Higher-order convolution identities for Bernoulli and Euler polynomials

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Generating functions (EGF)

- Bernoulli numbers & polynomials \((|t| < 2\pi)\)

\[
P(t) := \frac{t}{e^t - 1} = \sum_{n\geq 0} B_n t^n / n!
\]

\[
P(t, x) := \frac{te^{xt}}{e^t - 1} = \sum_{n\geq 0} B_n(x) t^n / n!
\]

Where \(B_n(0) = (-1)^n B_n(1) = B_n\) and \(B_n(1 - x) = (-1)^n B_n(x) (n \geq 0)\).

- Genocchi numbers & polynomials \((|t| < \pi)\)

\[
Q(t) := \frac{2t}{e^t + 1} = \sum_{n\geq 0} G_n t^n / n!
\]

\[
Q(t, x) := \frac{2te^{xt}}{e^t + 1} = \sum_{n\geq 0} G_n(x) t^n / n!
\]

Where \(G_n(0) = G_n\) and \(G_n(x) = 2B_n(x) - 2^{n+1} B_n(x/2)\), i.e. \(G_n = 2 (1 - 2^n) B_n\).

- Euler numbers & polynomials \((|t| < \pi)\)

\[
R(t) := \frac{2}{e^t + e^{-t}} = \sum_{n\geq 0} E_n t^n / n!
\]

\[
R(t, x) := \frac{2e^{xt}}{e^t + 1} = \sum_{n\geq 0} E_n(x) t^n / n!
\]

Where \(G_{n+1}(x) = (n + 1)E_n(x)\) and \(G_{n+1} = \frac{n+1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} E_k (n \geq 0)\).
Bernoulli numbers & polynomials convolution identities

- Euler type convolution identities [Euler, Ramanujan] & [Hansen-1975]

\[
\sum_{k=2}^{n-2} \binom{n}{k} B_k B_{n-k} = -(n + 1)B_n \quad (n \geq 4)
\]

and, for \( n \geq 1, \)

\[
\sum_{k=0}^{n} \binom{n}{k} B_k(x) B_{n-k}(y) = n(x + y - 1)B_{n-1}(x + y) - (n - 1)B_n(x + y)
\]


\[
\sum_{k=2}^{n-2} \frac{B_k B_{n-k}}{k(n - k)} = \sum_{n=2}^{n-2} \binom{n}{k} \frac{B_k B_{n-k}}{k(n - k)} + \frac{2}{n}H_n B_n \quad (n \geq 4)
\]

\[
\sum_{k=1}^{n-1} \frac{B_k(x) B_{n-k}(x)}{k(n - k)} = \frac{2}{n} \sum_{k=0}^{n-1} \binom{n}{k} B_k(x) \frac{B_{n-k}}{n - k}
\]

\[
+ B_{n-1}(x) + \frac{2}{n}H_{n-1} B_n(x) \quad (n \geq 1)
\]
Let \( F(t) := \sum_{n \geq 0} f_n t^n / n! \) and \( G(t) := \sum_{n \geq 0} g_n t^n / n! \) define two EGFs s.t.

\[
F(ut) G((1 - u)t) = \sum_{n \geq 0} h_n(u) t^n / n!, \quad u \in \mathbb{R}.
\]

Equating coefficients of \( t^n / n! \) in both sides and integrating w.r. to \( u \) from 0 to 1 yields

\[
\frac{1}{n+1} \sum_{k=0}^{n} f_k g_{n-k} = \int_{0}^{1} h_n(u) du.
\]

By using the beta integral (see (i) in the next lemma),

\[
\int_{0}^{1} u^k (1 - u)^{n-k} = \frac{\Gamma(k + 1) \Gamma(n - k + 1)}{\Gamma(n + 1)}.
\]
Crabb’s basic ideas for short proof (cont’d)

Lemma 1

Let $H_n$ be the $n$th harmonic number,

\[ i) \quad \int_0^1 u^{n-1}(1-u)^{k-1} = \frac{(n-1)!(k-1)!}{(n+k-1)!} \quad (n, k \geq 1) \]

\[ ii) \quad \frac{1}{2} \int_0^1 \frac{1-u^{n+1}(1-u)^{n+1}}{u(1-u)} = \int_0^1 \frac{1-u^n}{1-u} = H_n \quad (n \geq 1) \]

Functional relations satisfied e.g. by the EGF $P(t, x)$ (of Bernoulli polynomials).

