Asymptotics for connected graphs and irreducible tournaments

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Abstract. We compute the whole asymptotic expansion of the probability that a large uniform labeled graph is connected, and of the probability that a large uniform labeled tournament is irreducible. In both cases, we provide a combinatorial interpretation of the involved coefficients.

1 Introduction

Let us consider the Erdös-Rényi model of random graphs $G(n, 1/2)$, where for each integer $n \geqslant 0$, we endow the set of undirected simple graphs on the set $\{1, \ldots, n\}$ with the uniform probability: each graph appears with probability $1/2^{\binom{n}{2}}$. The probability p_n that such a random graph of size n is connected goes to 1 as n goes to ∞. In 1959, Gilbert [2] provided a more accurate estimation and proved that

$$p_n = 1 - \frac{2n}{2^n} + O\left(\frac{n^2}{2^{3n/2}}\right).$$

In 1970, Wright [8] computed the first four terms of the asymptotic expansion of this probability:

$$p_n = 1 - \binom{n}{1}\frac{1}{2^{n-1}} - 2\binom{n}{3}\frac{1}{2^{3n-6}} - 24\binom{n}{4}\frac{1}{2^{4n-10}} + O\left(\frac{n^5}{2^{5n}}\right).$$

The method can be used to compute more terms, one after another. However, it does not allow to provide the structure of the whole asymptotic expansion, since no interpretation is given to the coefficients $1, 2, 24, \ldots$.

The first goal of this paper is to provide such a structure: the kth term of the asymptotic expansion of p_n is of the form

$$i_k \; 2^{k(k+1)/2} \binom{n}{k} \frac{1}{2^{kn}},$$

where i_k counts the number of irreducible labeled tournaments of size k. A tournament is said *irreducible* if for every partition $A \sqcup B$ of the set of vertices there exist an edge from A to B and an edge from B to A. Equivalently, a tournament is irreducible if, and only if, it is strongly connected [6][7].

Theorem 1 (Connected graphs). *For any positive integer r, the probability p_n that a random graph of size n is connected satisfies*

$$p_n = 1 - \sum_{k=1}^{r-1} i_k \binom{n}{k} \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where i_k is the number of irreducible labeled tournaments of size k.

In particular, as there are no irreducible tournament of size 2, this explains why there is no term in $\binom{n}{2}\frac{1}{2^{2n}}$ in Wright's formula. This result might look surprising as it relates asymptotics of undirected objects with directed ones.

A similar development happened for irreducible tournaments. For $n \geqslant 0$, we endow the set of tournaments on the set $\{1, \ldots, n\}$ with the uniform probability: each tournament appears with probability $1/2^{\binom{n}{2}}$. In 1962, Moon and Moser [5] gave a first estimation of the probability q_n that a labeled tournament of size n is irreducible, which was improved in [4] into

$$q_n = 1 - \frac{n}{2^{n-2}} + O\left(\frac{n^2}{2^{2n}}\right).$$

In 1970, Wright [9] computed the first four terms of the asymptotic expansion of the probability that a labeled tournament is irreducible:

$$q_n = 1 - \binom{n}{1} 2^{2-n} + \binom{n}{2} 2^{4-2n} - \binom{n}{3} 2^{8-3n} - \binom{n}{4} 2^{15-4n} + O\left(n^5 2^{-5n}\right).$$

Here again, we provide the whole structure of the asymptotic expansion, together with a combinatorial interpretation of the coefficients (they are not all powers of two):

Theorem 2 (Irreducible tournaments). *For any positive integer r, the probability q_n that a random labeled tournament of size n is irreducible satisfies*

$$q_n = 1 - \sum_{k=1}^{r-1} \left(2i_k - i_k^{(2)}\right) \binom{n}{k} \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $i_k^{(2)}$ is the number of labeled tournaments of size k with two irreducible components.

We can notice that the coefficients cannot be interpreted as counting a single class of combinatorial objects, since the coefficient $2i_2 - i_2^{(2)} = 0 - 2$ is negative.

2 Notations, strategy and tools

Let us denote, for every integer n, g_n the number of labeled graphs of size n, c_n the number of connected labeled graphs of size n, t_n the number of labeled tournaments of size n, and i_n the number of irreducible labeled tournaments of size n. We have $p_n = c_n/g_n$ and $q_n = i_n/t_n$.

