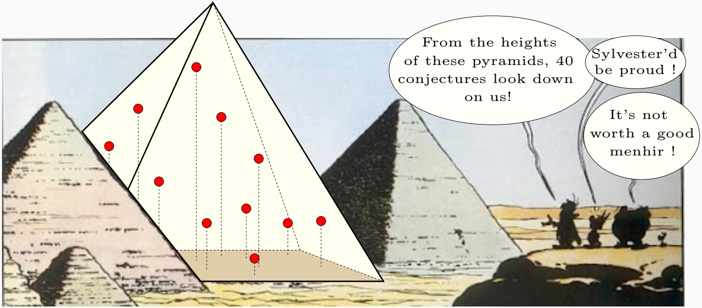
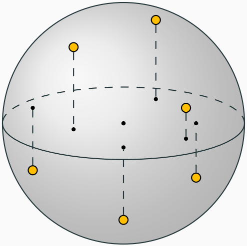
Sylvester's question: the flat floor problem



Ludovic Morin
Joint work with Jean-François Marckert
Friday, 8^{th} November

Notation

- $\mathbb{P}_K(n)$: probability that n points i.i.d. drawn uniformly in a compact convex domain $K \subset \mathbb{R}^d$ of volume 1 are in convex position.



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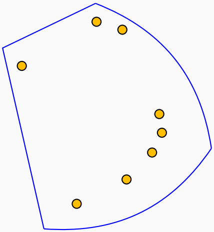
- The problem was widely studied in the case $d = 2$.



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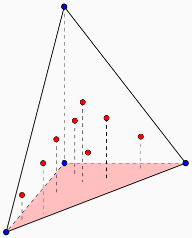
- The problem was widely studied in the case $d = 2$.
- In the end, we will be imposing a "floor" in K.



Sylvester's problem

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What is the probability that 4 points drawn "uniformly" in the plane are in convex position ?

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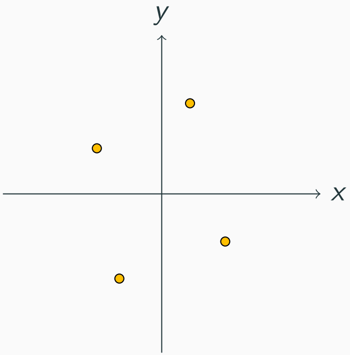

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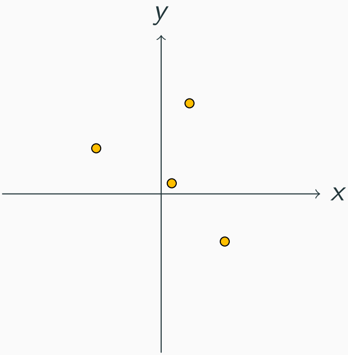


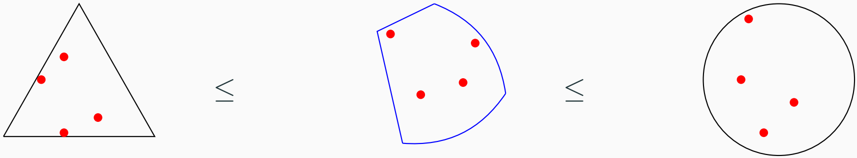
Not OK

A few important results

Theorem : Blaschke, 1917

For all compact convex domain $K \subset \mathbb{R}^2$ of area 1,

$$\frac{2}{3} = \mathbb{P}_{\triangle}(4) \leq \mathbb{P}_K(4) \leq \mathbb{P}_{\bigcirc}(4) = 1 - \frac{35}{12\pi^2}.$$



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Efron's formula :

