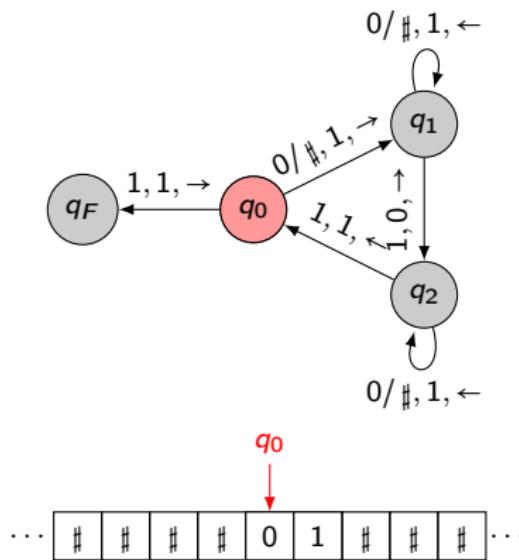
An example of model of computation:



...	#	#	#	#	#	0	1,	q_2	#	#	#	#	...
...	#	#	#	#	#	1,	q_1	1	#	#	#	#	...
...	#	#	#	#	#	1	#,	q_1	#	#	#	#	...
...	#	#	#	#	#	#,	q_0	#	#	#	#	#	...

Computation of Turing machine

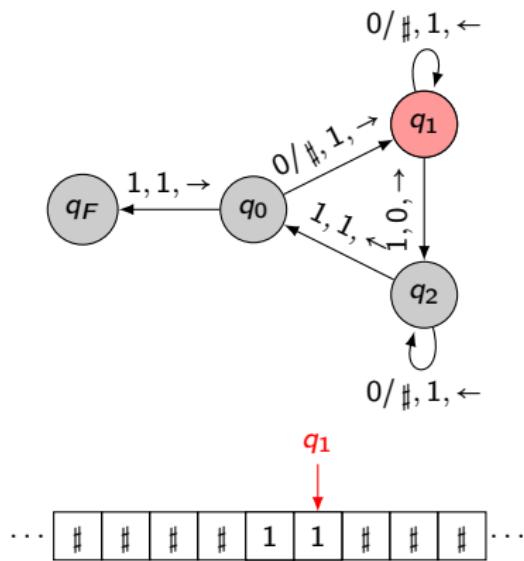
An example of model of computation:



...	#	#	#	#	#	#	0, <i>q0</i>	1	#	#	#	#	...
...	#	#	#	#	#	#	0	1, <i>q2</i>	#	#	#	#	...
...	#	#	#	#	#	#	1, <i>q1</i>	1	#	#	#	#	...
...	#	#	#	#	#	#	1	#, <i>q1</i>	#	#	#	#	...
...	#	#	#	#	#	#	, <i>q0</i>	#	#	#	#	#	...

Computation of Turing machine

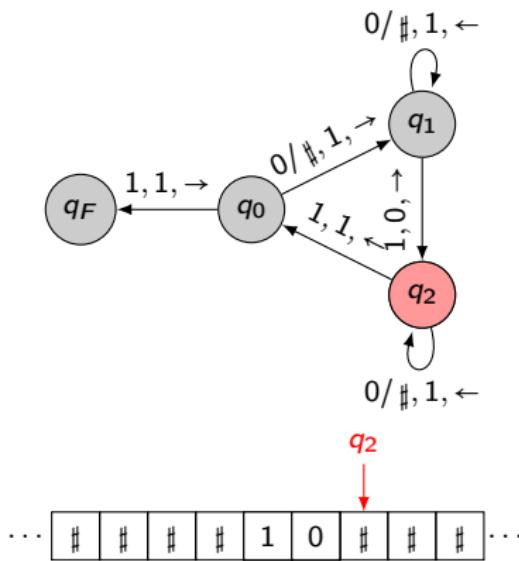
An example of model of computation:



...	#	#	#	#	#	#	1	1	q_1	#	#	#	#	...
...	#	#	#	#	#	#	0	q_0	1	#	#	#	#	...
...	#	#	#	#	#	#	0	1	q_2	#	#	#	#	...
...	#	#	#	#	#	#	1	q_1	1	#	#	#	#	...
...	#	#	#	#	#	#	1	#	q_1	#	#	#	#	...
...	#	#	#	#	#	#	#, q_0	#	#	#	#	#	#	...

Computation of Turing machine

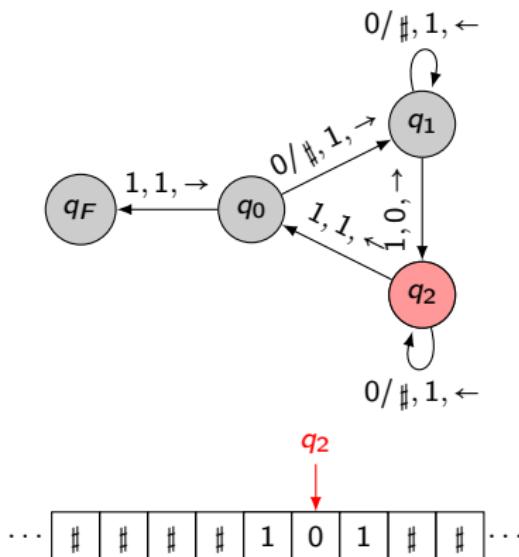
An example of model of computation:



...	#	#	#	#	#	1	0	#	q_2	#	#	#
...	#	#	#	#	#	1	1	q_1	#	#	#	#
...	#	#	#	#	#	0	q_0	1	#	#	#	#
...	#	#	#	#	#	0	1	q_2	#	#	#	#
...	#	#	#	#	#	1	q_1	1	#	#	#	#
...	#	#	#	#	#	1	#	q_1	#	#	#	#
...	#	#	#	#	#	#	q_0	#	#	#	#	#

Computation of Turing machine

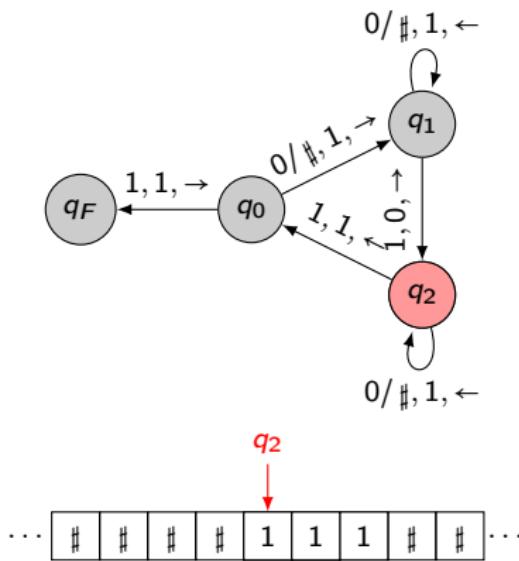
An example of model of computation:



...	#	#	#	#	#	#	1	0,	q_2	1	#	#	#	...
...	#	#	#	#	#	#	1	0,	q_2	#	#	#	#	...
...	#	#	#	#	#	#	1	1,	q_1	#	#	#	#	...
...	#	#	#	#	#	#	0,	q_0	1	#	#	#	#	...
...	#	#	#	#	#	#	0	1,	q_2	#	#	#	#	...
...	#	#	#	#	#	#	1,	q_1	1	#	#	#	#	...
...	#	#	#	#	#	#	1	#,	q_1	#	#	#	#	...
...	#	#	#	#	#	#	,	q_0	#	#	#	#	#	...