Looking for a proof of Theorem 1, we see two issues: finding a formal relation between connected graphs and irreducible tournaments, and proving the convergence. A tool to settle the first issue is the symbolic method: we associate to each integer sequence its exponential generating function:

$$G(z) = \sum_{n=0}^{\infty} g_n \frac{z^n}{n!}, \quad C(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}, \quad T(z) = \sum_{n=0}^{\infty} t_n \frac{z^n}{n!}, \quad I(z) = \sum_{n=0}^{\infty} i_n \frac{z^n}{n!}.$$

Since $g_n = t_n = 2^{\binom{n}{2}}$, we have

$$G(z) = T(z) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{z^n}{n!}. \tag{1}$$

Note that, while the number of labeled tournaments of size n is equal to the number of labeled graphs of size n, their associated species are not isomorphic: for $n = 2$, the two labeled tournaments are isomorphic (by swapping the vertices), while the two labeled graphs are not, so this equality is somewhat artificial.

Since every labeled graph can be uniquely decomposed as a disjoint union of connected labeled graphs, we have

$$G(z) = \exp(C(z)). \tag{2}$$

It remains to find a relation between tournaments and irreducible tournaments.

Lemma 1. *Any tournament can be uniquely decomposed into a sequence of irreducible tournaments.*

In terms of generating functions, Lemma 1 translates to

$$T(z) = \frac{1}{1 - I(z)}. \tag{3}$$

Hence, part of the work will be to let those expressions interplay.

Regarding asymptotics, we will rely on Bender's Theorem [1]:

Theorem 3 (Bender). *Consider a formal power series*

$$A(z) = \sum_{n=1}^{\infty} a_n z^n$$

and a function $F(x, y)$ *which is analytic in some neighborhood of* $(0, 0)$. *Define*

$$B(z) = \sum_{n=1}^{\infty} b_n z^n = F(z, A(z)) \qquad \text{and} \qquad D(z) = \sum_{n=1}^{\infty} d_n z^n = \frac{\partial F}{\partial y}(z, A(z)),$$

Assume that $a_n \neq 0$ for all $n \in \mathbb{N}$, and that for some integer $r \geqslant 1$ we have

(i) $\displaystyle \frac{a_{n-1}}{a_n} \to 0 \text{ as } n \to \infty;$ *(ii)* $\displaystyle \sum_{k=r}^{n-r} |a_k a_{n-k}| = O(a_{n-r}) \text{ as } n \to \infty.$

Then

$$b_n = \sum_{k=0}^{r-1} d_k a_{n-k} + O(a_{n-r}).$$

3 Proofs

Proof (Proof of Lemma 1).
Let T be a tournament. It is either irreducible, and all is done, or it consists of two nonempty parts A and B such that all edges between A and B are directed from A to B. Applying the same argumentation recursively to A and B, we obtain a decomposition of T into a sequence of subtournaments $T_1, \ldots, T_k$, such that each T_i is irreducible and for every pair $i < j$, all edges go from T_i to T_j (see Figure 1). Since T_i are also the strongly connected components of T, the decomposition is unique.

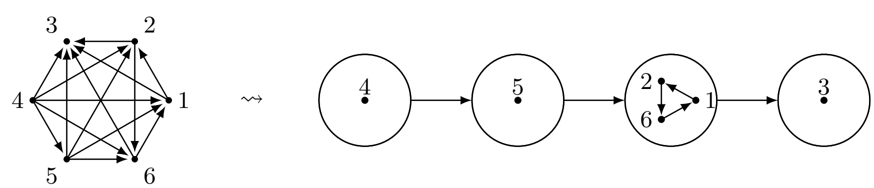

Fig. 1. Decomposition of a tournament as a sequence of irreducible components.

Proof (Proof of Theorem 1).
Let us apply Bender's theorem (Theorem 3) the following way: take

$$A(z) = G(z) - 1 \qquad \text{and} \qquad F(z, w) = \ln(1 + w).$$

Then, in accordance with formulas (1), (2) and (3),

$$B(z) = \ln(G(z)) = C(z) \qquad \text{and} \qquad D(z) = \frac{1}{G(z)} = \frac{1}{T(z)} = 1 - I(z).$$

Check the conditions of Theorem 3. In the case at hand, condition (i) has the form:

$$\frac{a_{n-1}}{a_n} = \frac{2^{\binom{n-1}{2}}}{(n-1)!} \frac{n!}{2^{\binom{n}{2}}} = \frac{n}{2^{n-1}} \to 0, \qquad \text{as} \quad n \to \infty.$$