For all compact convex domain $K \subset \mathbb{R}^2$ of area 1, and A, B, C uniformly distributed in K,

$$\mathbb{P}_K(4) = 1 - 4\mathbb{E}\left[\text{Area}_K(A, B, C)\right].$$

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Theorem : Marckert, Rahmani, 2021

$$\frac{11}{36} = \mathbb{P}_{\triangle}(5) \leq \mathbb{P}_K(5) \leq \mathbb{P}_{\bigcirc}(5) = 1 - \frac{305}{48\pi^2}.$$

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Conjecture (in the plane)

$$\mathbb{P}_{\triangle}(n) \leq \mathbb{P}_K(n) \leq \mathbb{P}_{\bigcirc}(n), \forall n \geq 6.$$

The long-standing d-Sylvester's conjecture

Conjecture (in dimension $d \geq 3$)

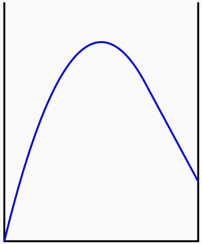
For all compact convex domain $K \subset \mathbb{R}^d$ of volume 1,

$$\mathbb{P}_{\triangle^d}(d+2) \leq \mathbb{P}_K(d+2) \leq \mathbb{P}_{\bigcirc^d}(d+2).$$

The bi-pointed (or 2d flat floor) problem

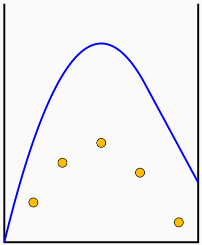
Around the bi-pointed problem

- Let $t \mapsto G(t)$ a concave map of integral 1 on $[0, 1]$



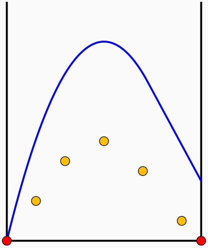
Around the bi-pointed problem

- Let $t \mapsto G(t)$ a concave map of integral 1 on $[0,1]$
- n points i.i.d. uniforms in the convex domain delineated by G



Around the bi-pointed problem

- Let $t \mapsto G(t)$ a concave map of integral 1 on $[0,1]$
- n points i.i.d. uniforms in the convex domain delineated by G
- $\mathbb{Q}_G(n)$: probability that the n points are in convex position together with $(0,0)$ and $(1,0)$.

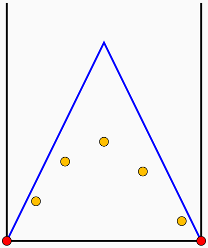


The bi-pointed triangle

Theorem : Bárány, Rote, Steiger, Zhang, 2000

For all $n \geq 1$,

$$\mathbb{Q}_{\triangle}(n) = \frac{2^n}{n!(n+1)!}$$



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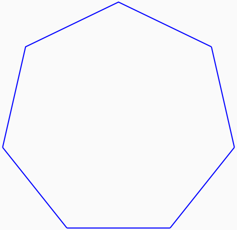
$$\mathbb{Q}_{\triangle}(n) = \frac{2^n}{n!(n+1)!}$$

Theorem : Buchta, 2009

Gives a formula for the probability that k points among n are on the convex hull.

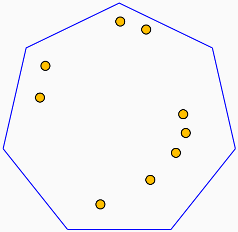
Notation

- Reg_κ : a regular convex κ-gon of area 1



Notation

- Reg_κ : a regular convex κ-gon of area 1
- $\mathbb{P}_\kappa(n) := \mathbb{P}_{\text{Reg}_\kappa}(n)$ the probability that n points i.i.d. drawn uniformly in Reg_κ are in convex position.



Why the bi-pointed problem ? : M.

