Chains, Antichains, and Complements in Infinite Partition Lattices

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Abstract. We consider the partition lattice $\Pi(\lambda)$ on any set of transfinite cardinality λ and properties of $\Pi(\lambda)$ whose analogues do not hold for finite cardinalities. Assuming AC, we prove: (I) the cardinality of any maximal well-ordered chain is always exactly λ; (II) there are maximal chains in $\Pi(\lambda)$ of cardinality $> \lambda$; (III) a regular cardinal λ is strongly inaccessible if and only if every maximal chain $\Pi(\lambda)$ has size at least λ; if λ is a singular cardinal and $\mu^{<\kappa} < \lambda \leq \mu^{\kappa}$ for some cardinals κ and (possibly finite) μ, then there is a maximal chain of size $< \lambda$ in $\Pi(\lambda)$; (IV) every non-trivial maximal antichain in $\Pi(\lambda)$ has cardinality between λ and 2^{λ}, and these bounds are realized. Moreover, there are maximal antichains of cardinality $\max(\lambda, 2^{\kappa})$ for any $\kappa \leq \lambda$; (V) all cardinals of the form λ^{κ} with $0 \leq \kappa \leq \lambda$ occur as the cardinalities of sets of complements to some partition $\mathcal{P} \in \Pi(\lambda)$, and only these cardinalities appear. Moreover, we give a direct formula for the number of complements to a given partition;

Under the GCH, the cardinalities of maximal chains, maximal antichains, and numbers of complements are fully determined, and we provide a complete characterization.

1. Introduction and results

For a set S, the lattice of all equivalence relations, and that of all partitions over S, will be denoted by $\mathrm{Equ}(S)$ and $\Pi(S)$, respectively. For a cardinal κ, $\mathrm{Equ}(\kappa)$ and $\Pi(\kappa)$ will denote $\mathrm{Equ}(S)$ and $\Pi(S)$, respectively, such that $|S| = \kappa$. By the standard correspondence between partitions and equivalence relations, the lattices $\mathrm{Equ}(S)$ and $\Pi(S)$ are isomorphic. It is well-known that the lattice $\mathrm{Equ}(S) \simeq \Pi(S)$ is algebraic, simple, semimodular, and relatively complemented, see, for example Birkhoff [Bir40, §8-9], Burris and Sankappanavar [BS81, Theorem 4.11], Grätzer [Grä03, Sec. IV.4], Nation [Nat, Sec. 4], or Stern [Ste99].

For properties depending on κ, only a few results exist in the literature for infinite κ. Czédli has proved that if there is no inaccessible cardinal $\leq \kappa$ then the following holds: If $\kappa \geq 4$, $\Pi(\kappa)$ is generated by four elements [Czé96a], and if $\kappa \geq 7$, $\Pi(\kappa)$ is (1+1+2)-generated [Czé99] (for $\kappa = \aleph_0$, slightly stronger results hold [Czé96b]). It appears that no further results are known, beyond those holding for all cardinalities, finite or infinite. The aim of the present

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work is to prove a number of results concerning $\Pi(\kappa)$ that depend on κ being an infinite cardinal.

We first deal with the length of maximal chains in partition lattices. We prove that a well-ordered maximal chain in the lattice $\Pi(\lambda)$ has cardinality λ. Then, given infinite cardinals $\kappa < \lambda$, we prove that there is a maximal chain of length $\leq \kappa$ in the lattice $\Pi(\lambda)$ if and only if $\Pi(\kappa)$ contains a maximal chain of length $\geq \lambda$, and both equivalent to the property $D(\kappa, \lambda)$, introduced by Baumgartner [Bau76, Sec. 2], that there exists a chain of size λ with a weakly dense subset of size κ. Furthermore, property $D(\kappa, \lambda)$ holds true if and only if there is a tree of size and height at most κ with at least λ branches. Besides we study the question whether a lattice $\Pi(\lambda)$ contains a maximal chain of size $> \lambda$ and $< \lambda$, respectively. The first follows from Theorem 3.19 and [Sie22] (see also [Bau76, Corollary 2.4]) while the latter is false for strong limit cardinals. For regular λ, this is the only case when the maximal chains of size $< \lambda$ do not exist. For singular cardinals we only provide some partial results.

Next we study possible cardinalities of maximal antichains in the partition lattice $\Pi(\lambda)$. We prove that a non-trivial (*i.e.*, distinct from $\{\bot_{\Pi(\lambda)}, \top_{\Pi(\lambda)}\}$) maximal antichain has size between λ and 2^λ. It is easy to see that both the bounds are realized by the antichain of all atoms and the antichain consisting of all co-atoms of $\Pi(\lambda)$, respectively. We prove that these might not be the only possible cardinalities of maximal antichains by constructing a maximal antichain in $\Pi(\lambda)$ of size 2^κ for every κ with $\lambda \leq 2^\kappa \leq 2^\lambda$.

Finally we prove that a non-trivial, *i.e.*, distinct from $\bot$ and $\top$, partition $\mathcal{P} \in \Pi(\lambda)$ has 2^λ complements unless $\mathcal{P}$ has a single block, say B, of size λ and $|\lambda \setminus B| < \lambda$. In this case the partition $\mathcal{P}$ has exactly $\lambda^{|\lambda \setminus B|}$ complements.

We conclude the paper with the summary of all the previous results under the assumption of GCH.

2. Preliminaries and notation

2.1. Set theory. We work in ZF with the Axiom of Choice (AC). As usual, a set S is *well-ordered* if and only if it is totally ordered and every non-empty subset of S has a least element. Throughout the paper, we use von Neumann's characterization of ordinals: a set S is an ordinal if and only if it is strictly well-ordered by $\subsetneq$ and every element of S is a subset of S. The *order type* of a well-ordered set S is the (necessarily unique) ordinal α that is order-isomorphic to S. In addition to standard notation such as ω and $\aleph_0$, cardinals and ordinals are denoted by lowercase Greek letters $\alpha, \beta, \gamma, \delta, \xi \dots$ for ordinals and $\kappa, \lambda, \mu, \nu, \dots$ for cardinals.

Cardinals are least ordinals of a given cardinality, and by $|\alpha|$ we denote the cardinality of α. The cardinality of a set is denoted $|S|$ and its powerset is denoted $\mathscr{P}(S)$. We denote by κ^+ the successor cardinal of κ.

Given sets S, T, we denote by ${}^{S}T$ the set of all maps $S \to T$ and by $|S|^{|T|}$ its cardinality. For an infinite cardinal κ, and a possibly finite cardinal μ, we set $\mu^{<\kappa} := \sum_{\alpha\in\kappa} \mu^{|\alpha|}$. By $\log_\mu \lambda$ we denote the least cardinal κ such that $\lambda \leq \mu^\kappa$.

We denote by $\mathrm{cf}(\kappa)$ the cofinality of an infinite cardinal κ. Recall that König's Theorem [Kön05] implies $\mathrm{cf}(2^\kappa) > \kappa$, and we additionally have $2^\kappa \leq 2^\lambda$ whenever $\kappa < \lambda$. By Easton's theorem [Eas70], these are the only two constraints on permissible values for 2^κ when κ is regular and when only ZFC is assumed. In contrast, when the Generalized Continuum Hypothesis (GCH) is assumed, cardinal exponentiation is completely determined.

Many standard results on cardinal arithmetic can be found in [HSW99], among other places, and are used frequently throughout the proofs.

2.2. Posets and chains. A *chain* in a poset $(\mathbf{P}, \leq)$ is a subset of $\mathbf{P}$ that is linearly ordered by $\leq$. Similarly, an *antichain* in $(\mathbf{P}, \leq)$ is a subset of $\mathbf{P}$ such that any two distinct elements of the subset are $\leq$-incomparable. By a *maximal chain* (resp. *antichain*) in $(\mathbf{P}, \leq)$ we mean a chain (resp. antichain) maximal with respect to inclusion. By the Maximal Chain Theorem [Hau14], every chain in a poset is contained in a maximal chain. Similarly, every antichain is contained in a maximal one.

A poset $(\mathbf{P}, \leq)$ is called *bounded* provided that it contains a bottom element, $\bot_\mathbf{P}$, and a top element, $\top_\mathbf{P}$. A subset of the poset $\mathbf{P}$ containing the bottom and the top element of $\mathbf{P}$ is called *bounded* (in $\mathbf{P}$). We say that a subset $\mathbf{X}$ is *complete in* $\mathbf{L}$ if it forms a complete sublattice of $\mathbf{L}$.

We use $[x, z] := \{\, y \in \mathbf{P} \mid x \leq y \leq z \,\}$ to denote the interval in a poset $\mathbf{P}$ with bounds $x \leq z$.

As usual, given elements x and y of a poset $(\mathbf{P}, \leq)$, we write $x \prec_\mathbf{P} y$ if $x < y$ and no $z \in \mathbf{P}$ exists such that $x < z < y$. Furthermore, $x \preceq_\mathbf{P} y$ denotes that either $x \prec y$ or $x = y$. Given a subset $\mathbf{X} \subseteq \mathbf{P}$, we write $x \prec_\mathbf{X} y$ if $x, y \in \mathbf{X}$ with $x < y$, and there exists no $z \in \mathbf{X}$ such that $x < z < y$. A subset $\mathbf{X} \subseteq \mathbf{P}$ is called *covering* (in $\mathbf{P}$) if $x \prec_\mathbf{X} y$ implies $x \prec_\mathbf{P} y$.

