Tverberg theorems over discrete sets of points

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Abstract. This paper discusses Tverberg-type theorems with coordinate constraints (i.e., versions of these theorems where all points lie within a subset $S \subset \mathbb{R}^d$ and the intersection of convex hulls is required to have a non-empty intersection with S). We determine the m-Tverberg number, when $m \geq 3$, of any discrete subset S of $\mathbb{R}^2$ (a generalization of an unpublished result of J.-P. Doignon). We also present improvements on the upper bounds for the Tverberg numbers of $\mathbb{Z}^3$ and $\mathbb{Z}^j \times \mathbb{R}^k$ and an integer version of the well-known positive-fraction selection lemma of J. Pach.

Introduction

Consider n points in $\mathbb{R}^d$ and a positive integer $m \geq 2$. If $n \geq (m-1)(d+1)+1$, the points can always be partitioned into m subsets whose convex hulls contain a common point. This is the celebrated theorem of Tverberg [**Tve66**], which has been the topic of many generalizations and variations since it was first proved in 1966 [**BS18, DLGMM19**]. In this paper we focus on new versions of Tverberg-type theorems where some of the coordinates of the points are restricted to discrete subsets of a Euclidean space. The associated discrete Tverberg numbers are much harder to compute than their classical real-version counterparts (see for instance the complexity discussion of [**Onn91**]).

We begin our work remembering the following unpublished Tverberg-type result of Doignon. Consider n points with coordinates in $\mathbb{Z}^2$ and a positive integer $m \geq 3$. If $n \geq 4m-3$, then the points can be partitioned into m subsets whose convex hulls contain a common point in $\mathbb{Z}^2$. According to Eckhoff [**Eck00**] this result was stated by Doignon in a conference.

A partition of points where the intersection of the convex hulls contains at least one lattice point is called an *integer m-Tverberg partition* and such a common point is an *integer Tverberg point* for that partition. Regarding the case $m = 2$, the integer 2-Tverberg partitions are called *integer Radon partitions* and the integer Tverberg points are called *integer Radon points*. Any configuration of at least six points in $\mathbb{Z}^2$ admits an integer Radon partition. This was proved by Doignon in his PhD thesis [**Doi75**] and later discovered independently by Onn [**Onn91**]. All these values for $\mathbb{Z}^2$ are optimal as shown by the following examples. The 5-point

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configuration $\{(0,0),(0,1),(2,0),(1,2),(3,2)\}$, exhibited by Onn in the cited paper, has no integer Radon partition. To address the optimality when $m \geq 3$, consider the set $\{(i,i),(i,-i+1)\colon i = -m+2,-m+3,\ldots,m-2,m-1\}$. (According to Eckhoff [**Eck00**], this set was proposed by Doignon during the aforementioned conference.) This set has $4m-4$ points and a moment of reflection might convince the reader that it has no integer m-Tverberg partition.

More generally, one can define the *Tverberg number* $\mathrm{Tv}(S,m)$ for any subset S of $\mathbb{R}^d$ and an integer $m \geq 2$ as the smallest positive integer n with the following property: Any multiset of n points in S admits a partition into m subsets $A_1, A_2, \ldots, A_m$ with

$$\left(\bigcap_{i=1}^{m} \mathrm{conv}(A_i)\right) \cap S \neq \varnothing.$$

Here, by "partition of a multiset", we mean that each element of a multiset A is contained in a number of sub-multisets $A_1,\ldots,A_m$ so that the sum of its multiplicities in the A_i is equal to its multiplicity in A. If no such number exists, we say that $\mathrm{Tv}(S,m)=\infty$. Note that Doignon's theorem, together with the discussion that follows, allows us to state

$$\mathrm{Tv}(\mathbb{Z}^2,m) = \begin{cases} 6 & \text{if } m=2, \\ 4m-3 & \text{otherwise.} \end{cases}$$

Our contributions. All our results deal with discrete versions of Tverberg's theorem. This latter theorem, in its classical version, and its variations have many applications, e.g., to the computation of simplicial data depth that is important in statistics [**RH99**], to data classification algorithms [**DLH19**], and to the study and computation of centerpoints whose applications in integer optimization are well-documented in [**BO16**]; see also the surveys [**BS18, DLGMM19**] and the references therein. We can thus expect that our results extend the range of these applications. Moreover, they also form a contribution to the theory of abstract convexity; see the book by van de Vel for an introduction to this theory [**vdV93**].

Our first main result generalizes Doignon's theorem. We determine the exact m-Tverberg number (when m is at least three) for any discrete subset S of $\mathbb{R}^2$, as considered in [**DLHRS17**]. Before stating this result we recall the *Helly number* $\mathrm{H}(S)$ of a discrete subset S of $\mathbb{R}^d$ as the smallest positive integer h with the following property: Suppose $\mathcal{F}$ is a finite family of convex sets in $\mathbb{R}^d$, and that $\bigcap \mathcal{G}$ (the intersection of all elements in $\mathcal{G}$) intersects S in at least one point for every subfamily $\mathcal{G}$ of $\mathcal{F}$ having at most h members. Then $\bigcap \mathcal{F}$ intersects S in at least one point. If no such integer exists, we say that $\mathrm{H}(S)=\infty$.

Then we have the following theorem. (The theorem is stated for S with finite Helly number, as any $S \subset \mathbb{R}^d$ with $\mathrm{H}(S)=\infty$ has $\mathrm{Tv}(S,m)=\infty$ for all $m \geq 2$ [**Lev51**].)

THEOREM 1. *Suppose S is a discrete subset of $\mathbb{R}^2$ with $\mathrm{H}(S) < \infty$.*

Then for all $m \geq 2$, we have

$$\mathrm{Tv}(S,m) = \mathrm{H}(S)(m-1)+1,$$

except for the case $m=2$, $\mathrm{H}(S)=4$, for which we have

$$5 \leq \mathrm{Tv}(S,2) \leq 6$$

and both values are possible.

