On the Zarankiewicz Problem for Intersection Hypergraphs*

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Abstract

Let d and t be fixed positive integers, and let $K^d_{t,\ldots,t}$ denote the complete d-partite hypergraph with t vertices in each of its parts, whose hyperedges are the d-tuples of the vertex set with precisely one element from each part. According to a fundamental theorem of extremal hypergraph theory, due to Erdős [6], the number of hyperedges of a d-uniform hypergraph on n vertices that does not contain $K^d_{t,\ldots,t}$ as a subhypergraph, is $n^{d-\frac{1}{t^{d-1}}}$. This bound is not far from being optimal.

We address the same problem restricted to *intersection hypergraphs* of $(d-1)$-dimensional *simplices* in $\mathbb{R}^d$. Given an n-element set $\mathcal{S}$ of such simplices, let $\mathcal{H}^d(\mathcal{S})$ denote the d-uniform hypergraph whose vertices are the elements of $\mathcal{S}$, and a d-tuple is a hyperedge if and only if the corresponding simplices have a point in common. We prove that if $\mathcal{H}^d(\mathcal{S})$ does not contain $K^d_{t,\ldots,t}$ as a subhypergraph, then its number of edges is $O(n)$ if $d=2$, and $O(n^{d-1+\epsilon})$ for any $\epsilon>0$ if $d\geq 3$. This is almost a factor of n better than Erdős's above bound. Our result is tight, apart from the error term ϵ in the exponent. In particular, for $d=2$, we obtain a theorem of Fox and Pach [7], which states that every $K_{t,t}$-free intersection graph of n *segments* in the plane has $O(n)$ edges. The original proof was based on a separator theorem that does not generalize to higher dimensions. The new proof works in any dimension and is simpler: it uses *size-sensitive cuttings*, a variant of random sampling.

1 Introduction

Let $\mathcal{H}$ be a d-uniform hypergraph on n vertices. One of the fundamental questions of extremal graph and hypergraph theory goes back to Turán and Zarankiewicz: What is the largest number $\mathrm{ex}^d(n,K)$ of hyperedges (or, in short, edges) that $\mathcal{H}$ can have if it contains no subhypergraph isomorphic to fixed d-uniform hypergraph K. (See Bollobás [2]). In the most applicable special case, $K=K^d(t,\ldots,t)$ is the complete d-partite hypergraph on the vertex set $V_1,\ldots,V_d$ with $|V_1|=\ldots=|V_d|=t$, consisting of all d-tuples that contain one point from each V_i. For graphs ($d=2$), it was proved by Erdős (1938) and Kővári-Sós-Turán [12] that

$$\mathrm{ex}^2(n,K^2_{t,t})\leq n^{2-1/t}.$$

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The order of magnitude of this estimate is known to be best possible only for $t = 2$ and 3 (Reiman [16]; Brown [3]). The constructions for which equality is attained are algebraic.

Erdős (1964) generalized the above statement to d-uniform hypergraphs for all $d \geq 2$. A hypergraph $\mathcal{H}$ is said to be K-*free* if it contains no copy of K as a (not necessarily induced) subhypergraph.

Theorem A ([6]). *The maximum number of hyperedges that a d-uniform, $K^d_{t,\dots,t}$-free hypergraph of n vertices can have satisfies*

$$\mathrm{ex}^d(n, K^d_{t,\dots,t}) \leq n^{d-\frac{1}{t^{d-1}}}.$$

It was further shown in [6] that this bound cannot be substantially improved. There exists an absolute constant $C > 0$ (independent of n, t, d) such that

$$\mathrm{ex}^d(n, K^d_{t,\dots,t}) \geq n^{d-\frac{C}{t^{d-1}}}.$$

In particular, for every $\epsilon > 0$, there exist $K^d_{t,\dots,t}$-free d-uniform hypergraphs with $t \approx (C/\epsilon)^{1/(d-1)}$, having at least $n^{d-\epsilon}$ edges. The construction uses the probabilistic method.

In certain geometric scenarios, better bounds are known. For instance, consider a bipartite graph with $2n$ vertices that correspond to n distinct points and n distinct lines in the plane, and a vertex representing a point p is connected to a vertex representing a line ℓ if and only if p is incident to ℓ (i.e., $p \in \ell$). Obviously, this graph is $K^2_{2,2}$-free (in short, $K_{2,2}$-free). Therefore, by the above result, it has at most $O(n^{3/2})$ edges. On the other hand, according to a celebrated theorem of Szemerédi and Trotter (1983), the number of edges is at most $O(n^{4/3})$, and the order of magnitude of this bound cannot be improved. In [13], this result was generalized to incidence graphs between points and more complicated curves in the plane.