If the poset $\mathbf{P}$ is clear from the context, we drop the subscript when denoting the top, the bottom element, or the covering relation in $\mathbf{P}$. We are going to make use of the following elementary lemmas:

Lemma 2.1. *A chain* $\mathbf{C}$ *in a complete lattice* $\mathbf{L}$ *is maximal if and only if it is bounded, complete and covering in* $\mathbf{L}$*.*

Proof. The only if implication is trivial. We prove the opposite one. Suppose, for contradiction, that the chain $\mathbf{C}$ is bounded, complete and covering in $\mathbf{L}$ but $\mathbf{C}$ is not maximal, *i.e.*, there is $x \in \mathbf{L} \setminus \mathbf{C}$ such that $\mathbf{C} \cup \{x\}$ is a chain. We set

$$\mathbf{C}_x^- := \{\, y \in \mathbf{C} \mid y < x \,\} \quad \text{and} \quad \mathbf{C}_x^+ := \{\, z \in \mathbf{C} \mid x < z \,\}.$$

Since $\mathbf{C}$ is bounded in $\mathbf{L}$, $\bot_{\mathbf{L}} \in \mathbf{C}_x^-$ and $\top_{\mathbf{L}} \in \mathbf{C}_x^+$, in particular, the sets $\mathbf{C}_x^-$ and $\mathbf{C}_x^+$ are non-empty. It follows that

$$\bigvee \mathbf{C}_x^- < x < \bigwedge \mathbf{C}_x^+. \tag{2.1}$$

Since $\mathbf{C}$ is complete in $\mathbf{L}$, both $\bigvee \mathbf{C}_x^-$ and $\bigwedge \mathbf{C}_x^+$ belong to $\mathbf{C}$. From $x \notin \mathbf{C}$ and $\mathbf{C} \cup \{x\}$ being a chain, we infer that $\mathbf{C} = \mathbf{C}_x^- \cup \mathbf{C}_x^+$, hence $\bigvee \mathbf{C}_x^- \prec_{\mathbf{C}} \bigwedge \mathbf{C}_x^+$. Since $\mathbf{C}$ is covering in $\mathbf{L}$, we conclude that $\bigvee \mathbf{C}_x^- \prec_{\mathbf{L}} \bigwedge \mathbf{C}_x^+$. This is in contradiction with (2.1). ☐

Lemma 2.2. *Let $\mathbf{C}$, $\mathbf{L}$ be complete lattices and $\varphi\colon \mathbf{C} \to \mathbf{L}$ be a map preserving arbitrary joins and meets. If $\mathbf{C}$ is a chain and*

$$x \prec_{\mathbf{C}} y \implies \varphi(x) \preceq_{\mathbf{L}} \varphi(y), \tag{2.2}$$

for all $x, y \in \mathbf{C}$, then the image $\varphi(\mathbf{C})$ forms a maximal chain in the interval $[\bigwedge \varphi(\mathbf{C}), \bigvee \varphi(\mathbf{C})]$.

Proof. Clearly, $\varphi(\mathbf{C})$ is bounded in $[\bigwedge \varphi(\mathbf{C}), \bigvee \varphi(\mathbf{C})]$. Since $\mathbf{C}$ is complete and the map φ preserves arbitrary meets and joins, the chain $\varphi(\mathbf{C})$ is complete in $\mathbf{L}$. It remains to prove that $\varphi(\mathbf{C})$ is covering in $\mathbf{L}$.

Let $a \prec_{\varphi(\mathbf{C})} b$ in $\varphi(\mathbf{C})$. Since $\mathbf{C}$ is a chain, $x := \bigvee \varphi^{-1}(a) \leq y := \bigwedge \varphi^{-1}(b)$. The map φ preserves arbitrary joins, and so we have that

$$\varphi(x) = \varphi(\bigvee \varphi^{-1}(a)) = \bigvee \varphi(\varphi^{-1}(a)) = a.$$

Similarly, φ preserving arbitrary joins implies that $\varphi(y) = b$. From $a \prec_{\varphi(\mathbf{C})} b$, we infer that $x < y$. If $x \leq z \leq y$ for some $z \in \mathbf{C}$, then $a = \varphi(x) \leq \varphi(z) \leq \varphi(y) = b$. Since $a \prec_{\varphi(\mathbf{C})} b$, either $\varphi(z) = a$, in which case $z = x$, or $\varphi(z) = b$, which implies that $z = y$. We conclude that $x \prec_{\mathbf{C}} y$, and so $a \prec_{\mathbf{L}} b$ due to (2.2). ☐

By a *tree* we mean a partially ordered set $\mathbf{T}$ such that for every $t \in \mathbf{T}$, the set $\downarrow t := \{\, x \in \mathbf{T} \mid x < t \,\}$ is well-ordered. The *height* of $t \in \mathbf{T}$, denoted by $\mathrm{ht}(t)$, is the order-type of $\downarrow t$. The *height* of $\mathbf{T}$ is the ordinal $\mathrm{ht}(\mathbf{T}) := \sup\{\, \mathrm{ht}(t) + 1 \mid t \in \mathbf{T} \,\}$. The *$\alpha$th level* of the tree $\mathbf{T}$ is the subset $\mathbf{T}_\alpha$ of all elements with height α.

A maximal well-ordered subset of $\mathbf{T}$ is called a branch. The height of a branch $\mathbf{b}$ is the ordinal $\mathrm{ht}(\mathbf{b}) := \sup\{\, \mathrm{ht}(t) \mid t \in \mathbf{b} \,\}$, *i.e.*, the order type of $\mathbf{b}$. An α-branch is a branch of height α. A $\mathrm{ht}(\mathbf{T})$-branch is called *cofinal.* We denote by $\mathbf{Br}(\mathbf{T})$ the set of all branches of the tree $\mathbf{T}$.

A *node* of the tree $\mathbf{T}$ is any block of the equivalence relation on $\mathbf{T}$ defined by $s \sim t$ if and only if $\downarrow s = \downarrow t$. The height of a node N is the ordinal $\mathrm{ht}(N) := \mathrm{ht}(t)$ where $t \in N$. The *support* of the tree $\mathbf{T}$, denoted by $\mathrm{supp}(\mathbf{T})$, is the set of all non-singleton nodes of $\mathbf{T}$.

2.3. Partitions and equivalences. Following [Grä03, Sec. IV.4] (see also [Nat, Sec. 4]) we sum up basic properties of the lattices $\Pi(S)$ (resp. $\mathrm{Equ}(S)$) of all partitions (resp. all equivalence relations) of a set S.

We denote binary relations, in particular equivalences, on a set S by Greek letters $\Theta, \Phi, \Psi \ldots$ and we view them as subsets of $S \times S$. We write $x \equiv_\Theta y$ if $(x, y) \in \Theta$.

We denote partitions by capital italic Roman letters $\mathcal{P}, \mathcal{Q}, \ldots$. We call elements of a partition *blocks*. For an element $x \in S$ we denote by $[x]_S$ the block of a partition $\mathcal{P}$ of a set S containing x. A partition $\mathcal{P} \in \Pi(S)$ naturally induces an equivalence relation, denoted by $\Theta_\mathcal{P}$ defined by $x \equiv_{\Theta_\mathcal{P}} y$ if and only if x and y belong to the same block of $\mathcal{P}$. Conversely, any equivalence relation corresponds to the partition whose blocks are the maximal sets of equivalent elements. This one-to-one correspondence allows us to consider a partition as its corresponding equivalence relation when convenient, and vice versa.

All equivalence relations on the set S form a complete lattice, denoted by $\mathrm{Equ}(S)$. The meet in $\mathrm{Equ}(S)$ corresponds to intersection while the join is given as a transitive closure of union, i.e, given $\mathbf{E} \subseteq \mathrm{Equ}(S)$, then $x \equiv_{\bigvee \mathbf{E}} y$ if and only if there is a natural number n and a sequence $x = z_0, z_1, \ldots, z_n = y$ of elements of S such that are equivalences $\Theta_1, \ldots, \Theta_n$ in $\mathbf{E}$, such that $z_{i-1} \equiv_{\Theta_i} z_i$, for all $i \in \{1, 2, \ldots, n\}$. In particular, if $\mathbf{E}$ is a chain, then $\bigvee \mathbf{E} = \bigcup \mathbf{E}$.

The description of (arbitrary) meets and joins in the lattice $\mathrm{Equ}(S)$ has a straightforward translation for partitions based on the equalities

$$\Theta_{\bigvee \mathbf{C}} = \bigvee_{\mathcal{P} \in \mathbf{C}} \Theta_\mathcal{P} \quad \text{and} \quad \Theta_{\bigwedge \mathbf{C}} = \bigwedge_{\mathcal{P} \in \mathbf{C}} \Theta_\mathcal{P}, \tag{2.3}$$

for all $\mathbf{C} \subseteq \Pi(S)$. It is straightforward to see from (2.3) that the blocks of $\bigwedge \mathbf{C}$ are all the nonempty intersections whose terms are exactly one block from every partition $\mathcal{P} \in \mathbf{C}$ and if $\mathbf{C}$ is a chain, then

$$[x]_{\bigvee \mathbf{C}} = \bigcup_{\mathcal{P} \in \mathbf{C}} [x]_\mathcal{P}\,, \tag{2.4}$$

for all $x \in S$.

It is easily seen that $\bot_{\Pi(S)} = \{\, \{x\} \mid x \in S \,\}$ and $\top_{\Pi(S)} = \{S\}$; that is, the set of all singleton subsets of S and the singleton set equal to S, respectively. If $\mathcal{P} \in \Pi(S)$ contains exactly one block B with $|B| \geq 2$ and the remaining blocks are all singletons, we call $\mathcal{P}$ a *singular* partition, following Ore [Ore42]. We will denote the singular partition with the non-singular block B by $\widehat{B}$.

Observe that $\mathcal{P} \prec \mathcal{Q}$ in $\Pi(S)$ if and only if $\mathcal{Q}$ can be obtained by merging exactly two distinct blocks of $\mathcal{P}$. It follows readily that singular partitions with a 2-element non-singleton block correspond to atoms of the lattice $\Pi(S)$ while co-atoms of $\Pi(S)$ are partitions with exactly two blocks.

Let $\mathcal{P} \leq \mathcal{Q}$ be partitions of a set S. We denote by $\mathcal{Q}/\mathcal{P}$ the partition of $\mathcal{P}$, *i.e.*, of the set $\{\, [x]_\mathcal{P} \mid x \in S \,\}$, such that

$$[x]_\mathcal{P} \equiv_{\Theta_{\mathcal{Q}/\mathcal{P}}} [y]_\mathcal{P} \iff x \equiv_{\Theta_\mathcal{Q}} y,$$

for all $x, y \in S$.

Observe that

$$[\mathcal{P}, \mathcal{Q}] \simeq [\bot, \mathcal{Q}/\mathcal{P}], \tag{2.5}$$

where $[\bot, \mathcal{Q}/\mathcal{P}]$ is the lower interval in the partition lattice $\Pi(\mathcal{P})$ on the set of blocks of $\mathcal{P}$, and that

$$[\bot, \widehat{B}] \simeq \Pi(B) \tag{2.6}$$

for a singular partition $\widehat{B}$.