In particular we present a proof of Doignon's theorem, the special case of Theorem 1 where $S = \mathbb{Z}^2$ (noting that $\mathrm{H}(\mathbb{Z}^d) = 2^d$ as established in [**Doi73**]).

> **Remark:** Theorem 1 shows that, except for an exceptional case, the Tverberg number of a planar set can be written as a function of its Helly number (see [**Ave13, AGS$^+$17**] and the references there). For the case $\mathrm{H}(S) = 4$, the bounds on $\mathrm{Tv}(S, 2)$ given above cannot be improved. For example, $S' = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\mathbb{Z}^2$ both have Helly number four, but $\mathrm{Tv}(\mathbb{Z}^2, 2) = 6$, while the pigeonhole principle implies that $\mathrm{Tv}(S', 2) = 5$.

Our second main result improves the upper bound on the integer Tverberg numbers for the three-dimensional case $S = \mathbb{Z}^3$.

THEOREM 2. *The following inequality holds for all $m \geq 2$:*

$$\mathrm{Tv}(\mathbb{Z}^3, m) \leq 24m - 31.$$

Our third main result is an inequality that will be used to derive improved bounds on S-Tverberg numbers when S is a product of a Euclidean space with some subset S' of a Euclidean space.

THEOREM 3. *Let $S' \subset \mathbb{R}^j$. Then for all integers $k \geq 1$ and all $m \geq 2$, we have*

$$\mathrm{Tv}(S' \times \mathbb{R}^k, m) \leq \mathrm{Tv}(S', \mathrm{Tv}(\mathbb{R}^k, m)).$$

For example, choosing $S' = \mathbb{Z}^j$ leads to the "mixed integer" case. Then Theorem 3 implies that for all positive integers j, k and all $m \geq 2$, we have

$$\mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) \leq \mathrm{Tv}(\mathbb{Z}^j, \mathrm{Tv}(\mathbb{R}^k, m)).$$

Moreover, we will use Theorem 3 to obtain the following bound:

$$2^j(m-1)(k+1) + 1 \leq \mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) \leq j2^j(m-1)(k+1) + 1. \tag{1}$$

Our fourth main result is a generalization of *Pach's positive-fraction selection lemma* [**Pac98**] (see [**KKP$^+$15**] for related bounds). Here is Pach's result: Given an integer d, there exists a constant c_d such that for any set P of n points in $\mathbb{R}^d$, there exists a point $\boldsymbol{q} \in \mathbb{R}^d$, and $d+1$ disjoint subsets of P, say $P_1, \ldots, P_{d+1}$, such that $|P_i| \geq c_d \cdot n$ for all i and every simplex defined by a transversal of $P_1, \ldots, P_{d+1}$ contains $\boldsymbol{q}$. (By "transversal", we mean a set containing exactly one element from each P_i.)

Unfortunately the point $\boldsymbol{q}$ need not be an integer point; furthermore, the proof uses the so-called "second selection lemma" that currently does not exist for integer points (see Pach [**Pac98**] and Matoušek [**Mat02**, Chapter 9]). In Section 4, we strengthen the above theorem, such that, as a consequence, the theorem now extends to the integer case—indeed, to any scenario where one has points of high half-space depth in the following sense:

Given a finite set P of points in $\mathbb{R}^d$ and a point $\boldsymbol{q} \in \mathbb{R}^d$, we say that $\boldsymbol{q}$ is *of half-space depth t with respect to P* if any half-space containing $\boldsymbol{q}$ contains at least t points of P (when the context is clear, we will simply say that $\boldsymbol{q}$ is of *depth t*). Then here is our theorem.

THEOREM 4. *For any integer $d \geq 1$ and real number $\alpha \in (0, 1]$, there exists a constant $c_{d,\alpha}$ such that the following holds. For any set P of n points in $\mathbb{R}^d$ and any point $\boldsymbol{q} \in \mathbb{R}^d$ of half-space depth at least $\alpha \cdot n$, there exist $d+1$ disjoint subsets of P, say $P_1, \ldots, P_{d+1}$, such that*

- *$|P_i| \geq c_{d,\alpha} \cdot n$ for $i = 1, \ldots, d+1$, and*
- *every simplex defined by a transversal of $P_1, \ldots, P_{d+1}$ contains $\boldsymbol{q}$.*

Remark: Our proof yields a constant $c_{d,\alpha}$ whose value is exponentially decreasing with the dimension d.

Note that the existence of integer points of high half-space depth (Lemma 1) together with Theorem 4 implies the following integer version of the positive-fraction selection lemma.

COROLLARY 1. *Let P be a set of $n \geq d+1$ points in $\mathbb{Z}^d$. Then there exists a point $\boldsymbol{q} \in \mathbb{Z}^d$, and $d+1$ disjoint subsets of P, say $P_1, \ldots, P_{d+1}$, such that $|P_i| \geq c_{d,2^{-d}} \cdot n$ for all $i = 1, \ldots, d+1$, and the simplex defined by every transversal of $P_1, \ldots, P_{d+1}$ contains $\boldsymbol{q}$.*

Remark: In particular, this implies that $\boldsymbol{q}$ belongs to many *distinct* Tverberg partitions—at least $\left(\lceil c_{d,2^{-d}} \cdot n \rceil !\right)^d$ *distinct* Tverberg partitions, with each such Tverberg partition containing $\lceil c_{d,2^{-d}} \cdot n \rceil$ sets. To see this, assume without loss of generality that the P_i are equally sized with cardinality equal to the lower bound given in our theorem, say $|P_i| = k$. Then consider them as columns of a matrix A. Each row of A is a transversal of the P_i and contains $\boldsymbol{q}$ in its convex hull. Thus, regardless of the ordering of the points in each column, we get a Tverberg partition of P into k subsets where each subset is a row of A. Matrices obtained by permuting independently the points in each column provide a same partition only if the same permutation is applied to each column, and thus there are at least $(k!)^d$ distinct Tverberg partitions in total.

