

Combinatorics of Local Search: An Optimal 4-Local Hall's Theorem for Planar Graphs*

Daniel Antunes¹, Claire Mathieu², and Nabil H. Mustafa¹

- 1 **Université Paris-Est, Laboratoire d'Informatique Gaspard-Monge, Equipe A3SI, ESIEE Paris. Email: {daniel.antunes, mustafan}@esiee.fr.**
- 2 **Department of Computer Science, CNRS, École Normale Supérieure and PSL Research University, Paris, France. Email: Claire.Mathieu@ens.fr.**

Abstract

Local search for combinatorial optimization problems is becoming a dominant algorithmic paradigm, with several papers using it to resolve long-standing open problems. In this paper, we prove the following '4-local' version of Hall's theorem for planar graphs: given a bipartite planar graph $G = (B, R, E)$ such that $|N(B')| \geq |B'|$ for all $|B'| \leq 4$, there exists a matching of size at least $\frac{|B|}{4}$ in G ; furthermore this bound is tight. Besides immediately implying improved bounds for several problems studied in previous papers, we find this variant of Hall's theorem to be of independent interest in graph theory.

1998 ACM Subject Classification F.2.2 Nonnumerical Algorithms and Problems

Keywords and phrases Combinatorial Optimization, Planar Graphs, Local Search, Hall's Theorem, Expansion

Digital Object Identifier 10.4230/LIPIcs.ESA.2017.9

1 Introduction

One of the exciting developments in the field of geometric algorithms in recent years has been the use of *local search* techniques to resolve several open problems in combinatorial optimization. Remarkably, all these following NP-hard problems are approximately solved by the same meta-algorithm:

1. *Minimum hitting set problem for pseudo-disks*¹ [16]. Given a set X of points and a set \mathcal{D} of pseudo-disks in the plane, compute a minimum size subset of X that hits all pseudo-disks in \mathcal{D} .
2. *Maximum independent set in the intersection graph of pseudo-disks* [1, 8]. Given a set \mathcal{D} of pseudo-disks in the plane, compute a maximum size pairwise disjoint subset of \mathcal{D} .
3. *Terrain guarding problem* [10]. Given a 1.5D terrain² T and two subsets $X, G \subseteq T$, compute a minimum size subset of G such that every point of X is visible from some point of G .
4. *Minimum dominating set in disk intersection graphs* [11]. Given a set \mathcal{D} of disks in the plane, compute a minimum size subset $\mathcal{D}' \subseteq \mathcal{D}$ such that each $D \in \mathcal{D}$ is either in \mathcal{D}' or intersects some disk in \mathcal{D}' .

* This work was supported by the grant ANR SAGA (JCJC-14-CE25-0016-01).

¹ A set of geometric objects in the plane are called pseudo-disks if the boundary of every pair of objects intersect at most twice.

² A 1.5D terrain T is an x -monotone chain of line segments in \mathbb{R}^2 .



5. *Minimum dominating set in pseudo-disk intersection graphs* [12]. Given a set \mathcal{D} of pseudo-disks in the plane, compute a minimum size subset \mathcal{D}' of \mathcal{D} such that each $D \in \mathcal{D}$ is either in \mathcal{D}' or intersects some pseudo-disk in \mathcal{D}' .
6. *Minimum set-cover problem for disks in the plane* [7, 15]. Given a set of points X and a set of disks \mathcal{D} in the plane, compute the minimum sized subset of \mathcal{D} that covers all the points of X . This problem can be reduced to the minimum hitting set problem for disks.

The Meta-Algorithm: Local Search

The meta-algorithm can be parameterized by an integer k representing the search radius. Abstractly, let X be a set of given base elements, and $\Pi : 2^X \rightarrow \{0, 1\}$ be a function that assigns feasibility to each subset of X with respect to the specific problem. Then the goal is to find a minimum/maximum sized subset of X for which $\Pi(\cdot)$ is feasible. The local-search algorithm proceeds as follows: start with any feasible solution $\mathcal{S} \subseteq X$, and iteratively improve \mathcal{S} by changing³ subsets of \mathcal{S} of size at most k , as long as the new solution is also feasible. We restrict the discussion below to instances of minimization problems; the maximization case is similar.

Local-Search Method With Search Radius k (minimization instance).

Let $\mathcal{S} \subseteq X$ be any feasible solution.
while *there exists \mathcal{S}' with $\pi(\mathcal{S}')$ feasible and where $|\mathcal{S}' \setminus \mathcal{S}| < |\mathcal{S} \setminus \mathcal{S}'| \leq k$* **do**
 \perp set $\mathcal{S} = \mathcal{S}'$.
return \mathcal{S}

The analysis of the approximation factor of a local search algorithm, assuming the problem has some planar features, usually proceeds as follows.

Recall that for a graph $G = (V, E)$ and a subset V' of V , $N_G(V') = \{v \in V : \exists u \in V', \{u, v\} \in E\}$ denotes the set of neighbors of V' in G .

► **Definition 1.** Let $k \geq 1$ be given. A bipartite graph $G = (B, R, E)$ satisfies a *local expansion property* if, for every subset B' of B of cardinality at most k , we have $|N_G(B')| \geq |B'|$. Then G is called a k -expanding graph. If $k = |B|$ then G is called an expanding graph.

► **Lemma 2.** [8, 16] *There is an absolute constant c_0 such that any planar bipartite k -expanding graph $G = (B, R, E)$ satisfies $|R| \geq (1 - \frac{c_0}{\sqrt{k}})|B|$.*

The analysis of local-search algorithm with search radius k proceeds by first constructing a certain bipartite planar graph $G = (\mathcal{S}, \mathcal{O}, E)$ on \mathcal{S} and \mathcal{O} , where \mathcal{S} is the local-search solution with radius k and \mathcal{O} is an (unknown) optimal solution, such that G is k -expanding.