3. Maximal chains in partition lattices

3.1. Well-ordered chains. For finite $\kappa = n$, it is immediate that any maximal chain in $\Pi(n)$ has cardinality n: Each step reduces the number of blocks by 1, whereby going from $\bot$ with n blocks to $\top$ with one block requires $n-1$ steps, hence n elements in the chain. For $n \geq 3$, maximal chains are not unique. In this section, we show that well-ordered maximal chains in $\Pi(\kappa)$ *always* have cardinality κ, whether κ is finite or infinite.

Let κ be an infinite cardinal. If a maximal chain in $\Pi(\kappa)$ is well-ordered of order type α, then α is a successor ordinal.[1]For clarity, we will write the order type of a maximal chain as $\alpha+1$ to emphasize this fact. Because $|\alpha| = |\alpha+1|$, this has no impact on the cardinality of the chain. Such chains can be written as $\mathbf{C} = \{\, \mathcal{P}_\beta \mid \beta \leq \alpha \,\}$ — or as $\mathbf{C} = \{\, \mathcal{P}_\beta \mid \beta < \alpha + 1 \,\}$ to emphasize the order type — with $\mathcal{P}_0 = \bot$ and $\mathcal{P}_\alpha = \top$.

Lemma 3.1. *Let κ be any cardinal. If a chain in $\Pi(\kappa)$ is well-ordered of order type α, then $|\alpha| \leq \kappa$.*

Note that we are here speaking of any well-ordered chain, not necessarily a maximal one, so its order type may be a limit ordinal.

Proof. The lemma holds trivially when κ is finite, as detailed above. Assume that κ is an infinite cardinal. Let $\mathbf{C} = \{\, \mathcal{E}_\beta \mid \beta < \alpha \,\}$ be an infinite well-ordered chain in $\mathrm{Equ}(\kappa)$. We view the equivalence relations on κ as subsets on $\kappa \times \kappa$. The set $\{\, \mathcal{E}_{\beta+1} \setminus \mathcal{E}_\beta \mid \beta + 1 < \alpha \,\}$ is formed by $|\alpha|$ nonempty pairwise disjoint subsets of $\kappa \times \kappa$. It follows that $|\alpha| \leq |\kappa \times \kappa| = \kappa$. □

Lemma 3.2. *Every well-ordered maximal chain of order type $\alpha + 1$ in $\Pi(\kappa)$ satisfies* $\mathrm{cf}(\kappa) \leq |\alpha|$.

Proof. Let $\mathbf{C} = \{\, \mathcal{P}_\beta \mid \beta < \alpha + 1 \,\}$ be a well-ordered maximal chain in $\Pi(\kappa)$ of order type $\alpha + 1$. Consider the partitions in this chain that have at least one block of cardinality κ. Since $\top = \mathcal{P}_\alpha$, there is at least one such partition. Let δ be the least ordinal such that $\mathcal{P}_\delta$ contains a block of cardinality κ. It

[1]Every maximal chain in $\Pi(\kappa)$ has a maximal element, namely $\top$. But since a well-ordered set of limit-ordinal type has no maximal element, this implies that the order type of a well-ordered maximal chain must be a successor-ordinal.

exists, because every non-empty set of ordinals has a least element. Let $\xi \in \kappa$ be an ordinal such that $\left|[\xi]_{\mathcal{P}_\delta}\right| = \kappa$.

First suppose that $\delta = \gamma + 1$ is a successor ordinal. Since $\mathbf{C}$ is a maximal chain, it is covering due to Lemma 2.1, hence $\mathcal{P}_\gamma \prec_{\Pi(\kappa)} \mathcal{P}_\delta$. It follows that the block $[\xi]_{\mathcal{P}_\delta}$ is a union of at most two blocks of $\mathcal{P}_\gamma$. This is impossible since $\left|[\xi]_{\mathcal{P}_\delta}\right| = \kappa$ while the size of all blocks of $\mathcal{P}_\gamma$ is less than κ.

It follows that δ is a limit ordinal. Since the chain $\mathbf{C}$ is maximal, it is a complete sublattice of $\Pi(\kappa)$ by Lemma 2.1, hence $\mathcal{P}_\delta = \bigvee_{\gamma<\delta} \mathcal{P}_\gamma$. Applying (2.4) we get that $[\xi]_{\mathcal{P}_\delta} = \bigcup_{\gamma<\delta} [\xi]_{\mathcal{P}_\gamma}$, hence

$$\kappa = \left|[\xi]_{\mathcal{P}_\delta}\right| \leq \sum_{\gamma<\delta} \left|[\xi]_{\mathcal{P}_\gamma}\right|. \tag{3.1}$$

Since $\left|[\xi]_{\mathcal{P}_\gamma}\right| < \kappa$ for every $\gamma < \delta$, we conclude from (3.1) that $\mathrm{cf}(\kappa) \leq \delta \leq |\alpha|$. □

Corollary 3.3. *Let κ be an infinite regular cardinal. Then every well-ordered maximal chain in $\Pi(\kappa)$ has size κ.*

Lemma 3.4. *Let $\mathbf{C}$ be a chain complete in $\Pi(\kappa)$ and $\mathcal{Q} \in \Pi(\kappa)$. The map $- \wedge \mathcal{Q}\colon \mathbf{C} \to \Pi(\kappa)$ given by the assignment $\mathcal{P} \mapsto \mathcal{P} \wedge \mathcal{Q}$ preserves arbitrary joins and meets.*

Proof. The map clearly preserves arbitrary meets. The partition lattice $\Pi(\kappa)$ is geometric [Grä03, Theorem IV.4.2], hence upper-continuous. It follows readily that the map $- \wedge \mathcal{Q}$ preserves arbitrary joins. □

Applying Lemma 2.2 we get that

Lemma 3.5. *Let $\mathbf{C}$ be a chain, complete in $\Pi(\kappa)$, and $\mathcal{Q} \in \Pi(\kappa)$. Suppose that*

$$\mathcal{P} \prec_{\mathbf{C}} \mathcal{R} \implies \mathcal{P} \wedge \mathcal{Q} \preceq_{\Pi(\kappa)} \mathcal{R} \wedge \mathcal{Q},$$

for all $\mathcal{P}, \mathcal{R} \in \mathbf{C}$. Then

$$\mathbf{C} \wedge \mathcal{Q} := \{\, \mathcal{P} \wedge \mathcal{Q} \mid \mathcal{P} \in \mathbf{C} \,\}$$

is a maximal chain in the interval $[\bot, \mathcal{Q}]$.

Furthermore, if $\mathbf{C}$ is a well-ordered chain, then $\mathbf{C} \wedge \mathcal{Q}$ is well-ordered as well.

To see that $\mathbf{C} \wedge \mathcal{Q}$ is well-ordered chain whenever $\mathbf{C}$ is well ordered, observe that the $- \wedge \mathcal{Q}$ preimages of an infinite set with no minimal element in $\mathbf{C} \wedge \mathcal{Q}$ would be an infinite set with no minimal element in $\mathbf{C}$.

Theorem 3.6. *If κ is any infinite cardinal, then every well-ordered maximal chain of order type $\alpha + 1$ in $\Pi(\kappa)$ satisfies $|\alpha| = \kappa$.*

Proof. Thanks to the previous results, we only need to check that $|\alpha| \geq \kappa$ whenever κ is singular. Indeed, the case of regular κ is handled by Corollary 3.3 and the upper bound is due to Lemma 3.1.

Since singular κ is a supremum of all regular $\lambda < \kappa$, it suffices to show that $\lambda \leq |\alpha|$ for every regular $\lambda < \kappa$.

If $\mathcal{P} \prec \mathcal{Q}$, then $\mathcal{Q}$ is obtained by replacing exactly two blocks, say B_1, B_2 of $\mathcal{P}$ by their union. It follows that $\mathcal{P} \wedge \widehat{\lambda} \prec \mathcal{Q} \wedge \widehat{\lambda}$ if both $B_1 \cap \lambda$ and $B_2 \cap \lambda$ are non-empty; and $\mathcal{P} \wedge \widehat{\lambda} = \mathcal{Q} \wedge \widehat{\lambda}$ otherwise. In any case, $\mathcal{P} \wedge \widehat{\lambda} \preceq \mathcal{Q} \wedge \widehat{\lambda}$. Applying Lemma 3.5, we infer that if $\mathbf{C}$ is a well-ordered maximal chain in $\Pi(\kappa)$ of order type $\alpha + 1$, then $\mathbf{C} \wedge \widehat{\lambda}$ is a well-ordered maximal chain in the interval $[\bot, \widehat{\lambda}]$.

Since $[\bot, \widehat{\lambda}] \simeq \Pi(\lambda)$ by (2.6), and λ is a regular cardinal, we conclude from Corollary 3.3 that $\lambda = \left|\mathbf{C} \wedge \widehat{\lambda}\right| \leq |\mathbf{C}|$. □

Remark 3.7. Notice that every ordinal $\alpha + 1$ with $|\alpha| = \kappa$ appears as the order type of some well-ordered maximal chain in $\Pi(\kappa)$. Indeed, the chain of singular partitions $\left\{ \widehat{\beta} \mid \beta \leq \alpha + 1 \right\}$ in $\Pi(\alpha + 1)$ works. In other words, it suffices to find a well-order of order type $\alpha + 1$ on κ (which exists by a cardinality argument) and add the elements to a single block in this order.

The *support* of a partition $\mathcal{P}$, denoted by $\operatorname{supp}(P)$, is the union of all non-singleton blocks of $\mathcal{P}$.