Given a set $\mathcal{S}$ of geometric objects, their *intersection graph* $\mathcal{H}(\mathcal{S})$ is defined as a graph on the vertex set $\mathcal{S}$, in which two vertices are joined by an edge if and only if the corresponding elements of $\mathcal{S}$ have a point in common. To better understand the possible intersection patterns of edges of a *geometric graph*, that is, of a graph drawn in the plane with possibly crossing straight-line edges, Pach and Sharir [13] initiated the investigation of the following problem. What is the maximum number of edges in a $K_{t,t}$-free intersection graph of n segments in the plane? The Kővári-Sós-Turán theorem (Theorem A for $d = 2$) immediately implies the upper bound $n^{2-1/t}$. Pach and Sharir managed to improve this bound to $O(n)$ for $t = 2$ and to $O(n \log n)$ for any larger (but fixed) value of t. They conjectured that here $O(n \log n)$ can be replaced by $O(n)$ for every t, which was proved by Fox and Pach [7]. They later extended their proof to *string graphs*, that is, to intersection graphs of arbitrary continuous arcs in the plane [8, 9]. Some weaker results were established by Radoičić and Tóth [15] by the "discharging method".

The aim of the present note is to generalize the above results to *d-uniform intersection hypergraphs of $(d-1)$-dimensional simplices* in $\mathbb{R}^d$, for any $d \geq 2$. The arguments used in the above papers are based on planar separator theorems that do not seem to allow higher dimensional extensions applicable to our problem.

Given an n-element set $\mathcal{S}$ of $(d-1)$-dimensional simplices in general position in $\mathbb{R}^d$, let $\mathcal{H}^d(\mathcal{S})$ denote the d-uniform hypergraph on the vertex set $\mathcal{S}$, consisting of all unordered d-tuples of elements $\{S_1, \dots, S_d\} \subset \mathcal{S}$

with $S_1 \cap \ldots \cap S_d \neq \emptyset$. We prove the following theorem, providing an upper bound on the number of hyperedges of a $K^d_{t,\ldots,t}$-free intersection hypergraph $\mathcal{H}^d(\mathcal{S})$ of $(d-1)$-dimensional simplices. This bound is almost a factor of n better than what we obtain using the abstract combinatorial bound of Erdős (Theorem A), and it does not depend strongly on t.

Theorem 1. *Let $d, t \geq 2$ be integers, let $\mathcal{S}$ be an n-element set of $(d-1)$-dimensional simplices in $\mathbb{R}^d$, and let $\mathcal{H}^d(\mathcal{S})$ denote its d-uniform intersection hypergraph.*

If $\mathcal{H}^d(\mathcal{S})$ is $K^d_{t,\ldots,t}$-free, then its number of edges is $O(n^{d-1+\epsilon})$ for any $\epsilon > 0$. For $d = 2$, the number of edges is at most $O(n)$.

To see that this bound is nearly optimal for every d, fix a hyperplane h in $\mathbb{R}^d$ with normal vector v, and pick $d-1$ sets $P_1, \ldots, P_{d-1}$, each consisting of $\frac{n-1}{d-1}$ parallel $(d-2)$-dimensional planes in h, with the property that any $d-1$ members of $P_1 \cup \ldots \cup P_{d-1}$ that belong to different P_is have a point in common. For each i $(1 \leq i \leq d-1)$, replace every plane $p_i \in P_i$ by a hyperplane h_i parallel to v such that $h_i \cap h = p_i$. Clearly, the d-uniform intersection hypergraph of these n hyperplanes, including h, is $K^d_{2,\ldots,2}$-free, and its number of edges is $(\frac{n-1}{d-1})^{d-1} = \Omega(n^{d-1})$. In each hyperplane, we can take a large $(d-1)$-dimensional simplex so that the intersection pattern of these simplices is precisely the same as the intersection pattern (i.e., the d-uniform intersection hypergraph) of the underlying hyperplanes.

The proof of Theorem 1 is based on a partitioning scheme, which was first formulated by Pellegrini [14]. Given an n-element set $\mathcal{S}$ of $(d-1)$-dimensional simplices (or other geometric objects) in $\mathbb{R}^d$, let m denote the number of hyperedges in their d-uniform intersection hypergraph, that is, the number of d-tuples of elements of $\mathcal{S}$ having a point in common. For a parameter $r \leq n$, a $\frac{1}{r}$*-cutting* with respect to $\mathcal{S}$ is a partition of $\mathbb{R}^d$ into simplices such that the interior of every simplex intersects at most n/r elements of $\mathcal{S}$. The *size* of a $\frac{1}{r}$-cutting is the number of simplices it consists of. (See Matoušek [11].)

