Density of sphere packings

Thomas Fernique

Outline

The problem

Motivations

Some results

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Conjecture (Kepler, 1610)

For equal spheres in $\mathbb{R}^3$, the maximal density is $\frac{\pi}{3\sqrt{2}} \approx 74\%$.

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Theorem (Viazovska, 2016)

For equal spheres in $\mathbb{R}^8$, *the maximal density is* $\frac{\pi^4}{384} \approx 25\%$.

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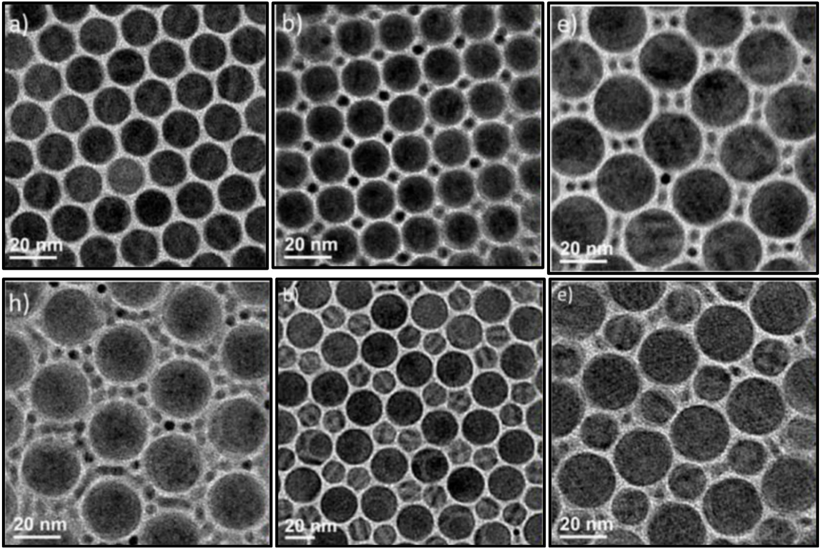
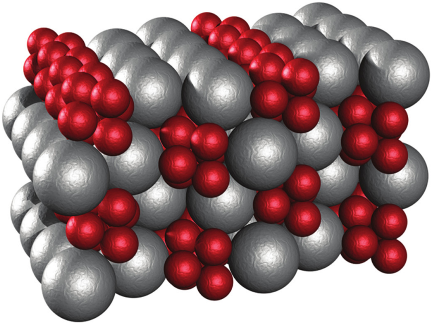
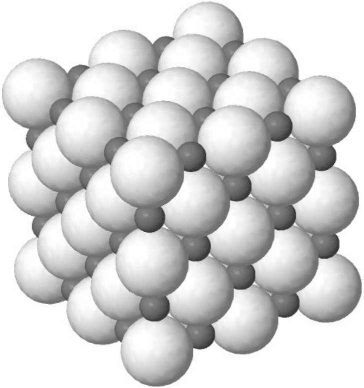


Materials science





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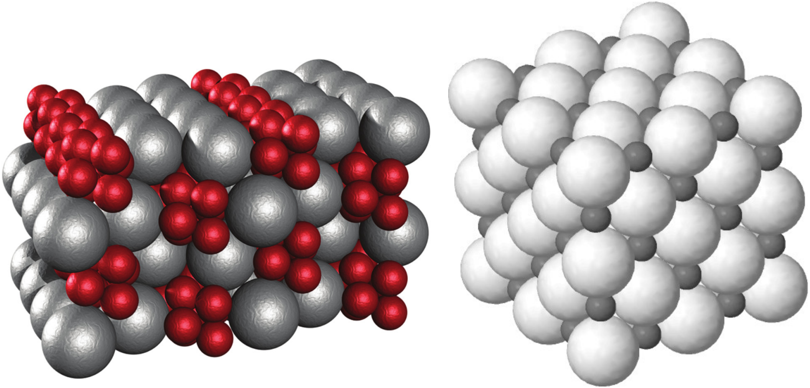

Slicing higher dimensional packings may also be interesting!