Theorem : M., 2023

Let $\kappa \geq 3$ an integer. We have

$$\mathbb{P}_\kappa(n) \underset{n\to+\infty}{\sim} C_\kappa \cdot \frac{e^{2n}}{4^n} \frac{\kappa^{3n} r_\kappa^{2n} \sin(\theta_\kappa)^n}{n^{2n+\kappa/2}},$$

where

$$C_\kappa = \frac{1}{\pi^{\kappa/2}\sqrt{d_\kappa}} \frac{\sqrt{\kappa}^{\kappa+1}}{4^\kappa(1+\cos(\theta_\kappa))^\kappa},$$

and

$$d_\kappa = \frac{\kappa}{3\cdot 2^\kappa}\left(2(-1)^{\kappa-1} + (2-\sqrt{3})^\kappa + (2+\sqrt{3})^\kappa\right).$$

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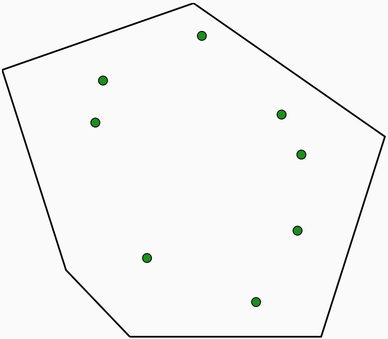
We extended this formula for any convex polygon ! (M.,24+)

A few elements of proof



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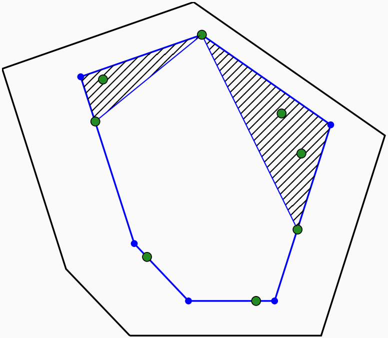

- Conditional on the "contact" points, the probability to be in convex position in K is the product of probabilities to be in convex position in each bi-pointed triangle !

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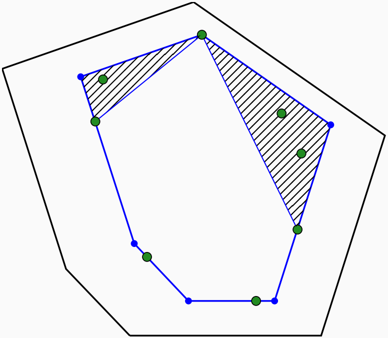
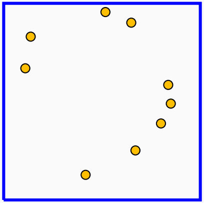


- Conditional on the "contact" points, the probability to be in convex position in K is the product of probabilities to be in convex position in each bi-pointed triangle !
- We integrate on all parallel polygons and all bi-pointed triangles possible inside to get the probability $\mathbb{P}_K(n)$.

Theorem : Valtr, 1995

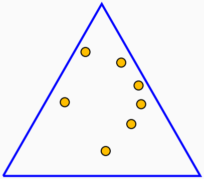
For all $n \geq 3$, $\quad \mathbb{P}_4(n) = \mathbb{P}_\square(n) = \frac{1}{(n!)^2}\binom{2n-2}{n-1}^2.$



Why the bi-pointed problem : Valtr's results

Theorem : Valtr, 1996

For all $n \geq 3$, $$\mathbb{P}_3(n) = \mathbb{P}_\triangle(n) = \frac{2^n(3n-3)!}{(2n)!((n-1)!)^3}.$$



Why the bi-pointed problem ? : Bárány's result

Theorem : Bárány, 1999

For all compact convex domain K with non empty interior,

$$\lim_{n \to +\infty} n^2 \left(\mathbb{P}_K(n)\right)^{\frac{1}{n}} = \frac{e^2}{4} \mathrm{AP}^*(K)^3,$$

where $\mathrm{AP}^*(K)$ is the supremum of affine perimeters of convex subsets of K.

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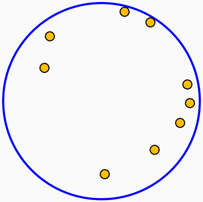
This limit gives a logarithmic equivalent of $\mathbb{P}_K(n)$ but hides lower order terms.