Related Results and Organization of the paper. The problem of computing the Tverberg number for $\mathbb{Z}^d$ with $d \geq 3$ seems to be challenging. It has been identified as an interesting problem since the 1970's [**GS79**] and yet the following inequalities are almost all that is known about this problem: for the general case, De Loera et al. [**DLHRS17**] proved

$$2^d(m-1)+1 \leq \mathrm{Tv}(\mathbb{Z}^d, m) \leq d2^d(m-1)+1, \quad \text{for } d \geq 1 \text{ and } m \geq 2. \tag{2}$$

Two special cases get better bounds:

$$\mathrm{Tv}(\mathbb{Z}^3, 2) \leq 17 \qquad \text{and} \qquad 5 \cdot 2^{d-2}+1 \leq \mathrm{Tv}(\mathbb{Z}^d, 2) \quad \text{for } d \geq 1. \tag{3}$$

The left-hand side inequality is due to Bezdek and Blokhuis [**BB03**] and the right-hand side was proved by Doignon in his PhD thesis (and rediscovered by Onn).

Previously established bounds for the "mixed integer" case include the bounds for the Radon number (2-Tverberg number) found by Averkov and Weismantel [**AW12**].

$$2^j(k+1)+1 \leq \mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, 2) \leq (j+k)2^j(k+1)-j-k+2.$$

Later, De Loera et al. [**DLHRS17**] gave the following general bound for all mixed integer Tverberg numbers:

$$\mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) \leq (j+k)2^j(m-1)(k+1)+1.$$

Note that (1) above is a simultaneous improvement of both of these if $k \geq 1$ or if $k = 0$ and $j \leq 1$.

Previous bounds and related work on more general S-Tverberg numbers can also be found in [**DLHRS17**], including the following bound for any discrete $S \subset \mathbb{R}^d$-and $m \geq 2$:

$$\mathrm{Tv}(S, m) \leq \mathrm{H}(S)(m-1)d + 1.$$

The following lemma about integer points of high half-space depth is used throughout the paper. See [**BO16**] for a proof and related results.

Lemma 1. *Consider a multiset A of points in $\mathbb{Z}^d$. If $|A| \geq 2^d(m-1)+1$ (counting multiplicities), then there is a point $\boldsymbol{q} \in \mathbb{Z}^d$ of half-space depth m in A.*

The paper is organized as follows. In Section 1, we prove Theorem 1 using a somewhat similar strategy to Birch's proof of the planar case of the original Tverberg theorem [**Bir59**]. In Section 2, we prove Theorem 2 using techniques reminiscent of those in [**DLHRS17**]. In Section 3, we prove Theorem 3 and collect some consequences of the main theorems presented above, including (1). Finally, in Section 4, we prove Theorem 4 by proving a new lemma and adapting the methods of Pach in [**Pac98**].

1. Tverberg Numbers over Discrete Subsets of $\mathbb{R}^2$: Proof of Theorem 1

We start with the proof of the special case $S = \mathbb{Z}^2$ (where $\mathrm{H}(\mathbb{Z}^2) = 4$) because it nicely illustrates the techniques of the more general proof of Theorem 1.

1.1. Proof of the special case $S = \mathbb{Z}^2$.

The lower bound for the theorem is given by Inequality (2) given in the introduction. The upper bound will follow easily from the following two lemmas, the first covering the case $m \geq 3$ and the second the case $m = 2$.

Lemma 2. *Consider a multiset A of points in $\mathbb{R}^2$ with $|A| \geq 4m-3$ and $m \geq 3$. If $\boldsymbol{p} \notin A$ is a point of depth m, then there is an m-Tverberg partition of A with $\boldsymbol{p}$ as Tverberg point.*

Lemma 3. *Consider a multiset A of points in $\mathbb{R}^2$ with $|A| \geq 6$. If $\boldsymbol{p} \notin A$ is a point of depth two, then there is a Radon partition of A with $\boldsymbol{p}$ as Radon point.*

Proof of Theorem 1 when $S = \mathbb{Z}^2$. The inequality $\mathrm{Tv}(\mathbb{Z}^2, m) \geq 4m-3$ is given by Inequality (2). The proof consists thus in establishing the upper bound.

We start with the case $m \geq 3$. Consider a multiset A of at least $4m-3$ points in $\mathbb{Z}^2$. By Lemma 1, A has an integer point $\boldsymbol{p}$ of depth m. If $\boldsymbol{p}$ is an element of A with multiplicity μ, then take the singletons $\{\boldsymbol{p}\}$ as μ of the sets in the Tverberg partition. Then $\boldsymbol{p}$ is a point of depth $m-\mu$ of the remaining $4m-\mu-3$ points. If $\mu \geq m$, we are done, and if $\mu = m-1$, the point $\boldsymbol{p}$ is in the convex hull of the remaining points and we take them to be the last set in the desired partition. If $\mu \leq m-3$, according to Lemma 2, there is an $(m-\mu)$-Tverberg partition of the remaining points with $\boldsymbol{p}$ as Tverberg point. There is thus an m-Tverberg partition of A with $\boldsymbol{p}$ as Tverberg point. The case $\mu = m-2$ is treated similarly with the help of Lemma 3 in place of Lemma 2.

For the case $m = 2$, we proceed similarly, except that we start with a set A of 6 points in $\mathbb{Z}^2$, in order to be able to apply Lemma 3. □

Proof of Lemma 2. Since $\boldsymbol{p}$ is not in A, up to a radial projection, we can assume that the points of A are arranged in a circle around $\boldsymbol{p}$. This is without loss of generality since points in A contain $\boldsymbol{p}$ in their convex hull if and only if their radial projections from $\boldsymbol{p}$ do so.

Define q and r to be respectively the quotient and the remainder of the Euclidean division of $|A|$ by m. Define moreover e to be $\lceil \frac{r}{q} \rceil$.