Lemma 3.8. *Let $\mathcal{P} \leq \mathcal{Q}$ in $\Pi(\kappa)$. There is a well-ordered maximal chain in the interval $[\mathcal{P}, \mathcal{Q}]$ of size at most $|\operatorname{supp}(\mathcal{Q}/\mathcal{P})|$.*

Proof. Since $[\mathcal{P}, \mathcal{Q}] \simeq [\bot, \mathcal{Q}/\mathcal{P}]$, due to (2.5), and $\mathcal{Q}/\mathcal{P} = \mathcal{Q}/\mathcal{P} \wedge \widehat{\operatorname{supp}(\mathcal{Q}/\mathcal{P})}$, we can without loss of generality assume that $\mathcal{P} = \bot$ and $\operatorname{supp}(\mathcal{Q}) = \kappa$. Let $\mathbf{C} := \left\{ \widehat{\beta} \mid \beta \in \kappa \right\}$ be a well-ordered maximal chain of singular partitions in $\Pi(\kappa)$. Observing that $\widehat{\beta} \wedge \mathcal{Q} \preceq_{\Pi(\kappa)} \widehat{\beta + 1} \wedge \mathcal{Q}$ for every ordinal $\beta \in \kappa$ and applying Lemma 3.5, we get that $\mathbf{C} \wedge \mathcal{Q} = \left\{ \widehat{\beta} \wedge \mathcal{Q} \mid \beta \in \kappa \right\}$ is a well-ordered maximal chain in $[\bot, \mathcal{Q}]$. The cardinality of $\mathbf{C} \wedge \mathcal{Q}$ is bounded by $|\mathbf{C}| = \kappa$. □

3.2. Short chains. We now turn to the question of whether there are maximal chains of cardinality strictly less than λ in $\Pi(\lambda)$. It is immediate that there are no maximal chains of finite cardinality in $\Pi(\aleph_0)$. Indeed, each step in a maximal chain merges exactly two blocks. Hence, starting from $\bot$ which has infinitely many blocks, after a finite number of steps it is impossible to reach $\top$ or any other partition that has only finitely many blocks. For larger cardinals, there is a general construction that proves existence of short chains. In the special case where GCH is assumed, it proves existence of a maximal chain of cardinality κ for every successor cardinal $\lambda = \kappa^+$.

Let $\mathbf{C}$ be a complete lattice. For each $c \in \mathbf{C}$, we set

$$c^* := \bigwedge \{ c' \in \mathbf{C} \mid c < c' \} .$$

Observe that $c \preceq_{\mathbf{C}} c^*$ and if $c \prec_{\mathbf{C}} c^*$, then c^* is the only cover of c. We put

$$\mathbf{C}^* := \{\, c \in \mathbf{C} \mid c < c^* \,\}.^2$$

Lemma 3.9. *Let $\mathbf{L}$ be a complete lattice and $\mathbf{C}$ a chain, complete in $\mathbf{L}$. Suppose that there is a maximal chain $\mathbf{M}_c \subset \mathbf{L}$ in the interval $[c, c^*]$ for every $c \in \mathbf{C}$. Then $\mathbf{M} = \bigcup_{c\in\mathbf{C}} \mathbf{M}_c$ is a maximal chain in $\mathbf{L}$.*

Proof. Suppose for the contradiction that $\{x\} \cup \mathbf{M}$ is a chain for some $x \in \mathbf{L} \setminus \mathbf{M}$. We set

$$c := \bigvee \{\, c' \in \mathbf{C} \mid c' < x \,\} \quad \text{and} \quad d := \bigwedge \{\, d' \in \mathbf{C} \mid x < d' \,\}.$$

By completeness, $c, d \in \mathbf{C}$ and because $x \notin \mathbf{C}$, we get that $c < x < d$. Since the union $\{x\} \cup \mathbf{C}$ is a chain, we have that $c \prec_{\mathbf{C}} d$, hence $d = c^*$. It follows that $x \cup \mathbf{M}_c$ is a chain in the interval $[c, c^*]$, which contradicts the maximality of $\mathbf{M}_c$ in $[c, c^*]$. □

Observe that if $\mathbf{M}$ is infinite, then the equality

$$|\mathbf{M}| = \sum_{c\in\mathbf{C}} |\mathbf{M}_c| = |\mathbf{C} \setminus \mathbf{C}^*| + \sum_{c\in\mathbf{C}^*} |\mathbf{M}_c| \tag{3.2}$$

holds true.[3]

A $\langle\kappa, \lambda\rangle$-*tree* is a tree $\mathbf{T}$ of size and height at most κ and with at least λ branches.

Lemma 3.10. *Let κ, λ be infinite cardinals. There is a $\langle\kappa, \lambda\rangle$-tree if and only if there is a tree $\mathbf{T}$ with $|\bigcup \operatorname{supp}(\mathbf{T})| \le \kappa$, $\operatorname{ht}(\mathbf{T}) = \kappa$, and at least λ cofinal branches.*

Proof. By extending branches with well-ordered chains, we can transform a $\langle\kappa, \lambda\rangle$-tree to a tree $\mathbf{S}$ with $|\bigcup \operatorname{supp}(\mathbf{S})| \le \kappa$, $\operatorname{ht}(\mathbf{S}) = \kappa$, and at least λ cofinal branches.

On the other hand, by removing singleton nodes from such a tree, we obtain a $\langle\kappa, \lambda\rangle$-tree. Indeed, let $\mathbf{S}$ be a tree with $|\bigcup \operatorname{supp}(\mathbf{S})| \le \kappa$, $\operatorname{ht}(\mathbf{S}) = \kappa$, and at least λ cofinal branches. We take the subtree $\mathbf{T}' := \bigcup \operatorname{supp}(\mathbf{S})$ of $\mathbf{S}$. From the properties of $\mathbf{S}$ we readily see that the size and the height of $\mathbf{T}'$ are at most κ. Let $\mathbf{b} \ne \mathbf{c}$ be branches of $\mathbf{S}$ and $\beta := \sup \{\, \alpha \in \kappa \mid \mathbf{b} \restriction \alpha = \mathbf{c} \restriction \alpha \,\}$. Then there is a node N of height β such that $\mathbf{b} \cap N \ne \mathbf{c} \cap N$. The node N is necessarily non-singleton, hence $\mathbf{b} \cap \mathbf{T}' \ne \mathbf{c} \cap \mathbf{T}'$ are distinct branches of $\mathbf{T}'$. It follows that the tree $\mathbf{T}'$ has at least λ branches, and so $\mathbf{T}'$ is a $\langle\kappa, \lambda\rangle$-tree. □

Existence of $\langle\kappa, \lambda\rangle$-trees, for $\kappa < \lambda$, implies the existence of short chains in $\Pi(\lambda)$.

Lemma 3.11. *Let $\kappa < \lambda$ be cardinals. If there is a $\langle\kappa, \lambda\rangle$-tree, then there is a maximal chain of size $\le \kappa$ in $\Pi(\lambda)$.*

---

[2]Note that $\mathbf{C}^*$ is the set of all completely meet-irreducible elements of $\mathbf{C}$.

[3]Observe that for $\mathbf{M}$ finite we trivially compute that $|\mathbf{M}| = \sum_{c\in\mathbf{C}} |\mathbf{M}_c| - |\mathbf{C}| + 2$.

Proof. Suppose that there is a $\langle\kappa,\lambda\rangle$-tree. By Lemma 3.10 there is a tree $\mathbf{S}$ with $|\bigcup \operatorname{supp}(\mathbf{S})| \leq \kappa$, $\operatorname{ht}(\mathbf{S}) = \kappa$, and at least λ cofinal branches. Let T be a set of cofinal branches of $\mathbf{S}$ with $|T| = \lambda$. Observe that $\mathbf{T} = \bigcup T \subset \mathbf{S}$ is a tree with $|\bigcup \operatorname{supp}(\mathbf{T})| \leq \kappa$, $\operatorname{ht}(\mathbf{T}) = \kappa$, $|\mathbf{Br}(\mathbf{T})| = \lambda$ and all branches of $\mathbf{T}$ are cofinal.

Given an ordinal $\alpha \leq \operatorname{ht}(\mathbf{T})$, we set

$$\mathbf{b} \restriction \alpha := \{\, t \in \mathbf{b} \mid \operatorname{ht}(t) < \alpha \,\},$$

for every branch $\mathbf{b} \in \mathbf{Br}(\mathbf{T})$, and we let $\mathcal{P}_\alpha$ be a partition of $\mathbf{Br}(\mathbf{T})$ defined via the corresponding equivalence relation as

$$\mathbf{b} \equiv_{\Theta_{\mathcal{P}_\alpha}} \mathbf{c} \iff \mathbf{b} \restriction \alpha = \mathbf{c} \restriction \alpha,$$

for all $\mathbf{b}, \mathbf{c} \in \mathbf{Br}(\mathbf{T})$. Observe that $\mathcal{P}_\alpha \geq \mathcal{P}_\beta$ whenever $\alpha \leq \beta < \operatorname{ht}(\mathbf{T})$ and

$$\mathcal{P}_\beta = \bigwedge_{\alpha<\beta} \mathcal{P}_\alpha, \tag{3.3}$$

for every limit ordinal $\beta \leq \operatorname{ht}(\mathbf{T})$. It follows that the chain

$$\mathbf{C} := \{\, \mathcal{P}_\alpha \mid \alpha \leq \operatorname{ht}(\mathbf{T}) \,\}$$

is complete in $\Pi(\mathbf{Br}(\mathbf{T}))$. By possibly removing all levels where the tree $\mathbf{T}$ does not branch, we can without loss of generality assume that $\mathcal{P}_\alpha > \mathcal{P}_\beta$ for all $\alpha < \beta \leq \operatorname{ht}(\mathbf{T})$. Observe that then $\mathbf{C}^* = \{\, \mathcal{P}_{\alpha+1} \mid \alpha < \operatorname{ht}(\mathbf{T}) \,\}$. Using Lemma 3.8, we construct a well-ordered maximal chain in each $[\mathcal{P}_\alpha, \mathbf{P}_{\alpha+1}]$. Applying Lemma 3.9, we extend the chain $\mathbf{C}$ to a maximal chain $\mathbf{M}$. The size of $\mathbf{M}$ is estimated as

$$|\mathbf{M}| \leq |\mathbf{C} \setminus \mathbf{C}^*| + \sum_{\alpha \in \operatorname{ht}(\mathbf{T})} |\operatorname{supp}(\mathcal{P}_\alpha/\mathcal{P}_{\alpha+1})|, \tag{3.4}$$

due to (3.2) and Lemma 3.8. It is straightforward to see that $|\mathbf{C}| \leq \operatorname{ht}(\mathbf{T}) \leq \kappa$ and $|\operatorname{supp}(\mathcal{P}_\alpha/\mathcal{P}_{\alpha+1})| \leq |\bigcup \operatorname{supp}(\mathbf{T})| \leq \kappa$, for every $\alpha < \operatorname{ht}(\mathbf{T})$. Substituting to (3.4) we conclude that $|\mathbf{M}| \leq \kappa + \operatorname{ht}(\mathbf{T}) \cdot \kappa = \kappa$. □

Let κ be a possibly finite cardinal and μ an infinite cardinal. We pick a sequence $\{\, \nu_\alpha \mid \alpha \in \operatorname{cf}(\mu) \,\}$ of cardinals with $\sum_{\alpha \in \operatorname{cf}(\mu)} \nu_\alpha = \mu$ and denote by $\mathbf{B}(\mu,\kappa)$ the tree consisting of all maps $\nu_\alpha \to \kappa$, $\alpha \in \operatorname{cf}(\mu)$, *i.e.*,

$$\mathbf{B}(\mu,\kappa) := \bigcup_{\alpha \in \operatorname{cf}(\mu)} {}^{\nu_\alpha}\kappa,$$

ordered by inclusion. Observe that $\mathbf{B}(\mu,\kappa)$ is a tree of height $\operatorname{cf}(\mu)$, size $\kappa^{<\mu} = \sum_{\alpha \in \operatorname{cf}(\mu)} \kappa^{\nu_\alpha}$, and with κ^μ branches, all of them cofinal. In particular $\mathbf{B}(\mu,\kappa)$ is a $\langle\kappa^{<\mu}, \kappa^\mu\rangle$-tree.

