Packing Unequal Disks in the Euclidean Plane

Thomas Fernique

After the eponymous survey
(Comput. Geom.: Theory Appl., 2025)

Outline

1. Compacity
2. Conjugacy
3. Density
4. Uniformity
5. Complexity

Outline

Compact packings of disks

Disk packing: set of interior disjoint disks in the Euclidean plane.

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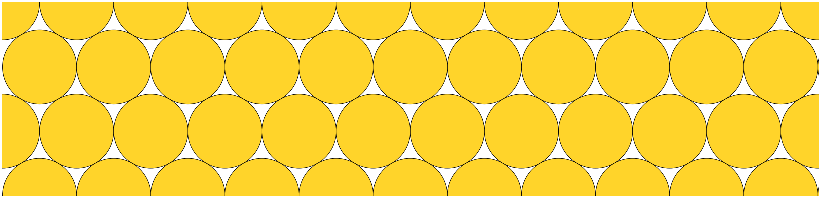

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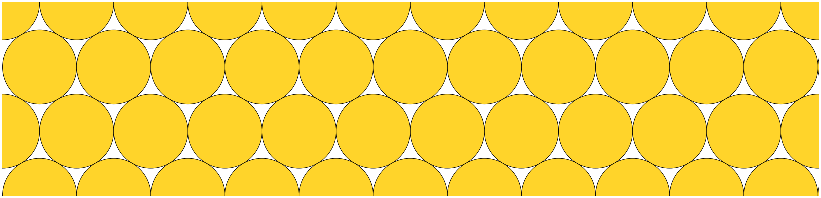

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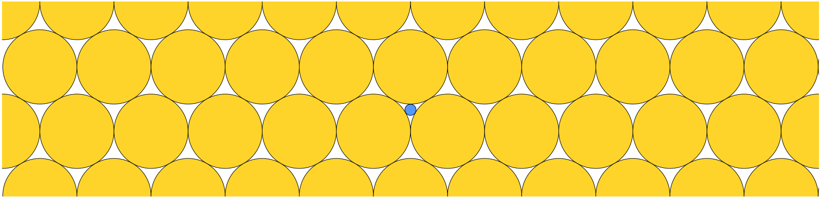

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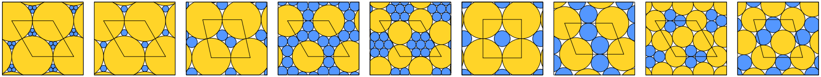
Two disks

Theorem (Kennedy, 2004)

9 values $r < 1$ allow a compact packing of disks of size 1 and r.



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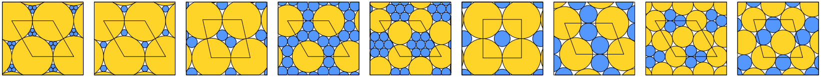

Proof sketch:

- at most 5 disks in an r-*corona* (other than r^6);

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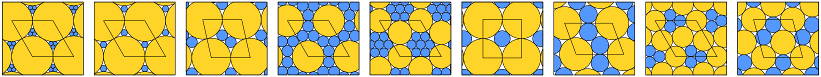

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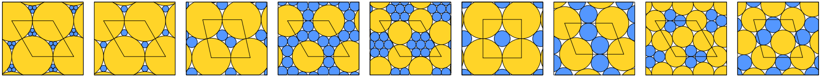

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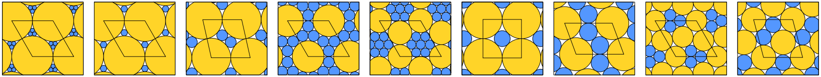

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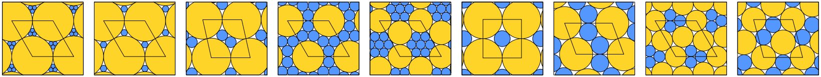

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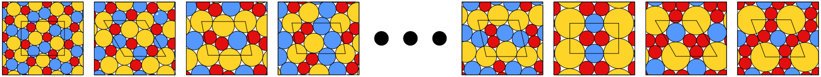
Theorem (F.-Hashemi-Sizova, 2018)

164 pairs $s < r < 1$ allow a compact packing of disks of size 1, r, s.