Now setting $k = \Theta(\frac{1}{\epsilon^2})$ and applying Lemma 2 to G implies that the local optimum \mathcal{S} has size $(1 + O(\epsilon))$ times the optimal size $|\mathcal{O}|$, hence near-optimality. A straightforward implementation of the local-search algorithm gives a running time of $n^{O(\frac{1}{\epsilon^2})}$, so this is a PTAS (polynomial-time approximation scheme). Note that as most of the problems listed earlier are $W[1]$ -hard [13, 14], it is unlikely that algorithms exist that do not have a dependency on $1/\epsilon$ in the exponent of n .

³ In case of a minimization problem, replace some k elements of \mathcal{S} with some $k - 1$ elements of X ; for a maximization problem replace some k elements of \mathcal{S} with some $k + 1$ elements of X .

Combinatorics of Local Search: Hall's Theorem for Planar Graphs

The reader will notice the resemblance between the Local Expansion Property and pre-conditions of Hall's theorem—Local Expansion Property is simply the pre-condition of Hall's theorem restricted to subsets of size at most k . And indeed, the statement of Lemma 2 can be re-cast as a 'local' version of Hall's theorem for planar graphs, as follows. One of the cornerstones of graph theory, Hall's theorem, can be rephrased as:

Hall's Theorem. *Let $G = (B, R, E)$ be a $|B|$ -expanding bipartite graph. Then there exists a matching in G of size $|B|$.*

Note that if we restrict the expanding subsets to be of size at most k for some integer k , then the theorem fails, as one cannot guarantee a matching of size more than k —e.g., take G to be the complete bipartite graph $K_{|B|,k}$. Interestingly, Lemma 2 implies that unlike the general graph case, a 'local' version of Hall's theorem is indeed true for planar graphs. We first observe that Lemma 2 can be used to get a local variant of Hall's theorem for planar graphs:

► **Theorem A (k -local Hall's Theorem for Planar Graphs).** Let $G = (B, R, E)$ be a k -expanding bipartite planar graph. Then there exists a matching in G of size at least $(1 - \frac{c_0}{\sqrt{k}})|B|$.

Proof. Let $B' \subseteq B$ for any subset of B . Observing that the subgraph of G induced by $B' \cup N_G(B')$ is planar, bipartite and k -expanding, we have $|N_G(B')| \geq (1 - \frac{c_0}{\sqrt{k}})|B'|$ by Lemma 2. Let S be a new set of $\frac{c_0|B|}{\sqrt{k}}$ dummy vertices. Construct a bipartite graph $G' = (B, R \cup S, E \cup E')$, where E' is the set of all $|B| \cdot |S|$ edges between B and S . Then G' satisfies the conditions of Hall's theorem, as for any $B' \subseteq B$, we have

$$|N_{G'}(B')| = |N_G(B')| + |S| \geq (1 - \frac{c_0}{\sqrt{k}})|B'| + \frac{c_0|B|}{\sqrt{k}} \geq |B'|.$$

Thus there is a matching of size $|B|$ in G' by Hall's theorem. Removing the vertices of S from this matching still leaves a matching of size at least $(1 - \frac{c_0}{\sqrt{k}})|B|$. ◀

Note that Theorem A is more general than Lemma 2, so it can be interpreted as a strengthening of Lemma 2. Summarizing this discussion, the above local version of Hall's theorem for planar graphs is the key combinatorial reason why local-search works for a wide variety of geometric optimization problems. The proof of Lemma 2 relies on separators in planar graphs, and there has been work in generalizing these ideas to classes of non-planar graphs which still have small separators (see [6, 2, 5]).

Our Results

While local-search with search radius $k = \Theta(\frac{1}{\epsilon^2})$ theoretically gives the best possible result in terms of approximation factors, these problems are far from being solved satisfactorily:

- As stated earlier, most of these problems are $W[1]$ -hard [13, 14]: therefore unless $W[1] = FPT$, there is no efficient polynomial-time approximation scheme for most of the listed problems; i.e., algorithms with running time $O(n^c)$, where c is a constant independent of $\frac{1}{\epsilon}$. This effectively restricts local search to small constant values of k .
- Furthermore, local-search is often the *only* approach known for these problems that yields good approximations. For example, the best approximation ratio for the hitting set problem for disks without using local-search is 13.4 [4] via the theory of ϵ -nets (see the chapter [17] for details); or $O(\log n)$ -approximation for dominating sets in disk intersection graphs [11]. Any effective solution to these problems entails examining closely the limits of efficiency and quality of local search for small values of k .

- While the construction of the graph is specific to the problem at hand, all these algorithms rely on the same Local Expansion Property of planar graphs, and thus the quantitative approximation bounds are the same across all the problems. The constants involved in Theorem A unfortunately make this result inefficient even for small values of k ; e.g., the current best work shows that setting k to get a 3-approximation implies a running time of $\Omega(n^{66})$ for the hitting set problem for disks [9].

Thus the natural way forward is to explore the limits of local search for small values of k . In this paper, we will consider the combinatorial aspect, and evaluate the quality of local-search—alternatively, the precise statement of local Hall’s theorem for planar graphs:

- $k = 1, 2$. The local Hall’s theorem *fails* (and so does local search) for the same reason as for general graphs— $K_{|B|,2}$ is a 2-expanding planar graph, but with a matching of size only 2.
- $k = 3$. An optimal local Hall’s theorem was shown in [3] by a short argument: any planar bipartite 3-expanding graph has a matching of size $\frac{|B|}{8}$ and this is tight.

The next fundamental case of local search that is open is for $k = 4$; the previous-best bound was $\frac{|B|}{5}$ and the resolution of the optimal bound was the main problem left open in [3]. In this paper we settle this question by presenting an optimal bound for local Hall’s theorem for 4-expanding planar graphs.