Let κ be a possibly finite cardinal and λ an infinite cardinal. Observe that $\log_\kappa \lambda$ is infinite provided that $\kappa < \lambda$. Therefore $\mathbf{B}(\log_\kappa \lambda, \kappa)$ is a $\langle\kappa^{<\log_\kappa \lambda}, \lambda\rangle$-tree (cf. [Bau76, Corollary 2.4]). Applying Lemma 3.11 we get that

Corollary 3.12. *Let λ be an infinite cardinal and $\kappa < \lambda$ a possibly finite cardinal. Then $\Pi(\lambda)$ contains a maximal chain of size at most $\kappa^{<\log_\kappa \lambda} < \lambda$.*

From Corollary 3.12 we conclude that for some infinite cardinals λ, there are maximal chains of size less than λ in the lattices $\Pi(\lambda)$.

Lemma 3.13. *Let λ be an infinite cardinal, $\mathbf{C}$ a maximal chain in $\Pi(\lambda)$, and $\kappa = |\mathbf{C}^*|$. Then there exists a $\langle\kappa, \lambda\rangle$-tree.*

Proof. Since the chain $\mathbf{C}$ is maximal, hence covering by Lemma 2.1, $\mathcal{C} \prec_{\Pi(\lambda)} \mathcal{C}^*$ for all $\mathcal{C} \in \mathbf{C}^*$. Let $\mathbf{C}^* := \{ \mathcal{C}_\alpha \mid \alpha \in \kappa \}$ be an ordering of $\mathbf{C}^*$. For each $\alpha \in \kappa$ we pick B_α to be one of the two blocks of $\mathcal{C}$ that are merged in $\mathcal{C}^*$. We assign to each $\beta \in \lambda$ the map $f_\beta \colon \kappa \to 2$ defined by

$$f_\beta(\alpha) := \begin{cases} 1 & : \beta \in B_\alpha, \\ 0 & : \beta \notin B_\alpha, \end{cases}$$

for all $\alpha \in \kappa$.

Claim 1. The map $\lambda \to {}^\kappa 2$ given by the correspondence $\beta \mapsto f_\beta$ is one-to-one.

Proof of Claim. Let $\beta \neq \gamma$ be ordinals from λ. We set

$$\begin{aligned} \mathcal{C} &:= \bigvee \{ \mathcal{P} \in \mathbf{C} \mid \beta \not\equiv_{\Theta_\mathcal{P}} \gamma \}, \\ \mathcal{D} &:= \bigwedge \{ \mathcal{Q} \in \mathbf{C} \mid \beta \equiv_{\Theta_\mathcal{Q}} \gamma \}. \end{aligned} \tag{3.5}$$

It follows from the definition that $\mathcal{C} \prec_{\mathbf{C}} \mathcal{D}$, hence $\mathcal{D} = \mathcal{C}^*$. In particular, we have $\mathcal{C} \in \mathbf{C}^*$. Therefore $\mathcal{C} = \mathcal{C}_\alpha$ for some $\alpha \in \kappa$. From $\beta \not\equiv_{\Theta_{\mathcal{C}_\alpha}} \gamma$ while $\beta \equiv_{\Theta_{\mathcal{C}_\alpha^*}} \gamma$, we conclude that $f_\beta(\alpha) \neq f_\gamma(\alpha)$. □ Claim 1.

We let $\mathbf{F} := \{ f_\beta \restriction \alpha \mid \alpha \in \kappa \text{ and } \beta \in \lambda \}$ be ordered by inclusion. It is clear, that $\mathbf{F}$ is a tree of height κ. Since the maps f_β correspond to its cofinal branches, the tree $\mathbf{F}$ has at least λ cofinal branches.

Claim 2. The tree $\mathbf{F}$ has at most one node of height α, for every $\alpha \in \kappa$.

Proof of Claim. Let $\alpha_1 \in \kappa$ and suppose that $\mathbf{F}$ has two distinct nodes, say N_0, N_1, of height β_1. We can pick $\{ \beta_{i,j} \mid i, j \in \{0,1\} \} \subseteq \lambda$ such that $f_{\beta_{i,j}} \restriction \alpha_1 \in N_i$ and $f_{\beta_{i,j}}(\alpha_1) = j$, for all $i, j \in \{0, 1\}$. It follows that

$$f_{\beta_{0,0}} \restriction \alpha_1 = f_{\beta_{0,1}} \restriction \alpha_1 \neq f_{\beta_{1,0}} \restriction \alpha_1 = f_{\beta_{1,1}} \restriction \alpha_1.$$

Let α_0 be the least ordinal in κ such that $f_{\beta_{0,0}}(\alpha_0) \neq f_{\beta_{1,1}}(\alpha_0)$. Since $f_{\beta_{0,0}} \restriction \alpha_1 \neq f_{\beta_{1,1}} \restriction \alpha_1$, we have that $\alpha_0 < \alpha_1$. We can without loss of generality assume that $f_{\beta_{0,0}}(\alpha_0) = 0$ while $f_{\beta_{1,1}}(\alpha_0) = 1$ (otherwise we swich N_0 and N_1). We infer that

$$f_{\beta_{i,j}}(\alpha_0) = i \quad \text{and} \quad f_{\beta_{i,j}}(\alpha_1) = j,$$

for all $i, j \in \{0, 1\}$. It follows that

$$\beta_{0,1} \in B_{\alpha_1} \setminus B_{\alpha_0},\ \beta_{1,0} \in B_{\alpha_0} \setminus B_{\alpha_1},\ \text{and } \beta_{1,1} \in B_{\alpha_0} \cap B_{\alpha_1}. \tag{3.6}$$

Since B_{α_0} and B_{α_1} are blocks of comparable partitions $\mathcal{C}_{\alpha_0}$ and $\mathcal{C}_{\alpha_1}$, respectively, they are either comparable or disjoint. Therefore either $B_{\alpha_1} \setminus B_{\alpha_0} = \emptyset$ or $B_{\alpha_0} \setminus B_{\alpha_1} = \emptyset$, provided that they are comparable, or $B_{\alpha_0} \cap B_{\alpha_1} = \emptyset$. Each of the three cases violates (3.6). □ Claim 2.

We infer from Claim 2 that the tree $\mathbf{F}$ has at most κ non-singleton nodes, each of size 2, hence $|\bigcup \operatorname{supp}(\mathbf{F})| \leq \kappa$. Applying Lemma 3.10 we conclude that there is a $\langle \kappa, \lambda \rangle$-tree. □

Observe that a tree of size κ has at most 2^κ branches. Thus, we have the following corollary.

Corollary 3.14. *Let λ be an infinite cardinal. If $\mathbf{C}$ a maximal chain in $\Pi(\lambda)$, then $\lambda \leq 2^{|\mathbf{C}^*|}$.*

>From Corollary 3.14 we get immediatelly that

Proposition 3.15. *If λ is a strong limit cardinal, then each maximal chain in $\Pi(\lambda)$ has size at least λ.*

Clearly,

$$2^{<\log_2 \lambda} = \sup \{ 2^\kappa \mid 2^\kappa < \lambda \} \leq \lambda, \tag{3.7}$$

for every infinite cardinal λ. For a regular cardinal λ the equality $2^{<\log_2 \lambda} = \lambda$ happens if and only if λ is a strong limit cardinal, *i.e.*, $\log_2 \lambda = \lambda$. Note that in this case the cardinal λ is strongly inaccessible. For a singular λ, the situation is less obvious. It can happen that $\log_2 \lambda < \lambda$ and still $2^{<\log_2 \lambda} = \lambda$. However observe that there is a strong limit cardinal $\log_2 \lambda \leq \mu < \lambda$, then $2^{<\log_2 \lambda} \leq \mu < \lambda$. We conclude that

Theorem 3.16. *Let λ be an infinite cardinal. If $\lambda = \aleph_0$ or λ is a strongly inaccessible cardinal, then each maximal chain in $\Pi(\lambda)$ has size at least λ. Otherwise $2^{<\log_2 \lambda} < \lambda$ and there is a maximal chain in $\Pi(\lambda)$ of size at most $2^{<\log_2 \lambda}$*

For a singular cardinal λ, if there is a possibly finite cardinal $\kappa < \lambda$ satisfying $\kappa^{<\log_\kappa \lambda} < \lambda$, then there is a maximal chain in $\Pi(\lambda)$ of size $< \lambda$. In particular, the assumption is satisfied when there is a strong limit cardinal $\log_2 \lambda \leq \mu < \lambda$.