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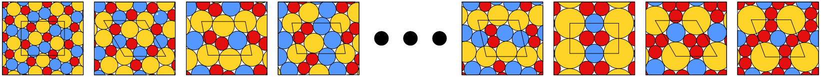

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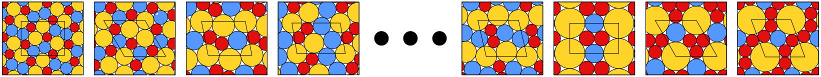

Proof sketch:

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- arbitrarily many s-disks in an r-corona?

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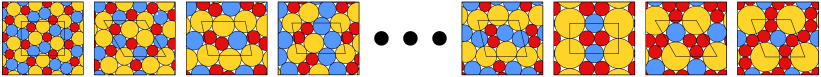

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- at most 5 disks in an s-corona (other than s^6);
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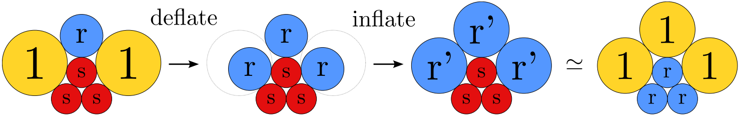

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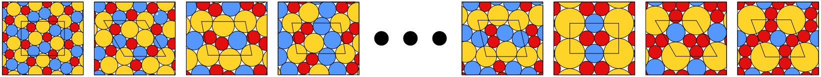

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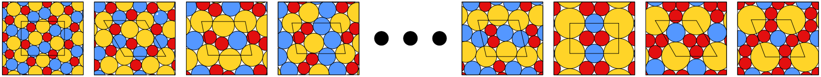

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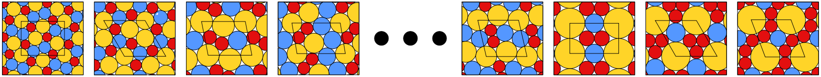

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Disk sizes $r_1 < \ldots < r_n = 1$ that allow a compact packing?

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- at most 5 disks in an r_1-corona (trivial);
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"Uterus lemma" (F.-Pchelina-Sizova, 2025): $r_2/r_3 \geq 0.054$.

- at most 572 disks in an r_3-corona.

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For $n = 4$:
At most 89 408 704 197 429 760 000 polynomial systems in r_1, r_2, r_3.
Current progress: stuck on the mere computation of polynomials.

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Theorem (Messerschmidt, 2021)

Finitely many $r_1 < \ldots < r_n = 1$ allow a compact packing.

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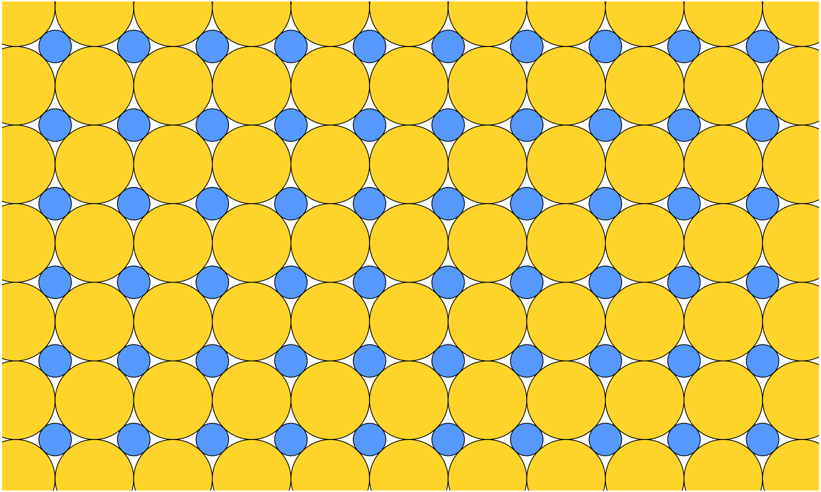
Motivation



Disk sizes that allow a compact packing.