Computation of Turing machine

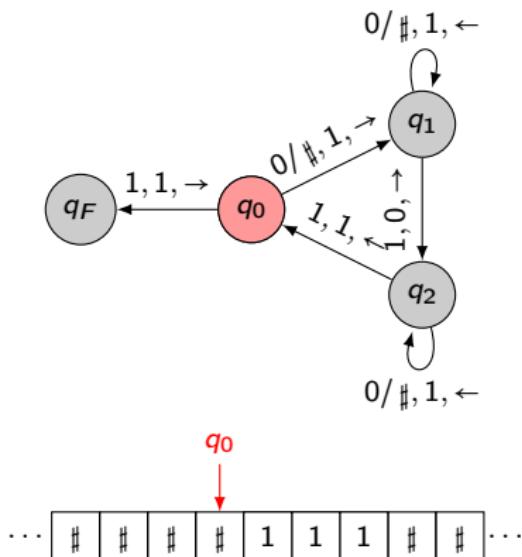
An example of model of computation:



...	#	#	#	#	#	#	1, <i>q₂</i>	1	1	#	#	#	...
...	#	#	#	#	#	#	1 0, <i>q₂</i>	1	#	#	#	#	...
...	#	#	#	#	#	#	1 0, <i>q₂</i>	#	#	#	#	#	...
...	#	#	#	#	#	#	1 1, <i>q₁</i>	#	#	#	#	#	...
...	#	#	#	#	#	#	0, <i>q₀</i>	1	#	#	#	#	...
...	#	#	#	#	#	#	0 1, <i>q₂</i>	#	#	#	#	#	...
...	#	#	#	#	#	#	1, <i>q₁</i>	1	#	#	#	#	...
...	#	#	#	#	#	#	1 #, <i>q₁</i>	#	#	#	#	#	...
...	#	#	#	#	#	#	, <i>q₀</i>	#	#	#	#	#	...

Computation of Turing machine

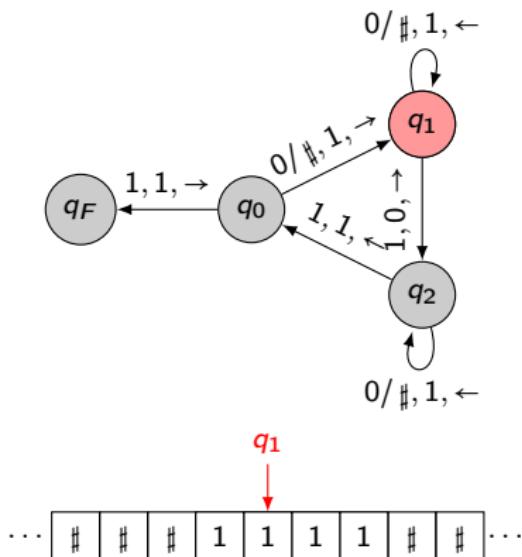
An example of model of computation:



...	#	#	#	#	#	, q_0	1	1	1	#	#	#
...	#	#	#	#	#	, q_2	1	1	1	#	#	#
...	#	#	#	#	#	1, q_2	1	0, q_2	1	#	#	#
...	#	#	#	#	#	1, q_2	0, q_2	1, q_2	1	#	#	#
...	#	#	#	#	#	1, q_2	1, q_1	1, q_1	1	#	#	#
...	#	#	#	#	#	0, q_0	1	1, q_1	1	#	#	#
...	#	#	#	#	#	0, q_0	1, q_1	1, q_1	1	#	#	#
...	#	#	#	#	#	1, q_1	1, q_1	1, q_1	1	#	#	#
...	#	#	#	#	#	, q_0	1, q_1	1, q_1	1	#	#	#
...	#	#	#	#	#	, q_0	1, q_1	1, q_1	1	#	#	#

Computation of Turing machine

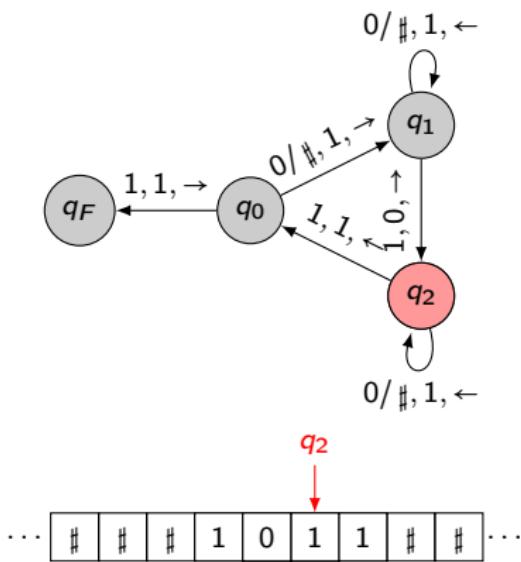
An example of model of computation:



...	#	#	#	#	1	1	q_1	1	1	#	#	#	...
...	#	#	#	#	,	q_0	1	1	1	#	#	#	...
...	#	#	#	#		1,	q_2	1	1	#	#	#	...
...	#	#	#	#			1 0,	q_2	1	#	#	#	...
...	#	#	#	#			1 0,	q_2	#	#	#	#	...
...	#	#	#	#			1 1,	q_1	#	#	#	#	...
...	#	#	#	#		0,	q_0	1	#	#	#	#	...
...	#	#	#	#		0 1,	q_2	#	#	#	#	#	...
...	#	#	#	#			1,	q_1	1	#	#	#	...
...	#	#	#	#			1 #,	q_1	#	#	#	#	...
...	#	#	#	#			,	q_0	#	#	#	#	#

Computation of Turing machine

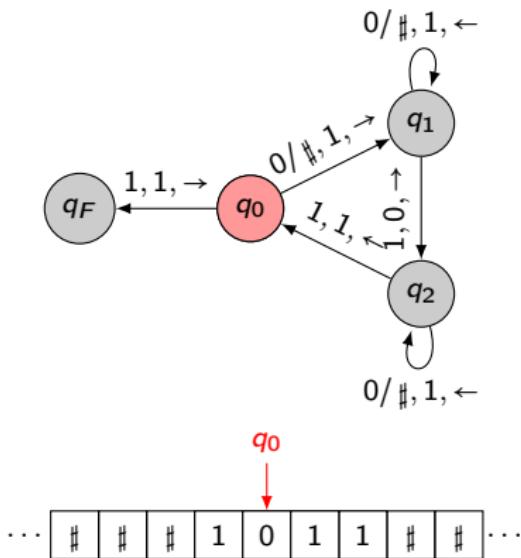
An example of model of computation:



...	:	:	:	:	:	:	:	:	:	:	:	:	:	...
#	#	#	#	1	1	1	1	1	1	#	#	#	#	...
#	#	#	#	,	q_0	1	1	1	1	#	#	#	#	...
#	#	#	#		1,	q_2	1	1	1	#	#	#	#	...
#	#	#	#			1	0,	q_2	1	#	#	#	#	...
#	#	#	#				1	0,	q_2	#	#	#	#	...
#	#	#	#					1	q_1	#	#	#	#	...
#	#	#	#					0,	q_0	1	#	#	#	...
#	#	#	#					0	1,	q_2	#	#	#	...
#	#	#	#					1,	q_1	1	#	#	#	...
#	#	#	#					1	#,	q_1	#	#	#	...
#	#	#	#						,	q_0	#	#	#	...

Computation of Turing machine

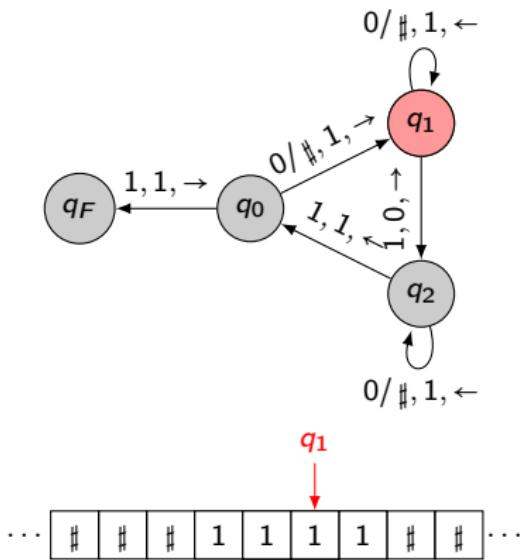
An example of model of computation:



...	#	#	#	#	1	1	q_1	1	1	#	#	#	...
...	#	#	#	#	,	q_0	1	1	1	#	#	#	...
...	#	#	#	#		1,	q_2	1	1	#	#	#	...
...	#	#	#	#			1 0,	q_2	1	#	#	#	...
...	#	#	#	#			1 0,	q_2	#	#	#	#	...
...	#	#	#	#			1 1,	q_1	#	#	#	#	...
...	#	#	#	#		0,	q_0	1	#	#	#	#	...
...	#	#	#	#		0 1,	q_2	#	#	#	#	#	...
...	#	#	#	#		1,	q_1	1	#	#	#	#	...
...	#	#	#	#		1 #,	q_1	#	#	#	#	#	...
...	#	#	#	#		,	q_0	#	#	#	#	#	...