We do not know whether $\Pi(\lambda)$ contains a maximal chain of size $< \lambda$ when λ is a singular cardinal such that $\log_2 \lambda < \lambda$ and $\kappa^{<\log_\kappa \lambda} = \lambda$ for all cardinals κ satisfying $2 \leq \kappa < \lambda$. Note however that $\kappa^{<\log_\kappa \lambda} = \lambda$ implies $\operatorname{cf}(\log_\kappa \lambda) = \operatorname{cf}(\lambda)$. Clearly, $\operatorname{cf}(\kappa) = \operatorname{cf}(\aleph_\kappa)$, and so for an infinite regular cardinal κ, the limit cardinal $\aleph_\kappa$ is the second least cardinal of cofinality κ. It follows that

Corollary 3.17. *For an infinite regular cardinal cardinal κ, either $\aleph_\kappa$ is a strong limit cardinal, or $2^\kappa = \aleph_\kappa$ or the lattice $\Pi(\aleph_\kappa)$ contains a maximal chain of length $< \aleph_\kappa$.*

3.3. Long chains. Finally we prove that the lattice $\Pi(\kappa)$ can contain maximal chains of cardinality $> \kappa$.

Recall that a subset D of a chain $\mathbf{C}$ is *weakly dense* in $\mathbf{C}$ provided that for all $x, y \in \mathbf{C}$ with $x < y$ there is $z \in D$ such that $x \leq z \leq y$. Let $\kappa \leq \lambda$ be cardinals. A $\langle \kappa, \lambda \rangle$*-chain* is a chain $\mathbf{C}$ of size $\geq \lambda$ with a weakly dense subset of size $\leq \kappa$.

We will call a subset D of a chain $\mathbf{C}$ *left dense* if for all $x, y \in \mathbf{C}$ with $x < y$ there is $z \in D$ such that $x \leq z < y$. Clearly, every left dense subset of $\mathbf{C}$ is weakly dense. On the other hand every weakly dense subset D of the chain $\mathbf{C}$ contains at least one of the elements c, c^* for every $c \in \mathbf{C}^*$. Putting $D' = D \cup \mathbf{C}^*$, we obtain a left dense subset of $\mathbf{C}$ of size $|D'| \leq 2\,|D|$. In particular, for infinite $\kappa \leq \lambda$, every $\langle \kappa, \lambda \rangle$-chain contains a left dense subset of size $\leq \kappa$.

Lemma 3.18. *Let $\kappa < \lambda$ be infinite cardinals. The lattice $\Pi(\kappa)$ contains a chain of size $\geq \lambda$ if and only if there is $\langle \kappa, \lambda \rangle$-chain.*

Proof. ($\Leftarrow$) Let $\mathbf{C}$ be a $\langle \kappa, \lambda \rangle$-chain. As discussed above, $\mathbf{C}$ contains a left dense subset, D, of size $\leq \kappa$. Note that D is necessarily infinite. Pick an enumeration $\{\, \nu_\alpha \mid 1 < \alpha \in |D| \,\}$ of D and set $D(c) := \{0, 1\} \cup \{\, \alpha \mid c < \nu_\alpha \,\}$ for each $c \in \mathbf{C}$. Since D is left dense in $\mathbf{C}$, we have that $c < c'$ implies $D(c) \subsetneq D(c')$. It follows that the singular partitions $\widehat{D(c)}$, $c \in \mathbf{C}$, form a chain of size $|\mathbf{C}|$, which is $\geq \lambda$.

($\Rightarrow$) Suppose that $\Pi(\kappa)$ contains a chain $\mathbf{C}$ of size $\geq \lambda$. We can without loss of generality assume that $\mathbf{C}$ is a maximal chain in $\Pi(\kappa)$, and so $\mathbf{C}$ is complete in $\Pi(\kappa)$ due to Lemma 2.1. For each $\alpha < \beta \in \kappa$, we set

$$\mathcal{D}_{\alpha,\beta} := \bigvee \{\, \mathcal{C} \in \mathbf{C} \mid \alpha \not\equiv_{\Theta_\mathcal{C}} \beta \,\}\,.$$

Since the chain $\mathbf{C}$ is complete in $\Pi(\kappa)$, we have $\mathcal{D}_{\alpha,\beta} \in \mathbf{C}$, for all $\alpha < \beta \in \kappa$.

Let $\mathcal{P} < \mathcal{Q}$ in $\mathbf{C}$. We pick $\alpha < \beta \in \kappa$ such that $\alpha \not\equiv_{\Theta_\mathcal{P}} \beta$ while $\alpha \equiv_{\Theta_\mathcal{Q}} \beta$. It follows that $\mathcal{P} \leq \mathcal{D}_{\alpha,\beta} < \mathcal{Q}$, and so

$$D := \{\, \mathcal{D}_{\alpha,\beta} \mid \alpha < \beta \in \kappa \,\}\,,$$

is a weakly dense[4] subset of $\mathbf{C}$ of size at most κ. Therefore $\mathbf{C}$ is a $\langle \kappa, \lambda \rangle$-chain. □

We are ready to prove the main theorem of this section.

Theorem 3.19. *Let $\kappa \leq \lambda$ be infinite cardinals. The following properties are equivalent:*

(1) *There is a $\langle \kappa, \lambda \rangle$-tree.*
(2) *There is a $\langle \kappa, \lambda \rangle$-chain.*
(3) *There is a chain of size $\geq \lambda$ in $(\mathscr{P}(\kappa), \subseteq)$.*
(4) *There is a chain of size $\geq \lambda$ in $\Pi(\kappa)$.*
(5) *There is a maximal chain of size $\leq \kappa$ in $\Pi(\lambda)$.*

Proof. Properties (1), (2), and (3) are equivalent due to [Bau76, Theorem 2.1] (cf. [Mit72]). Equivalence ($2 \Leftrightarrow 4$) follows from Lemma 3.18. Finally ($1 \Rightarrow 5$) holds true due to Lemma 3.11 and ($1 \Leftarrow 5$) follows from Lemma 3.13. □

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[4] In fact left dense.

By a result of Sierpiński [Sie22] (see also [Har05, Thm. 4.7.35]), there is a chain $\mathbf{C}$ in the poset $(\mathscr{P}(\kappa), \subseteq)$ of cardinality $\lambda > \kappa$. Applying Theorem 3.19 we conclude that

Corollary 3.20. *For every infinite cardinal κ, there is a chain of cardinality $> \kappa$ in $\Pi(\kappa)$.*

Rational numbers form a countable weakly dense subset of the set $\mathbb{R}$ of all real numbers. It follows from Lemma 3.18 that $\Pi(\aleph_0)$ contains a chain of cardinality $2^{\aleph_0}$.

We can refine Corollary 3.20 applying [Bau76]. Observe that for an infinite cardinal λ and μ with $2 \leq \mu < \lambda$ we have that

$$\mu^{<\log_\mu \lambda^+} \leq \lambda < \lambda^+ \leq \mu^{\log_\mu \lambda^+} \tag{3.8}$$

By [Bau76, Corollary 2.4] there is a $\langle \lambda, \mu^{\log_\mu \lambda^+} \rangle$-tree for every $2 \leq \mu < \lambda$; in fact, $\mathbf{B}(\lambda, \mu)$ is an example. From Theorem 3.19 and (3.8) we conclude that

Corollary 3.21. *Let λ be an infinite cardinal. Then $\lambda < \lambda^+ \leq \mu^{\log_\mu \lambda^+}$ for every μ satisfying $2 \leq \mu < \lambda$ and $\Pi(\lambda)$ contains a chain of size $\mu^{\log_\mu \lambda^+}$.*

Remark 3.22. Following [Bau76], we write $D(\kappa, \lambda)$ if the equivalent properties of Theorem 3.19 are satisfied. The question whether there is a $\langle \kappa, 2^\kappa \rangle$-chain, thus implying that $D(\kappa, 2^\kappa)$ holds true for every infinite cardinal κ, is due to Malitz [Mal68]. This is clearly the case if we assume GCH. On the other hand, Mitchell constructed by Cohen forcing a cardinal preserving extension of a contable transitive model $\mathcal{M}$ of **ZFC** with an arbitrarily picked cardinal κ with $\mathrm{cf}(\kappa)^{\mathcal{M}} > \aleph_0$ such that $\neg D(\kappa, 2^\kappa)$ [Mit72, Theorem 4.2]. Given regular cardinals $\mu < \kappa$ in the model $\mathcal{M}$, another forcing construction led to a cardinal preserving extension such that $2^\mu = \kappa^+$, $2^\kappa = \aleph_{\kappa^+}$ and $\neg D(\kappa, 2^\kappa)$ [Mit72, Theorem 4.4]. These in the particular case $\kappa = \aleph_1$ prove the consistency of

(i) $2^{\aleph_0} = \aleph_{\omega_1}$ and $2^{\aleph_1} = \aleph_{\omega_1+1}$,
(ii) $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_1} = \aleph_{\omega_2}$,

and $\neg D(\aleph_1, 2^{\aleph_1})$, respectively. The cases (i) and (ii) are the smallest possible where $\neg D(\kappa, 2^\kappa)$ due to Baumgartner's analysis of possible cases [Bau76, p. 414]. In particular, $D(\aleph_0, 2^{\aleph_0})$ holds for a trivial reason and Baumgartner proved that if $\neg D(\aleph_1, 2^{\aleph_1})$, then either $\aleph_1 = \mathrm{cf}(2^{\aleph_0}) < 2^{\aleph_0} < 2^{\aleph_1}$, or $\aleph_1 < 2^{\aleph_0} = \mathrm{cf}(2^{\aleph_1}) < 2^{\aleph_1}$. Clearly, (i) is the smallest instance of the first and (ii) of the latter.

The cardinal $\mathrm{ded}(\kappa) := \sup \{ \lambda \mid D(\kappa, \lambda) \}$ was recently studied by Chernikov, Kaplan, and Shelah [CKS16, CS16] in connection with model theory. By Theorem 3.19 we can view $\mathrm{ded}(\kappa)$ as the supremum of the lengths of chains in $\Pi(\kappa)$. Note that it need not be attained in general and that $\kappa < \mathrm{ded}(\kappa) \leq \mathrm{ded}(\kappa)^{\aleph_0} \leq 2^\kappa$ (cf. [CS16]). It follows from the previous paragraph, that $\mathrm{ded}(\kappa) < 2^\kappa$ is consistent with **ZFC** whenever $\mathrm{cf}(\kappa) > \aleph_0$. The consistency of $\mathrm{ded}(\kappa) < \mathrm{ded}(\kappa)^{\aleph_0}$ is due to [CKS16, Corollary 6.11]. Whether $\mathrm{ded}(\kappa) < \mathrm{ded}(\kappa)^{\aleph_0} < 2^\kappa$ can happen is not known. Some interesting estimations of ded(κ) are obtained by PCF-theory in [CKS16], in particular, it is proved that $2^\kappa \leq \text{ded}(\text{ded}(\text{ded}(\text{ded}(\kappa))))$ [CS16, Theorem 2.10].

