Towards a theory of computational complexity for concurrent processes

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From functions to behaviors

- The original perspective of recursion theory:

  a program computes a function.

- Albeit enormously complex, an operating system is still a program. What function does it compute?

- The more modern perspective of concurrency theory:

  a program has a behavior.

Formalizing behaviors: labelled transition systems

- A widely accepted way of formalizing the notion of "behavior" is that of *labelled transition system*.

- Labels are of the form

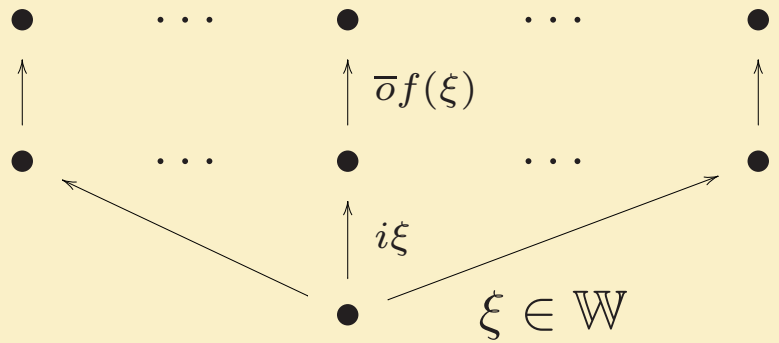
  – $i\xi$ (input action)
  – $\overline{o}\xi$ (output action)
  – τ (internal action)

  where i, o range over *channels* and ξ may in principle belong to any set. We restrict to $\xi \in \mathbb{W} = \{0,1\}^*$. As usual, $|\xi|$ is the length of ξ.

- We formally define a *behavior* as an equivalence class (with respect to *coupled similarity*) of labelled transition systems.

Functions as behaviors

- Of course every function $f : \mathbb{W} \to \mathbb{W}$ may be seen as a behavior:



  which we may write in CCS notation as $i(x).\overline{o}\langle f(x)\rangle$.

- Needless to say, this is far from exhausting all possible behaviors. . .

Computing behaviors

- Not every function is computable; of course, neither is every behavior.
- Process calculi may be used to define a notion of *effective behavior*, in analogy with recursive functions.
- However, process calculi are not quite *computational models*. In particular, they lack a clear *resource semantics*, or *cost model*.
- Our proposal is to define a sensible *process machine*.

A process calculus

- String and boolean expressions ($x \in \mathcal{V}$ and $\xi \in \mathbb{W}$):

$$\begin{array}{rcl} E & ::= & x \mid \xi \mid \mathsf{0}(E) \mid \mathsf{1}(E) \mid \mathrm{tail}(E) \\ B & ::= & \mathrm{tt} \mid \mathrm{ff} \mid \mathsf{0}_?(E) \mid \varepsilon_?(E). \end{array}$$

- Processes (O (resp. I) is either E or an output (resp. input) channel):

$$P, Q \quad ::= \quad \mathbf{0} \mid A\langle E_1, \ldots, E_n \rangle \mid \overline{O}\langle E \rangle.P \mid I(E).P \mid B.(P, Q) \mid P \mid Q,$$

  where A ranges over *process variables*, each with a definition of the form

$$A(x_1, \ldots, x_n) \stackrel{\text{def}}{=} P.$$

The process machine

- A finite, read-only *code memory*, where a finite sequence of definitions $A_0 \stackrel{\text{def}}{=} P_0, A_1(\vec{x}_1) \stackrel{\text{def}}{=} P_1, \ldots, A_n(\vec{x}_n) \stackrel{\text{def}}{=} P_n$ is stored;

- an unbounded number of *processors*, each with a *pointer register*, pointing to the code memory, and an unbounded *private memory* in which assignments of the form $x := \xi$ are stored;

- an unbounded number of *internal channels*, each identified by a string and with an unbounded capacity to queue up strings, through which processors may communicate;

- an *interface*, with separate input and output channels, from which strings may be read/sent from/to the external world.