Computation of Turing machine

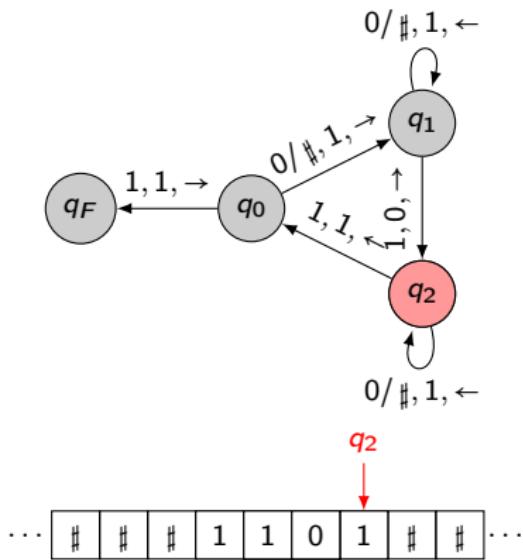
An example of model of computation:



\vdots												
$\#$	$\#$	$\#$	$\#$	1	1	q_1	1	1	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	q_0	1	1	1	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	1	q_2	1	1	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	1	0	q_2	1	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	1	0	q_2	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	1	1	q_1	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	0	q_0	1	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	0	1	q_2	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	1	q_1	1	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	1	$\#$	q_1	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	$\#$	q_0	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$

Computation of Turing machine

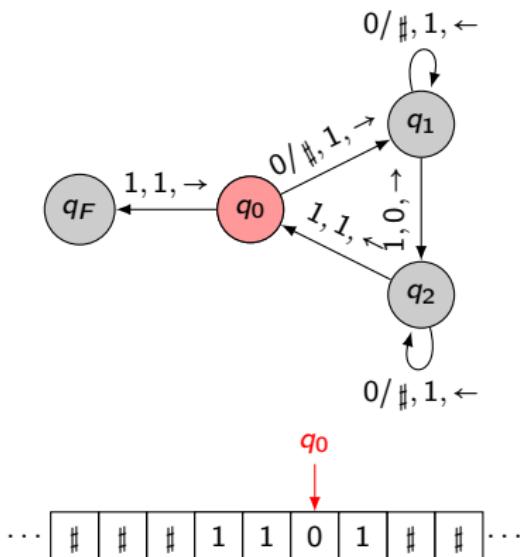
An example of model of computation:



...	:	:	:	:	:	:	:	:	:	:	:	:	:	:	...
...	#	#	#	#	1	1	q_1	1	1	#	#	#	#	#	...
...	#	#	#	#	,	q_0	1	1	1	#	#	#	#	#	...
...	#	#	#	#	,	q_2	1	1	1	#	#	#	#	#	...
...	#	#	#	#	,	q_2	1	0	q_2	#	#	#	#	#	...
...	#	#	#	#	,	q_2	1	0	,	#	#	#	#	#	...
...	#	#	#	#	,	q_1	1	1	,	#	#	#	#	#	...
...	#	#	#	#	,	q_0	1	0	,	#	#	#	#	#	...
...	#	#	#	#	,	q_2	0	1	,	#	#	#	#	#	...
...	#	#	#	#	,	q_1	1	1	,	#	#	#	#	#	...
...	#	#	#	#	,	q_0	1	1	,	#	#	#	#	#	...

Computation of Turing machine

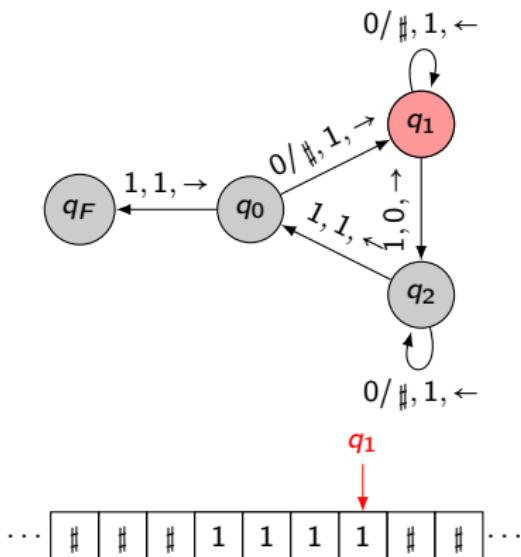
An example of model of computation:



...	#	#	#	#	1	1	q_1	1	1	#	#	#	...
...	#	#	#	#	,	q_0	1	1	1	#	#	#	...
...	#	#	#	#		1,	q_2	1	1	#	#	#	...
...	#	#	#	#			1 0,	q_2	1	#	#	#	...
...	#	#	#	#			1 0,	q_2	#	#	#	#	...
...	#	#	#	#			1 1,	q_1	#	#	#	#	...
...	#	#	#	#			0,	q_0	1	#	#	#	...
...	#	#	#	#			0 1,	q_2	#	#	#	#	...
...	#	#	#	#			1,	q_1	1	#	#	#	...
...	#	#	#	#			1 #,	q_1	#	#	#	#	...
...	#	#	#	#			,	q_0	#	#	#	#	...

Computation of Turing machine

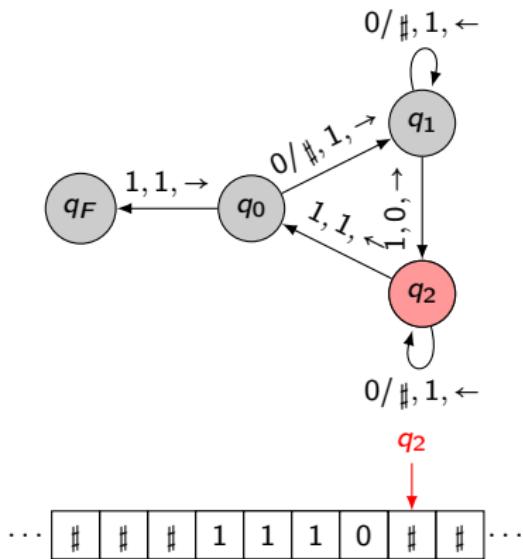
An example of model of computation:



\vdots													
$\#$	$\#$	$\#$	$\#$	1	1	q_1	1	1	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	q_0	1	1	1	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	1	q_2	1	1	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	1	0	q_2	1	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	1	0	q_2	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	1	1	q_1	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	0	q_0	1	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	0	1	q_2	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	1	q_1	1	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	1	$\#$	q_1	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	q_0	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$

Computation of Turing machine

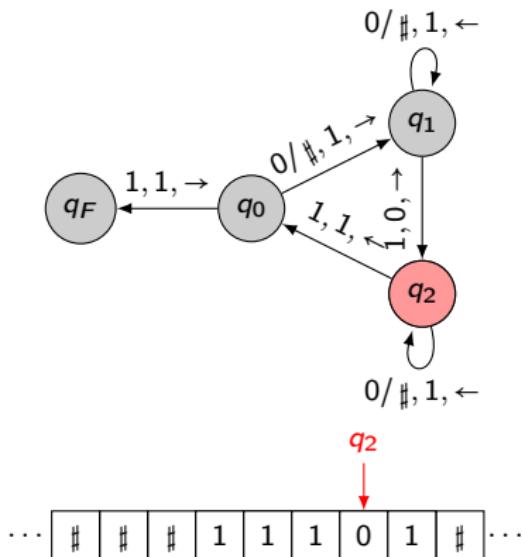
An example of model of computation:



\dots	$\#$	$\#$	$\#$	$\#$	1	1	1	q_1	1	1	$\#$	$\#$	$\#$	\dots
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	q_0	1	1	1	$\#$	$\#$	$\#$	$\#$	\dots
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	1	q_2	1	1	$\#$	$\#$	$\#$	$\#$	\dots
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	1	0	q_2	1	$\#$	$\#$	$\#$	$\#$	\dots
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	1	0	q_2	$\#$	$\#$	$\#$	$\#$	$\#$	\dots
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	1	1	q_1	$\#$	$\#$	$\#$	$\#$	$\#$	\dots
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	0	q_0	1	$\#$	$\#$	$\#$	$\#$	$\#$	\dots
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	0	1	q_2	$\#$	$\#$	$\#$	$\#$	$\#$	\dots
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	1	q_1	1	$\#$	$\#$	$\#$	$\#$	$\#$	\dots
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	1	$\#$	q_1	$\#$	$\#$	$\#$	$\#$	$\#$	\dots
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$	q_0	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$	\dots

Computation of Turing machine

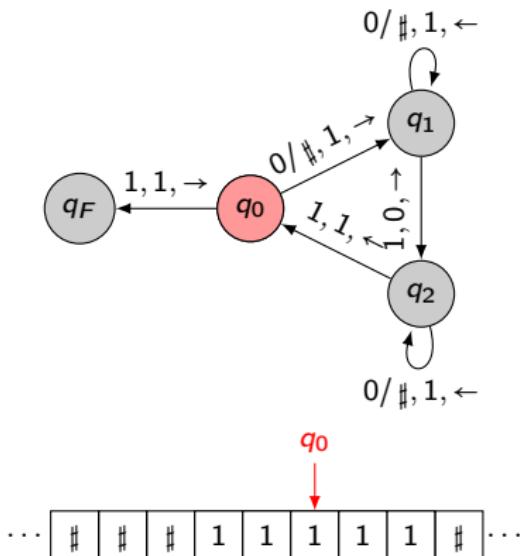
An example of model of computation:



\vdots													
$\#$	$\#$	$\#$	$\#$	1	1	1	0	1	$\#$	$\#$	$\#$	$\#$	$\#$
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$	1	1	1	1	$\#$
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$	q_0	1	1	1	$\#$	$\#$
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$	1	q_1	1	1	$\#$	$\#$	$\#$
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	1	0	q_2	1	$\#$	$\#$	$\#$	$\#$
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	1	0	q_2	$\#$	$\#$	$\#$	$\#$	$\#$
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	1	1	q_1	$\#$	$\#$	$\#$	$\#$	$\#$
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	0	q_0	1	$\#$	$\#$	$\#$	$\#$	$\#$
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	0	1	q_2	$\#$	$\#$	$\#$	$\#$	$\#$
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	1	q_1	1	$\#$	$\#$	$\#$	$\#$	$\#$
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	1	$\#$	q_1	$\#$	$\#$	$\#$	$\#$	$\#$
\dots	$\#$	$\#$	$\#$	$\#$	$\#$	q_0	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$

Computation of Turing machine

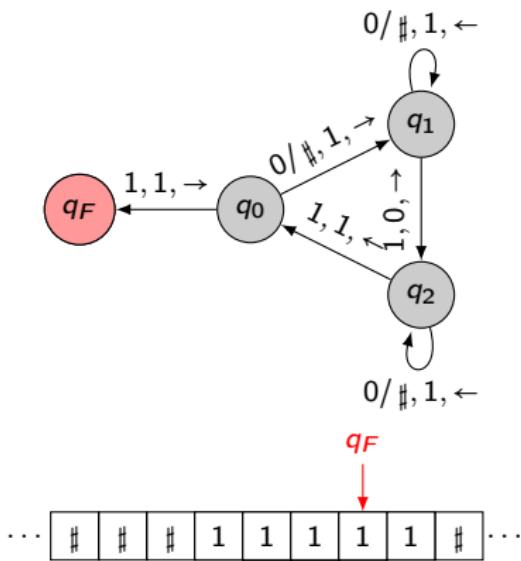
An example of model of computation:



					1	1, <i>q</i> ₁	1	1				
						, <i>q</i> ₀	1	1	1			
						1, <i>q</i> ₂	1	1				
							1	0, <i>q</i> ₂	1			
							1	0	, <i>q</i> ₂			
								1	1, <i>q</i> ₁			
							0	, <i>q</i> ₀	1			
								0	1, <i>q</i> ₂			
								1, <i>q</i> ₁	1			
									1, <i>q</i> ₁			
									, <i>q</i> ₀			

Computation of Turing machine

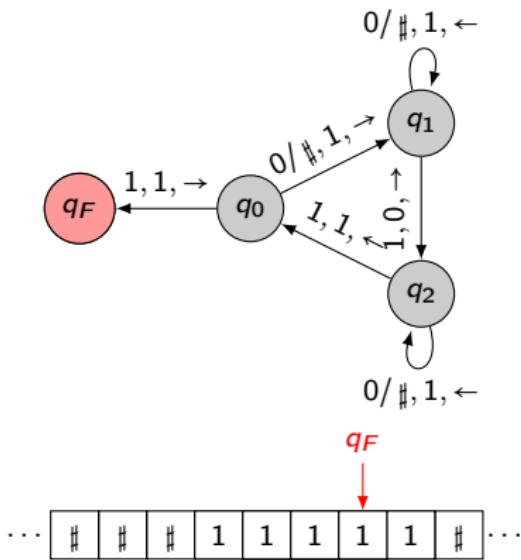
An example of model of computation:



					1	1, <i>q</i> ₁	1	1
						, <i>q</i> ₀	1	1
						1, <i>q</i> ₂	1	1
						1	0, <i>q</i> ₂	1
						1	0	, <i>q</i> ₂
						1	1, <i>q</i> ₁	
						0	, <i>q</i> ₀	1
						0	1, <i>q</i> ₂	
						1	, <i>q</i> ₁	1
						1	, <i>q</i> ₀	

Computation of Turing machine

An example of model of computation:

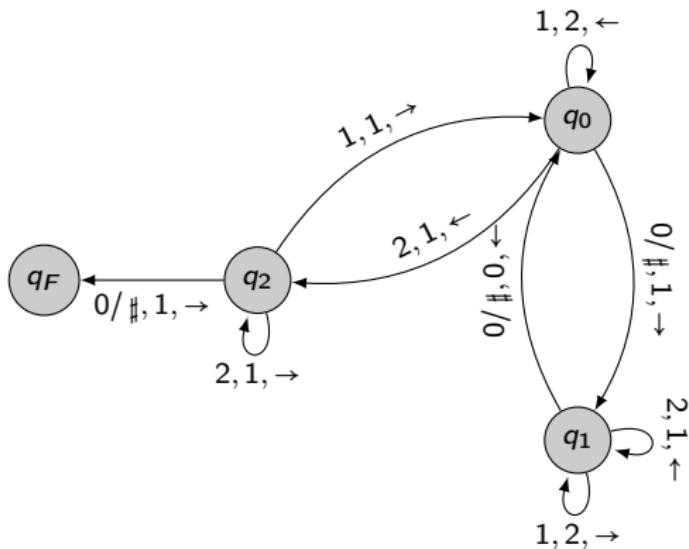


...	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	...
...	#	#	#	#	1	1	<i>q₁</i>	1	1	#	#	#	#	#	#	#	#	#	...
...	#	#	#	#	,	<i>q₀</i>	1	1	1	#	#	#	#	#	#	#	#	#	...
...	#	#	#	#	1	<i>q₂</i>	1	1	#	#	#	#	#	#	#	#	#	#	...
...	#	#	#	#	1	0	<i>q₂</i>	1	#	#	#	#	#	#	#	#	#	#	...
...	#	#	#	#	1	0	<i>q₂</i>	<i>q₂</i>	#	#	#	#	#	#	#	#	#	#	...
...	#	#	#	#	1	1	<i>q₁</i>	#	#	#	#	#	#	#	#	#	#	#	...
...	#	#	#	#	0	<i>q₀</i>	1	#	#	#	#	#	#	#	#	#	#	#	...
...	#	#	#	#	0	1	<i>q₂</i>	#	#	#	#	#	#	#	#	#	#	#	...
...	#	#	#	#	1	<i>q₁</i>	1	#	#	#	#	#	#	#	#	#	#	#	...
...	#	#	#	#	1	#	<i>q₁</i>	#	#	#	#	#	#	#	#	#	#	#	...
...	#	#	#	#	,	<i>q₀</i>	#	#	#	#	#	#	#	#	#	#	#	#	...