Main Theorem. *Let $G = (B, R, E)$ be a bipartite planar graph on vertex sets R and B , such that G is 4-expanding; i.e., for all $B' \subseteq B$ with $|B'| \leq 4$, $|N_G(B')| \geq |B'|$. Then there exists a matching in G of size at least $\frac{|B|}{4}$. Furthermore, this bound is tight up to lower-order terms.*

► **Corollary 3.** *The local search algorithm with parameter $k = 4$ gives a 4-approximation to these problems in geometric combinatorial optimization:*

1. *Minimum hitting set problem for pseudo-disks in the plane.*
2. *Maximum independent set problem in the intersection graph of pseudo-disks.*
3. *Terrain guarding problem.*
4. *Minimum dominating set in the pseudo-disk intersection graphs.*
5. *Minimum set-cover problem for disks in the plane.*

Tightness.

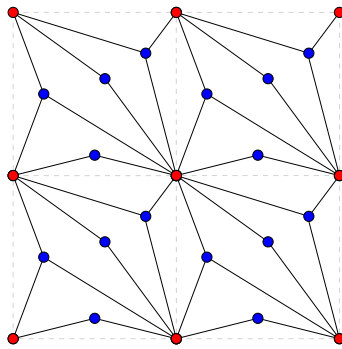
The optimality of the bound follows from the example shown in Figure 1, where R consists of n vertices of a $\sqrt{n} \times \sqrt{n}$ grid, and each ‘grid cell’ contains 4 vertices of B connected to the four red vertices of that cell. It is easy to verify that there is no matching of size greater than $\frac{|B|}{4} + O(\sqrt{|B|})$ (this is trivial, as $|B| = 4n - O(\sqrt{n})$), and the graph is planar and bipartite.

Finally, the fact that it is 4-expanding follows from the observation that, except at the grid boundary, any set of two vertices of B of degree 3 or any set of three vertices of B of degree 2 has at least 4 neighbors in R .

The proof of the upper-bound relies on the following key lemma, presented in Section 2:

► **Lemma 4.** *Let $G = (B, R, E)$ be a bipartite planar graph on vertex sets R and B , such that G is 4-expanding. Then $|R| \geq \frac{|B|}{4}$.*

Lemma 4 can be seen as a version of Lemma 2 for $k = 4$ and $c_0 = \frac{3}{2}$, leading to the Main Theorem via an argument identical to the proof of Theorem A.



■ **Figure 1** A lower-bound construction for 4-expanding bipartite planar graphs.

2 Proof of Lemma 4

The proof, at its core, uses the *discharging method* [18] of combinatorial geometry. Henceforth, a graph satisfying 4-expanding property is said to satisfy 4L.

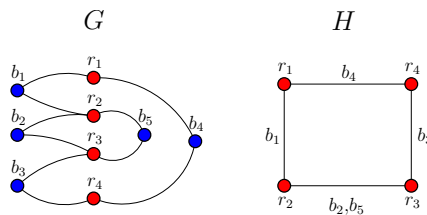
First note that no vertex in B can have degree zero, as otherwise the neighborhood of such a vertex would violate 4L. Moreover, it can be assumed that every vertex in B has degree at least two, since it is always possible to add edges to all vertices of B which have degree one in G while maintaining the planarity and bipartiteness of the graph (as any such vertex v must lie in a face which has at least two vertices of R , at least one of which is not adjacent to v).

Let $B_{=i} \subseteq B$ be the subset of vertices of B of degree exactly i , and $B_{\geq i} \subseteq B$ the set of vertices of degree at least i .

For the remainder of the proof, we fix a planar embedding of G .

Let $H(R, E)$ be a planar graph on R constructed from G as follows: two vertices $r_1 \in R$ and $r_2 \in R$ are adjacent in H iff there is at least one vertex $b \in B_{=2}$ which is adjacent to both r_1 and r_2 in G . Note that H is planar since G is planar, and the edges between r_1 and r_2 can be routed via one such vertex b . Note also that vertices in $B_{=3}$ lie in the interior of faces of H . Vertices of R will be called the *red* vertices, and vertices of B the *blue* vertices.

Note that for a fixed pair $\{r_1, r_2\} \subseteq R$, there cannot be three distinct vertices $b_1, b_2, b_3 \in B_{=2}$ adjacent to both r_1 and r_2 , since in this case the neighborhood of set $\{b_1, b_2, b_3\}$ is of size two and the graph G would violate 4L. Therefore, each edge of H corresponds to one or two vertices in $B_{=2}$. Edges corresponding to a single vertex in $B_{=2}$ are called *single edges* and the set of all such edges is denoted by E_1 , while edges mapped to two vertices in $B_{=2}$ are called *double edges* and its set is denoted by E_2 . In Figure 2, $\{r_1, r_2\}$ is a single edge and $\{r_2, r_3\}$ is a double edge. In later figures, the numbers 1 and 2 will be used to indicate whether an edge is single or double. When referring to a particular face f , ∂f will denote its set of edges while E_1^f and E_2^f will denote the set of single and double edges of f , respectively.



■ **Figure 2** A bipartite planar graph $G(B, R, E)$ and its corresponding graph $H(R, E)$.

For the rest of the proof, fix an embedding of H as well as the counter-clockwise ordering on $\partial_V f$ for each $f \in F$, where $\partial_V f$ denotes the vertices of f . Let F_i be the set of faces of H with exactly i edges on its boundary, and let F be the set of all faces of H . A face in F_3 will be called a *triangular* face and a face in F_4 a *rectangular* face. If ∂f is a cycle then f is called a *face cycle*. An edge e on the boundary of two different faces is called a *boundary edge*; it is called a *cut edge* otherwise.

In proceeding with the proof, we now encounter a technical difficulty: H need not be 2-connected, and so the structure of the faces can be arbitrarily complex. We first prove, in the next subsection, Lemma 4 for the case when H is 2-connected. Then we show how to handle the general case by reducing it to the 2-connected case.

2.1 Case: $H(R, E)$ is 2-connected.

If H is 2-connected then all its faces are face cycles; in particular, each edge of H is a boundary edge, and there are no cut edges.