States

- We may represent the state of a processor of the machine by a pair (P, M) where P is a process and M a finite function $\mathcal{V} \to \mathbb{W}$.

- The state of channels may be represented by a finite *queue function* Θ from $\mathbb{W}$ to finite sequences of $\mathbb{W}$.

- Then, a *state* of the machine is represented by a pair

$$[\Gamma]\Theta$$

  where $\Gamma = (P_1, M_1), \ldots, (P_n, M_n)$ is a finite multiset of processor states (the active processors) and Θ is a queue function.

Executing processes (1)

Nil: $[(\mathbf{0}, M), \Gamma]\Theta \xrightarrow{\tau} [\Gamma]\Theta$;

Rec: if $A(x_1, \ldots, x_n) \stackrel{\text{def}}{=} P$,

$$[(A\langle E_1, \ldots, E_n\rangle, M), \Gamma]\Theta \xrightarrow{\tau} [(P, \{x_1 \mapsto E_1^M, \ldots, x_n \mapsto E_n^M\}), \Gamma]\Theta;$$

Snd: $[(\overline{E_1}\langle E_2\rangle.P, M), \Gamma]\Theta \xrightarrow{\tau} [(P, M), \Gamma]\Theta'$,
where Θ' is equal to Θ everywhere except in E_1^M, where it is equal to $\Theta(E_1^M) \cdot E_2^M$;

Rcv: $[(E(x).P, M), \Gamma]\Theta \xrightarrow{\tau} [(P, M \cup \{x \mapsto \xi\}), \Gamma]\Theta'$,
where we must have $\Theta(E^M) = \xi \cdot q$, and then Θ' is equal to Θ everywhere except in E^M, where it is equal to q;

Executing processes (2)

Out: $[(\overline{o}\langle E\rangle.P, M), \Gamma]\Theta \xrightarrow{\overline{o}E^M} [(P, M), \Gamma]\Theta$;

Inp: $[(i(x).P, M), \Gamma]\Theta \xrightarrow{i\xi} [(P, M \cup \{x := \xi\}), \Gamma]\Theta$;

Cnd: $[(B.(P, Q), M), \Gamma] : \Theta \xrightarrow{\tau} \begin{cases} [(P, M), \Gamma] : \Theta, & \text{if } B^M = \mathrm{tt} \\ [(Q, M), \Gamma] : \Theta, & \text{if } B^M = \mathrm{ff} \end{cases}$

Spn: $[(P \mid Q, M), \Gamma] : \Theta \xrightarrow{\tau} [(P, M), (Q, M), \Gamma] : \Theta$.

Traces

- Observe how, given a process P, the transitions of the machine from $[(P, \emptyset)]\emptyset$ define a labelled transition system $[P]$. With this, we may define effective behaviors.

- Moreover, we may define a *trace* as a sequence of transitions.

- Two transitions are *orthogonal* if they are not both *send*, *receive*, *output*, or *input* transitions on the same channel.

- Orthogonal transitions *commute*, defining an equivalence relation $\sim$ on traces. This will be essential for defining the *temporal* cost model of our machine.

Causality

- Trace equivalence yields an *event structure*. In particular, we have a pre-order $\lesssim$ between traces ($f \lesssim g$ if $g \sim fg'$), which becomes a *causal order* $\leq$ between equivalence classes of traces.

- An equivalence class represents a transition t when all of its traces are of the form ft and, for every trace g, $g \sim ft$ implies $g = g't$ with $g' \sim f$.

- Given a transition t, we may speak of the transitions which are *causally necessary* for it to occur, *i.e.*, $t{\downarrow} = \{t' \mid t' \leq t\}$.

What is space?