Theorem (*Turing 1937*)

The halting problem is undecidable (but it is semi-decidable).

Combinatory monster



Starting from an empty tape, this Turing machine writes 374×10^6 letters in 119×10^{15} steps of computation.

From the behavior of a Turing machine to SFT

Completion problem

Given a SFT \mathbf{T} and a pattern p , it is possible to find $x \in \mathbf{T}$ such that $p \sqsubset x$?

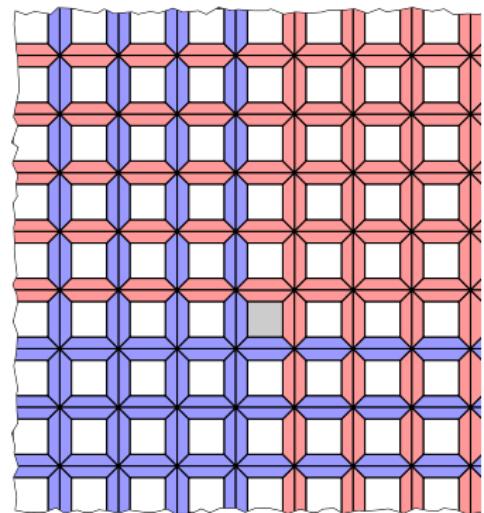
From the behavior of a Turing machine to SFT

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Consider

$$\mathbf{T}(\mathcal{F}_{\leq 1}) \subset \mathcal{A}_{\leq 1}$$



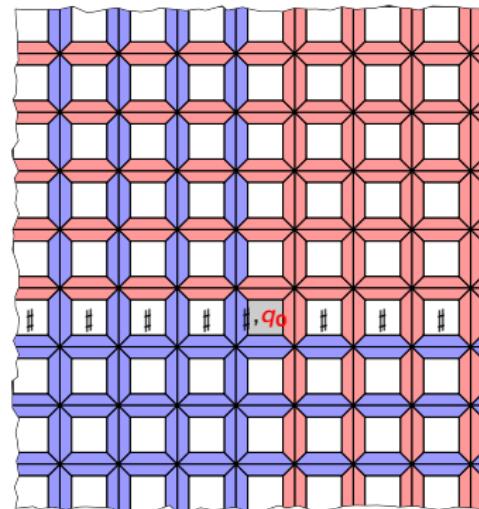
From the behavior of a Turing machine to SFT

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Given a SFT \mathbf{T} and a pattern p , it is possible to find $x \in \mathbf{T}$ such that $p \sqsubset x$?

Consider

$$\mathbf{T}(\mathcal{F}_{\leq 1} \cup \mathcal{F}_{\text{Ini}}) \subset \mathcal{A}_{\leq 1} \times \mathcal{A}_{\mathcal{M}}$$



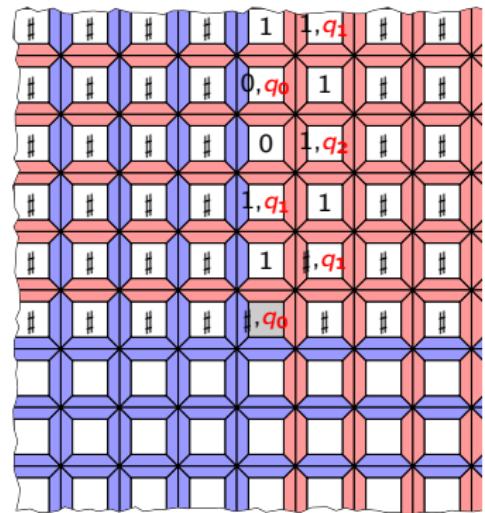
From the behavior of a Turing machine to SFT

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From the behavior of a Turing machine to SFT

Completion problem

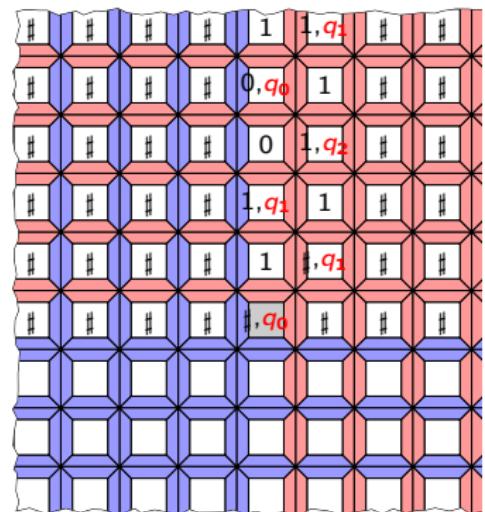
Given a SFT \mathbf{T} and a pattern p , it is possible to find $x \in \mathbf{T}$ such that $p \sqsubset x$?

Consider

$$\mathbf{T}(\mathcal{F}_{\leq 1} \cup \mathcal{F}_{\text{Ini}} \cup \mathcal{F}_M \cup \{q_F\}) \subset \mathcal{A}_{\leq 1} \times \mathcal{A}_M$$



can be completed in $\mathbf{T} \iff M$ does not halt



From the behavior of a Turing machine to SFT

Completion problem

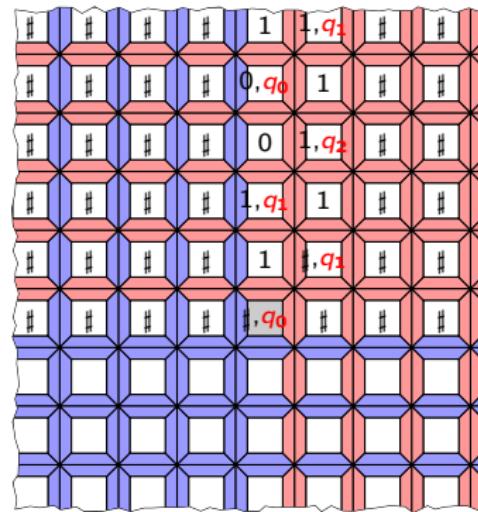
Given a SFT \mathbf{T} and a pattern p , it is possible to find $x \in \mathbf{T}$ such that $p \sqsubset x$?

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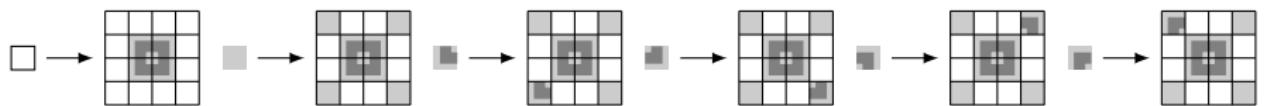


Theorem (Wang 1961)

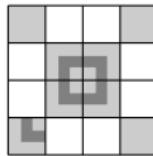
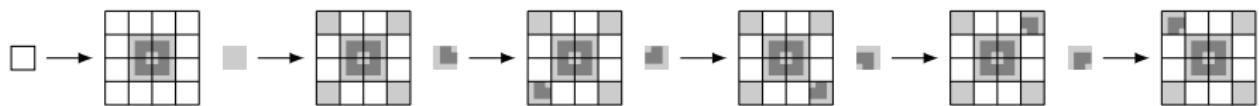
The completion problem is undecidable.

There is no links with the domino problem: by compacity there is no subshift such that a tile appear exactly one times.