Suppose first that $\boldsymbol{p}$ is a point of depth $m+e$. Since $qe \geq r$, we can choose k_i with $i \in [q]$, and $0 \leq k_i \leq e$, such that $k_1 + k_2 + \cdots + k_q = r$. Then we arbitrarily select a first point in A, and label clockwise the points with elements in $[m]$ according to the following pattern:

$$1, 2, \ldots, m, 1, 2, \ldots, k_1, 1, 2, \ldots, m, 1, 2, \ldots, k_2, \ldots, 1, 2, \ldots, m, 1, 2, \ldots, k_q.$$

Each half-plane delimited by a line passing through $\boldsymbol{p}$ contains at least $m+e$ consecutive points in this pattern and thus has at least one point with each of the m different labels. Partitioning the points so that each subset consists of all points with a fixed label, we therefore obtain an m-Tverberg partition with $\boldsymbol{p}$ as Tverberg point.

Suppose now that $\boldsymbol{p}$ is not a point of depth $m+e$. There is thus a closed half-plane H_+, delimited by a line passing through $\boldsymbol{p}$, with $|H_+ \cap A| < m+e$. The complementary closed half-plane to H_+, which we denote by H_-, is such that $|H_- \cap A| > 4m-3-(m+e)$. Define ℓ to be $|H_- \cap A|$. Since $e \leq \frac{m}{3}$, we have $\ell \geq 2m$. Denote the points in $H_- \cap A$ by $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_\ell$, where the indices are increasing when we move clockwise. We label $\boldsymbol{x}_i$ with $r+i$ from $\boldsymbol{x}_1$ to $\boldsymbol{x}_{m-r}$, and then label $\boldsymbol{x}_{m-r+j}$ with j from $\boldsymbol{x}_{m-r+1}$ to $\boldsymbol{x}_m$. We then continue labeling the points of A, still moving clockwise, using labels $1, 2, \ldots, m, \ldots, 1, 2, \ldots m, 1, 2, \ldots r$. See Figure 1 for an illustration of the labeling scheme.

The labeling pattern is such that any sequence of m consecutive points either has all m labels, or contains the two consecutive points $\boldsymbol{x}_m$ and $\boldsymbol{x}_{m+1}$. Let us prove that any closed half-plane H delimited by a line passing through $\boldsymbol{p}$ contains at least one point with each label. Once this is proved, the conclusion will be immediate by taking as subsets of points those with same labels, as above.

If such an H does not simultaneously contain $\boldsymbol{x}_m$ and $\boldsymbol{x}_{m+1}$, then H contains at least one point with each label. Consider thus a closed half-plane H delimited by a line passing through $\boldsymbol{p}$ and containing $\boldsymbol{x}_m$ and $\boldsymbol{x}_{m+1}$. Note that according to Farkas' lemma ([**Sch03**] Theorem 5.3) , $\boldsymbol{x}_{m+1}$ cannot be separated from $\boldsymbol{x}_1$ and $\boldsymbol{x}_\ell$ by a line passing through $\boldsymbol{p}$, since they are all in H_-. This means that either

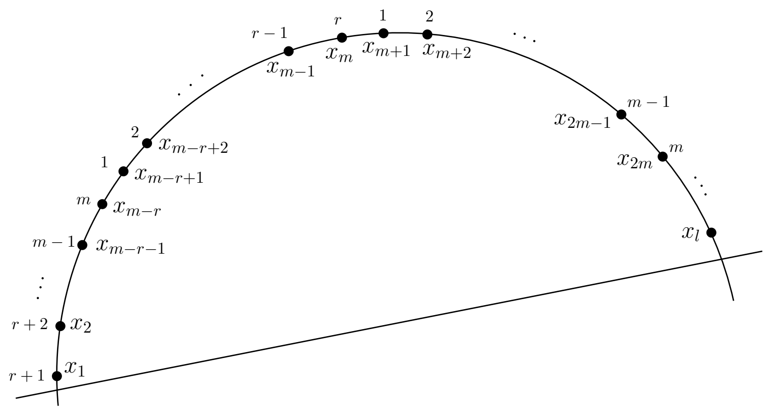

FIGURE 1. Labeling of the points in the half-plane H_-.

H contains $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{m+1}$, or H contains $\boldsymbol{x}_{m+1}, \boldsymbol{x}_{m+2}, \ldots, \boldsymbol{x}_\ell$. In any case, H contains a point with each label. □

PROOF OF LEMMA 3. As before, we assume that the points in A are arranged on a circle centered at $\boldsymbol{p}$. If $|A|$ is even, it clearly suffices to label the points in order, alternating between 1 and 2. We may therefore assume that $|A|$ is odd, and thus $|A| \geq 7$. If $\boldsymbol{p}$ is a point of depth three, it suffices to label the points alternating labels between 1 and 2, except with two consecutive points labeled 1. If $|A|$ is odd but $\boldsymbol{p}$ is not a point of depth three, then $|A| \geq 7$ and there is a half-plane H_+ containing $\boldsymbol{p}$ with $|H_+ \cap A| = 2$. The complementary half-plane H_- has $|H_- \cap A| \geq 5$ and we follow a similar strategy as in the second half of Lemma 2. Namely, we denote the points in $H_- \cap A$ by $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_\ell$, where the indices are increasing when we move clockwise. Then we label $\boldsymbol{x}_1$ with 2, $\boldsymbol{x}_2$ with 1, $\boldsymbol{x}_3$ with 1, and $\boldsymbol{x}_4$ with 2. We continue this pattern for $\alpha \geq 5$, labeling $\boldsymbol{x}_\alpha$ with 1 if α is odd, and $\boldsymbol{x}_\alpha$ with 2 if α is even. For the remaining points in A we continue labeling clockwise, alternating between the labels 1 and 2.

The labeling pattern is such that any sequence of 2 consecutive points either has both labels, or contains the two consecutive points $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$. As in Lemma 2 it suffices to show that any closed half-plane H delimited by a line passing through $\boldsymbol{p}$ contains at least one point with each label.

If such an H does not simultaneously contain $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$, then H contains at least one point with each label. Consider thus a closed half-plane H delimited by a line passing through $\boldsymbol{p}$ and containing $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$. Note that according to Farkas' lemma, $\boldsymbol{x}_3$ cannot be separated from $\boldsymbol{x}_1$ and $\boldsymbol{x}_4$ by a line passing through $\boldsymbol{p}$, since they are all in H_-. This means that either H contains $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$, or H contains $\boldsymbol{x}_3$ and $\boldsymbol{x}_4$. In any case, H contains a point with each label. □

1.2. Proof of the general case.

The proof of the general case is split into three lemmas addressing the lower bound, the upper bound for $\mathrm{H}(S) \geq 4$, and the upper bound for $\mathrm{H}(S) \leq 3$, respectively.