Theorem B ([5, 14])**.** *Let $d \geq 2$ be an integer, let $\mathcal{S}$ be an n-element set of $(d-1)$-dimensional simplices in $\mathbb{R}^d$, and let m denote the number of d-tuples of simplices in $\mathcal{S}$ having a point in common.*

Then, for any $\epsilon > 0$ and any $r \leq n$, there is a $\frac{1}{r}$-cutting with respect to $\mathcal{S}$ of size at most

$$C_2 \cdot (r + \frac{mr^2}{n^2}) \quad \text{if} \;\; d = 2, \;\; \text{and}$$

$$C_{d,\epsilon} \cdot (r^{d-1+\epsilon} + \frac{mr^d}{n^d}) \;\; \text{if} \;\; d \geq 3.$$

Here C_2 is an absolute constant and $C_{d,\epsilon}$ depends only on d and ϵ.

To construct a $\frac{1}{r}$-cutting, we have to take a *random sample* $\mathcal{R} \subseteq \mathcal{S}$, where each element of $\mathcal{S}$ is selected with probability r/n. It can be shown that, for every $k > 0$, the expected value of the total number of k-dimensional faces of all cells of the cell decomposition induced by the elements of $\mathcal{R}$ is $O(r^{d-1})$, while the expected number of vertices (0-dimensional faces) is $m(r/n)^d$. This cell decomposition can be further subdivided to obtain a partition of $\mathbb{R}^d$ into simplices that meet the requirements. The expected number of elements of $\mathcal{S}$ that intersect a given cell is at most n/r.

Cuttings have been successfully used before, e.g., for an alternative proof of the Szemerédi-Trotter theorem [4]. In our case, the use of this technique is somewhat unintuitive, as the size of the cuttings we construct depends on the number of intersecting d-tuples, that is, on the parameter we want to bound. This sets up an unusual recurrence relation, where the required parameter appears on both sides, but nonetheless, whose solution implies Theorem 1.

2 Proof of Theorem 1

For any element $S_k \in \mathcal{S}$, let $\mathrm{supp} S_k$ denote the supporting hyperplane of S_k. By slightly perturbing the arrangement, if necessary, we can assume without loss of generality that the supporting hyperplanes of the elements of $\mathcal{S}$ are in *general position*, that is,

(a) no $d-j+1$ of them have a j-dimensional intersection ($0 \leq j \leq d-1$), and

(b) the intersection of any d of them is empty or a point that lies in the relative interior of these d elements.

Theorem 1 is an immediate corollary of the following lemma.

Lemma 1. *Let $d, t \geq 2$ be fixed integers. Let $\mathcal{S}$ be an n-element set of $(d-1)$-dimensional simplices in general position in $\mathbb{R}^d$. Assume that their d-uniform intersection hypergraph $\mathcal{H}^d(\mathcal{S})$ has m edges and is $K^d_{t,\ldots,t}$-free.*

If, for suitable constants $C \geq 1$ and u, there exists a $\frac{1}{r}$-cutting of size at most $C(r^u + \frac{mr^d}{n^d})$ with respect to $\mathcal{S}$, consisting of full-dimensional simplices, then

$$m \leq C' \cdot n^u,$$

where C' is another constant (depending on d, t, C, and u).

Proof. For some value of the parameter r to be specified later, construct a $\frac{1}{r}$-cutting $\{\Delta_1, \ldots, \Delta_k\}$ with respect to $\mathcal{S}$, where $k \leq C(r^u + mr^d/n^d)$. Using our assumption that the elements of $\mathcal{S}$ are in general position, we can suppose that all cells Δ_i are full-dimensional.

For every i ($1 \leq i \leq k$), let $\mathcal{S}_i^{\mathrm{int}} \subseteq \mathcal{S}$ denote the set of all elements in $\mathcal{S}$ that intersect the interior of Δ_i. As $\{\Delta_1, \ldots, \Delta_k\}$ is a cutting with respect to $\mathcal{S}$, we have $|\mathcal{S}_i^{\mathrm{int}}| \leq n/r$. Let $\mathcal{S}_i^{\mathrm{bd}} \subseteq \mathcal{S}$ be the set of all elements $S_l \in \mathcal{S}$ such that the supporting hyperplane $\mathrm{supp} S_l$ of S_l contains a j-dimensional face of Δ_i for some j ($0 \leq j \leq d-1$). Set $\mathcal{S}_i = \mathcal{S}_i^{\mathrm{int}} \cup \mathcal{S}_i^{\mathrm{bd}}$.