Why the bi-pointed problem : Marckert's result

Theorem : Marckert, 2016

Gives a recursive formula for $\mathbb{P}_{\bigcirc}(n)$;



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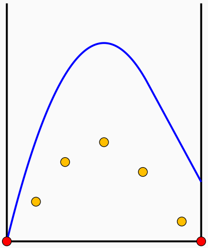
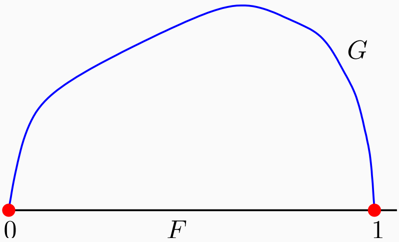


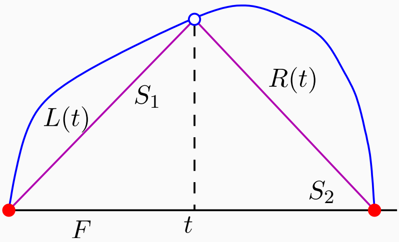
Back to the bi-pointed problem

Theorem

Recursive formula for the bi-pointed case :

$$\mathbb{Q}_G(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} \int_0^1 G(t)\mathbb{Q}_{L(t)}(k)\mathbb{Q}_{R(t)}(n-1-k)L(t)^k R(t)^{n-1-k} dt.$$



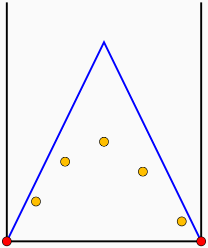


A few bi-pointed computations

Theorem : Bárány, Rote, Steiger, Zhang, 2000

For all $n \geq 1$,

$$\mathbb{Q}_{\triangle}(n) = \frac{2^n}{n!(n+1)!}.$$

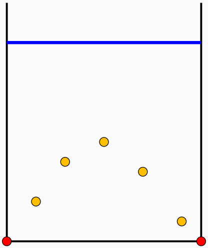


A few bi-pointed computations

Theorem : Marckert, M., 24+

For all $n \geq 1$,

$$\mathbb{Q}_{\square}(n) = \frac{1}{n!(n+1)!}\binom{2n}{n}.$$

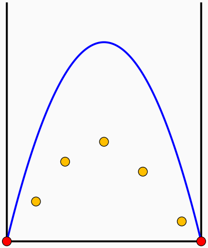


A few bi-pointed computations

Theorem : Marckert, M., 24+

For all $n \geq 1$,

$$\mathbb{Q}_{\text{Parabola}}(n) = \frac{2 \cdot 12^n}{(2n+2)!}.$$

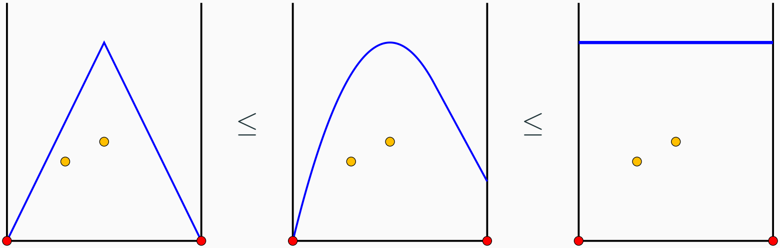


Sylvester's bi-pointed problem

Theorem : Marckert, M., 24+

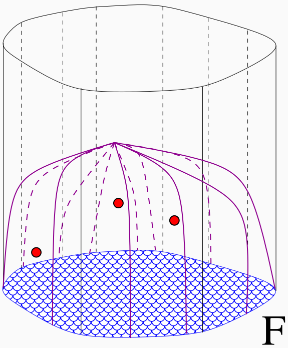
For all concave map G of area 1,

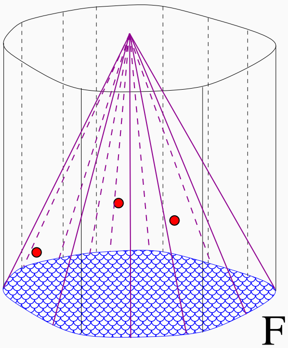
$$\mathbb{Q}_{\triangle}(2) \leq \mathbb{Q}_G(2) \leq \mathbb{Q}_{\square}(2).$$