LEMMA 4. *For any discrete set $S \subset \mathbb{R}^2$ with finite Helly number $\mathrm{H}(S)$ and for all $m \geq 2$, we have $\mathrm{Tv}(S,m) \geq \mathrm{H}(S)(m-1)+1$.*

PROOF. It suffices to exhibit a subset $R \subseteq S$, of cardinality $|R| = \mathrm{H}(S)(m-1)$, with the property that no point in S is of half-space depth m with respect to R. By Lemma 2.6 in [**ADLS17**], there exists a set R' of $\mathrm{H}(S)$ points in S in convex position with the property that $\mathrm{conv}(R') \cap S = R'$. Let R be the multiset given by taking each point in R' with multiplicity $m-1$, so $|R| = \mathrm{H}(S)(m-1)$. No points of $S - R$ are in $\mathrm{conv}(R)$. Since R' was taken to be in convex position, for any point in R, there exists a line such that one side of that line has at most $m-1$ points in R. Thus S cannot contain a point of half-space depth m with respect to R. □

LEMMA 5. *For any discrete set $S \subset \mathbb{R}^2$ with finite Helly number $\mathrm{H}(S) \geq 4$ and for all $m \geq 2$, we have $\mathrm{Tv}(S,m) \leq \mathrm{H}(S)(m-1)+1$, except when $m = 2$ and $\mathrm{H}(S) = 4$ simultaneously, in which case we only have $\mathrm{Tv}(S,2) \leq 6$.*

PROOF. The proof of Lemma 5 is the same as the proof of Theorem 1 for $S = \mathbb{Z}^2$, except that we use the following result (Theorem 2 in [**BO16**] with μ being the uniform probability measure on A) in place of Lemma 1: For any discrete subset S of a Euclidean space with finite Helly number $\mathrm{H}(S)$, and any set $A \subseteq S$ with $|A| \geq \mathrm{H}(S)(m-1)+1$, there exists a point $\boldsymbol{p} \in S$ that is of depth m with respect to A. □

LEMMA 6. *For a discrete set $S \subset \mathbb{R}^2$ with finite Helly number $\mathrm{H}(S) \leq 3$ and for all $m \geq 2$, we have $\mathrm{Tv}(S,m) \leq \mathrm{H}(S)(m-1)+1$.*

PROOF. The case $\mathrm{H}(S) = 1$ implies that S consists of a single point, so the result trivially follows. If $\mathrm{H}(S) = 2$, it must be that all points in S are collinear (as any set containing a non-degenerate triangle has Helly number at least 3), and thus we can take the median of any set with at least $2(m-1)+1$ points in S as the desired m-Tverberg point. Thus for the remainder of the proof we assume that $\mathrm{H}(S) = 3$.

Given any set A of $\mathrm{H}(S)(m-1)+1 = 3m-2$ points in S, there exists an m-Tverberg partition, say $\mathcal{P}$ by the classical Tverberg theorem. We denote by $K_1, \ldots, K_m$ the m convex hulls of the subsets in $\mathcal{P}$. As $\bigcap_{1\leq i\leq m} K_i$ is a nonempty polygon, say Q, (possibly just a point or line segment) we pick an arbitrary vertex $\boldsymbol{q}$ of Q.

It suffices to show that $\boldsymbol{q} \in S$. We can assume that $\boldsymbol{q}$ is not a vertex of any K_i, since otherwise $\boldsymbol{q} \in A \subseteq S$.

Since $\boldsymbol{q}$ is a vertex of Q, it must be contained in a one dimensional face F_1 of at least one K_i. Since $\boldsymbol{q}$ is not a vertex of any K_i, in fact $\boldsymbol{q}$ is in the relative interior of F_1. For $\boldsymbol{q}$ to be a vertex of Q, it must also be in another one dimensional face, say F_2, of some other K_i, such that F_1 is not parallel to F_2. Moreover, $\boldsymbol{q}$ must be in the relative interior of F_2, and we also have $F_1 \cap F_2 = \{\boldsymbol{q}\}$.

Denote by $\{\boldsymbol{a},\boldsymbol{b}\}$ and $\{\boldsymbol{c},\boldsymbol{d}\}$ the vertices of F_1 and F_2 respectively. We have that $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d} \in S$ are the vertices of a convex quadrilateral with diagonals intersecting at $\boldsymbol{q}$, by the assumption that F_1 and F_2 are non parallel. Out of the four triangles $\mathrm{conv}(\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}), \mathrm{conv}(\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{d}\}), \mathrm{conv}(\{\boldsymbol{a},\boldsymbol{c},\boldsymbol{d}\}), \mathrm{conv}(\{\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}\})$, any three have at least one vertex in common, and therefore intersect in S. Since $\mathrm{H}(S) = 3$, the four triangles therefore all intersect in S. This intersection point is $\boldsymbol{q}$, the point where the diagonals of the quadrilateral intersect. $\square$

2. Tverberg Numbers over $\mathbb{Z}^3$: Proof of Theorem 2

As in Section 1, the proof of Theorem 2 will follow from some lemmas.

We state the following lemma without proof; it is a consequence, upon close inspection of the argument, of the proof of the main theorem in the already mentioned paper by Bezdek and Blokhuis [**BB03**].

LEMMA 7. *Consider a multiset A of at least 17 points in $\mathbb{R}^3$ and a point $\boldsymbol{p}$ of depth three in A. There is a bipartition of A into two subsets whose convex hulls contain $\boldsymbol{p}$.*

The next lemma will be proved later in this section.