Let $u, v \in \mathbb{R} \setminus \{0\}$ with $u + v \neq 0$, put $\xi := (ux + vy)/(u + v)$

\[ P(ut, x)P(vt, xy) = \frac{uv}{u+v} tP\left((u+v)t, \xi\right) + \frac{v}{u+v} tP\left((u+v)t, \xi\right)P(ut) \]

\[ - \frac{u}{u+v} tP\left((u+v)t, \xi\right)P(vt). \]
Bernoulli polynomial versions of Matiyasevich and Gessel

Set \( y = x \) and \( v = 1 - u (\neq 0) \): \( \xi := (ux + vy)/(u + v) = x \).

Two Bernoulli polynomial versions of Matiyasevich’s [Matiyasevich-1997] and Gessel’s [Gessel-2005] Bernoulli convolutions identities for \( n \geq 1 \):

\[
\sum_{k=1}^{n-1} B_k(x) B_{n-k}(x) = \frac{2}{n-2} \sum_{k=1}^{n-1} \binom{n+2}{k} B_k(x) B_{n-k} + \frac{n(n+1)}{6} B_{n-1}(x) + (n-1)B_n(x)
\]

\[
\sum_{k=1}^{n-1} \frac{B_k(x)}{k} \frac{B_{n-k}(x)}{n-k} = \frac{2}{n} \sum_{k=0}^{n-1} \binom{n}{k} B_k(x) \frac{B_{n-k}}{n-k} + \frac{2}{n} H_{n-1}B_n(x)
\]

For \( x = 0 \) or \( x = 1/2 \), one gets (respectively) Miki’s identity or a variant involving \( \overline{B}_i = ((1 - 2^{i-1})/2^{i-1}) B_i \) \( (i \geq 0) \) [Faber–Pandharipande–Zagier-2000] (FPZ).

The most convenient variants of Miki–Gessel identities [Miki-1978] & [FPZ-2000] take advantage of the definition of the values of Bernoulli numbers: \( B_0 = 1, \ B_1 = -1/2 \) and \( B_n = 0 \) for odd integers \( n \geq 3 \).

\[
\sum_{k=1}^{n-1} \frac{B_{2k}B_{2n-2k}}{2k(2n-2k)} = \sum_{k=1}^{n-1} \binom{2n}{2k} \frac{B_{2k}B_{2n-2k}}{2k(2n-2k)} + \frac{B_{2n}}{n} H_{2n} \quad (n \geq 2)
\]

\[
\sum_{k=1}^{n-1} \frac{\overline{B}_{2k}\overline{B}_{2n-2k}}{2k(2n-2k)} = \frac{1}{n} \sum_{k=1}^{n-1} \binom{2n}{2k} + \frac{B_{2k}\overline{B}_{2n-2k}}{2k} + \frac{\overline{B}_{2n}}{n} H_{2n-1} \quad (n \geq 2).
\]

\( H_j \) denotes the \( j \)th harmonic number \( H_j := \sum_{i=1}^{j} 1/i = \psi(j+1) + \gamma \) (where \( \psi(z) = \Gamma'(z)/\Gamma(z) \) is the digamma function and \( \gamma \) the Euler–Mascheroni constant) and \( \overline{B}_n := \left( \frac{1-2^{n-1}}{2^{n-1}} \right) B_n \).
Simpler proofs by generating functions (GFs)

Dunne–Schubert’s method stems from perturbative quantum field theory at the 2nd order (“two loop” level), where the Bernoulli numbers appear naturally, e.g. through the asymptotics

$$\text{tr} \left( \partial_P^{-2n} \right) \sim \sum_{k \geq 1} \frac{1}{k^{2n}} = \zeta(2n) \quad (n \to \infty) \quad \text{with}$$

$$\zeta(2n) = \frac{(-1)^{n+1}}{2} \frac{(2\pi)^{2n}}{(2n)!} B_{2n} \quad \text{(Euler’s identity)}$$