Larger dimensions

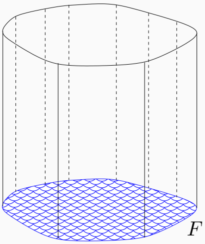
Sylvester's flat floor problem in dimension d





Sylvester's flat floor subprism problem in dimension d

- Pick a convex domain $F \subset \mathbb{R}^{d-1} \times \{0\}$ with $\mathrm{Vol}_{d-1}(F) = 1$



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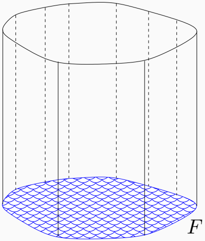
- A prism with base F is a convex domain of the form $F \times [0, h]$ for some $h > 0$.



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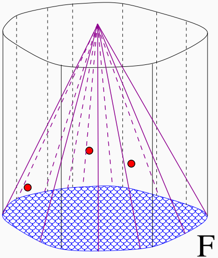
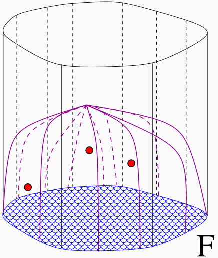
- A prism with base F is a convex domain of the form $F \times [0, h]$ for some $h > 0$.
- A mountain with floor F and apex z (with positive last coordinate $z_d \geq 0$), is the compact convex set

$$\mathrm{Mo}_F(z) = \mathrm{CH}(\{z\} \cup F).$$

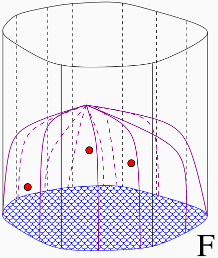


- We will be looking at convex domains K having floor F, contained in a prism with floor F.



The class SubPrism(F)

- We will be looking at convex domains K having floor F, contained in a prism with floor F.
- Each element $K \in \text{SubPrism}(F)$ is characterized by its top function G_K defined by
$G_K(z) := \sup\{y \in \mathbb{R}, (\underbrace{z_1, \cdots, z_{d-1}}_{\in F}, y) \in K\}$



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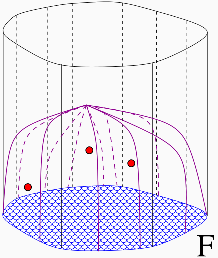
- $Q_K(n)$: probability that n points uniform under G_K are in convex position together with F.



Sylvester's flat floor subprism problem

Theorem : Marckert, M., 24+

For all $K \in \mathrm{SubPrism}(F)$, we have

$$Q_{\mathrm{Mo}_F}(2) \leq Q_K(2) \leq Q_{\mathrm{Prism}(F)}(2).$$

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Proof.

A uniform point in K has coordinates $U = (Z, H_K)$ where $Z \in F$, and H_K is its height "above" F.

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Proof.

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$$\begin{aligned} Q_K(2) &= 1 - 2\mathbb{E}(H_K/d) \\ &= 1 - \int_F \frac{G_K^2(z)}{d} dz \end{aligned}$$

□

The lower bound

Let us write $\mathbb{E}(H_K) = \int_{\mathbb{R}} \mathbb{P}(H_K \geq t)dt$.

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Proof.