To establish condition (ii), consider $x_k = n!a_k a_{n-k} = \binom{n}{k} 2^{\binom{k}{2}+\binom{n-k}{2}}$, where $r \leqslant k \leqslant n-r$. Then (x_k) decreases for $r \leqslant k \leqslant n/2$ and increases symmetrically for $n/2 \leqslant k \leqslant n-r$. Bounding each summand by the first term is not enough, but bounding each summand (except for the first and last) by the second term gives the following:

$$\sum_{k=r}^{n-r} a_k a_{n-k} \leqslant \frac{1}{n!}\binom{n}{r} 2^{\binom{r}{2}+\binom{n-r}{2}+1} + \frac{n-2r-1}{n!}\binom{n}{r+1} 2^{\binom{r+1}{2}+\binom{n-r-1}{2}}$$
$$= O\left(\frac{2^{\left(n^2-(2r+1)n\right)/2}}{(n-r)!}\right) + O\left(\frac{2^{\left(n^2-(2r+3)n\right)/2}}{(n-r-2)!}\right) = O(a_{n-r}).$$

Hence, Bender's theorem implies

$$b_n = \frac{c_n}{n!} = \frac{2^{\binom{n}{2}}}{n!} - \sum_{k=1}^{r-1} \frac{i_k}{k!} \frac{2^{\binom{n-k}{2}}}{(n-k)!} + O\left(\frac{2^{\binom{n-r}{2}}}{(n-r)!}\right).$$

Dividing by $g_n/n! = 2^{\binom{n}{2}}/n!$, we get

$$p_n = \frac{c_n}{g_n} = 1 - \sum_{k=1}^{r-1} i_k \binom{n}{k} \frac{2^{\binom{n-k}{2}}}{2^{\binom{n}{2}}} + O\left(\frac{n^r}{2^{nr}}\right).$$

Proof (Proof of Theorem 2).

Let us apply Bender's theorem (Theorem 3) for

$$A(z) = T(z) - 1 \qquad \text{and} \qquad F(z,w) = -\frac{1}{1+w}.$$

Then, in accordance with formula (3),

$$B(z) = -\frac{1}{T(z)} = -1 + I(z) \qquad \text{and} \qquad D(z) = \frac{1}{\left(T(z)\right)^2} = \left(1 - I(z)\right)^2.$$

Since $\left(I(z)\right)^2$ is the generating function for the class of labeled tournaments which can be decomposed into a sequence of two irreducible tournaments, we can rewrite the latter identity in the form

$$D(z) = 1 - \sum_{n=1}^{\infty} \left(2i_k - i_k^{(2)}\right) \frac{z^n}{n!}.$$

In the case at hand, the conditions that are needed to apply Theorem 3 are the same as in the proof of Theorem 1, since the sequence (a_n) is the same. Hence,

$$b_n = \frac{i_n}{n!} = \frac{2^{\binom{n}{2}}}{n!} - \sum_{k=1}^{r-1} \frac{2i_k - i_k^{(2)}}{k!} \frac{2^{\binom{n-k}{2}}}{(n-k)!} + O\left(\frac{2^{\binom{n-r}{2}}}{(n-r)!}\right).$$

Dividing by $t_n/n! = 2^{\binom{n}{2}}/n!$, we get

$$q_n = \frac{i_n}{t_n} = 1 - \sum_{k=1}^{r-1} \left(2i_k - i_k^{(2)}\right) \binom{n}{k} \frac{2^{\binom{n-k}{2}}}{2^{\binom{n}{2}}} + O\left(\frac{n^r}{2^{nr}}\right).$$

4 Further results

With a bit more work, we can compute the probability that a random graph of size n has exactly m connected components, and the probability that a random tournament of size n has exactly m irreducible components as n goes to ∞.

In another direction, we can also generalize Theorem 1 to the Erdös-Rényi model $G(n,p)$, where the constant 2 in the formulas is replaced by $\rho = 1/(1-p)$ and the sequence (i_k) is replaced by a sequence of polynomials $(P_k(\rho)) = 1, \rho - 2, \rho^3 - 6\rho + 6, \rho^6 - 8\rho^3 - 6\rho^2 + 36\rho - 24, \ldots$ with an explicit combinatorial interpretation.

The methods presented here can also be extended to some geometrical context where connectedness questions appear. In particular, we will provide asymptotics for combinatorial maps, square tiled surfaces, constellations, random tensor model [3]. In some of the models, the coefficients in the asymptotic expansions show connections with indecomposable tuples of permutations and perfect matchings.

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