2.1.1 Structural properties of H .

► **Claim 2.1.** For $i \geq 4$, let $f \in F_i$. Then $|E_2^f| \leq \lfloor \frac{i}{2} \rfloor$. A triangular face has no double edges.

Proof. Let f be a triangular face with vertices $\{r_1, r_2, r_3\}$, and with, say, $\{r_1, r_2\} \in E_2^f$. Recall that edges of H are associated with vertices of $B_{=2}$. Thus the two single edges and one double edge of f correspond to a set $B' \subseteq B$ of four vertices, with $N(B') = \{r_1, r_2, r_3\}$, violating 4L. For $i \geq 4$, if a face $f \in F_i$ has $|E_2^f| > \lfloor \frac{i}{2} \rfloor$, then there must exist two double edges incident to the same vertex of f and 4L is again violated. ◀

For a face $f \in F_i$, f is called a *full face* if $|E_2^f| = \lfloor \frac{i}{2} \rfloor$. Let $B_{=3}^f$ denote the set of $B_{=3}$ vertices lying in the interior of f . Note that due to planarity, for a fixed face f in the embedding of H , each vertex $v \in B_{=3}^f$ can be written uniquely (up to rotation) as an ordered triple $v = (r_1, r_2, r_3)$, where $r_1, r_2, r_3 \in R$ are vertices of f in counter-clockwise order with $\{v, r_i\} \in E(G)$ for $i = 1, 2, 3$.

► **Claim 2.2.** For $i \geq 4$, let $f \in F_i$. Then $|B_{=3}^f| \leq (i - 2)$.

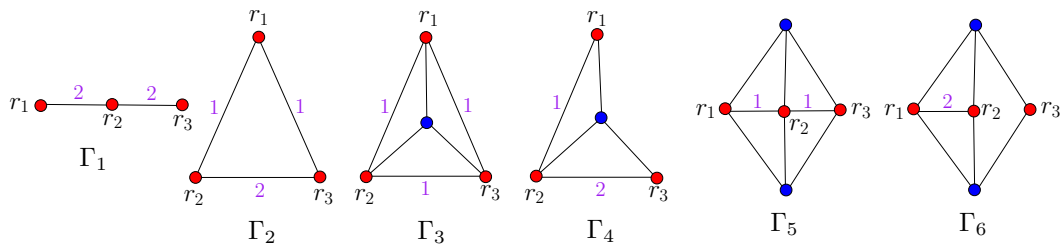
Proof. Note that we can assume that $|E_2^f| = 0$, as a double edge can only make it harder to ‘pack’ more vertices of $B_{=3}$ into f . Define a chain τ of f to be a consecutive set of vertices of $\partial_V f$. The size $|\tau|$ of a chain is equal to its number of vertices, and define $B_{=3}^\tau$, in the natural way, as the set of vertices of $B_{=3}^f$ with edges only to vertices of τ . We show that for a chain τ of size n , $|B_{=3}^\tau| \leq n - 2$. The proof will be by induction on the size of τ . For $|\tau| = 2$, $|B_{=3}^\tau| = 0$, trivially. For $|\tau| = j$, any fixed $v \in B_{=3}^\tau$ divides τ into three distinct sub-chains, τ_1, τ_2, τ_3 , with $|\tau_1| + |\tau_2| + |\tau_3| = j + 3$. Applying the induction hypothesis on each sub-chain,

$$|B_{=3}^\tau| \leq (|\tau_1| - 2) + (|\tau_2| - 2) + (|\tau_3| - 2) + 1 = j + 3 - 6 + 1 = j - 2.$$

For the next steps, we will need the list of ‘forbidden’ substructures in graphs satisfying 4L. ◀

► **Claim 2.3.** H satisfies 4L if and only if it does not contain the structures shown in Figure 3.

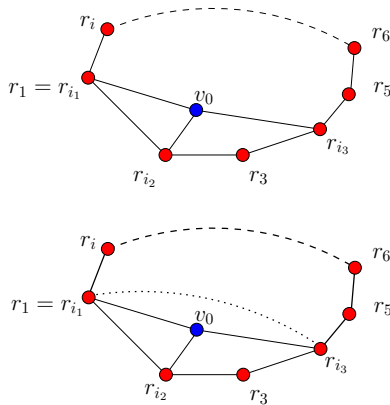
For the next claim, we will need the following independent property for planar graphs.



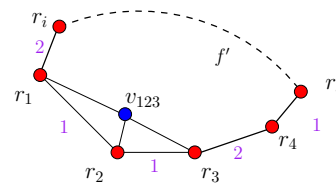
■ **Figure 3** Forbidden structures for a graph H satisfying 4L.

► **Claim 2.4.** Let G be a planar graph consisting of one (external) cycle $C = \langle r_1, \dots, r_i \rangle$ of i vertices and a set V of internal vertices, such that each vertex of V has exactly three neighbors, all in C , with these three neighbors not being consecutive vertices of C . Then $|V| \leq i - 4$.

Proof. The proof is inductive. For $i = 4$, we have $|V| = 0 = i - 4$, as there cannot exist a vertex not adjacent to three consecutive vertices of C . Consider the case where $i \geq 5$. By an extremal argument, there must exist a vertex $v_0 \in V$, say connected to $\{r_{i_1}, r_{i_2}, r_{i_3}\}$ where we can assume without loss of generality that $1 = i_1 < i_2 < i_3$, such that the two regions—one with boundary vertices $\langle v_0, r_{i_1}, r_{i_1+1}, \dots, r_{i_2} \rangle$ and the other with boundary vertices $\langle v_0, r_{i_2}, r_{i_2+1}, \dots, r_{i_3} \rangle$ —are both empty of vertices of V (see Figure 4). Furthermore, by the assumption that v does not have edges to three consecutive vertices of C , we have $(i_3 - i_1) \geq 3$. If there exists a vertex, other than v_0 , in V with edges to both r_{i_1} and r_{i_3} , call it v_1 (note that due to planarity, there can exist only one such vertex). Consider a new cycle $C' = \langle r_1, r_{i_3}, r_{i_3+1}, \dots, r_i \rangle$ of size $i - (i_3 - i_1) + 1 \leq (i - 2)$, and set $V' = V \setminus \{v_0, v_1\}$ to be a subset of vertices lying inside C' . It is easy to see that no vertex of V' can have edges to three consecutive vertices of C' , and thus by induction, we have $|V'| \leq |C'| - 4 \leq (i - 2) - 4$, and thus $|V| \leq |V'| + 2 \leq (i - 4)$.