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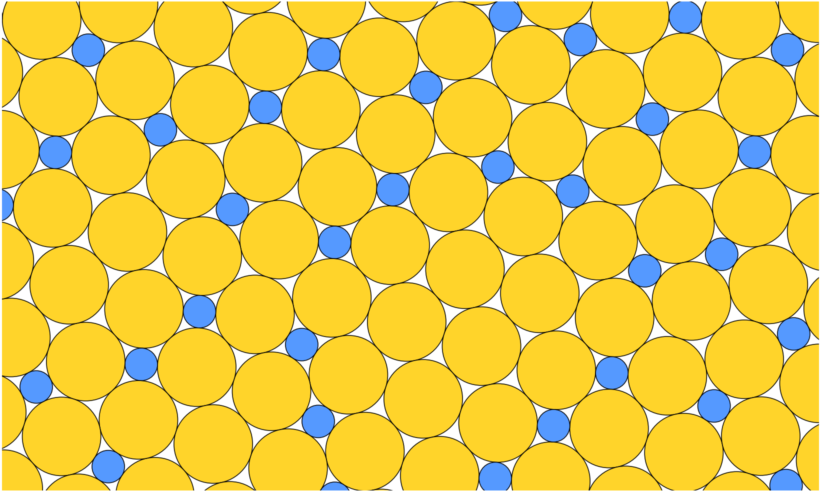
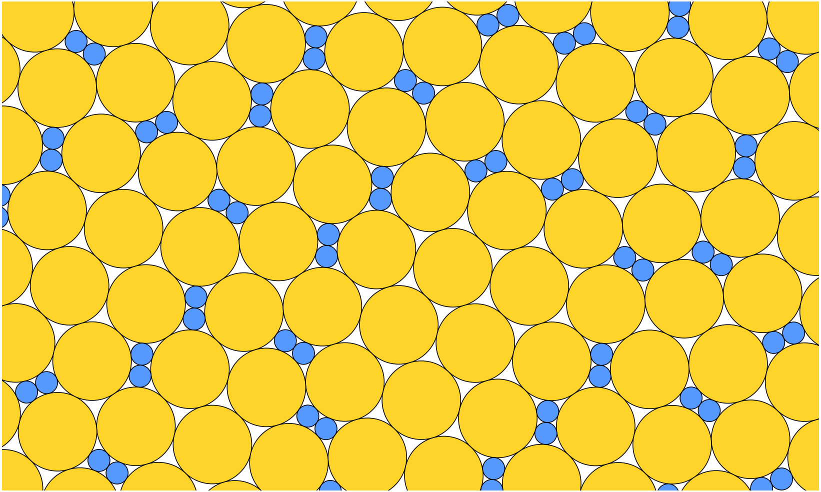


They may allow other compact packings. Can we describe the set?

Motivation



Or show it is similar to some set defined by different sizes of disks?

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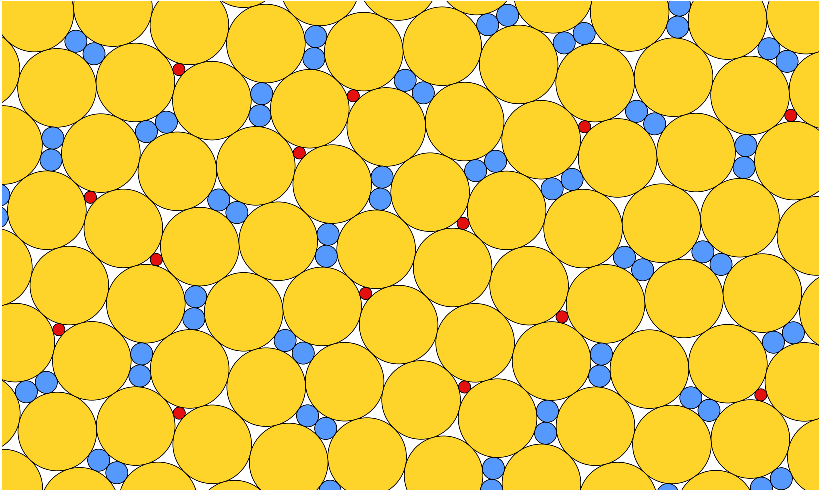
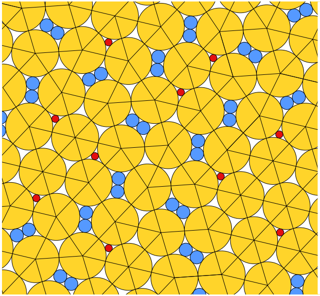
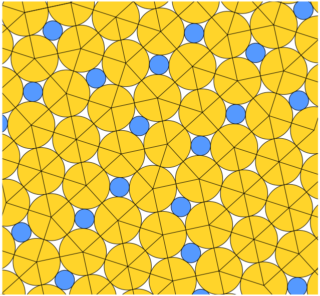


Possibly up to adding or removing some “decorations”?