4. Maximal antichains

An *antichain* in $(\mathbf{P}, \leq)$ is a subset $\mathbf{A} \subseteq \mathbf{P}$ in which no two distinct elements of $\mathbf{P}$ are $\leq$-comparable. An antichain is *maximal* if adding an element to it results in a set that is not an antichain. Observe that in a poset, the *trivial* antichains $\{\bot\}$ and $\{\top\}$ are always maximal antichains.

There is no known tight bound on the cardinality of maximal antichains in $\Pi(n)$ for finite n, but some asymptotic results are known [Can98, BH02]. The cardinality of a maximal antichain in $\Pi(n)$ is $\Theta\left(n^a (\log n)^{-a-1/4} S(n, K_n)\right)$ where $a = (2 - e\log 2)/4$ and $S(n, K_n) = \max_k \left\{ {n \atop k} \right\}$ is the largest Stirling number of the second kind for fixed n.

It is straightforward that the atoms, respectively co-atoms, form maximal antichains in a partition lattice. If λ is an infinite cardinal, the sizes of these antichains in $\Pi(\lambda)$ are λ, respectively 2^λ. Since $\Pi(\lambda)$ itself has cardinality 2^λ, there is no antichain of a greater size. The proof that there is no maximal antichain shorter than λ is more involved.

Theorem 4.1. *For infinite λ, the cardinality of a non-trivial maximal antichain in $\Pi(\lambda)$ is at least λ.*

Proof. Let $\kappa < \lambda$ and $\mathbf{A}$ be a non-trivial antichain of cardinality κ in $\Pi(\lambda)$. We will show that $\mathbf{A}$ is not maximal by building a partition $\mathcal{P} \notin \mathbf{A}$ that is not comparable to any of its elements.

Recall that $\text{supp}(\mathcal{P})$ is the union of all non-singleton blocks of $\mathcal{P}$. We call a partition $\mathcal{P} \in \Pi(\lambda)$ *small* if $|\text{supp}(\mathcal{P})| < \aleph_0 \cdot \kappa^+$. There are two cases to consider.

First, suppose that there are no small partitions in $\mathbf{A}$. Since the antichain $\mathbf{A}$ is non-trivial, it does not contain $\top_{\Pi(\lambda)} = \widehat{\lambda}$. Therefore, for each $\mathcal{A} \in \mathbf{A}$ we can pick an element $\xi_\mathcal{A}$ that is not in the same block as 0. Let $\mathcal{P}$ be the singular partition with the non-singleton block $\{0\} \cup \{\, \xi_\mathcal{A} \mid \mathcal{A} \in \mathbf{A} \,\}$. Since $0 \equiv_{\Theta_\mathcal{P}} \xi_\mathcal{A}$ while $0 \not\equiv_{\Theta_\mathcal{A}} \xi_\mathcal{A}$, we have that $\mathcal{P} \nleq \mathcal{A}$, for all $\mathcal{A} \in \mathbf{A}$. On the other hand, $\text{supp}(\mathcal{P}) = \{0\} \cup \{\, \xi_\mathcal{A} \mid \mathcal{A} \in \mathbf{A} \,\}$, hence the partition $\mathcal{P}$ is small, and so it is not above any non-small partition. In particular, $\mathcal{A} \nleq \mathcal{P}$, for all $\mathcal{A} \in \mathbf{A}$. We conclude that $\mathcal{P} \notin \mathbf{A}$ and $\mathbf{A} \cup \{\mathcal{P}\}$ is an antichain, which contradicts the maximality of $\mathbf{A}$.

Second, assume that there are small partitions in $\mathbf{A}$. We set

$$S := \bigcup \{\, \text{supp}(\mathcal{A}) \mid \mathcal{A} \in \mathbf{A} \text{ and } \mathcal{A} \text{ is small} \,\}.$$

Observe that S is finite when κ is finite, and $|S| < \kappa \cdot \kappa^+ = \kappa^+ \leq \lambda$ otherwise. In both cases we have that $|\lambda \setminus S| = \lambda$, and so we can pick pairwise distinct elements $\nu_\mathcal{A}$, $\mathcal{A} \in \mathbf{A}$, in $\lambda \setminus S$.

Given a partition $\mathcal{A} \in \mathbf{A}$, observe that $S \nsubseteq [\nu_{\mathcal{A}}]_{\mathcal{A}}$, for otherwise the partition $\mathcal{A}$ would properly contain all small partitions from $\mathbf{A}$. It follows that we can pick an element $\xi_{\mathcal{A}} \in S$ such that $\xi_{\mathcal{A}} \not\equiv_{\Theta_{\mathcal{A}}} \nu_{\mathcal{A}}$, for all $\mathcal{A} \in \mathbf{A}$. Let $\mathcal{P}$ be the least partition from $\Pi(\lambda)$ such that $\xi_{\mathcal{A}} \equiv_{\Theta_{\mathcal{P}}} \nu_{\mathcal{A}}$, for all $\mathcal{A} \in \mathbf{A}$.

Since $\xi_{\mathcal{A}} \not\equiv_{\Theta_{\mathcal{A}}} \nu_{\mathcal{A}}$ while $\xi_{\mathcal{A}} \equiv_{\Theta_{\mathcal{P}}} \nu_{\mathcal{A}}$, we get that $\mathcal{P} \nleq \mathcal{A}$, for all $\mathcal{A} \in \mathbf{A}$. Observe that

$$\operatorname{supp}(P) = \{\, \xi_{\mathcal{A}} \mid \mathcal{A} \in \mathbf{A} \,\} \cup \{\, \nu_{\mathcal{A}} \mid \mathcal{A} \in \mathbf{A} \,\},$$

hence $|\operatorname{supp}(P)| \leq \kappa + \kappa < \aleph_0 \cdot \kappa^+$, whence $\mathcal{P}$ is small. It follows that $\mathcal{A} \nleq \mathcal{P}$ for all $\mathcal{A} \in \mathbf{A}$ that are not small. Since the elements $\nu_{\mathcal{A}}$ are pairwise distinct, $\mathcal{P} \wedge \widehat{S} = \bot_{\Pi(\lambda)}$. It follows that $\mathcal{A} \nleq \mathcal{P}$ for every small partition $\mathcal{A} \in \mathbf{A}$. We conclude that $\mathcal{P} \notin \mathbf{A}$ and that $\mathbf{A} \cup \{\mathcal{P}\}$ is an antichain, which is a contradiction. □

Note that, while we here restrict our attention to infinite partition lattices, for finite n, maximal non-trivial antichains in $\Pi(n)$ can be proved to have cardinality at least n by induction on n.

We have proved that every maximal antichain in $\Pi(\lambda)$ has cardinality between λ and 2^λ and since both of the bounds occur as cardinality of a maximal antichain, of all atoms and co-atoms of $\Pi(\lambda)$, respectively, they are as tight as possible. However, the following construction shows that, in some models where GCH is violated, there are maximal antichains of cardinalities between these bounds. Specifically, it is true whenever there exists $\lambda < 2^\kappa < 2^\lambda$ for some infinite cardinals $\kappa < \lambda$.

Proposition 4.2. *Let $\kappa < \lambda$ be infinite cardinals. If $\mathbf{A}$ is a maximal antichain in $\Pi(\kappa)$, then there is a maximal antichain of size $|\mathbf{A}| + \lambda$ in $\Pi(\lambda)$.*

Proof. For each partition $\mathcal{A} \in \Pi(\kappa)$, we define its extension as

$$\mathcal{A}^e := \mathcal{A} \cup \{\, \{\xi\} \mid \kappa \leq \xi < \lambda \,\},$$

and we put $\mathbf{A^e} := \{\, \mathcal{A}^e \mid \mathcal{A} \in \mathbf{A} \,\}$. Let

$$\mathbf{B} := \left\{ \widehat{\{\xi, \nu\}} \mid \xi \neq \nu \text{ and } \kappa \leq \nu \right\}$$

be the set of all atoms of $\Pi(\lambda)$ that are not extensions of atoms of $\Pi(\kappa)$.

Let $\mathcal{P} \in \Pi(\lambda)$ be a partition. If $\mathcal{P} \nleq \widehat{\kappa}$, then $\mathcal{P}$ is above some atom from $\mathbf{B}$. On the other hand, $\mathcal{P} \nleq \widehat{\kappa}$ implies that $\mathcal{P}$ is comparable to $\mathcal{A}^e$, for some $\mathcal{A} \in \mathbf{A}$, because $\mathbf{A}$ is a maximal antichain in κ. It follows that $\mathbf{A^e} \cup \mathbf{B}$ is a maximal antichain in $\Pi(\lambda)$. Clearly,

$$|\mathbf{A^e} \cup \mathbf{B}| = |\mathbf{A}| + |\mathbf{B}| = |\mathbf{A}| + \lambda.$$

□

Corollary 4.3. *Let $\kappa < \lambda$ be infinite cardinals such that $\lambda < 2^\kappa < 2^\lambda$. Then there is a maximal antichain of size 2^κ in $\Pi(\lambda)$.*

Proof. As noted at the beginning of the section, co-atoms of κ form a maximal antichain of size 2^κ. Applying Proposition 4.2, we construct an antichain of size $2^\kappa + \lambda = 2^\kappa$ in $\Pi(\lambda)$. □

We do not yet know whether all cardinalities between the bounds always occur as antichains. Certainly, we can construct non-GCH models where there exist cardinals between κ and 2^κ that cannot be written as 2^λ. It is not yet known whether they can be realized as the cardinality of a maximal antichain through another construction.

5. Complements

Recall that in a bounded lattice L, elements $a, b \in L$ are *complements* if and only if $a \vee b = \top$ and $a \wedge b = \bot$. We denote by $\operatorname{compl}(\mathcal{P})$ the set of all complements to $\mathcal{P}$ in $\Pi(\kappa)$. For finite $\kappa = n$, counting the number of elements in $\operatorname{compl}(\mathcal{P})$ is a difficult combinatorial problem. The best known estimate, due to Grieser [Gri91], is that if $\mathcal{P} = \{B_1, \dots, B_m\}$ is a partition in $\Pi(n)$, then the number of complements $\mathcal{Q}$ of $\mathcal{P}$ satisfying $|\mathcal{Q}| = n - m + 1$ is $\prod_{i=1}^{m} |B_i| \cdot (n - m + 1)^{m-2}$.