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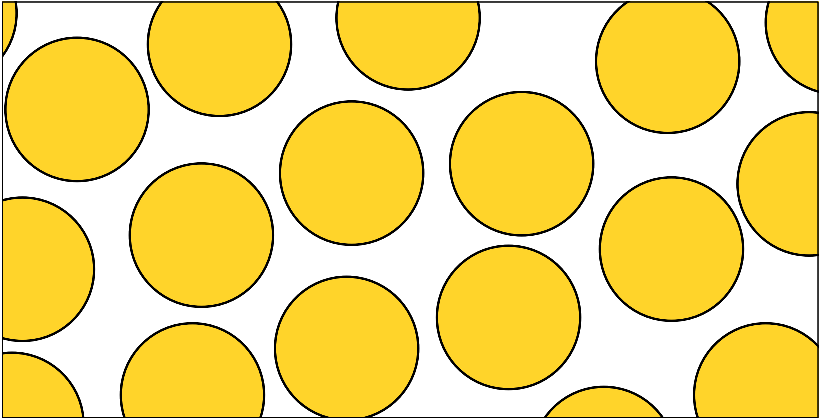
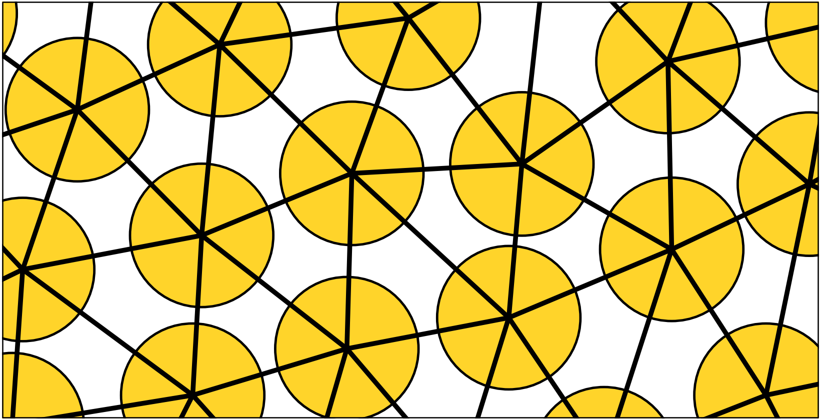
Equal disks



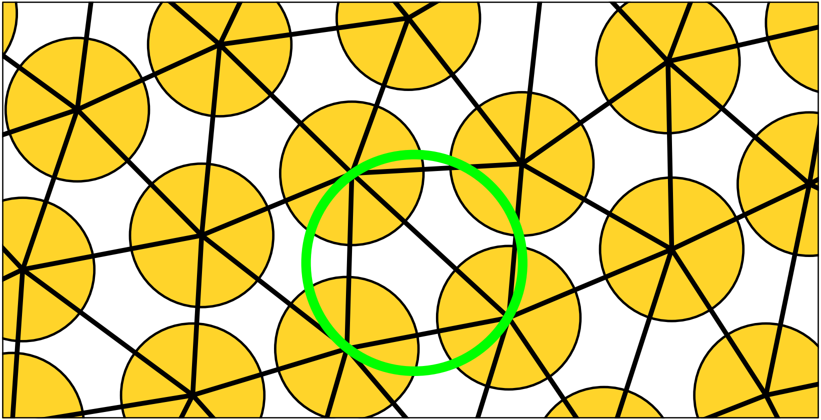
Consider a packing of unit disks.

Equal disks



Consider the *Delaunay triangulation* of the disk centers.