Symbolic dynamics to the rescue!

Compact packings are *tilings* by triangles (or more clever tiles).

X_τ denotes the set of tilings by a tile set τ.

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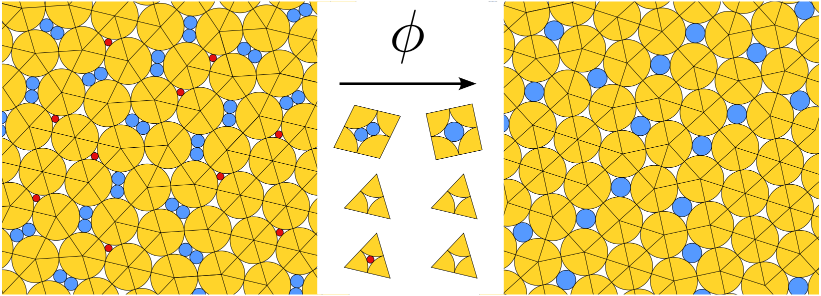

X_τ denotes the set of tilings by a tile set τ.

X_τ factors on X_σ if there is a factor map $\phi : \tau \to \sigma$ such that:

- for every tiling $\mathcal{S} \in X_\sigma$, there is an isomorphic tiling $\mathcal{T} \in X_\tau$;
- every tile t of $\mathcal{T}$ corresponds to a tile $\phi(t)$ in $\mathcal{S}$.

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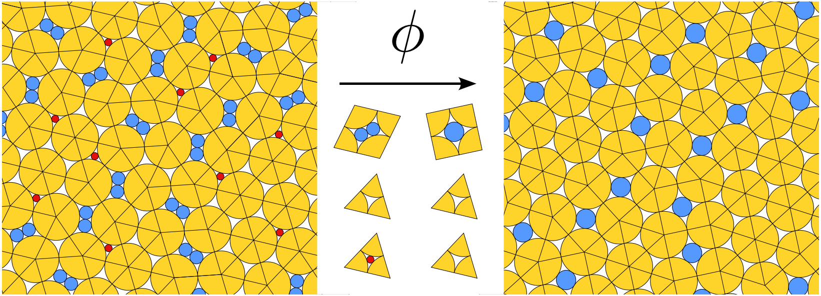

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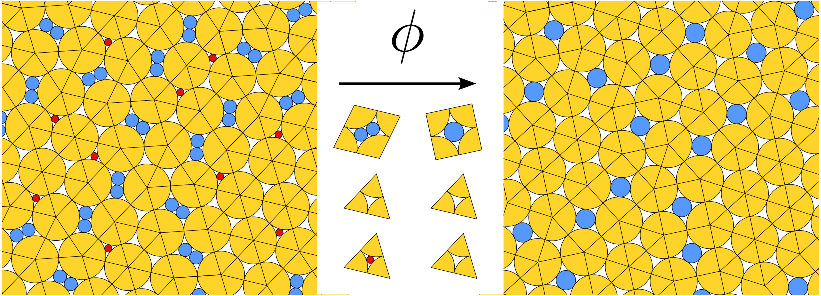
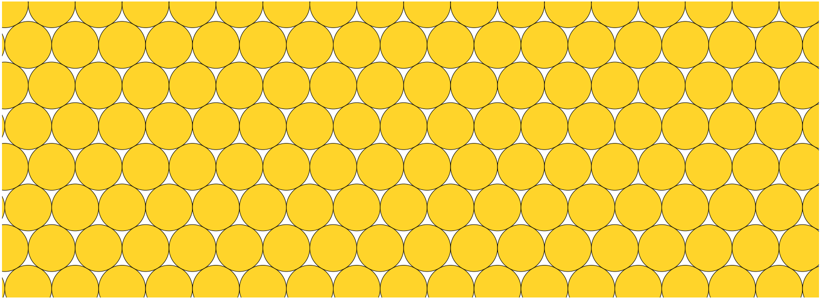


X_τ denotes the set of tilings by a tile set τ.

