Errata Corrige for "Non-Linearity as the Metric Completion of Linearity"

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The Metric d Does Not Yield the Uniformity $\mathcal{FB}$

The paper [Maz13] mentioned in the title contains a false claim: the metric d introduced at the beginning of Sect. 3 *does not* induce the uniformity of uniform convergence on finitely branching trees (call it $\mathcal{FB}$). The rest of the paper should be read with this latter uniform structure in mind, disregarding the metric d. For instance, when the paper invokes the completion of the *metric space* $(\Lambda_{\mathrm{p}}, d)$, one should consider the completion of the *uniform space* $(\Lambda_{\mathrm{p}}, \mathcal{FB})$.

The metric d alone does not yield the desired results because the completion of $(\Lambda_{\mathrm{p}}, d)$ introduces terms which, as trees, are not well-founded. As an example, take any finite term u and consider the sequence obtained by setting, for $n \in \mathbb{N}$,

$$t_0 := u\langle\rangle,$$
$$t_{n+1} := u\langle\overbrace{t_n, \dots, t_n}^{n+1}\rangle.$$

A straightforward induction on n shows that, for all $p > 0$, $d(t_n, t_{n+p}) = 2^{-n-1}$, so the sequence is Cauchy. But this sequence tends to the non-well-founded infinitary term T verifying the equation

$$T = u\langle T, T, T, \dots\rangle.$$

In fact, the metric d is similar to the 001-metric of [KKSdV97]. If one considers the quotient of the completion of $(\Lambda_{\mathrm{p}}, d)$ with respect to the partial equivalence relation introduced later in [Maz13] (Definition 1), one obtains exactly the infinitary λ-calculus Λ^{001} of [KKSdV97], whose normal forms correspond to Böhm trees.

The metric d is already used in my first paper on the infinitary affine λ-calculus [Maz12], in combination with the "height pseudometric" ρ which is defined by setting $\rho(t, t') = 1$ if t, t' have different height and $\rho(t, t') = 0$ otherwise. Indeed, the paper [Maz12] uses the metric $\max(d, \rho)$, which eliminates non-well-foundedness "drastically", because Cauchy sequences w.r.t. that metric necessarily consist of terms which have ultimately the same height, so the completion of $(\Lambda_{\mathrm{p}}, \max(d, \rho))$ only contains terms of finite height (but whose width may be infinite). However, as observed in the errata corrige of that paper (available on my web page), β-reduction in the space $(\Lambda_{\mathrm{p}}, \max(d,\rho))$ does not have the property of being Cauchy-continuous. Although this does not compromise the key result motivating my work (the isomorphism theorem, *i.e.*, that the full λ-calculus may be seen as a quotient of the completion of the affine λ-calculus, according to a certain metric), Cauchy-continuity of β-reduction seems to be a natural property to ask. This is what pushed me to find an alternative metric according to which β-reduction is Cauchy-continuous and the isomorphism theorem still holds. The uniformity $\mathcal{FB}$, although not a metric, is a solution to this problem and I mistakenly thought that the metric d alone (without being "maxed" with ρ) induced precisely $\mathcal{FB}$.

The Uniformity $\mathcal{FB}$ is not Metrizable

It is interesting to observe that the mismatch between the metric d and the uniformity $\mathcal{FB}$ is essential, because it turns out that the latter is not metrizable. Embarrassingly enough, this goes against the very title of the paper: according to the given definitions, non-linearity is the *Hausdorff completion* (in the sense of uniform spaces) of linearity, not the *metric* completion. Of course, the paper [Maz12] shows that there exists a metric according to which the title of the paper is correct (the metric $\max(d,\rho)$ discussed above), but it seems that this is incompatible with β-reduction being topologically well-behaved.

That $\mathcal{FB}$ is not metrizable may be seen by showing that its topology is not first-countable. First-countability actually fails in a strong way: *no* term admits a countable basis of neighborhoods. We remind that an open basis for the topology induced by $\mathcal{FB}$ is defined by the sets of the form

$$V_A(t) := \{t' \mid \forall a \in A,\, t'(a) = t(a)\},$$

where t is an arbitrary term and $A \in \mathrm{FBT}$. In particular, the family $(V_A(t))_{A\in\mathrm{FBT}}$ is a neighborhood basis of t (uncountable, of course).

Let t be a term and let $(U_i)_{i\in\mathbb{N}}$ be a countable family of open neighborhoods of t. We will show that there exists an open neighborhood V of t that is not generated by the family, *i.e.*, such that $U_i \not\subseteq V$ for all $i \in \mathbb{N}$.

We start by observing that, by virtue of $(V_A(t))_{A\in\mathrm{FBT}}$ being a neighborhood basis of t, we have a family $(B_i)_{i\in\mathbb{N}}$ of finitely branching trees such that $V_{B_i}(t) \subseteq U_i$ for all $i \in \mathbb{N}$. Since each B_i is finitely branching, there certainly exists $m_i \in \mathbb{N}$ such that, for all $a \in B_i$ and $j \in \mathbb{N}$, $|a| = i$ and $j \geq m_i$ imply $a \cdot j \notin B_i$ (we denote by $|a|$ the length of the sequence a and by $a \cdot j$ the sequence obtained by adjoining j to a). In other words, m_i strictly bounds the "maximum branching index" of the immediate descendants of B_i at level i (the root being at level 0). We define $B \in \mathrm{FBT}$ to be such that each node at level i has exactly $m_i + 1$ immediate descendants (indexed by $0, \ldots, m_i$).

Now, it is not hard to define a sequence of terms $(t_i)_{i\in\mathbb{N}}$ such that, for all $i \in \mathbb{N}$:

- $t_i(b) = t(b)$ for all $b \in B_i$;
- there exists $a \in \mathbb{N}^*$ such that $|a| = i$ and $t_i(a \cdot m_i) \neq t(a \cdot m_i)$.

In other words, t_i and t coincide on B_i but differ on a position which is the immediate successor of a node of level i of branching index m_i.

By construction, $V_B(t)$ is the open neighborhood V we were seeking: it is an open neighborhood of t and yet, by definition, for all $i \in \mathbb{N}$ we have $U_i \not\subseteq V_B(t)$, because $t_i \in V_{B_i}(t) \setminus V_B(t)$.

References

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