The spectrum of the ordinary derivative operator $\partial_P$ with periodic boundary conditions indeed consists of the integers. The expansion of the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$ is naturally linked to $\zeta(2n)$ and Miki–Gessel type convolution identities in the form

$$\sum_{k=1}^{n-1} \frac{B_{2k}}{2k} \frac{B_{2n-2k}}{2n-2k}, \quad \sum_{k=1}^{n-1} \frac{\overline{B}_{2k}}{2k} \frac{\overline{B}_{2n-2k}}{2n-2k}, \quad \sum_{k=1}^{n-1} \frac{B_{2k}}{2k} \frac{\overline{B}_{2n-2k}}{2n-2k}, \quad \text{etc.}$$
1 – GF proof for Miki’s convolution identity

Consider the asymptotic expansion of the digamma function \( (z \to \infty) \),

\[
\psi(z) = \ln z - \frac{1}{2z} - \sum_{k=1}^{m} \frac{B_{2k}}{2k} \frac{1}{z^{2k}} + O(z^{-2m}) \quad (|\arg(z)| \leq \pi - \epsilon, \; \epsilon > 0),
\]

and take the GF (renormalization)

\[
\tilde{\psi}(z) := \psi(z) - \ln z + \frac{1}{2z}.
\]

Then,

\[
\tilde{\psi}(z) \sim - \sum_{k \geq 1} \frac{B_{2k}}{2k} \frac{1}{z^{2k}} \quad (z \to \infty)
\]

and

\[
\tilde{\psi}(z)^2 \sim \sum_{n \geq 2} \frac{1}{z^{2n}} \sum_{k=1}^{n-1} \frac{B_{2k}}{2k} \frac{B_{2n-2k}}{2n-2k}.
\]

The Cauchy product \( \tilde{\psi}(z)^2 \) involves naturally the l.h.s. of Miki’s identity.
The r.h.s. of Miki’s identity is shown by comparing \( \tilde{\psi}(z)^2 \) with the square of the integral representation of \( \tilde{\psi}(z) \) (see e.g. [Erđelyi-1981]):

\[
\tilde{\psi}(z) = - \int_{0}^{\infty} \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-zt} dt \quad (| \arg(z) | \leq \pi/2 - \epsilon)
\]

\[
= - \int_{0}^{\infty} \left( \coth t - \frac{1}{t} \right) e^{-2zt} dt.
\]

Then, by squaring,

\[
\tilde{\psi}(z)^2 = \int_{0}^{\infty} e^{-2zy} dy \int_{0}^{1} dx \left\{ -1 + 2 \left( \coth y - \frac{1}{y} \right) \left( \coth xy - \frac{1}{xy} \right) \right.
\]

\[
- \frac{2}{y(1-x)} \left[ x \left( \coth xy - \frac{1}{xy} \right) - \left( \coth y - \frac{1}{y} \right) \right] \left\}
\]

after using trigonometric identity, symmetry and appropriate changes of variables. ■

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2 – GF proof for FPZ’s convolution identity

In view of $B_n$, define $\overline{\psi}(z) := \psi(z + \frac{1}{2}) - \ln z$ instead of $\tilde{\psi}(z)$.

The duplication formula for $\psi$ [Abramowitz–Stegun-72],

$$\psi(2z) = \frac{1}{2} \psi(z) + \frac{1}{2} \psi(z + \frac{1}{2}) + \ln 2,$$

yields the expansion of $\overline{\psi}(z)$ ($z \to \infty$ in $|\arg(z)| \leq \pi - \epsilon, \epsilon > 0$):

$$\overline{\psi}(z) \sim - \sum_{k \geq 1} \frac{B_{2k}}{2k} \frac{1}{z^{2k}} \quad \text{with} \quad \overline{B}_n := \left( \frac{1 - 2^{n-1}}{2^{n-1}} \right) B_n.$$

Along the same lines as in 1., the convolution product $\overline{\psi}(z)^2$ is the GF for the l.h.s. of FPZ’s and the GF for the r.h.s. is obtained by squaring the integral representation of $\overline{\psi}(z)$:

$$\overline{\psi}(z)^2 \sim \sum_{n \geq 2} \frac{1}{z^{2n}} \sum_{k=1}^{n-1} \frac{B_{2k}}{2k} \frac{B_{2n-2k}}{2n-2k}$$

$$= \int_0^\infty \left( \frac{1}{\sinh x} - \frac{1}{x} \right) dx \int_0^\infty \left( \frac{1}{\sinh y} - \frac{1}{y} \right) e^{-2z(x+y)} dy.$$

New crossed Miki–FPZ convolution identity

- Such an identity can be derived from the GF $\tilde{\psi}(z)\overline{\psi}(z)$ by using the above method of summation & integral representation.