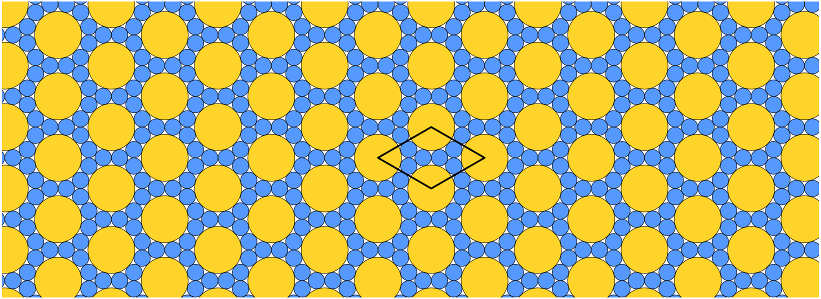
X_τ factors on X_σ if the tilings of X_σ are "rebranded" tilings of X_τ.
If X_σ also factors on X_τ, then X_τ and X_σ are conjugated.

Conjugacy classes



For one disk size: hexagonal compact packing (HCP).

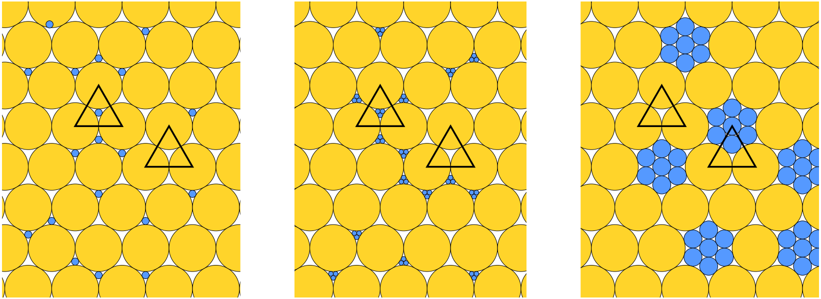
Conjugacy classes



For two disk sizes:

- A set conjugated to HCP;

Conjugacy classes



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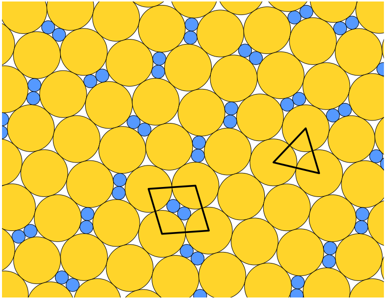

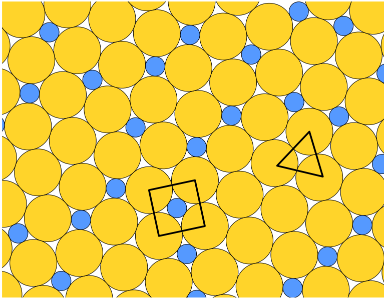

For two disk sizes:

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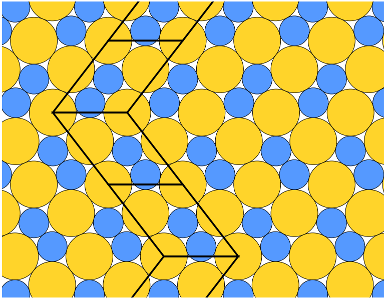
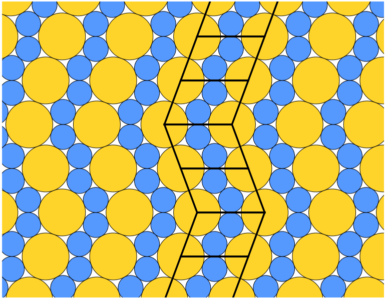
- Two conjugated sets of *square-triangle* tilings;

Conjugacy classes





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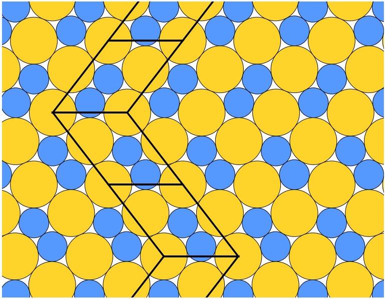