Equal disks



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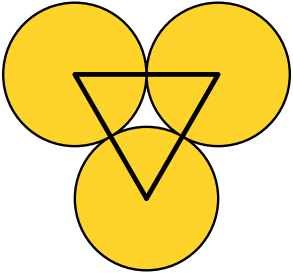
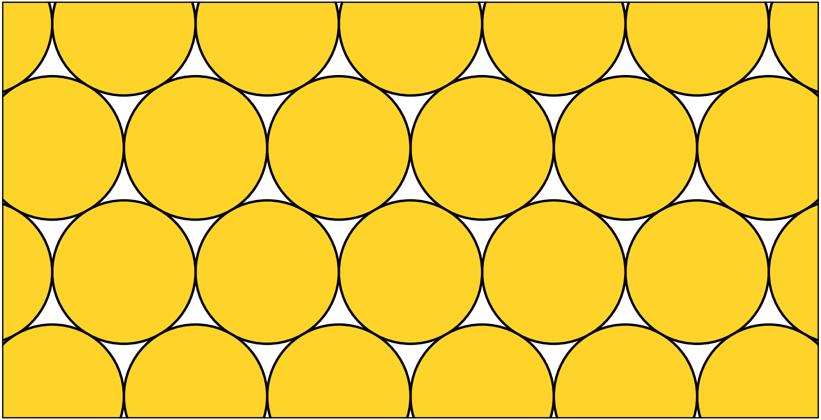


Lemma (Chan-Wang, 2010)

Densest possible triangle: three pairwise tangent disks.

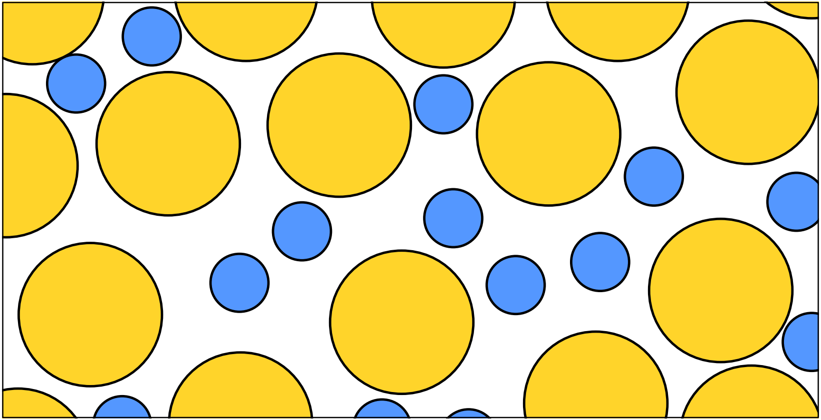
Equal disks



Theorem (Thue 1910, Tóth 1943)

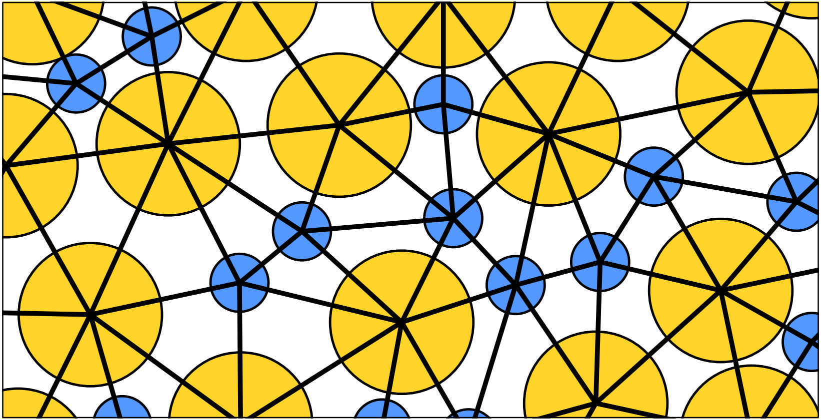
For equal disks, the maximal density is $\frac{\pi}{2\sqrt{3}} \approx 91\%$.

Unequal disks



Consider a packing of disks with, e.g., two sizes.