■ **Figure 4** The vertex v_0 divides the graph in two regions.



■ **Figure 5** An odd full face. There exist two consecutive single edges.

► **Claim 2.5.** For $i \geq 4$, let $f \in F_i$ be a full face. If i is even then $|B_{=3}^f| \leq (i - 4)$. If i is odd then $|B_{=3}^f| \leq (i - 3)$.

Proof. Let $f \in F_i$ and $|E_{=2}^f| = \lfloor \frac{i}{2} \rfloor$. Label the vertices of $\partial_V f$ as $\langle r_1, \dots, r_i \rangle$ in the assumed counter-clockwise ordering.

i is even: Note that as f is full, the edges around f alternate between single and double edges. Therefore 4L implies that there does not exist a $v \in B_{=3}^f$ with edges to three consecutive vertices of $\partial_V f$. Claim 2.4 applied to f shows that $|B_{=3}^f| \leq i - 4$.

i is odd: For $i \geq 5$, as f is a full face, the edges around f alternate between single and double edges—with one exception where two adjacent edges are both single. Say these adjacent single edges are $\{r_1, r_2\}$ and $\{r_2, r_3\}$ (see Figure 5). 4L implies that there does not exist a $v \in B_{=3}^f$ with edges to three consecutive vertices of $\partial_V f$, *except* possibly there could exist a single vertex $v_{123} \in B_{=3}^f$ with edges to $\{r_1, r_2, r_3\}$. Claim 2.4 applied to f shows that $|B_{=3}^f \setminus \{v_{123}\}| \leq i - 4$, and thus $|B_{=3}^f| \leq i - 3$. ◀

2.1.2 Bounding $|B|$.

We first observe that to bound the size of B , it suffices to bound the number of vertices of degree 2 and 3 in B . We will need the following fact on planar graphs.

► **Fact 2.1.** Let $G = (V, E)$ be a simple, connected, planar bipartite graph. Then $|E| \leq 2|V| - 4$.

► **Claim 2.6.** $|B| \leq |B_{=2}| + \frac{|B_{=3}|}{2} + |R|$.

Proof. We count the number of edges in G in two ways—first by summing up the degrees of the vertices in B (recall that G is a bipartite graph), and secondly by using the upper-bound on the number of edges of planar bipartite graphs from Fact 2.1:

$$2 \cdot |B_{=2}| + 3 \cdot |B_{=3}| + \sum_{i=4} i \cdot |B_{=i}| = |E(G)| \leq 2(|R| + |B|) - 4.$$

Simplifying,

$$2 \cdot |B_{=2}| + 3 \cdot |B_{=3}| + \sum_{i=4} i \cdot |B_{=i}| \leq 2 \cdot \left(|R| + |B_{=2}| + |B_{=3}| + \sum_{i=4} |B_{=i}| \right).$$

Re-arranging the terms,

$$\begin{aligned} \sum_{i=4} (i-2) \cdot |B_{=i}| \leq 2|R| - |B_{=3}| &\implies 2 \sum_{i=4} |B_{=i}| \leq 2|R| - |B_{=3}| \\ |B_{\geq 4}| \leq |R| - \frac{|B_{=3}|}{2}. \end{aligned} \tag{1}$$

Now one can get an upper-bound on $|B|$ from inequality (1):

$$|B| = |B_{=2}| + |B_{=3}| + |B_{\geq 4}| \leq |B_{=2}| + |B_{=3}| + |R| - \frac{|B_{=3}|}{2} = |B_{=2}| + \frac{|B_{=3}|}{2} + |R|.$$

Thus it remains to bound $|B_{=2}| + \frac{|B_{=3}|}{2}$. Towards this, a charging intuition leads one to classify the contribution of a face $f \in F$ as $2 \cdot |E_2^f| + |E_1^f| + \frac{|B_{=3}^f|}{2}$. It turns out that the right discharging function is slightly different; define the *weight* of a face $f \in F$ to be

$$w(f) = |E_2^f| + \frac{|E_1^f|}{2} + \frac{|B_{=3}^f|}{2},$$

and the *weight* of the graph H to be

$$w(H) = |B_{=2}| + \frac{|B_{=3}|}{2}.$$

Note that as each edge is part of the boundary of precisely two faces and each vertex of $B_{=3}$ lies in precisely one face, we have

$$\sum_{f \in F} w(f) = \sum_{f \in F} \left(|E_2^f| + \frac{|E_1^f|}{2} + \frac{|B_{=3}^f|}{2} \right) = 2 \cdot |E_2| + |E_1| + \frac{|B_{=3}|}{2} = |B_{=2}| + \frac{|B_{=3}|}{2} = w(H). \quad (2)$$

► **Claim 2.7.** $w(H) \leq \frac{1}{4} \sum_{i \geq 3} (5i - 6) |F_i| - \frac{1}{4} \sum_{i \text{ is odd}} |F_i| - \frac{1}{2} |F_4| - \frac{1}{2} |F_3|.$

Proof. Let $I_f \in \{0, 1\}$ be an indicator variable such that $I_f = 1$ if and only if f is a full face. For $f \in F_i$ and i an even number, by applying the upper bounds in Claims 2.1, 2.2 and 2.5,

$$\begin{aligned} w(f) &= |E_2^f| + \frac{|E_1^f|}{2} + \frac{|B_{=3}^f|}{2} \leq \left(\frac{i}{2} - 1 + I_f \right) + \frac{i - \left(\frac{i}{2} - 1 + I_f \right)}{2} + \frac{(i-2) - 2I_f}{2} \\ &= \frac{5i - 6 - 2I_f}{4} \leq \frac{5i - 6}{4}. \end{aligned}$$