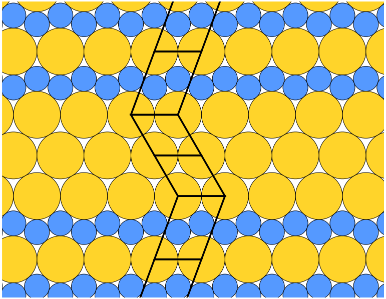

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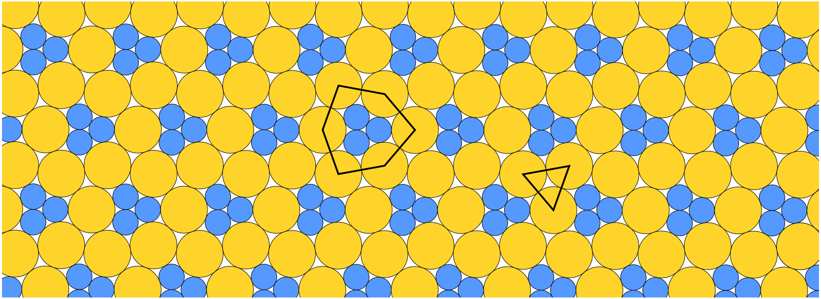

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- A set conjugated to HCP; three sets that factor onto HCP;
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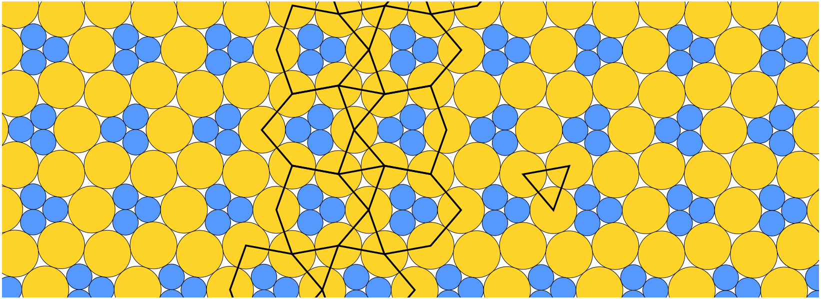

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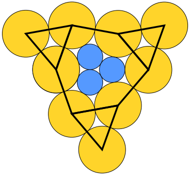

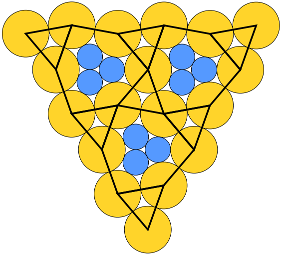

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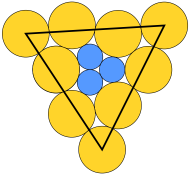

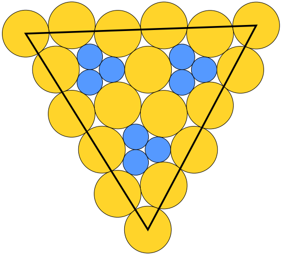

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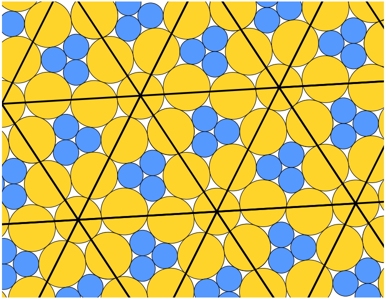 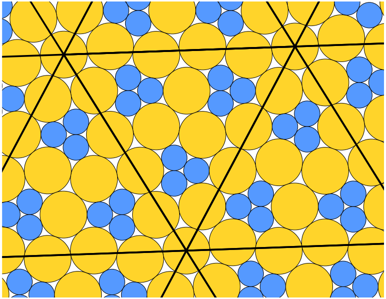

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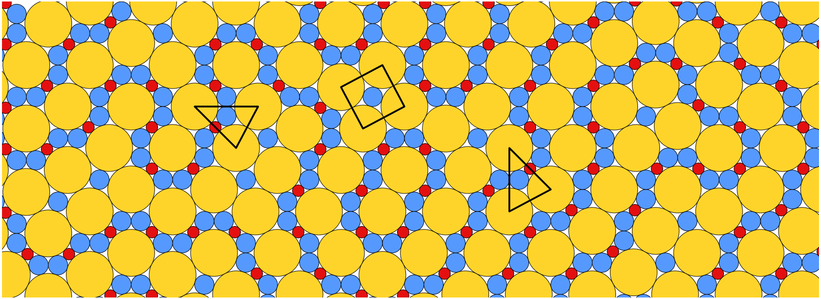

For three disk sizes: only one new class among the 164 cases!