- Even simpler way: use the duplication formula for $\psi(z)$,

$$\tilde{\psi}(z) + \overline{\psi}(z) = 2\tilde{\psi}(2z), \quad \text{and square}$$

$$2\tilde{\psi}(z)\overline{\psi}(z) = 4\tilde{\psi}(2z)^2 - \tilde{\psi}(z)^2 - \overline{\psi}(z)^2.$$ 

Asymptotic expansions ($z \to \infty$) of the functions on the l.h.s. and of the squares of their integral representations on the r.h.s. yields a new more symmetrical form of a crossed Miki–FPZ convolution identity for $n \geq 2$ [Dunne–Schubert-2013]:

$$\sum_{k=1}^{n-1} \frac{B_{2k}B_{2n-2k}}{2k(2n - 2k)} = \frac{1}{n} \sum_{k=1}^{n} \binom{2n}{2k} \left( \frac{1 - 2^{2k-1}}{2^{2n-1}} \right) \frac{B_{2k}B_{2n-2k}}{2k} + \frac{1}{n} \frac{B_{2n}H_{2n-1}}{2^{2n}}$$
1 – Generalization of Miki’s identity by deriving $\tilde{\psi}(z)$

The asymptotic expansion ($z \to \infty$) and the integral representation of the $p$th derivative of the GF $\tilde{\psi}(z)$ (with $p \in \mathbb{N}$) is [Dunne–Schubert-2013]

$$\tilde{\psi}(p)(z) \sim (-1)^{p+1} \sum_{n \geq 1} \frac{B_{2n} \Gamma(2n + p)}{2n \Gamma(2n)} \frac{1}{z^{2n+p}}$$

$$\tilde{\psi}(p)(z) = -(-2)^p \int_0^\infty t^p \left( \coth t - \frac{1}{t} \right) e^{-2zt} \, dt.$$

Squaring each $\tilde{\psi}(p)(z)$ and comparing, for any integer $p \geq 0$ and $n \geq 2$

$$\sum_{k=1}^{n-1} \frac{B_{2k}B_{2n-2k}}{2k(2n-2k)} \frac{\Gamma(2k + p)\Gamma(2n - 2k + p)}{\Gamma(2k)\Gamma(2n - 2k)} =$$

$$2\Gamma(p + 1) \sum_{k=1}^{n} \frac{B_{2k}B_{2n-2k}}{(2k)!(2n-2k)!} \frac{\Gamma(2k + p)\Gamma(2n + 2p)}{\Gamma(2p + 2k + 1)}$$

$$+ 2 \frac{B_{2n}\Gamma(2n + 2p)}{(2n)!} \sum_{k=1}^{2n-1} \beta(p + k, p + 1),$$

where $\beta(x, y) := \int_0^1 t^{x-1}(1 - t)^{y-1} \, dt = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ ($\Re(x), \Re(y) > 0$) (Euler beta function).
2 – Generalization of FPZ identity by derivation of $\overline{\psi}(z)$

Similarly for FPZ convolution identity. The $p$th derivative of the GF $\overline{\psi}(z)$ (with $p \in \mathbb{N}$) is

$$
\overline{\psi}^{(p)}(z) \sim (-1)^{p+1} \sum_{n \geq 1} \frac{B_{2n}}{2n} \frac{\Gamma(2n+p)}{\Gamma(2n)} \frac{1}{z^{2n+p}}
$$

$$
\overline{\psi}^{(p)}(z) = -(-2)^p \int_0^\infty t^p \left( \frac{1}{\sinh t} - \frac{1}{t} \right) e^{-2zt} \, dt.
$$