For $i = 4$, a better bound is possible. For a face $f \in F_4$, let $\alpha^f = |E_2^f|$. Then

$$w(f) \leq \alpha^f + \frac{4 - \alpha^f}{2} + \frac{(4-2) - \alpha^f}{2} = \frac{4 \cdot 4 - 4}{4} = \frac{5 \cdot 4 - 6}{4} - \frac{1}{2} = \frac{5i - 6}{4} - \frac{1}{2}. \quad (3)$$

For i an odd number,

$$w(f) \leq \left(\frac{i-1}{2} - 1 + I_f \right) + \frac{i - \left(\frac{i-1}{2} - 1 + I_f \right)}{2} + \frac{(i-2) - I_f}{2} = \frac{5i - 7}{4}. \quad (4)$$

For $i = 3$, note that for a face $f \in F_3$, $|E_2^f| = 0$, $|E_1^f| = 3$ and $|B_{=3}^f| = 0$, since f cannot have neither a $B_{=3}$ vertex in its interior nor a double edge, as otherwise the forbidden structures Γ_3 or Γ_2 would be present. Then,

$$w(f) = |E_2^f| + \frac{1}{2} |E_1^f| + \frac{|B_{=3}^f|}{2} = \frac{3}{2} = \frac{5i - 7}{4} - \frac{1}{2}. \quad (5)$$

By Equations (2)–(5),

$$\begin{aligned} w(H) &= \sum_{f \in F} w(f) = \sum_{f \in F_3} w(f) + \sum_{f \in F_4} w(f) + \sum_{\substack{i \geq 5 \\ i \text{ is odd}}} \sum_{f \in F_i} w(f) + \sum_{\substack{i \geq 6 \\ i \text{ is even}}} \sum_{f \in F_i} w(f) \\ &\leq \left(\frac{5 \cdot 3 - 7}{4} - \frac{1}{2} \right) |F_3| + \left(\frac{5 \cdot 4 - 6}{4} - \frac{1}{2} \right) |F_4| + \sum_{\substack{i \geq 5 \\ i \text{ is odd}}} \left(\frac{5i - 7}{4} \right) |F_i| + \sum_{\substack{i \geq 6 \\ i \text{ is even}}} \left(\frac{5i - 6}{4} \right) |F_i| \\ &= \frac{1}{4} \sum_{\substack{i \geq 3 \\ i \text{ is odd}}} (5i - 7) |F_i| + \frac{1}{4} \sum_{\substack{i \geq 4 \\ i \text{ is even}}} (5i - 6) |F_i| - \frac{1}{2} |F_4| - \frac{1}{2} |F_3| \\ &= \frac{1}{4} \sum_{i \geq 3} (5i - 6) |F_i| - \frac{1}{4} \sum_{i \text{ is odd}} |F_i| - \frac{1}{2} |F_4| - \frac{1}{2} |F_3|. \end{aligned}$$

◀

Finally we can bound the number of vertices of B of degree 2 and 3.

► **Lemma 5.** $w(H) = |B_{=2}| + \frac{|B_{=3}|}{2} \leq 3|R|$.

Proof. Let F^{odd} be the set of faces of H with an odd number of edges. By Claim 2.7,

$$\begin{aligned} w(H) &\leq \frac{1}{4} \sum_{i \geq 3} (5i - 6)|F_i| - \frac{1}{4} \sum_{i \text{ is odd}} |F_i| - \frac{1}{2}|F_4| - \frac{1}{2}|F_3| \\ &= \frac{5}{4} \sum_{i \geq 3} i|F_i| - \frac{3}{2} \sum_{i \geq 3} |F_i| - \frac{1}{4} \sum_{i \text{ is odd}} |F_i| - \frac{1}{2}|F_4| - \frac{1}{2}|F_3| \\ &= \frac{5}{2}|E| - \frac{3}{2}|F| - \frac{1}{4}|F^{\text{odd}}| - \frac{1}{2}|F_4| - \frac{1}{2}|F_3| = \hat{w}(H). \end{aligned} \tag{6}$$

Now note that the last quantity— $\hat{w}(H)$ as defined in Equation (6)—is maximized when H is a triangulation. To see this, consider an index i and a face $f \in F_i$ of H . Then decompose f into a face $f' \in F_{i-1}$ and a triangular face, resulting in a graph H' . Then comparing the bounds of Equation (6) for H and H' :

- Case $i = 4$: $\hat{w}(H') \geq \hat{w}(H) + 1 - \frac{2}{4} + \frac{1}{2} - \frac{2}{2} = \hat{w}(H)$.
- Case $i \geq 5$ and i is odd: $\hat{w}(H') \geq \hat{w}(H) + 1 - \frac{1}{2} - \frac{1}{2} = \hat{w}(H)$.
- Case $i \geq 6$ and i is even: $\hat{w}(H') \geq \hat{w}(H) + 1 - \frac{2}{4} - \frac{1}{2} = \hat{w}(H)$.

