Finite blocking property in billiards and translation surfaces

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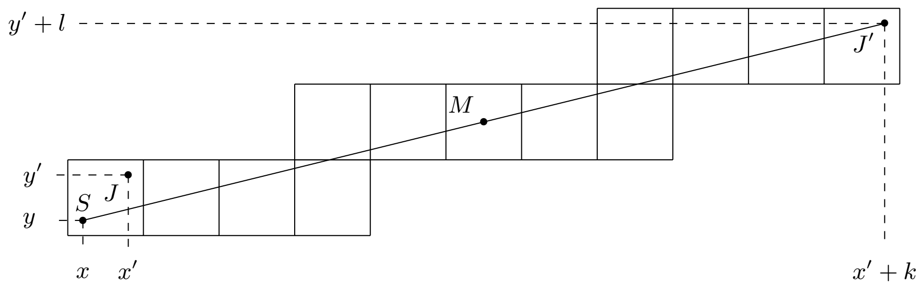
The following problem was stated for the Leningrad's Olympiad of 1989 [2]:

> "Professor Smith stands in a square hall with mirrored walls. Professor Jones intends to arrange several students in the hall so that Smith can't see his own reflection. Can Jones reach her goal? (Professor Smith and the students are considered points; students can be arranged by the walls and in the corners)."

Note that there are infinitely many light (billiard) trajectories between Jones and Smith. The square billiard table can be unfolded into a flat torus $\mathbb{R}^2/\mathbb{Z}^2$. A translation surface T is said to have the *finite blocking property* (FBP) if, for every pair (S, J) of points in T, there exists a finite number of "blocking" points $B_1, \ldots, B_n$ (different from S and J) such that every geodesic from S to J meets one of the B_i's. Let us solve the Olympiad's problem by showing that $\mathbb{R}^2/\mathbb{Z}^2$ has the FBP.

Let us write the professors' positions in coordinates: $S = (x, y)$, $J = (x', y')$. Any trajectory between S and J can be unfolded in $\mathbb{R}^2$ into a line between S and $J' = (x' + k, y' + l)$ for some $(k, l) \in \mathbb{Z}^2$.



The middle of the trajectory is $M = ((x + x' + k)/2, (y + y' + l)/2)$. If we project M back to $\mathbb{R}^2/\mathbb{Z}^2$, we get a point $\tilde{M} = ((x+x')/2, (y+y')/2) + (k/2, l/2) \bmod \mathbb{Z}^2$. Since $(k/2, l/2) \bmod \mathbb{Z}^2$ can only take four values, the infinite set of trajectories between S and J in $\mathbb{R}^2/\mathbb{Z}^2$ is blocked by at most four points (in some particular cases, some of the four points could correspond to J or S and should be removed from the blocking configuration). $\square$

Since the FBP is stable under branched coverings and under the action of $SL(2, \mathbb{R})$, we just saw that **any torus branched covering has the FBP**. If we try to generalise the previous construction to another surface T, " $\bmod \mathbb{Z}^2$ " should be replaced by " $\bmod G$ ", where G is the group generated by the translations used to identify the pairs of edges in some representation of T by a glued polygon. The previous construction of a finite set of points $\tilde{M}$ back in T works when G is discrete.

The easiest way to make G non-discrete is to have two adjacent parallel cylinders of uncommensurable perimeters. It turns out that in such a situation, the surface fails to have the FBP [6], hence we have a local criterion to start a classification. Any periodic orbit in a translation surface can be thickened into a cylinder. Unfortunately, the set of translation surfaces that contains two parallel cylinders with uncommensurable perimeters has zero measure, so this local criterion cannot be often directly used.

In [8], we proved that a translation surface with the finite blocking property is completely periodic. If we merge this result with the local criterion, we proved that **any translation surface with the FBP is purely periodic**, where a translation surface T is said to be *purely periodic* if, for any direction $\theta \in \mathbb{S}^1$, the existence of a (non-singular) periodic orbit in the direction θ implies that the directional flow ϕ_θ is periodic (*i.e.* there exists $t > 0$ such that $\phi_\theta^t = Id_S$ a.e.). Indeed the periodicity of the flow ϕ_θ is equivalent to the existence of a decomposition of T into cylinders of commensurable perimeters in the direction θ.

The geodesic flow on a translation surface T is defined on its unit tangent bundle $T \times \mathbb{S}^1$, it admits two subflows depending on whether we fix the direction $\theta \in \mathbb{S}^1$ (directional flow) or the starting point $J \in T$ (exponential flow). Hence, the previous results establish a surprising relation between three notions on translation surfaces, the first involving the global geometry of the surface (being a torus branched covering), the second involving the exponential flow (the FBP) and the third involving the directional flow (the pure periodicity). It would be nice to have an equivalence between those three notions, hence we would like the pure periodicity to imply being a torus branched covering.

Torus branched coverings can be characterised using *translational holonomy*: any curve $\gamma : [0,1] \to T$ on a translation surface T can be lifted as a planar curve $\tilde{\gamma}$ which is defined up to translation so that $hol(\gamma) = \tilde{\gamma}(1) - \tilde{\gamma}(0)$ is well defined. Restricted to the closed curves, the map hol induces a morphism from $H_1(T,\mathbb{Z})$ to $\mathbb{R}^2$. The "unfolding group" G previously introduced is actually $hol(H^1(T,\mathbb{Z}))$. A translation surface is a torus branched covering if, and only if, $hol(H^1(T,\mathbb{Z}))$ is a lattice.

For purely periodic translation surfaces, the J-invariant introduced in [5] can be computed through any pair of periodic directions, and the fact that it does not depend on such a pair implies that the holonomy of any three periodic orbits are rationally dependent. Hence, if $P(T)$ denotes the subgroup of $H^1(T,\mathbb{Z})$ generated by the periodic orbits (considered as closed curves), then $hol(P(T))$ is a lattice [7]. Hence, the previous three notions are equivalent when the periodic orbits of the translation surface generates its homology.

In a nutshell, the three notions are known to be equivalent:

- on a dense open subset of full measure in every stratum,
- for Veech surfaces, and more generally for surfaces whose Veech group contains two non-commuting parabolic elements,
- in genus 2 (using the classification of completely periodic surfaces [1]),
- for surfaces that admit a representation by a convex glued polygon, and more generally for surfaces which are named face-to-face surfaces.

A natural challenge is therefore to **describe the surfaces whose homology is not generated by the periodic orbits of their geodesic flow**. The eierlegende Wollmilchsau [4] and the translation surface introduced in [3] constitute the first examples. Indeed, in both cases, the two horizontal cylinders are homologous. Moreover, the Veech group of those two surfaces is equal to $SL(2,\mathbb{Z})$, hence the vertical and horizontal cylinders generate all cylinders (by making successive twists along both directions), hence the periodic orbits generate only a subgroup of dimension 2 in $H^1(T,\mathbb{Z})$. Those two examples are torus branched coverings, we do not know any primitive example.

Note that

- the set of translation surfaces that do not admit a strictly convex pattern,
- the non face-to-face surfaces,
- the set of translation surfaces whose homology is not generated by periodic orbits (and some variations on the dimension of the space generated by the periodic orbits)

are closed $SL(2,\mathbb{R})$-invariant spaces (containing each other).

References

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