LEMMA 8. *Consider a multiset A of points in $\mathbb{R}^3$ with $|A| \geq 4m+9$ and $m \geq 2$. If $\boldsymbol{p} \notin A$ is a point of depth $3m-3$, then there is an m-Tverberg partition of A with $\boldsymbol{p}$ as Tverberg point.*

The proof of Theorem 2 follows then from these two lemmas.

PROOF OF THEOREM 2. Consider a multiset A of $24m - 31$ points in $\mathbb{Z}^3$. For the case $m = 2$, note that Lemma 1 yields a point of depth three, and we can then apply Lemma 8 to obtain the result. Assume that $m \geq 3$. Applying Lemma 1, A has an integer point $\boldsymbol{p}$ of depth $3m-3$. If $\boldsymbol{p}$ is an element of A with multiplicity μ, then take the singletons $\{\boldsymbol{p}\}$ as μ of the sets in the Tverberg partition.

If $\mu \geq m$, we are done. If $\mu = m-1$, the point $\boldsymbol{p}$ is still in the convex hull of points in A, and thus we are done. And if $\mu \leq m-2$, the point $\boldsymbol{p}$ is still a point of depth $3m - \mu - 3 \geq 3(m-\mu) - 3$ of the remaining $24m - \mu - 31 \geq 24(m-\mu) - 31$ points. Thus, we may apply Lemma 8 to get an $(m-\mu)$-Tverberg partition of the remaining points, with $\boldsymbol{p}$ as Tverberg point, and conclude the result. $\square$

PROOF OF LEMMA 8. Since $\boldsymbol{p}$ is not an element of A, after radially projecting the points in A, we can assume without loss of generality that the points of A are located on a sphere centered at $\boldsymbol{p}$, as in the proof of Lemma 2.

We claim that there exist pairwise disjoint subsets $X_1, X_2, \ldots, X_{m-2}$ of A, each having $\boldsymbol{p}$ in its convex hull and each being of cardinality at most 4. (Here "pairwise disjoint" means that each element of A is present in a number of X_i's that does not exceed its multiplicity in A.) We proceed by contradiction. Suppose that we can find at most $s < m-2$ such subsets X_i's. Then, by Carathéodory's theorem, $\boldsymbol{p}$ is not in the convex hull of the remaining points in A. Therefore there is a half-space H_+ delimited by a plane containing $\boldsymbol{p}$ such that $H_+ \cap A \subseteq \bigcup_{i=1}^{s} X_i$. On the other hand, since each X_i contains $\boldsymbol{p}$ in its convex hull (and we can assume the X_i are minimal with respect to containing $\boldsymbol{p}$), we have $|H_+ \cap X_i| \leq 3$ for all $i \in [s]$. Therefore $|H_+ \cap A| \leq |H_+ \cap (\bigcup_{i=1}^{s} X_i)| \leq 3s < 3(m-2)$, which is a contradiction since $\boldsymbol{p}$ is a point of depth $3m-3$ in A. There are thus $m-2$ disjoint subsets $X_1, X_2, \dots, X_{m-2}$ as claimed.

Let X denote $\bigcup_{i=1}^{m-2} X_i$. Consider an arbitrary half-space H_+ delimited by a plane containing $\boldsymbol{p}$. Since $|H_+ \cap X_i| \le 3$ for all i, we have $|H_+ \cap X| \le 3(m-2)$. Furthermore $|H_+ \cap A| \ge 3m-3$, so $|H_+ \cap (A \setminus X)| \ge 3$. Since H_+ is arbitrary, $\boldsymbol{p}$ is a point of depth 3 of $A \setminus X$. Also, $|A \setminus X| \ge |A| - 4(m-2) \ge 17$, so Lemma 7 implies that $A \setminus X$ can be partitioned into two sets whose convex hulls contain $\boldsymbol{p}$. With the subsets X_i, we have therefore an m-Tverberg partition of A, with $\boldsymbol{p}$ as Tverberg point. □

3. Tverberg Numbers over $S' \times \mathbb{R}^k$: Proof of Theorem 3

In this section, we prove Theorem 3. We adapt an approach by Mulzer and Werner [**MW13**, Lemma 2.3] and show how the results of our paper can be combined to improve known bounds and to determine new exact values for the Tverberg number in the mixed integer case, as well as better bounds for certain S-Tverberg numbers.

Proof of Theorem 3. Let $t = \mathrm{Tv}(\mathbb{R}^k, m) = (m-1)(k+1)+1$. Choose a multiset A in $S' \times \mathbb{R}^k$ with $|A| \ge \mathrm{Tv}(S', t)$. It suffices to prove that A can be partitioned into m subsets whose convex hulls contain a common point in $S' \times \mathbb{R}^k$.

Let A' be the multiset projection of A onto S' (so that $|A'| = |A|$). Since $|A'| \ge \mathrm{Tv}(S', t)$, there is a partition of A' into t submultisets $Q'_1, \dots, Q'_t$ whose convex hulls contain a common point $\boldsymbol{q}$ in S'. The Q'_i are the projections onto S' of t disjoint subsets $Q_i \subseteq A$ forming a partition of A. For each $i \in [t]$, we can find a point $\boldsymbol{q}_i \in \mathrm{conv}(Q_i)$ projecting onto $\boldsymbol{q}$.

The t points $\boldsymbol{q}_1, \dots, \boldsymbol{q}_t$ belong to $\{\boldsymbol{q}\} \times \mathbb{R}^k$. As $t = \mathrm{Tv}(\mathbb{R}^k, m)$, there exists a partition of $[t]$ into $I_1, \dots, I_m$ and a point $\boldsymbol{p} \in \{\boldsymbol{q}\} \times \mathbb{R}^k$ such that $\boldsymbol{p} \in \mathrm{conv}\left(\bigcup_{i \in I_\ell} \boldsymbol{q}_i\right)$ for all $\ell \in [m]$. For each $\ell \in [m]$, define A_ℓ to be $\bigcup_{i \in I_\ell} Q_i$. We have, for each $\ell \in [m]$

$$\boldsymbol{p} \in \mathrm{conv}\left(\bigcup_{i \in I_\ell} \boldsymbol{q}_i\right) \subseteq \mathrm{conv}\left(\bigcup_{i \in I_\ell} \mathrm{conv}(Q_i)\right) = \mathrm{conv}(A_\ell)$$

and the A_ℓ form the desired partition. □

Here are the new bounds and exact values we get:

(1) $\mathrm{Tv}(\mathbb{Z} \times \mathbb{R}^k, m) = 2(m-1)(k+1)+1$.
(2) $\mathrm{Tv}(\mathbb{Z}^2 \times \mathbb{R}^k, m) = 4(m-1)(k+1)+1$.
(3) $\mathrm{Tv}(\mathbb{Z}^3 \times \mathbb{R}^k, m) \le 24(m-1)(k+1)-7$.
(4) $2^j(m-1)(k+1)+1 \le \mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) \le j2^j(m-1)(k+1)+1$.
(5) If $S' \subset \mathbb{R}^2$ has finite Helly number $\mathrm{H}(S')$, then

$$\mathrm{Tv}(S' \times \mathbb{R}^k, m) \le \mathrm{H}(S')(m-1)(k+1)+1.$$

The lower bound in (4) is obtained by repeated applications of Proposition 1 below. The upper bounds follow from Theorem 3, combined with the fact that $\mathrm{Tv}(\mathbb{Z}, m) = 2m-1$, Theorem 1 for $S = \mathbb{Z}^2$, Theorem 2, the upper bound in Equation (2), and Theorem 1 respectively.

Proposition 1. *Let j and k be two non-negative integers. Then we have*

$$\operatorname{Tv}(\mathbb{Z}^{j+1}\times\mathbb{R}^k, m) \geq 2\operatorname{Tv}(\mathbb{Z}^j\times\mathbb{R}^k, m) - 1.$$

We prove Proposition 1 by following the idea of the proof of Proposition 2.1 in [**Onn91**].

Proof of Proposition 1. Assume toward a contradiction that

$$\operatorname{Tv}(\mathbb{Z}^{j+1}\times\mathbb{R}^k, m) \leq 2\operatorname{Tv}(\mathbb{Z}^j\times\mathbb{R}^k, m) - 2.$$

Choose A to be a set of $\operatorname{Tv}(\mathbb{Z}^j\times\mathbb{R}^k, m)-1$ points in $\mathbb{Z}^j\times\mathbb{R}^k$ with no m-Tverberg partition. Let $A_i = \{(i, \boldsymbol{a})\colon \boldsymbol{a}\in A\}$, for $i\in\{0,1\}$. Since $A_0\cup A_1\subset\mathbb{Z}^{j+1}\times\mathbb{R}^k$ has cardinality $2\operatorname{Tv}(\mathbb{Z}^j\times\mathbb{R}^k, m)-2$, there exists an m-Tverberg partition $Y_1, Y_2, \ldots, Y_m$ of $A_0\cup A_1$ with $\boldsymbol{p}\in\bigcap_{i\in[m]}\operatorname{conv}(Y_i)$. Furthermore $\boldsymbol{p}$ is in $\mathbb{Z}^{j+1}\times\mathbb{R}^k$. That implies either $\boldsymbol{p}\in\operatorname{conv}(A_0)$ or $\boldsymbol{p}\in\operatorname{conv}(A_1)$. In either case A_0 or A_1 has an m-Tverberg partition, a contradiction with our choice of A. □

4. A Generalized Fraction Selection Lemma: Proof of Theorem 4

Our proof relies on the simplicial partition theorem of Matoušek, used in a similar manner as in [**MR17**], which states the following.

Theorem 5 ([**Mat92**]; see also [**Cha00**]). *Given an integer $d\geq 1$ and a parameter r, there exists a constant $c_d\geq 1$ such that for any set P of n points in $\mathbb{R}^d$, there exists an integer s and a partition $\{P_1,\ldots,P_s\}$ of P such that*

- *for each $i=1,\ldots,s$, $\frac{n}{r}\leq |P_i|\leq\frac{2n}{r}$, and*
- *any hyperplane intersects the convex hull of less than $c_d\cdot r^{1-\frac{1}{d}}$ sets of the partition.*

The constant c_d is independent of P and depends only on d.

We now prove the following key lemma.

Lemma 9. *For any integer $d\geq 1$, there exists a constant c_d such that the following holds. For any set P of n points in $\mathbb{R}^d$ and a real number $\alpha\in(0,1]$, there exists a partition $\mathcal{P}=\{P_1,\ldots,P_r\}$, $r=\left\lceil\left(\frac{4c_d}{\alpha}\right)^d\right\rceil$, of P such that*

- *$\frac{n}{2r}\leq|P_i|\leq\frac{2n}{r}$ for each $i=1,\ldots,r$, and*
- *the convex hull of any transversal Q of $\mathcal{P}$ contains all points in $\mathbb{R}^d$ of half-space depth at least $\alpha\cdot n$.*

Proof. Apply the simplicial partition theorem (Theorem 5) to P with $r=\left\lceil\left(\frac{4c_d}{\alpha}\right)^d\right\rceil$, and let the resulting partition be $\{P_1',\ldots,P_s'\}$. Note that as $\frac{n}{r}\leq|P_i'|\leq\frac{2n}{r}$ for each $i=1,\ldots,s$, we have $\frac{r}{2}\leq s\leq r$. Now partition arbitrarily each of the $r-s$ most numerous sets in $\{P_1',\ldots,P_s'\}$ into two equal parts, and let the resulting partition be $\{P_1,\ldots,P_r\}$. Clearly each set of this partition has size in the interval $\left[\frac{n}{2r},\frac{2n}{r}\right]$. This proves the first part. Note also that each hyperplane intersects the convex hull of at most twice as many sets, i.e., less than $2c_d\cdot r^{1-\frac{1}{d}}$ sets of the partition $\{P_1,\ldots,P_r\}$.