Consider any triangulation H' of H . Then,

$$\begin{aligned} w(H) &\leq \hat{w}(H) \leq \hat{w}(H') = \frac{5}{2}|E_{H'}| - \frac{3}{2}|F_{H'}| - \frac{1}{4}|F_{H'}| - \frac{1}{2}|F_{H'}| = \frac{5}{2}|E_{H'}| - \frac{9}{4}|F_{H'}| \\ &= \frac{5}{2}|E_{H'}| - \frac{9}{4} \cdot \frac{2}{3}|E_{H'}| = \frac{5}{2}|E_{H'}| - \frac{3}{2}|E_{H'}| = |E_{H'}|. \end{aligned}$$

By using Euler's formula for planar graphs,

$$|R| - |E_{H'}| + \frac{2}{3}|E_{H'}| = 2 \implies |R| = 2 + \frac{1}{3}|E_{H'}|.$$

Therefore,

$$\frac{w(H)}{|R|} \leq \frac{|E_{H'}|}{2 + \frac{1}{3}|E_{H'}|} \leq 3,$$

implying that $w(H) \leq 3|R|$ and we're done. ◀

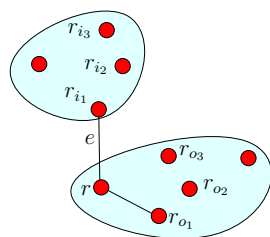
Now, Claim 2.6 and Lemma 5 imply the proof of the required Lemma 4.

2.2 Case: $H(R, E)$ is not 2-connected.

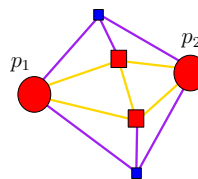
Now we deal with the case when H is not 2-connected. The general idea will be to transform each such planar graph H to a 2-connected planar graph H' while respecting the 4L property as well as planarity. Consider a straight-line embedding of H in the plane. If H is not 2-connected, there exists a *cut edge* e , say $e = \{r_{i_1}, r\}$. Let $I = \{r_{i_1}, r_{i_2}, \dots\}$ be the vertices in the connected component of r_{i_1} once e is removed. These vertices are called the *inner vertices*. Let $O = \{r, r_{o_1}, r_{o_2}, \dots, r_{o_m}\}$ be the vertices in the connected component of r . These vertices are called the *outer vertices*. Further assume that $r_{o_1} \in O$ is the first vertex after r_{i_1} , in the clockwise order, that is adjacent to r (see Figure 6).

Our goal is to connect an inner vertex in I to an outer vertex in O iteratively until H becomes 2-connected. In order to achieve that, we will apply the following transformation:

Clustering operation on $\{p_1, p_2\}$, where p_1 is an inner vertex and p_2 is an outer vertex: add a set Q of two new red vertices to H . Furthermore, add sets B''_2 of 5 new



■ **Figure 6** Inner vertices component and outer vertices component connected by cut edge e .



■ **Figure 7** Gadget used for the clustering operation.

degree-2 and B''_3 of 2 new degree-3 blue vertices. Connect these vertices as shown in Figure 7. Note that p_1 and p_2 are not adjacent in H .

We are going to argue that it is always possible to execute this while respecting planarity and 4L.

First we show that upper-bounding $w(\cdot)$ after a clustering operation gives an upper-bound for the original problem.

► **Claim 2.8.** Let $H'(B', R', E')$ be the graph resulting from an application of the clustering operation on a graph $H(B, R, E)$. If $w(H') \leq 3|R'|$ then $w(H) \leq 3|R|$.

Proof. More generally, assume we add b_2 new degree-two vertices to H' , b_3 degree-three vertices and r red vertices. Then from assumption, we have

$$w(H') = |B_{=2}| + b_2 + \frac{|B_{=3}|}{2} + \frac{b_3}{2} \leq 3(|R| + r),$$

which implies that

$$w(H) = |B_{=2}| + \frac{|B_{=3}|}{2} \leq 3|R| + 3r - b_2 - \frac{b_3}{2} \leq 3|R|,$$

assuming $3r \leq b_2 + \frac{b_3}{2}$. This condition is satisfied for the clustering operation, where $r = 2$, $b_2 = 5$ and $b_3 = 2$. ◀

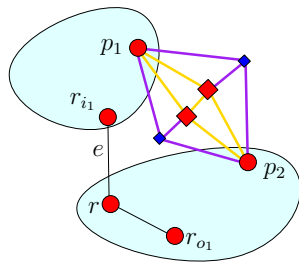
Next we show that a clustering operation does not violate the 4L condition.

► **Claim 2.9.** The clustering operation preserves the 4L property.

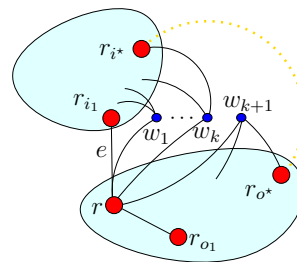
Proof. Let p_1 be any inner and p_2 be any outer vertex. Then add a set Q of two red vertices, a set B''_2 of 5 blue degree-2 vertices and a set B''_3 of 2 blue degree-3 vertices (see Figure 8). Let $B' \cup B''$ be any subset of size at most 4, where $B' \subseteq B$ and $B'' \subseteq B''_2 \cup B''_3$. We need to show that then $|N(B' \cup B'')| \geq |B' \cup B''|$.

1. $|B''| = 0$. Then $|N(B' \cup B'')| = |N(B')| \geq |B'|$, as H satisfies 4L.
2. $|B''| = 1$. As any vertex of B'' has at least one neighbor in Q , we have $|N(B' \cup B'')| \geq |N(B')| + 1 \geq |B'| + 1 = |B' \cup B''|$.
3. $|B''| = 2, 3$. As any two vertices of B'' have at least three neighbors in $Q \cup \{p_1, p_2\}$, and any vertex of B' must have at least one neighbor not in $Q \cup \{p_1, p_2\}$ (recall that p_1 and p_2 are not adjacent in H !), we get that $|N(B' \cup B'')| \geq |N(B'')| + 1 \geq 4$.
4. $|B''| = 4$. It can be verified that any set of 4 vertices of B'' have the set $Q \cup \{p_1, p_2\}$ of size 4 as its neighbor.

Finally, we show that there exists an inner and an outer vertex which can be connected via a clustering operation while maintaining planarity. ◀



■ **Figure 8** Clustering operation on p_1 and p_2 .



■ **Figure 9** A planar path between r_{i^*} to r_{o^*} .