And, for any integer $p \geq 0$ and $n \geq 2$,

$$
\sum_{k=1}^{n-1} \frac{B_{2k}B_{2n-2k}}{2k(2n-2k)} \frac{\Gamma(2k+p)\Gamma(2n-2k+p)}{\Gamma(2k)\Gamma(2n-2k)} = 2\Gamma(p+1) \sum_{k=1}^{n} \frac{B_{2k}B_{2n-2k}}{(2k)!(2n-2k)!} \frac{\Gamma(2k+p)\Gamma(2n+2p)}{\Gamma(2p+2k+1)}
$$

$$
+ 2 \frac{B_{2n}\Gamma(2n+2p)}{(2n)!} \sum_{k=1}^{2n-1} \beta(p+k,p+1).
$$
Differentiating $p$ times the relation between $\tilde{\psi}$ and $\overline{\psi}$ obtained via the duplication formula for $\psi$, gives

$$\tilde{\psi}^{(p)}(z) + \overline{\psi}^{(p)}(z) = 2^{p+1}\tilde{\psi}^{(p)}(2z).$$

Then squaring this last equation brings to a generalized crossed identity for any integer $p \geq 0$ and $n \geq 2$ [Dunne–Schubert-2013],

$$\sum_{k=1}^{n-1} \frac{B_{2k}B_{2n-2k}}{2k(2n-2k)} \frac{\Gamma(2k+p)\Gamma(2n-2k+p)}{\Gamma(2k)\Gamma(2n-2k)} =$$

$$2\Gamma(p+1)\sum_{k=1}^{n} \frac{B_{2k}B_{2n-2k}}{(2k)!(2n-2k)!} \left(1 - \frac{2^{2k-1}}{2^{2n-1}}\right) \frac{\Gamma(2k+p)\Gamma(2n+2p)}{\Gamma(2p+2k+1)}$$

$$+ 2 \frac{B_{2n}\Gamma(2n+2p)}{(2n)!2^{2n-1}} \sum_{k=1}^{2n-1} \beta(p+k,p+1).$$
Comments around the last three generalization results

- When $p = 0$, the Miki’s, FPZ and crossed identities are recovered.
- When $p = 1$, convolution identities involving just the Bernoulli numbers appears on the l.h.s. (for $n \geq 1$):

\[
\sum_{k=1}^{n} B_{2k}B_{2n-2k} = \frac{1}{n+1} \sum_{k=1}^{n} \binom{2n+2}{2k+2} B_{2k}B_{2n-2k} + 2nB_{2n}
\]

\[
\sum_{k=1}^{n} \overline{B}_{2k}\overline{B}_{2n-2k} = \frac{1}{n+1} \sum_{k=1}^{n} \binom{2n+2}{2k+2} B_{2k}\overline{B}_{2n-2k} + 2n\overline{B}_{2n}
\]

\[
\sum_{k=1}^{n} B_{2k}\overline{B}_{2n-2k} = \frac{1}{n+1} \sum_{k=1}^{n} \binom{2n+2}{2k+2} \left( \frac{1 - 2^{2k-1}}{2^{2n-1}} \right) B_{2k}B_{2n-2k} + (2n - 1) \frac{B_{2n}}{2^{2n}}.
\]

- The positiveness condition on $p$ is used only to avoid singularities. Moreover, all relations still hold true for non-integer positive $p$, as can be seen using the integral representations for $\tilde{\psi}$ and $\overline{\psi}$ and the trigonometric identities on $\coth t$ and $\sinh t$ within the integrals.
Gessel [Gessel-2005] shows the existence of a tower of convolution identities with multiple products of Bernoulli numbers, of which Miki’s identity is just the lowest order one.

Dunne–Schubert’s technique of expanding $\tilde{\psi}(z)^N$ (resp. $\overline{\psi}(z)^N$) and rewrite the $N$-fold integral representation provides $(N - 1)$-fold products of Bernoulli numbers for $N$-order Miki’s identity (resp. FPZ’s, with $B_{2k_i}$ replaced by $\overline{B}_{2k_i}$) proves rather easy and efficient.