Now we write

$$\begin{aligned}\mathbb{P}(H_K \geq t) &= \int_t^{\infty} L_K(s)ds \\ &= 1 - \int_0^t L_K(s)ds\end{aligned}$$

so that it suffices to prove

$$\int_0^t (L_K(s) - L_{\mathrm{Mo}}(s))ds \geq 0 \quad \forall t > 0.$$

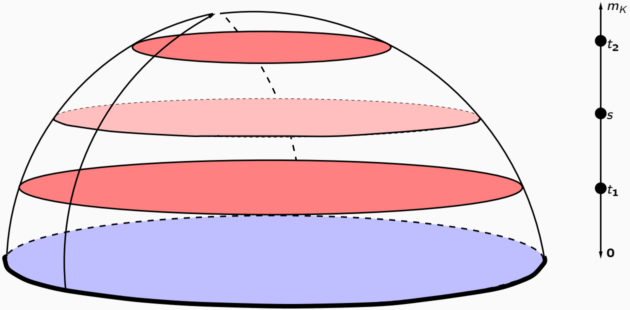
□

The lower bound

- We look for the sign of $t \mapsto L_K(t) - L_{\mathrm{Mo}}(t)$ to obtain the variations of $t \mapsto \int_0^t (L_K(s) - L_{\mathrm{Mo}}(s))ds$.

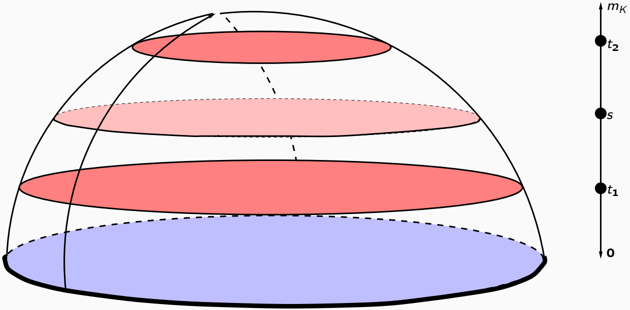
The lower bound

- We look for the sign of $t \mapsto L_K(t) - L_{\text{Mo}}(t)$ to obtain the variations of $t \mapsto \int_0^t (L_K(s) - L_{\text{Mo}}(s))ds$.
- The idea is to prove that $t \mapsto L_K(t)^{1/(d-1)} - L_{\text{Mo}}(t)^{1/(d-1)}$ is concave.



We can see that

$$\mathrm{Layer}_K(s) \supset \frac{t_2 - s}{t_2 - t_1}\mathrm{Layer}_K(t_1) + \frac{s - t_1}{t_2 - t_1}\mathrm{Layer}_K(t_2),$$



We can see that

$$\mathrm{Layer}_K(s) \supset \frac{t_2 - s}{t_2 - t_1}\mathrm{Layer}_K(t_1) + \frac{s - t_1}{t_2 - t_1}\mathrm{Layer}_K(t_2),$$

so that by the Brunn-Minkowski inequality,

$$L_K(s)^{\frac{1}{d-1}} \geq \frac{t_2 - s}{t_2 - t_1}L_K(t_1)^{\frac{1}{d-1}} + \frac{s - t_1}{t_2 - t_1}L_K(t_2)^{\frac{1}{d-1}}.$$

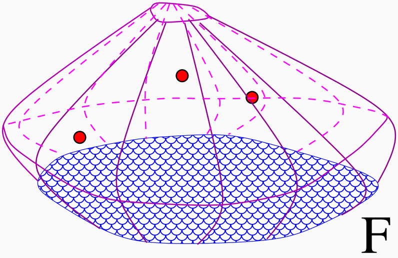
What if we remove the subprism hypothesis ?

Theorem : Marckert, M., 24+

For all domain K with floor F, we have

$$Q_{\mathrm{Mo}_F}(2) \leq Q_K(2) < 1,$$

and for all $\alpha > 0$ small enough, there exists a domain K with floor F such that $Q_K(2) \geq 1 - \alpha$.