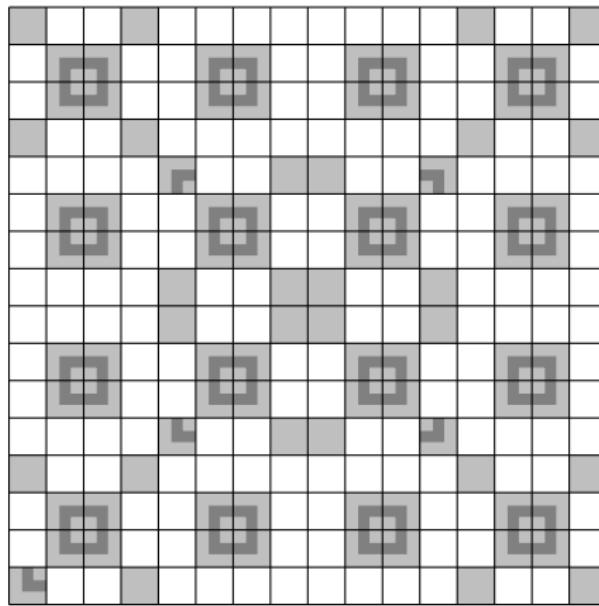
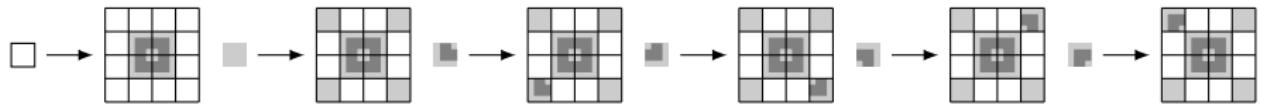
Computation zone



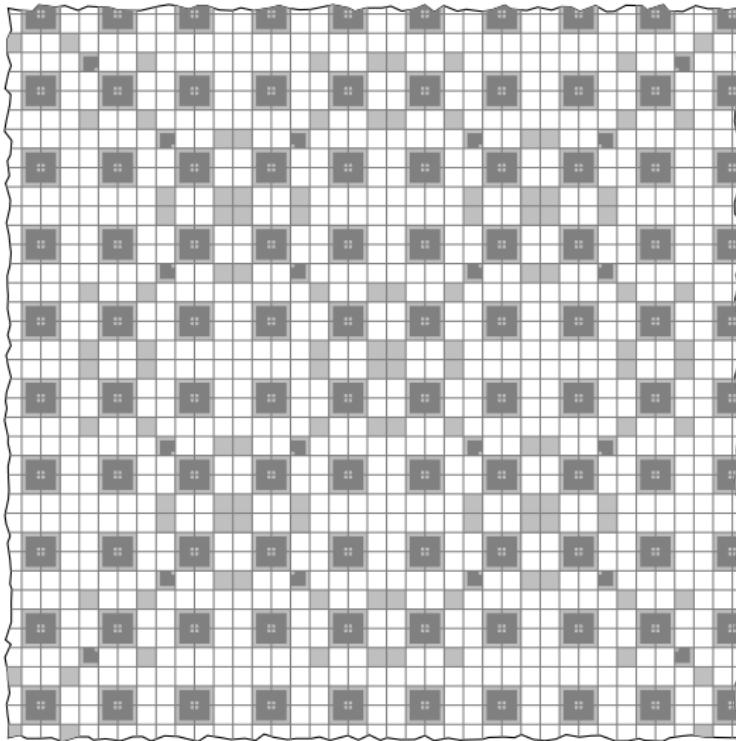
Computation zone



Computation zone



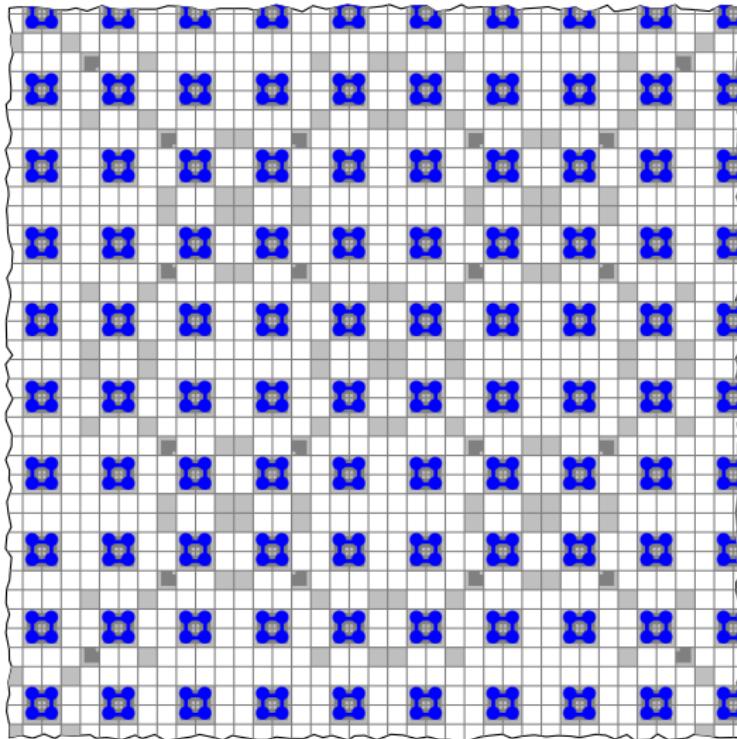
Computation zone



Space time diagram of a Turing machine:

1, q_2	1	1	#	#	#	#	#	#	#
1 0, q_2	1	#	#	#	#	#	#	#	#
1 0, q_2	#	#	#	#	#	#	#	#	#
1 1, q_1	#	#	#	#	#	#	#	#	#
0, q_0	1	#	#	#	#	#	#	#	#
0 1, q_2	#	#	#	#	#	#	#	#	#
1, q_1	1	#	#	#	#	#	#	#	#
1, q_1	#	#	#	#	#	#	#	#	#
, q_0	#	#	#	#	#	#	#	#	#

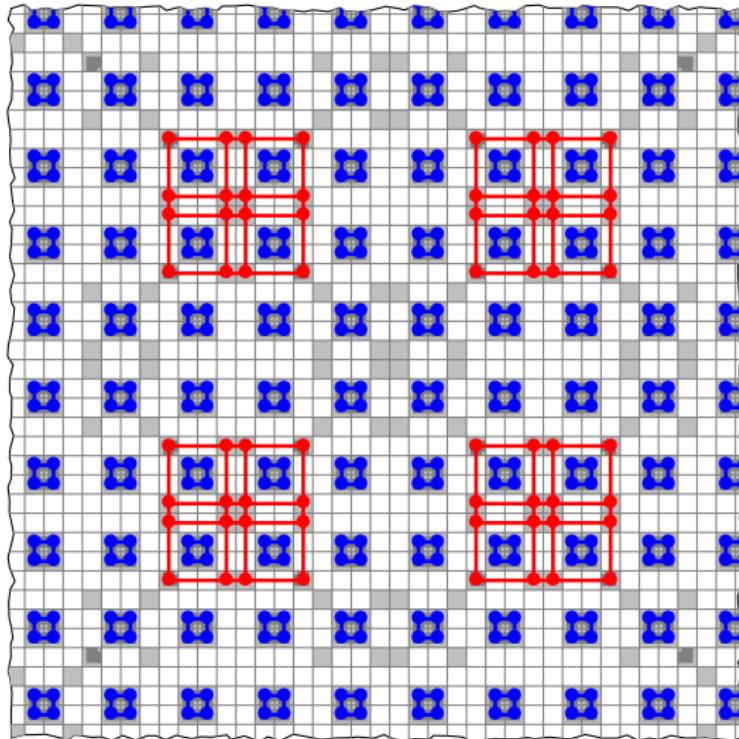
Computation zone



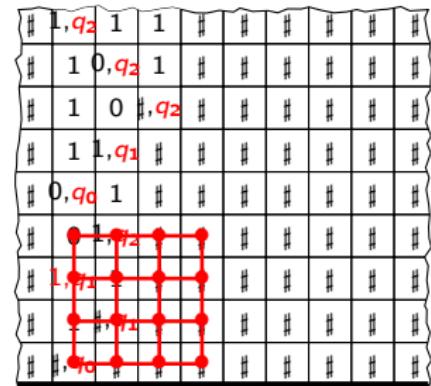
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1, q_2	1	1	#	#	#	#	#	#	#
1 0, q_2	1	#	#	#	#	#	#	#	#
1 0, q_2	#	#	#	#	#	#	#	#	#
1 1, q_1	#	#	#	#	#	#	#	#	#
0, q_0	1	#	#	#	#	#	#	#	#
0 1, q_2	#	#	#	#	#	#	#	#	#
1, q_1	1	#	#	#	#	#	#	#	#
1, q_1	#	#	#	#	#	#	#	#	#
, q_0	#	#	#	#	#	#	#	#	#