To see the second part, let $\boldsymbol{c}$ be any point of half-space depth at least $\alpha \cdot n$, and Q any transversal of $\mathcal{P}$. For contradiction, assume that $\boldsymbol{c} \notin \operatorname{conv}(Q)$. Then there exists a hyperplane H containing $\boldsymbol{c}$ in one of its two open half-spaces, say H^-, and containing $\operatorname{conv}(Q)$ in the half-space H^+. We will show that then there exists an index $i \in \{1, \dots, r\}$ such that $P_i \subseteq H^-$. But then $P_i \cap Q = \varnothing$, a contradiction to the fact that Q is a transversal of $\mathcal{P}$.

It remains to show the existence of a set $P_i \in \mathcal{P}$ such that $P_i \subseteq H^-$. Towards this, we bound $|P \cap H^-|$. Each point of P lying in H^- belongs to a set $P' \in \mathcal{P}$ such that either

- $P' \subseteq H^-$, in which case we are done, or
- $P' \nsubseteq H^-$. As H^- contains at least one point of P', we must have $\operatorname{conv}(P') \cap H \neq \varnothing$. As argued earlier, there are less than $2c_d \cdot r^{1-\frac{1}{d}}$ such sets.

Thus we have

$$\left|P \cap H^-\right| < 2c_d \cdot r^{1-\frac{1}{d}} \cdot \frac{2n}{r} = \frac{4c_d \cdot n}{\left\lceil \left(\frac{4c_d}{\alpha}\right)^d \right\rceil^{\frac{1}{d}}} \leq \alpha \cdot n. \tag{4}$$

On the other hand, as $\boldsymbol{c}$ has half-space depth at least $\alpha \cdot n$ and $\boldsymbol{c} \in H^-$, we have $|P \cap H^-| \geq \alpha n$, a contradiction to inequality (4). □

Remark: In particular, for $r = \left\lceil \left(\frac{4c_d}{\alpha}\right)^d \right\rceil$, there exist at least $\left(\frac{n}{2r}\right)^r$ r-sized subsets, each of whose convex hull contains all integer points of depth at least $\alpha \cdot n$.

Proof of Theorem 4. Given the point set P in $\mathbb{R}^d$ and a point $\boldsymbol{q} \in \mathbb{R}^d$ of half-space depth $\alpha \cdot n$, apply Lemma 9 with P and α to get a partition consisting of $r \geq d+1$ sets, where $r = \left\lceil \left(\frac{4c_d}{\alpha}\right)^d \right\rceil$. By discarding at most $\frac{n}{2}$ points of P, we can derive a partition on the remaining points of P, say $\mathcal{P} = \{P_1, \dots, P_r\}$, such that the P_i's are equal-sized disjoint subsets of P, i.e., $|P_i| = \frac{n}{2r}$ for all $i = 1, \dots, r$. Furthermore, every transversal of $\mathcal{P}$ contains all points in $\mathbb{R}^d$ of half-space depth at least αn, and thus $\boldsymbol{q}$.

For each transversal Q of $\mathcal{P}$, the point $\boldsymbol{q}$ lies in the convex hull of Q, and by Carathéodory's theorem, there exists a $(d+1)$-sized subset of Q whose convex hull also contains $\boldsymbol{q}$. By the pigeonhole principle, there must exist $d+1$ sets of $\mathcal{P}$, say the sets $P_1, \dots, P_{d+1}$, such that at least

$$\frac{\left(\frac{n}{2r}\right)^r}{\binom{r}{d+1}\left(\frac{n}{2r}\right)^{r-(d+1)}} \geq \frac{\left(\frac{n}{2r}\right)^{d+1}}{\left(\frac{er}{d+1}\right)^{d+1}} = \frac{1}{\left(\frac{er}{d+1}\right)^{d+1}} \cdot \prod_{i=1}^{d+1} |P_i|. \tag{5}$$

distinct transversals of $\{P_1, \dots, P_{d+1}\}$ contain $\boldsymbol{q}$.

The rest of the proof follows the one of Pach [**Pac98**]. In brief, we view the P_i's as parts of a $(d+1)$-partite hypergraph with vertices corresponding to points in P and a hyperedge corresponding to each transversal of $\mathcal{P}$ containing $\boldsymbol{q}$. As there are $\Omega\left(n^{d+1}\right)$ such transversals by inequality (5), we apply a weak form of the hypergraph version of Szemerédi's regularity lemma (see [**Mat92**] Theorem 9.4.1) to derive the existence of constant-fraction sized subsets $P_1' \subseteq P_1, \ldots, P_{d+1}' \subseteq P_{d+1}$ such that the following is true, for some constant c_d':

> for *any* $P_1'' \subseteq P_1', \ldots, P_{d+1}'' \subseteq P_{d+1}'$, with $|P_i''| \geq c_d' \cdot |P_i'|$ for $i = 1, \ldots, d+1$, we have the property that there exists at least one transversal of $\left\{P_1'', \ldots, P_{d+1}''\right\}$ whose convex hull contains $\boldsymbol{q}$.

Then the so-called same-type lemma ([**BV98**] Theorem 2) applied to $\left\{P_1', \ldots, P_{d+1}', \{\boldsymbol{q}\}\right\}$ gives constant-fraction sized subsets $X_1 \subseteq P_1', \ldots, X_{d+1} \subseteq P_{d+1}'$ such that either the convex hull of *each* transversal of $\{X_1, \ldots, X_{d+1}\}$ contains $\boldsymbol{q}$ or none of them do.

We can set up the parameters for the same-type lemma and the weak regularity lemma such that $|X_i| \geq c_d' \cdot |P_i'|$, for all $i = 1, \ldots, d+1$. Then the weak regularity lemma implies that there exists at least one transversal of $\{X_1, \ldots, X_{d+1}\}$ whose convex hull contains $\boldsymbol{q}$. This implies that the convex hull of *each* transversal of $\{X_1, \ldots, X_{d+1}\}$ contains $\boldsymbol{q}$. These are the required subsets.

The size of each X_i is a constant-fraction of n, say $|X_i| \geq c_{d,\alpha} \cdot n$, where the constant $c_{d,\alpha}$ depends on the constants in inequality (5), in the weak regularity lemma and in the same-type lemma. All of these depend only on α and d. □

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