- Let $[(P_1, M_1), \ldots, (P_n, M_n)]\Theta$ be a state of the machine. Its *size* is the sum of the sizes of all strings appearing in the codomain of $M_1, \ldots, M_n$ and Θ (counting multiplicities).

- If f is a trace, we define $\mathrm{space}(f)$ to be the maximum size of the states visited by f.

- $f \sim f'$ implies $\mathrm{space}(f) = \mathrm{space}(f')$.

- Therefore, the *space cost* $\mathrm{space}(t)$ of a transition t is uniquely defined as $\mathrm{space}(ft)$, where ft is any trace in the equivalence class representing t.

What is time?

- Each transition t is assigned a *weight* $\$t$:

  Nil: constant;
  Rec: the time it takes to evaluate the expressions, plus a constant;
  Snd,Rcv,Out,Inp: the time it takes to evaluate the expressions, plus the length of the string sent/received, plus a constant;
  Cnd: the time it takes to evaluate the Boolean expression, plus a constant;
  Spn: the time it takes to duplicate the memory, plus a constant.

- The *time cost* of a transition t is

$$\mathrm{time}(t) = \max_{C \in \mathrm{tot}(t\!\downarrow)} \sum_{t' \in C} \$t'.$$

  where $\mathrm{tot}(t\!\downarrow)$ denotes the set of all chains below t.

Input size

- Let t be a transition. We define $\mathrm{inp}(t)$ to be the subset of $t{\downarrow}$ consisting all transitions whose label is of the form $i\xi$. We say that the *size* of such an input transition t' is $|\xi|$, and we denote it by $|t'|$.

- We define the *input size* of a transition t as

$$\|t\| = \sum_{t' \in \mathrm{inp}(t)} |t'|.$$

Complexity classes

- Let f, g be proper complexity functions.
- We define $\mathbf{BTS}(f, g)$ as the set of behaviors $\mathfrak{b}$ such that:
  - there exists a process P such that $[P] \in \mathfrak{b}$;
  - for every output transition t generated in the execution of P,

$$\begin{aligned} \mathrm{time}(t) &\leq f(\|t\|), \\ \mathrm{space}(t) &\leq g(\|t\|). \end{aligned}$$

Sanity check: the functional case

- We denote by FUN the set of functional behaviors. If $\mathfrak{b} \in \mathrm{FUN}$, we may associate with it a language (subset of $\mathbb{W}$) $\mathrm{lang}\mathfrak{b}$. Then, we define

$$\mathbf{FUNTS}(f,g) = \mathrm{lang}(\mathbf{BTS}(f,g) \cap \mathrm{FUN}).$$

- Some results:
  - $\mathbf{TIME}(f(n)) \subseteq \mathbf{FUNTS}(f(n), f(n))$;
  - $\mathbf{FUNTS}(f(n), g(n)) \subseteq \mathbf{TIME}(f(n)(g(n)+n)^2)$;
  - $\mathbf{ATIME}(f(n)) \subseteq \mathbf{FUNTS}(f(n), 2^{f(n)})$.

- These imply:
  - $\mathbf{P} = \bigcup_{k<\omega} \mathbf{FUNTS}(n^k, n^k)$, $\mathbf{EXP} = \bigcup_{k<\omega} \mathbf{FUNTS}(2^{n^k}, 2^{n^k})\ldots$
  - $\mathbf{PSPACE} \subseteq \bigcup_{k<\omega} \mathbf{FUNTS}(n^k, 2^{n^k})$.

Perspectives

- The relationship with standard (functional) complexity classes must be explored further. We know that, at present, we can only hope to capture superlinear time and space (*i.e.*, we cannot capture $\mathbf{NC}$ or $\mathbf{L}$).

- However, our framework is extremely flexible; we have used very little of it for the moment. For instance, for what concerns extra-functional behaviors, we know that the way the complexity is to be measured must be given *with the behavior*.

- This allows to capture some interesting extra-functional behaviors, such as *servers* and *streams*.