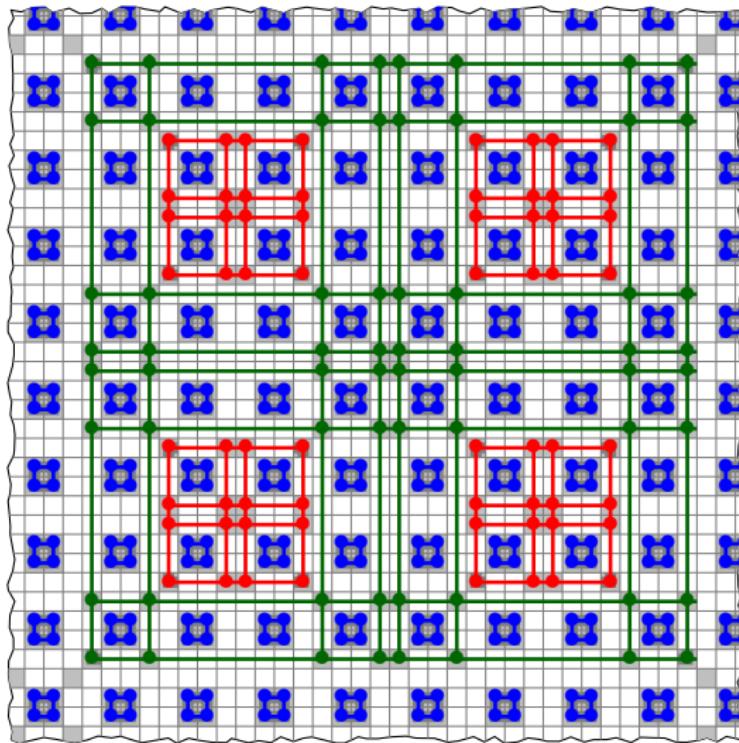
Computation zone



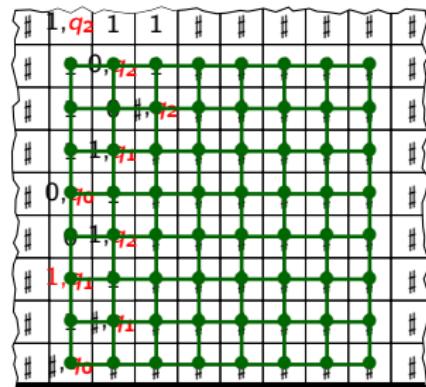
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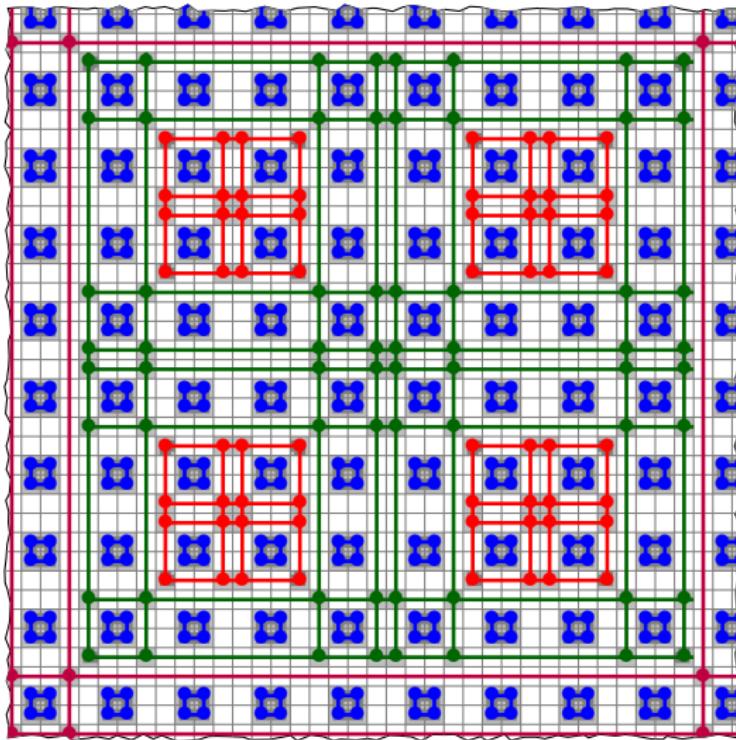
Computation zone



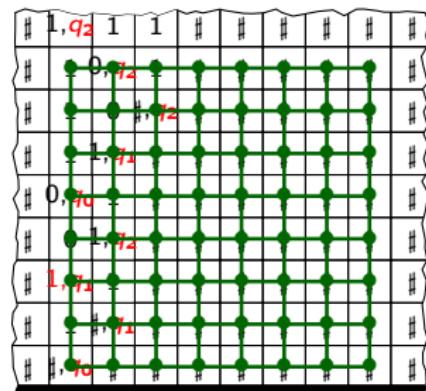
Space time diagram of a Turing machine:



Computation zone



Space time diagram of a Turing machine:



Theorem: Undecidability of the domino problem

The tile set without q_F tiles the plane. \iff The Turing machine does not halt.

Links between effective subshift and local rules

Realization of effective subshifts by sofic

$$\mathbf{T}_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \text{patterns of } \mathcal{F} \text{ does not appear in } x \right\} \subseteq \mathcal{A}^{\mathbb{Z}^d}$$

Some classes of subshifts invariant by conjugacy

\mathbf{T} subshift of finite type $\iff \exists \mathcal{F}$ finite set such that $\mathbf{T} = \mathbf{T}_{\mathcal{F}}$

\mathbf{T} subshift sofic $\iff \exists \mathcal{F}$ finite set and $\pi: \mathcal{A} \rightarrow \mathcal{B}$ such that $\mathbf{T} = \pi(\mathbf{T}_{\mathcal{F}})$

\mathbf{T} effective subshift $\iff \exists \mathcal{F}$ recursively enumerable such that $\mathbf{T} = \mathbf{T}_{\mathcal{F}}$

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Theorem (Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)

Let $\Sigma \subset \mathcal{A}^{\mathbb{Z}^d}$ be an effective subshift, then the following subshift is sofic:

$$\Sigma^\uparrow = \left\{ x \in \mathcal{A}^{\mathbb{Z}^{d+1}} : \exists y \in \Sigma, \forall i \in \mathbb{Z} x_{(\mathbb{Z}, i)} = yz \right\}.$$

An example

Let $\Sigma = \mathbf{T}(\{a, b, \$\}, 1, \{ba, \beta a^n b^m \alpha : n \neq m, \alpha \neq a, \beta \neq b\})$. Consider the subshift

$$\Sigma^\uparrow = \{x \in (\{a, b, \$\})^{\mathbb{Z}^2} : \exists y \in \Sigma \text{ tel que } x_{(.,j)} = y \text{ such that } j \in \mathbb{Z}\}$$

\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
\$	a	a	b	b	\$	\$	a	a	a	a	b	b	b	b	\$	\$	\$	a	b	\$	\$	\$
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	c	c	c	a	b	c	c	c

An example

Let $\Sigma = \mathbf{T}(\{a, b, \$\}, 1, \{ba, \beta a^n b^m \alpha : n \neq m, \alpha \neq a, \beta \neq b\})$. Consider the subshift

$$\Sigma^\uparrow = \{x \in (\{a, b, \$\})^{\mathbb{Z}^2} : \exists y \in \Sigma \text{ tel que } x_{(\cdot, j)} = y \text{ such that } j \in \mathbb{Z}\}$$