Unequal disks



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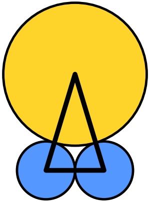
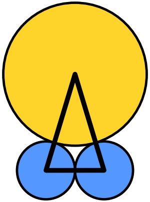


Theorem (Florian, 1960)

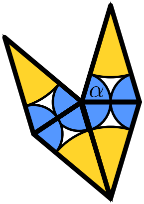
Densest triangle: two small and one large pairwise tangent disks.

Unequal disks



Frustration: Florian's triangles do not tile the plane!

Unequal disks



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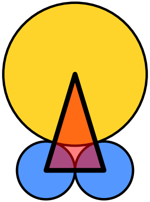
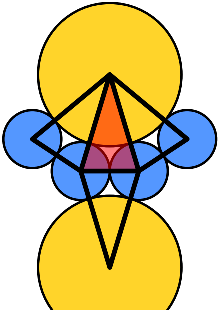
Around a small disk: $k > 4$ angles α, k even $\Rightarrow k \geq 6 \Rightarrow \alpha \leq \frac{\pi}{3}$.

Unequal disks



Strategy: local redistribution of density excesses...

Unequal disks



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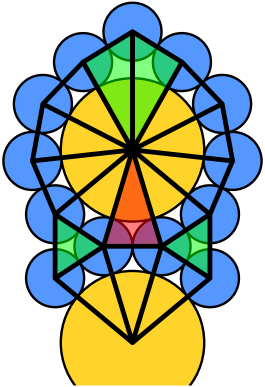

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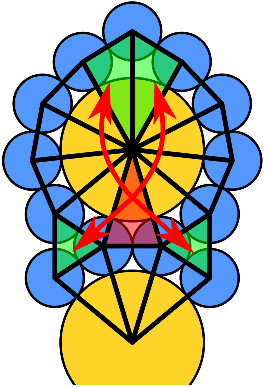

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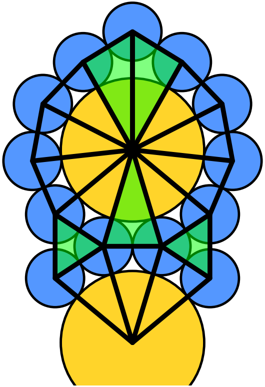

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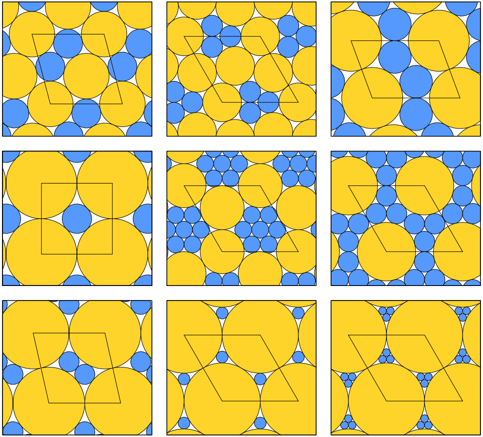

Theorem (Bédaride-F., 2022)

Each of these nine (periodic) packings maximizes the density.

Equal spheres

Theorem (Rogers, 1958)

Density upper bound: tetrahedron of pairwise tangent spheres.

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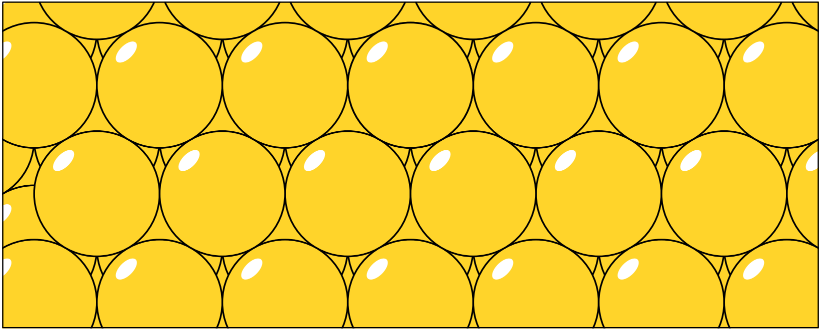