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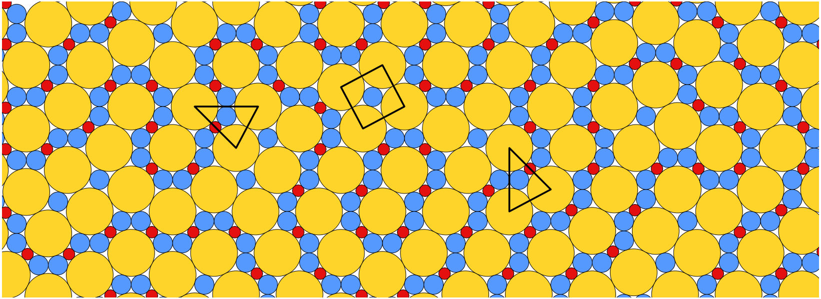

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But are all the previous conjugacy classes really pairwise different?

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Theorem (Heppes, 2003, Kennedy, 2004, F.-Bédaride, 2020)

For each of the 9 pairs of disk sizes that allow a compact packing, the maximal density is reached by a compact packing.

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For one size of disk, the maximal density is reached by HCP.

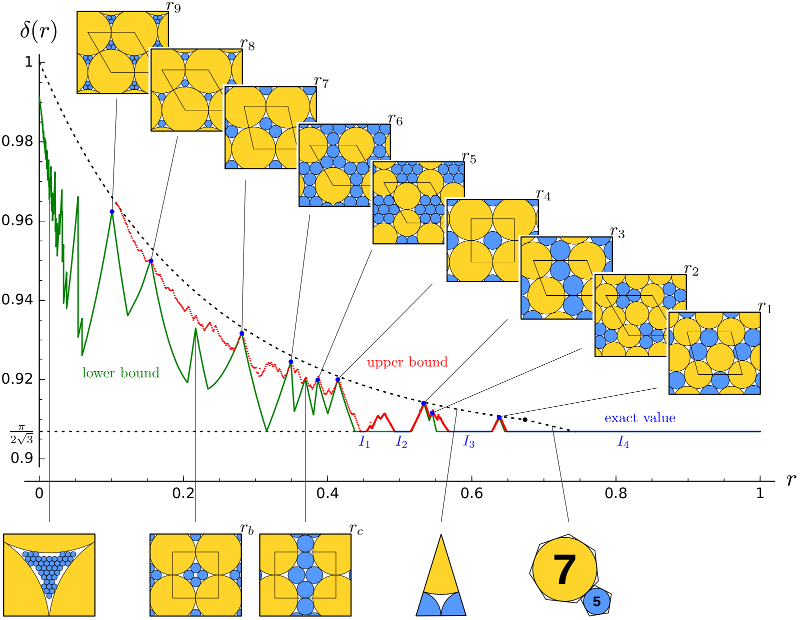
Theorem (Heppes, 2003, Kennedy, 2004, F.-Bédaride, 2020)

For each of the 9 pairs of disk sizes that allow a compact packing, the maximal density is reached by a compact packing.

Theorem (F.-Pchelina, 2022)

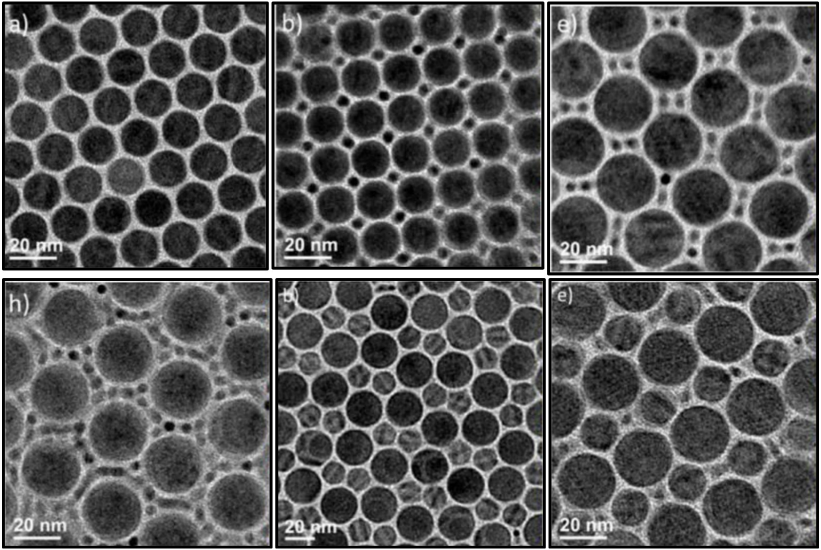
Among the 164 triples of disk sizes that allow a compact packing, the maximal density is reached by a compact packing in 31 cases and reached by a non-compact packing in 40 cases (93 open cases).