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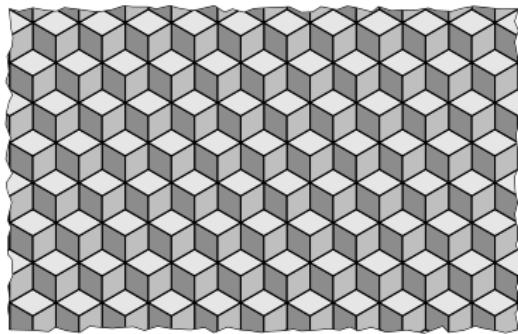
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\$	a'	a	b	b'	\$	\$	a	a	a	a''	a	b	b'	b	b	b	\$	\$	a''	b''	\$	\$	\$	
\$	a	a''	b''	b	\$	\$	a	a	a	a	a''	b	b	b'	b	b	b	\$	\$	a''	b''	\$	\$	\$
\$	a'	a	b	b'	\$	\$	a	a	a	a	a	b	b	b'	b	b	b	\$	\$	a''	b''	\$	\$	\$
\$	a	a''	b''	b	\$	\$	a	a	a	a'	a	a	b	b	b	b	b'	b	\$	\$	a''	b''	\$	\$
\$	a'	a	b	b'	\$	\$	a	a'	a	a	a	b	b	b	b	b	b'	b	\$	\$	a''	b''	\$	\$
\$	a	a''	b''	b	\$	\$	a'	a	a	a	a	b	b	b	b	b	b''	b	\$	\$	a''	b''	\$	\$
\$	a'	a	b	b'	\$	\$	a	a''	a	a	a	b	b	b	b	b	b''	b	\$	\$	a''	b''	\$	\$
\$	a	a''	b''	b	\$	\$	a	a	a	a	a''	a	b	b	b	b	b''	b	\$	\$	a''	b''	\$	\$
\$	a'	a	b	b'	\$	\$	a	a	a	a	a''	a	b	b''	b	b	b	\$	\$	a''	b''	\$	\$	\$
\$	a	a''	b''	b	\$	\$	a	a	a	a	a	a''	b''	b	b	b	b	\$	\$	a''	b''	\$	\$	\$

Which vector spaces admit local rules?

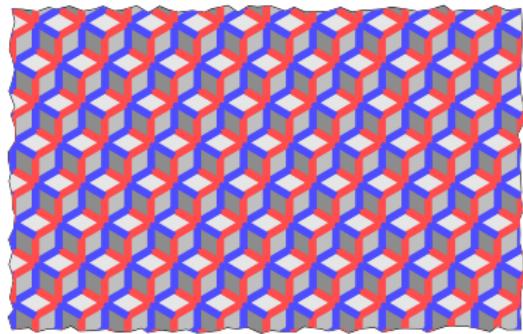
Undecorated local rules

$$\mathcal{F} = \left\{ \begin{array}{c} \text{diamond} \\ \text{triangle} \\ \text{square} \end{array} \right\}$$



Decorated local rules

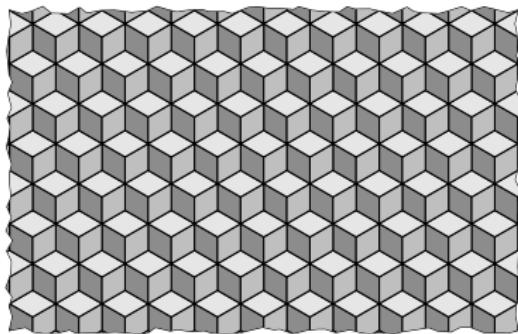
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Which vector spaces admit local rules?

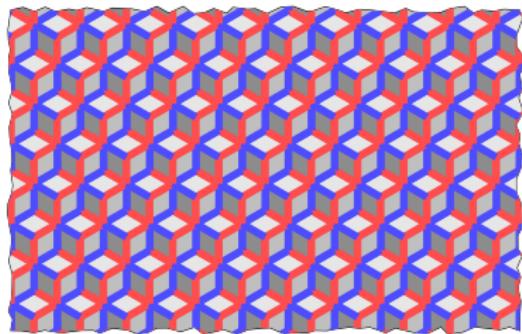
Undecorated local rules

$$\mathcal{F} = \left\{ \begin{array}{c} \text{diamond} \\ \text{triangle} \\ \text{bar} \end{array} \right\}$$

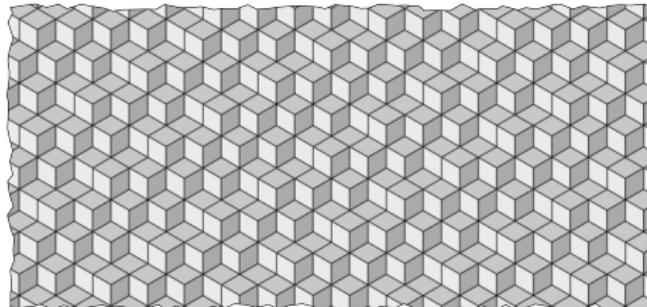


Decorated local rules

$$\mathcal{F} = \left\{ \begin{array}{c} \text{diamond} \\ \text{triangle} \\ \text{bar} \end{array} \right\} \text{ with colors}$$

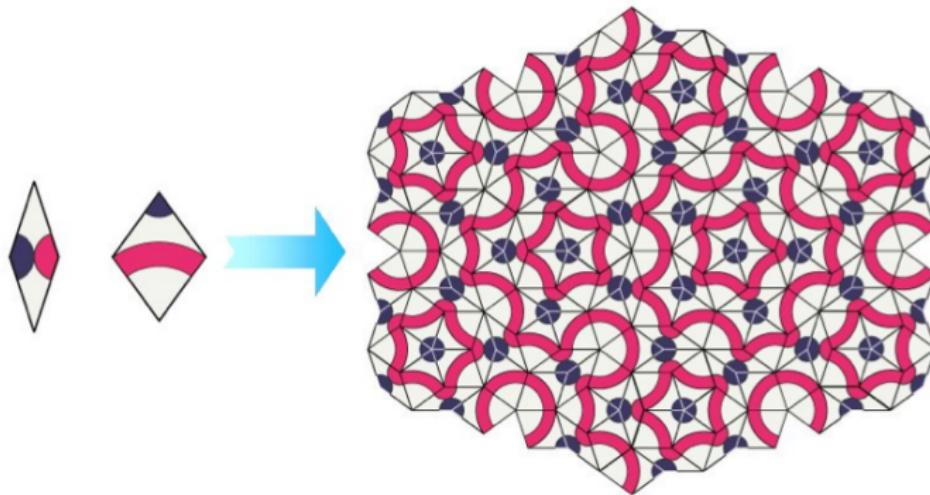


There is local rules if the normal vector is not rational?



Which vector space admits local rules?

- Some suffisant conditions suffisantes of algebraic nature:
Penrose-74, Burkov-88, Levitov-88, Socolar-89, Le-92...



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Theorem of realisation (*Fernique-Sablik-12*)

The tiling associated to $V \subset \mathbb{R}^n$ admits local rules iff V is computable.

How to realize such structure?

Structural problems

- Alphabet of huge cardinal
- It can be difficult to complete a patterns.
- We need a hierarchical structure

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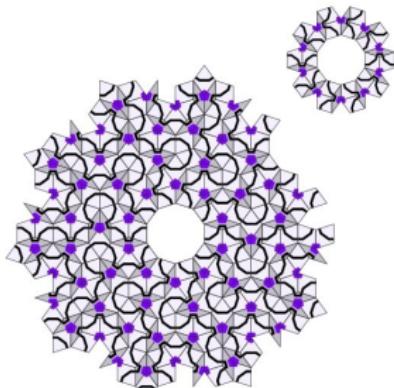
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Formation of aperiodic structures

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- Self assembly



Soclar 1991.

Which rules admits this type of assembly?

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- It can be difficult to complete a patterns.
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Formation of aperiodic structures

- What happens under dynamical constraints?
- Self assembly
- Stocha-flip