Kepler's packings: stacked layers of spheres on a triangular grid

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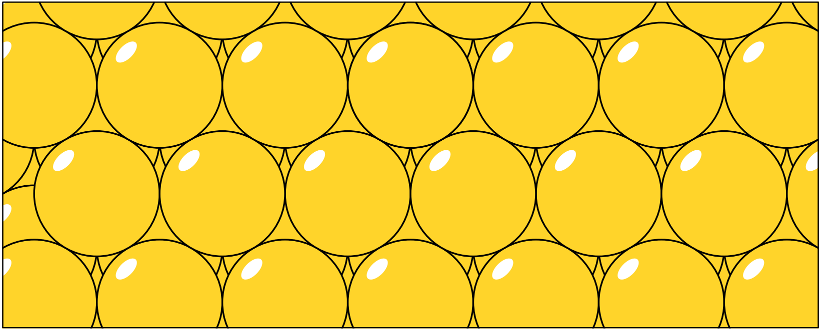

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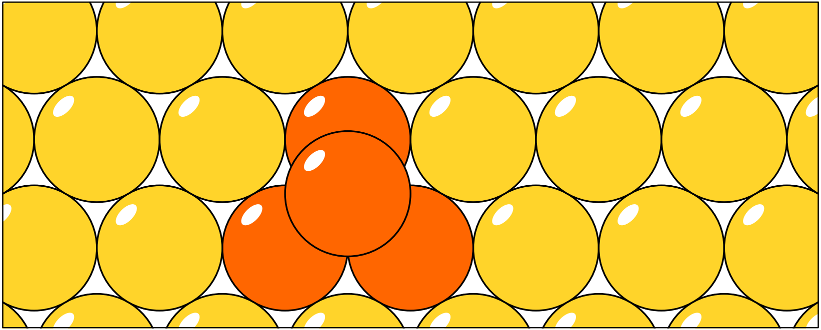

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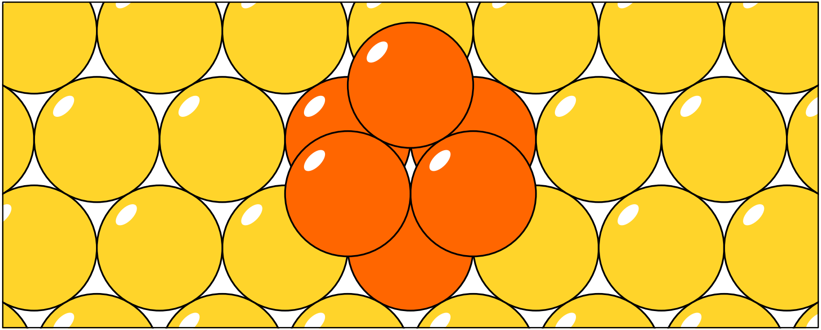

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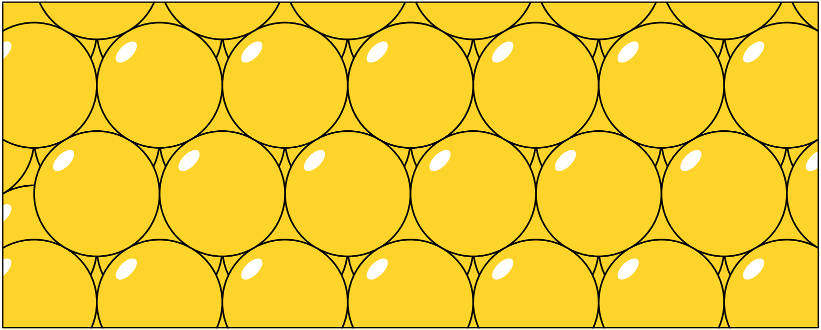

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Theorem (Hales-Ferguson, 1998-2014)