Bounds for two disk sizes



Outline

Motivation



Compact packings of disks of different sizes appear in nanocrystals.

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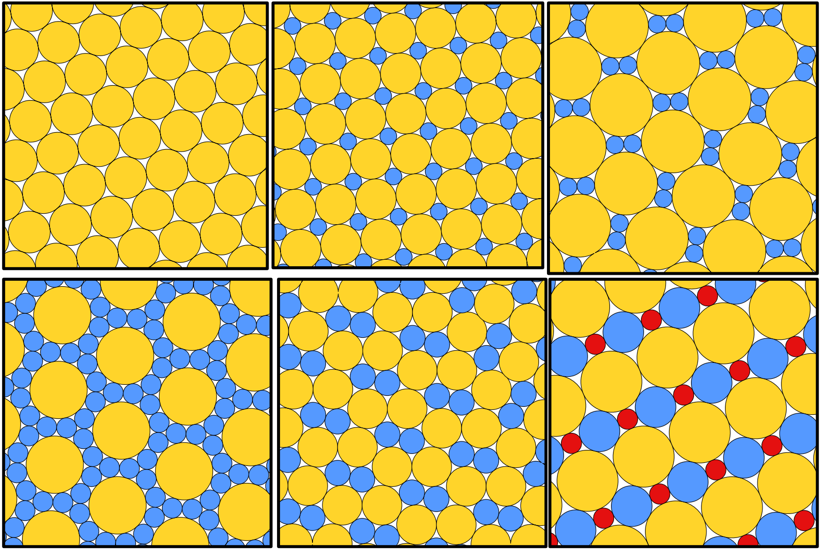

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What is the most uniform dense packing (denser than HCP)?

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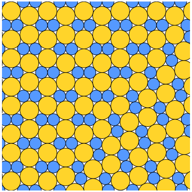

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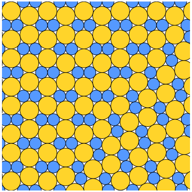

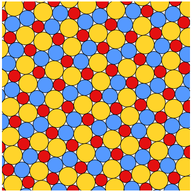

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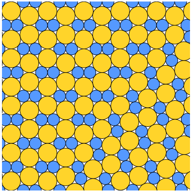

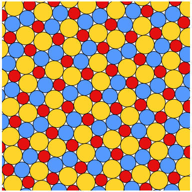

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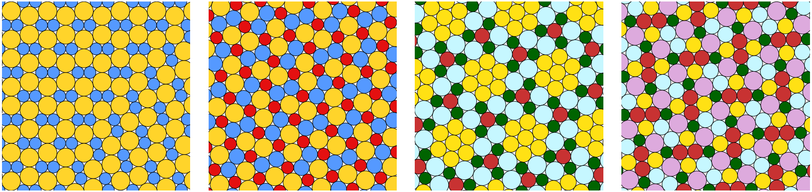
- Conjecture (Connelly, 2019): record achieved for 3 sizes.



Outline

1. Compacity
2. Conjugacy
3. Density
4. Uniformity
5. Complexity

Searching for compact packings

Recall the strategy to find a compact packing:

- bound from below the smallest/largest size ratio;
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But what if some sizes allow only *non-periodic* compact packings?

Decidability

Domino problem (Wang, 1961): is it possible to tile the plane with a given set of unit squares with colored edges (colors must match)?

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What about compact packings?

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- We have to exclude HCP. Consider only the *densest* packings?
- Does it mean the problem is undecidable?

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