Heterodox Exponential Modalities
in Linear Logic

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The Perfect World (or, Linear Logic without Exponential Modalities)

**classical**

\(\otimes, 1, \exists, \bot\)

\(&, \top, \oplus, 0\)

*-autonomous categories
with fin. products
(e.g. Vect\(_k\))

**intuitionistic**

\(\otimes, 1, \rightarrow\)

\(&, \top, \oplus, 0\)

symmetric closed monoidal cats
with fin. prods and fin. coprods
(e.g. CMon)

Everything is decidable:

- the space of proof search is finite;
- the size of proofs shrinks under cut-elimination (not quite in MALL...).

<table>
<thead>
<tr>
<th>provability</th>
<th>(untyped) cut-elimination</th>
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<tbody>
<tr>
<td>MLL</td>
<td>NP-complete</td>
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<tr>
<td>MALL</td>
<td>PSPACE-complete</td>
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<tr>
<td></td>
<td>P-complete</td>
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<tr>
<td></td>
<td>coNP-complete</td>
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Imperfection (or, Orthodox Exponential Modalities)

\[
\begin{array}{c}
\text{cat with fin. prods} & \xrightarrow{\text{strong}} & M & \xleftarrow{\text{lax symm. mon.}} & L & \text{model of (l)M(A)LL}
\end{array}
\]

Examples:
- \((-)^* : \text{Set} \rightleftarrows \text{CMon : } U\)
- \(U : \text{Set} \rightleftarrows \text{Rel : } P\)

Infinity steps in:

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<tr>
<td>MELL</td>
<td>???</td>
<td>(undecidable) non-elementary</td>
</tr>
<tr>
<td>LL</td>
<td>undecidable</td>
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Not so “God-given”:
- who had heard of LNL adjunctions before linear logic?
- Not determined by \(\otimes\) (consider \(U : M\text{Rel} \rightleftarrows \text{Rel : } M_{\text{fin}}\))
An Alternative Presentation of Linear Logic

Sequents with an “exponential part”: \( \vdash \Theta; \Gamma \)

\[
\begin{align*}
\vdash \Theta; A^\perp, A & \quad \vdash \Theta; \Gamma, A^\perp & \quad \vdash \Theta; \Delta, A \\
\vdash \Theta; \Gamma, \Delta & \\
\vdash \Theta; \Gamma, A & \quad \vdash \Theta; \Gamma, \Delta, B & \quad \vdash \Theta; \Gamma, \perp \\
\vdash \Theta; \Gamma, A \otimes B & \quad \vdash \Theta; \Gamma, A, B & \quad \vdash \Theta; \Gamma, A \parr B \\
\vdash \Theta; \Gamma, \top & \quad \vdash \Theta; \Gamma, A & \quad \vdash \Theta; \Gamma, B \\
\vdash \Theta; \Gamma, A & \quad \vdash \Theta; \Gamma, A \& B & \quad \vdash \Theta; \Gamma, A_i & \quad \vdash \Theta; \Gamma, A_1 \oplus A_2 \\
& \quad i \in \{1, 2\} \\
\vdash \Theta; A & \quad \vdash \Theta, A; \Gamma, A & \quad \vdash \Theta; A; \Gamma \\
\vdash \Theta, !A & \quad \vdash \Theta; \Gamma, ?A
\end{align*}
\]

(First considered by Andreoli for proof search).
The Polynomial Structure of Exponential Modalities

Decorate exponential part with $P_i \in \mathbb{N}[X]$: $\vdash P_1 \cdot A_1, \ldots, P_n \cdot A_n; \Gamma$

$\vdash \vec{0} \cdot \Theta; A^\perp, A$

$\vdash \vec{P} \cdot \Theta; \Gamma, A^\perp \vdash \vec{Q} \cdot \Theta; \Delta, A$

$\vdash \vec{P} \cdot \Theta; \Gamma, A \vdash \vec{Q} \cdot \Theta; \Delta, B$

$\vdash \vec{P} + \vec{Q} \cdot \Theta; \Gamma, \Delta, A \otimes B$

$\vdash \Theta; \Gamma \vdash \Theta; \Gamma, \perp \vdash \Theta; \Gamma, A \otimes B$

Making structure explicit yields graded modalities (bounded LL & co.).
A Family of Heterodox Exponential Modalities

We obtain a subsystem of LL by restricting the shape of $P$ in

$$
\vdash \Theta, P \cdot A; \Gamma
\vdash \Theta; \Gamma, \, ?A
$$

**Theorem.** For every submonoid $M$ of $(\mathbb{N}[X], \circ, X)$, the subsystem of LL defined by restricting the above rule to $P \in M$ enjoys $\eta$-expansion and cut-elimination (also, $!(-)$ is always lax monoidal). Moreover, if we define

\[
\begin{align*}
0(A) & := 1 \\
1(A) & := A \\
(P + Q)(A) & := P(A) \otimes Q(A) \\
(PQ)(A) & := P(Q(A)) \\
X(A) & := !A
\end{align*}
\]

then $P \in M$ implies $!A \rightarrow P(A)$ provable in the subsystem.
Examples of Systems with Heterodox Modalities

LL
- 4LL
  - \{\deg P \geq 1 \text{ or } P = 0\}
- TLL
  - \{\deg P \leq 1\}
- ELL
  - \{nX\}
- PLL
  - \{aX + n \text{ with } a \in \{0, 1\}\}
- LLL
  - \{X\} \cup \mathbb{N}
- "light logics"
- parsimonious logic

\(\neg\) not monoidal
Main Properties

- **4LL, TLL**: [Danos, Joinet 2003]. Stream computation in 4LL [Dal Lago 2016].

- **Light logics**: enjoy untyped normalization.
  - **ELL**: [Girard 1998] [Danos, Joinet 2003] characterizes elementary time.
  - **SLL**: [Lafont 2004] characterizes polynomial time.
  - **LLL**: [Girard 1998] [Danos, Joinet 2003] characterizes polynomial time.

- **PLL**: [M. 2014] Turing-complete if untyped. With $!A \equiv A \otimes !A$:
  - **propositional**: characterizes logspace [M. 2015];
  - **linear 2nd order**: characterizes polytime [M. and Terui 2015].

- Two different approaches to control complexity:
  - **stratification** (light logics) vs. **local control** (parsimony);
  - parsimony enables *non-uniform complexity* via approximations.
Characterizing Complexity Classes: What and How

Typical Theorem. For some types Str and Bool, terms of type
Str → Bool
decide exactly the problems in the complexity class C.

Typical proof.

Soundness: (decidable by a term implies in C)
Find a parameter \( d \) such that:
- terms of size \( s \) and parameter \( d \) normalize in \( O(f(d, s)) \) time/space;
- terms of type Str have constant parameter \( d \) and size \( O(n) \) where \( n \) is the
  length of the represented string;
- for constant \( k \), the bound \( O(f(k, n)) \) ensures membership in C.

The proof may be combinatorial or semantic.
For light logics, \( d \) does not depend on the type of the term.
For logspace, use the GoI (normalization via traveling pointers).

Completeness: (in C implies decidable by a term)
A programming exercise (maybe non-trivial). □
Approximations (or, Exponential Modalities are Limits)

Relation $t \sqsubseteq M$ between simple programs and programs (and between simple computations and computations) with cost maps

Such that

$$u \sqsubseteq u$$

iff

$$M \xrightarrow{\rho} N \quad M \xrightarrow{\rho} N$$

$$c_1(\rho) = c_0(t)$$
Conclusions

- Light logics are dead, long live heterodox exponentials!
- Categorical models?
- Limit constructions and approximations?
- Where do approximations come from?