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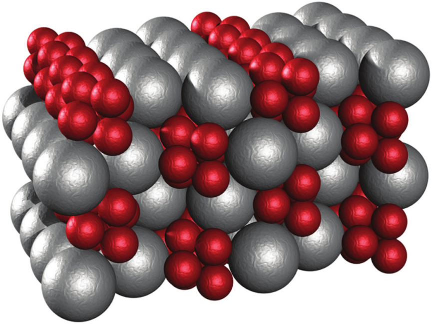
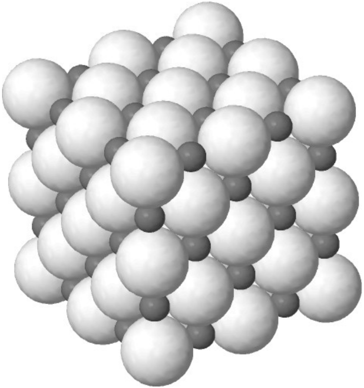
Unequal spheres

No tight bound, though very interesting for materials science. . .





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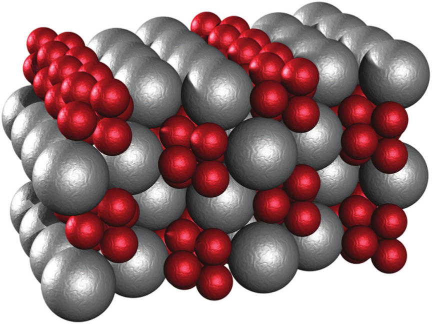

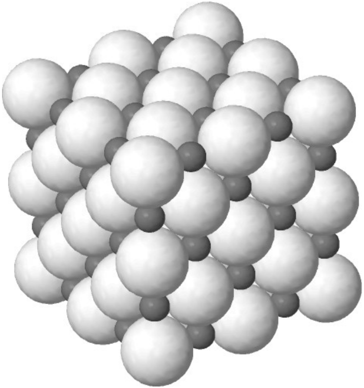

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Only equal spheres packing have been considered.

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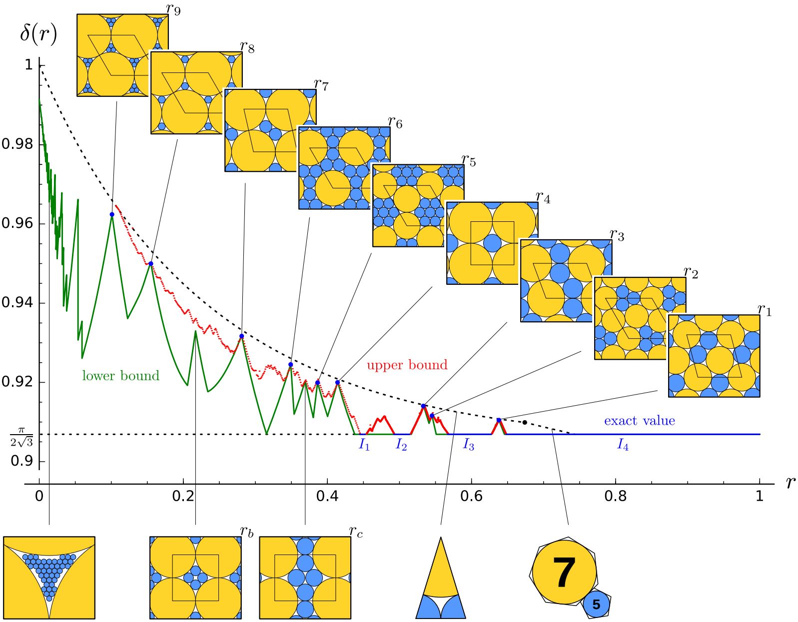
Theorem (Kabatianskiy-Levenshtein, 1978)

Any packing of unit spheres in $\mathbb{R}^n$ has density at most $1/2^{0.599n}$.

Outline

Density plot for two disks



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Given $f : \mathbb{R} \to \mathbb{R}$ and an interval X, computes an interval f_X s.t.

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