Lemma 5.1. *Let κ be an infinite cardinal, and $\mathcal{P} \notin \{\bot, \top\}$ a partition of κ. Then*

$$|\operatorname{compl}(\mathcal{P})| \geq 2^{|\mathcal{P}|}$$

Proof. If $|\mathcal{P}|$ is finite, then by the pigeonhole principle $\mathcal{P}$ contains at least one block, say B, of cardinality κ. Any singular partition whose non-singleton block contains exactly one element from each block of $\mathcal{P}$ is a complement to $\mathcal{P}$. Since there are κ choices for the element in B, there are at least κ complements to $\mathcal{P}$. Thus $|\operatorname{compl}(\mathcal{P})| \geq \kappa > 2^{|\mathcal{P}|}$.

Next assume that $|\mathcal{P}|$ is infinite. Since $\mathcal{P} \neq \bot_{\Pi(\lambda)}$, there are $\alpha < \beta \in \kappa$ with $\alpha \equiv_{\Theta_\mathcal{P}} \beta$. We set $\mathcal{Q} := \mathcal{P} \setminus \{[\alpha]_\mathcal{P}\}$ and pick an element ξ_B from each block $B \in \mathcal{Q}$. For a subset $\mathcal{X} \subseteq \mathcal{Q}$, let $\Psi_\mathcal{X}$ be the equivalence relation on κ generated by $\{\, \langle \alpha, \xi_B \rangle \mid B \in \mathcal{X} \,\} \cup \{\, \langle \beta, \xi_B \rangle \mid B \in \mathcal{Q} \setminus \mathcal{X} \,\}$. Clearly, $\Psi_\mathcal{X}$ is a complement of $\Theta_\mathcal{P}$ for every $\mathcal{X} \subseteq \mathcal{Q}$. Since $\Psi_\mathcal{X} \neq \Psi_\mathcal{Y}$, whenever $\mathcal{X} \neq \mathcal{Y}$ are subsets of $\mathcal{Q}$, we conclude that $\operatorname{compl}(\mathcal{P}) \geq 2^{|\mathcal{Q}|} = 2^{|\mathcal{P}|}$. □

Theorem 5.2. *Let λ be an infinite cardinal and $\mathcal{P} \notin \{\bot, \top\}$ a partition in $\Pi(\lambda)$. Then*

(1) *if $\mathcal{P}$ contains no block of cardinality λ, then $|\operatorname{compl}(\mathcal{P})| = 2^\lambda$,*
(2) *if $\mathcal{P}$ contains a block B of cardinality λ, then $|\operatorname{compl}(\mathcal{P})| = \lambda^{|\lambda \setminus B|}$.*

Proof. (1) First assume that $\mathcal{P}$ has no block of cardinality λ. We put $\kappa := |\mathcal{P}|$ and let $\mathcal{P} = \{\, B_\alpha \mid \alpha \in \kappa \,\}$, be an enumeration of all blocks of $\mathcal{P}$. For each $\alpha \in \kappa$, we pick a bijection $j_\alpha \colon |B_\alpha| \to B_\alpha$.

We can assume that $|\mathcal{P}| < \lambda$ since otherwise Lemma 5.1 applies. It follows that the set $S_\beta := \{\, \alpha \in \kappa \mid \beta < |B_\alpha| \,\}$ is infinite[5], for all $\beta < \lambda$. Let δ_β denote the least ordinal from S_β, for each $\beta < \lambda$. We infer that

$$\bigl|\lambda \setminus \bigl(\{\, j_\alpha(0) \mid \alpha \in \kappa \,\} \cup \{\, j_{\delta_\beta}(\beta) \mid \beta < \lambda \,\}\bigr)\bigr| = \lambda,$$

hence there are 2^λ maps $f\colon \lambda \to \{0,1\}$ satisfying $f(j_\alpha(0)) = f(j_{\delta_\beta}(\beta)) = 1$ for all $\alpha \in \kappa$ and $\beta < \lambda$. Let $C_{\mathcal{P}}$ denote the set of all such maps.

With each map $f \in C_{\mathcal{P}}$ we associate a partition $\mathcal{Q}_f$ with possibly non-singleton blocks

$$\bigl[j_{\delta_\beta}(\beta)\bigr]_{\mathcal{Q}_f} = \{\, j_\alpha(\beta) \mid \alpha \in S_\beta \text{ and } f(j_\alpha(\beta)) = 1 \,\}, \quad \text{where } \beta < \lambda.$$

Let $f \in C_{\mathcal{P}}$. First observe that

$$[\delta_\beta]_{\mathcal{Q}_f} \cap B_\alpha = \begin{cases} \{j_\alpha(\beta)\} & : f(j_\alpha(\beta)) = 1, \\ \emptyset & : f(j_\alpha(\beta)) = 0, \end{cases}$$

for all $\alpha \in \kappa$ and $\beta < \lambda$. It follows that $\mathcal{P} \wedge \mathcal{Q}_f = \bot_{\Pi(\lambda)}$.

Since, by the definition, $f(j_\alpha(0)) = 1$ for each $\alpha \in \kappa$, the block $[\delta_0]_{\mathcal{Q}_f}$ has a non-empty intersection with each block of $\mathcal{P}$. Therefore $\mathcal{P} \vee \mathcal{Q}_f = \top_{\Pi(\lambda)}$.

We have proved that all the partitions $\mathcal{Q}_f$, $f \in C_{\mathcal{P}}$, are complements of $\mathcal{P}$. Since

$$f^{-1}\{1\} = \bigcup_{\beta \in \lambda} \bigl[j_{\delta_\beta}(\beta)\bigr]_{\mathcal{Q}_f},$$

for all $f \in C_{\mathcal{P}}$, the correspondence $f \mapsto \mathcal{Q}_f$ is one-to-one. We conclude that the partition $\mathcal{P}$ has 2^λ complements.

(2) Second assume that $\mathcal{P}$ contains a block B of cardinality λ. We set $X := \lambda \setminus B$ and $\kappa := |X|$. Each injective map $f\colon X \to B$ corresponds to a partition whose non-singleton blocks are exactly $\{\xi, f(\xi)\}$, $\xi \in X$. It is straightforward to see that such partitions are complements of $\mathcal{P}$. Distinct maps clearly correspond to distinct partitions and since there is $|B|^{|X|} = \lambda^\kappa$ injective maps $X \to B$, there are at least λ^κ complements of $\mathcal{P}$.

Let $\mathcal{Q}$ be a complement of $\mathcal{P}$. The intersection of each block of $\mathcal{Q}$ and B is empty or a singleton. It follows that $|\mathrm{supp}(\mathcal{Q})| \leq 2\kappa$. Since λ is infinite, there are $\lambda^{2\kappa} = \lambda^\kappa$ subsets of λ of size 2κ. For each $S \subseteq \lambda$ there are at most $|\mathscr{P}(S \times S)| = 2^{|S|\cdot|S|}$ partitions with support S, hence there are at most $\lambda^\kappa \cdot 2^{2\kappa\cdot 2\kappa} = \lambda^\kappa$ partitions $\mathcal{Q} \in \Pi(\lambda)$ with $|\mathrm{supp}(\mathcal{Q})| \leq 2\kappa$. We conclude that $|\mathrm{compl}(\mathcal{P})| \leq \lambda^\kappa$. □

We get readily from Theorem 5.2 that

Corollary 5.3. *Let λ be infinite and $\mathcal{P} \notin \{\bot_{\Pi(\lambda)}, \top_{\Pi(\lambda)}\}$ be a partition of λ. If $\mathcal{P}$ contains two or more blocks of cardinality κ, then $|\mathrm{compl}(\mathcal{P})| = 2^\lambda$.*

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[5]Observe that $|S_\beta| \geq \mathrm{cf}(\lambda)$.

Theorem 5.2 provides a complete characterization of the cardinals that can be realized as complement counts for infinite cardinals λ: it is precisely those cardinals of the form λ^κ, with $0 \leq \kappa \leq \lambda$ (and $1 \leq \kappa \leq \lambda$ when considering only non-trivial partitions).

6. Results under GCH

Under the Generalized Continuum Hypothesis, the results from the previous sections all simplify greatly, and it is possible to determine possible cardinals of maximal chains, antichains, and sets of complements of single partitions.

Theorem 6.1. *Under GCH, when λ is an infinite cardinal:*

(1) *A maximal chain in $\Pi(\lambda)$ has cardinality*
  - $\log_2 \lambda$, λ, *or* λ^+ *(and all three are always achieved) if λ is a successor cardinal; and*
  - *either λ or λ^+ (and both are achieved) if λ is a limit cardinal.*

(2) *A non-trivial maximal antichain in $\Pi(\kappa)$ has cardinality either λ or λ^+, and both are achieved.*

(3) *Let $\mathcal{P} \notin \{\bot_{\Pi(\lambda)}, \top_{\Pi(\lambda)}\}$ be a partition of $\Pi(\lambda)$. If $\mathcal{P}$ contains a block B with $|\lambda \setminus B| < \mathrm{cf}(\lambda)$, then $\mathcal{P}$ has exactly λ complements, otherwise $\mathcal{P}$ has λ^+ complements.*

Proof. Under GCH every limit cardinal is strong limit and $\lambda = (\log_2 \lambda)^+ = 2^{\log_2 \lambda}$ for a successor cardinal λ. Applying Theorem 3.16 and Corollary 3.20, we conclude that (1) holds true. Part (2) follows from Theorem 4.1 and the fact that atoms and co-atoms of $\Pi(\lambda)$ form maximal antichains of sizes λ and $2^\lambda = \lambda^+$, respectively. Finally, (3) follows from Theorem 5.2 and the equality

$$\lambda^\kappa = \begin{cases} \lambda & : 0 < \kappa < \mathrm{cf}(\lambda), \\ \lambda^+ & : \mathrm{cf}(\lambda) \leq \kappa \leq \lambda, \end{cases}$$

which follows easily from GCH and König's Theorem. □

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