1 The asymptotic expansion step ($z \to \infty$) generates the $(N - 1)$-fold convolution on the l.h.s. of the identity

$$\tilde{\psi}(z)^N \sim \sum_{n \geq N} \frac{1}{z^{2n}} \sum \prod_{i=1}^{N} \frac{B_{2k_i}}{2k_i}.$$
Higher order Miki–Gessel–FPZ identities (cont’d)

The integral representation step of $\tilde{\psi}(z)^N$ (resp. $\overline{\psi}(z)^N$) generates the $(N - 1)$-fold convolution on the r.h.s. of the identity. After simplifications, the Miki’s identity (resp. FPZ’s) of order $N$ can be carried out...

For example, the generalized Miki–Gessel triple product identity writes for $n \geq 3$,

$$\sum_{k+\ell+m=n \atop k,\ell,m \geq 1} \frac{B_{2k}B_{2\ell}B_{2m}}{(2k)(2\ell)(2m)} = \frac{3}{2n} \sum_{k+\ell+m=n \atop k,\ell,m \geq 1} \binom{2n}{2k,2\ell,2m} \frac{B_{2k}B_{2\ell}B_{2m}}{(2k)(2\ell)}$$

$$+ \frac{3}{n} H_{2n} \sum_{k=1}^{n-1} \binom{2n}{2k} \frac{B_{2k}B_{2n-2k}}{(2k)} + 6H_{2n,2} \frac{B_{2n}}{2n} - \left(n^2 - \frac{3}{2}n + \frac{5}{4}\right) \frac{B_{2n-2}}{(2n-2)},$$

where $H_{2n,2} := \sum_{1 \leq i \leq j \leq 2n} \frac{1}{ij} = \sum_{\ell=1}^{2n-1} \frac{H_{\ell}}{\ell + 1}$.
Symmetrically, the generalized FPZ triple product identity writes for \( n \geq 3 \),

\[
\sum_{k, \ell, m \geq 1} \frac{B_{2k}B_{2\ell}B_{2m}}{(2k)(2\ell)(2m)} = \frac{3}{2n} \sum_{k, \ell, m \geq 1} \binom{2n}{2k, 2\ell, 2m} \frac{B_{2k}B_{2\ell}B_{2m}}{(2k)(2\ell)} \\
+ \frac{3}{n} H_{2n} \sum_{k=1}^{n-1} \binom{2n}{2k} \frac{B_{2k}B_{2n-2k}}{(2k)} \binom{2n}{2k} \frac{B_{2k}}{2k} \left( B_{2n-2k} - B_{2n-2k} \right) \\
+ \frac{3}{2n^2} H_{2n-1} \left( B_{2n} - B_{2n} \right) + 6H_{2n,2} \frac{B_{2n}}{2n} - \frac{2n-1}{4} B_{2n-2}.
\]
Conclusion and further results

- Dunne–Schubert’s GFs method helps derive convolution identities for Bernoulli (resp. Euler or Gennochi) numbers (or polynomials) at any order with little effort. Consider for instance the GF

\[ g(z) = \int_0^\infty \text{sech} \, s \, e^{-2zs} \, ds \sim \sum_{n \geq 0} \frac{E_{2n}}{(2z)^{2n+1}}. \]

It leads to

\[ g(z)^2 = 2 \int_0^\infty \frac{(\ln \cosh t)}{\sinh t} \, e^{-2zt} \, dt, \]

whereby the identity

\[ \sum_{k=1}^{n} E_{2k-2} E_{2n-2k} = \frac{2}{n} \sum_{k=1}^{n} \binom{2n}{2k} 2^{2k-1} \left( 2^{2k} - 1 \right) \left( 1 - 2^{2n-2k-1} \right) \frac{B_{2k} B_{2n-2k}}{k}. \]

- Several new mixed-crossed identities of high order can also be found by using other GFs, e.g. modification of the GFs \( \tilde{\psi}, \psi, \) etc.

- Generalization to families of (bi)-multivariate multiple convolution identities for Bernoulli, Euler, Gennochi, etc. polynomials at any order and tight links with \( L \) zeta functions (Hurwitz’s, Lerch’s, polyzetas, etc.) strengthen the case of an ubiquitous role of Bernoulli numbers/polynomials.