► **Claim 2.10.** It is always possible to find an inner vertex r_{i^*} and an outer vertex r_{o^*} such that there exists a path that connects them without violating planarity.

Proof. Denote by $B_{=3}^r$ the set of degree-3 vertices adjacent to vertex r . If $B_{=3}^r$ is empty, then clearly there exists a path from the inner vertex r_{i_1} to the outer vertex r_{o_1} . Similarly, if there exists a vertex $w \in B_{=3}^r$ with one edge to an inner vertex and one to an outer vertex (other than the edge to r), then there exists a planar path between these inner and outer vertices by following the path along the edges of w .

Otherwise, sort the vertices in $B_{=3}^r$ clockwise by the order of their edges around r , say labeled w_1, \dots, w_t . If w_1 has both edges (other than to r) to outer vertices, then clearly there is a planar path from r_{i_1} to one of these outer vertices (see Figure 9). Similarly, if w_t has both edges (other than to r) to inner vertices, then there is a planar path from r_{o_1} to one of these inner vertices. Now by a parity argument, there must exist two vertices, say w_k and w_{k+1} , such that w_k has both neighbors to inner vertices, and w_{k+1} has both neighbors to outer vertices. Then there exists a path from one of inner vertices adjacent to w_k to one of the outer vertices adjacent to w_{k+1} . ◀

► **Lemma 6.** Let H be a 1-connected planar graph. Then $w(H) \leq 3|R|$.

Proof. Claim 2.10 implies that—as long as the current graph H is not 2-connected—it is always possible to do a clustering operation between an inner vertex and an outer vertex while maintaining planarity. By Claim 2.9, the resulting graph H' still satisfies the condition 4L. Crucially, note that each new edge introduced by the clustering operation is not a cut edge in the derived graph H' , and further, the edge e which was a cut edge in H is no longer a cut edge in H' . Thus the clustering operation reduces the total number of cut edges, and so the process terminates after a finite number of steps. Apply this iteratively to get a 2-connected graph H' , which, by Lemma 5, satisfies $w(H') \leq 3|R'|$. Then $w(H) \leq 3|R|$ follows by Claim 2.8. ◀

References

- 1 Pankaj K. Agarwal and Nabil H. Mustafa. Independent set of intersection graphs of convex objects in 2D. *Comput. Geom.*, 34(2):83–95, 2006.
- 2 Rom Aschner, Matthew J. Katz, Gila Morgenstern, and Yelena Yuditsky. Approximation schemes for covering and packing. In *Proceedings of the 7th International Workshop on Algorithms and Computation (WALCOM)*, pages 89–100, 2013.
- 3 Norbert Bus, Shashwat Garg, Nabil H. Mustafa, and Saurabh Ray. Limits of local search: Quality and efficiency. *Discrete & Computational Geometry*, 57(3):607–624, 2017.

- 4 Norbert Bus, Nabil H. Mustafa, and Saurabh Ray. Geometric hitting sets for disks: Theory and practice. In *Proceedings of the 23rd Annual European Symposium on Algorithms (ESA)*, pages 903–914, 2015.
- 5 Sergio Cabello and David Gajser. Simple PTAS’s for families of graphs excluding a minor. *Discrete Applied Mathematics*, 189:41–48, 2015.
- 6 Timothy M. Chan. Polynomial-time approximation schemes for packing and piercing fat objects. *J. Algorithms*, 46(2):178–189, 2003.
- 7 Timothy M. Chan and Elyot Grant. Exact algorithms and APX-hardness results for geometric packing and covering problems. *Comput. Geom.*, 47(2):112–124, 2014.
- 8 Timothy M. Chan and Sariel Har-Peled. Approximation algorithms for maximum independent set of pseudo-disks. *Discrete & Computational Geometry*, 48(2):373–392, 2012.
- 9 Robert Fraser. *Algorithms for Geometric Covering and Piercing Problems*. PhD thesis, University of Waterloo, 2012.
- 10 Matt Gibson, Gaurav Kanade, Erik Krohn, and Kasturi R. Varadarajan. Guarding terrains via local search. *JoCG*, 5(1):168–178, 2014.
- 11 Matt Gibson and Imran A. Pirwani. Algorithms for dominating set in disk graphs: Breaking the $\log n$ barrier. In *Proceedings of the 18th Annual European Symposium on Algorithms (ESA)*, pages 243–254, 2010.
- 12 Sathish Govindarajan, Rajiv Raman, Saurabh Ray, and Aniket Basu Roy. Packing and covering with non-piercing regions. In *Proceedings of the 22nd Annual European Symposium on Algorithms (ESA)*, pages 47:1–47:17, 2016.
- 13 Dániel Marx. Efficient approximation schemes for geometric problems? In *Proceedings of the 13th Annual European Symposium on Algorithms (ESA)*, pages 448–459, 2005.
- 14 Dániel Marx. Parameterized complexity of independence and domination on geometric graphs. In *Proceedings of the 2nd International Workshop on Parameterized and Exact Computation (IWPEC)*, pages 154–165, 2006.
- 15 Nabil H. Mustafa, Rajiv Raman, and Saurabh Ray. Quasi-polynomial time approximation scheme for weighted geometric set cover on pseudodisks and halfspaces. *SIAM J. Comput.*, 44(6):1650–1669, 2015.
- 16 Nabil H. Mustafa and Saurabh Ray. Improved results on geometric hitting set problems. *Discrete & Computational Geometry*, 44(4):883–895, 2010.
- 17 Nabil H. Mustafa and K. Varadarajan. Epsilon-approximations and Epsilon-nets. In J. E. Goodman, J. O’Rourke, and C. D. Tóth, editors, *Handbook of Discrete and Computational Geometry*. CRC Press LLC, 2017.
- 18 Radoš Radoičić and Géza Tóth. The discharging method in combinatorial geometry and the Pach-Sharir conjecture. In *Contemporary Mathematics: Surveys on Discrete and Computational Geometry*, pages 319–342. American Mathematical Society, 2008.