Bounds in dimension 3

Theorem : Marckert, M., 24+

Take any floor F (compact convex subset of $\mathbb{R}^2 \times \{0\}$ with area 1), any unit mountain Mo_F with floor F. For all $n \geq 0$,

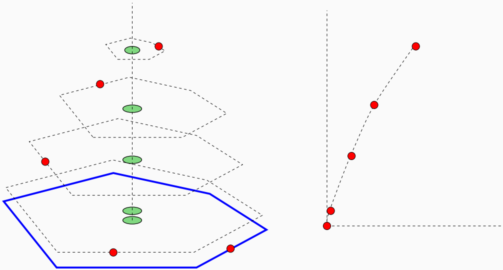
$$Q_{\mathrm{Mo}_F}(n) \geq Y_n := \frac{2^n}{n!} \prod_{j=1}^{n} \frac{1}{3j-1}.$$

The first terms of the sequence (Y_n), for $n \geq 0$, are the following :

$$1, 1, \frac{1}{5}, \frac{1}{60}, \frac{1}{1320}, \frac{1}{46200}, \frac{1}{2356200}, \frac{1}{164934000}, \frac{1}{15173928000}$$

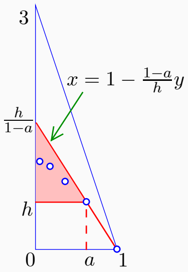
Lemma

For some valid floor F, consider a sequence of points such $z_1, \cdots, z_n$ in $F \times [0,3]$ such that that $0 \leq \pi_3(z_1) \leq \cdots \leq \pi_3(z_n)$ (their third coordinates are non decreasing). If the points $(1,0)$, $(a(z_1), \pi_3(z_1)), \cdots, (a(z_n), \pi_3(z_n))$ are in convex position in the 2D-rectangle $[0,1] \times [0,3]$, then the z_i together with the floor F, are in convex position in $\mathbb{R}^3$.



Lemma

For some valid floor F, consider a sequence of points such $z_1, \cdots, z_n$ in $F \times [0,3]$ such that that $0 \leq \pi_3(z_1) \leq \cdots \leq \pi_3(z_n)$ (their third coordinates are non decreasing). If the points $(1,0)$, $(a(z_1), \pi_3(z_1)), \cdots, (a(z_n), \pi_3(z_n))$ are in convex position in the 2D-rectangle $[0,1] \times [0,3]$, then the z_i together with the floor F, are in convex position in $\mathbb{R}^3$.



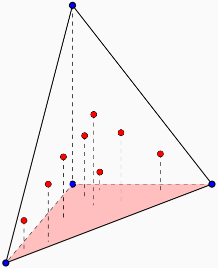
The tri-pointed tetrahedron

Theorem : Marckert, M., 24+

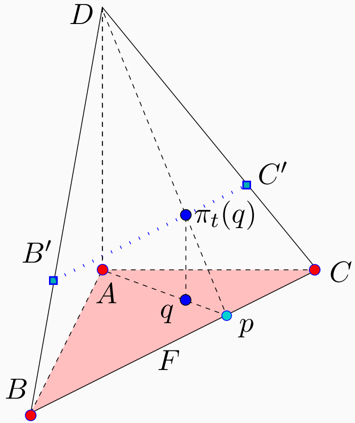
Consider a tetrahedron $\triangle^3$ with vertices $A = (0,0,0)$, $B = (1,0,0)$, $C = (0,1,0)$, $D = (0,0,6)$, and floor $F = \mathrm{CH}(A, B, C)$. We have

$$\ell_n \le Q_{\triangle^3}(n) \le u_n, \quad \text{with the sequences}$$

$$\ell_n = 6\frac{(n-1)!\,n!}{(2n+1)!}\sum_{k=0}^{n-1}\ell_k\,\ell_{n-1-k}, \qquad u_n = \frac{6}{(n+2)(n+1)n}\sum_{k=0}^{n-1}u_k\,u_{n-1-k}.$$



The example of the lower bound



Thanks for
your attention!