Using the general position assumption, we obtain that every j-dimensional face F of Δ_i is contained in the supporting hyperplanes of at most $d-j \leq d$ elements of $\mathcal{S}$ ($0 \leq j \leq d-1$). The total number of proper faces of Δ_i of all dimensions is smaller than 2^{d+1}, and each is contained in at most d elements of $\mathcal{S}$. Therefore, we have

$$|\mathcal{S}_i| < |\mathcal{S}_i^{\mathrm{int}}| + |\mathcal{S}_i^{\mathrm{bd}}| \leq n/r + d2^{d+1}.$$

Fix an intersection point $q = S_1 \cap \ldots \cap S_d$, and let $\mathcal{S}(q) = \{S_1, \ldots, S_d\} \subseteq \mathcal{S}$. Then either

1. q lies in the interior of some Δ_i, in which case $\mathcal{S}(q) \subseteq \mathcal{S}_i^{\mathrm{int}} \subseteq \mathcal{S}_i$, or

2. q lies at a vertex (0-dimensional face) F or in the relative interior of a j-dimensional face F of some Δ_i, where $1 \leq j \leq d-1$. Take any $S_l \in \mathcal{S}(q)$. If $F \subset \mathrm{supp} S_l$, then $S_l \in \mathcal{S}_i^{\mathrm{bd}} \subseteq \mathcal{S}_i$. If $F \not\subset \mathrm{supp} S_l$, then S_l intersects the interior of Δ_i, and since q lies in the relative interior of S_l, we have that $S_l \in \mathcal{S}_i^{\mathrm{int}} \subseteq \mathcal{S}_i$.

In both cases, $\mathcal{S}(q) \subseteq \mathcal{S}_i$.

This means that if for each i we bound the number of d-wise intersection points between the elements of $\mathcal{S}_i$, and we add up these numbers, we obtain an upper bound on $\mathcal{H}^d(\mathcal{S})$, the number of d-wise intersection points between the elements of $\mathcal{S}$. Within each $\mathcal{S}_i$, we apply the abstract hypergraph-theoretic bound of Erdős (Theorem A) to conclude that $\mathcal{H}^d(\mathcal{S}_i)$ has at most $|\mathcal{S}_i|^{d-1/t^{d-1}}$ edges. Hence,

$$m \quad \leq \quad \sum_{i=1}^{k} |\mathcal{S}_i|^{d-1/t^{d-1}}.$$

As $|\mathcal{S}_i| \leq n/r + d2^{d+1}$, substituting the bound on k, we get

$$\begin{aligned} m \quad &\leq \quad C \cdot (r^u + \frac{mr^d}{n^d}) \cdot (d2^{d+1} + \frac{n}{r})^{d-1/t^{d-1}} \\ &\leq \quad 2^d C \cdot (r^u + \frac{mr^d}{n^d}) \cdot (\frac{n}{r})^{d-1/t^{d-1}}, \end{aligned}$$

provided that $\frac{n}{r} \geq d2^{d+1}$. Setting $r = \frac{n}{C_0}$, where $C_0 = (2^{d+1}C)^{t^{d-1}}$, we obtain

$$\begin{aligned} m \quad &\leq \quad 2^d C \left(\frac{n^u}{C_0^u} + \frac{m}{C_0^d} \right) C_0^{d-1/t^{d-1}} \\ m \quad &\leq \quad \frac{C_0^{d-u}}{2} n^u + \frac{m}{2}, \end{aligned}$$

which implies that $m \leq C_0^{d-u} n^u$, as required. $\square$

Now Theorem 1 follows from Theorem B, as one can choose $u = 1$ if $d = 2$ and $u = d - 1 + \epsilon$ if $d \geq 3$. This completes the proof.

Remark 1. Every d-uniform hypergraph $\mathcal{H}$ has a *d-partite* subhypergraph $\mathcal{H}'$ that has at least $\frac{d!}{d^d}$ times as many hyperedges as $\mathcal{H}$. Therefore, if K is d-partite, the maximum number of hyperedges that a *d-partite* K-free hypergraph on n vertices can have is within a factor of $\frac{d!}{d^d}$ from the same quantity over *all* K-free hypergraphs on n vertices. If instead of abstract hypergraphs, we restrict our attention to intersection graphs or hypergraphs of geometric objects, the order of magnitudes of the two functions may substantially differ. Given two sets of segments $\mathcal{S}$ and $\mathcal{T}$ in the plane with $|\mathcal{S}| = |\mathcal{T}| = n$, let $\mathcal{B}(\mathcal{S}, \mathcal{T})$ denote their *bipartite* intersection graph, in which the vertices representing $\mathcal{S}$ and $\mathcal{T}$ form two independent sets, and a vertex representing a segment in $\mathcal{S}$ is joined to a vertex representing a segment in $\mathcal{T}$ are joined by an edge if and only if they intersect. It was shown in [10] that any such $K_{2,2}$-free bipartite intersection graph $\mathcal{B}(\mathcal{S}, \mathcal{T})$ of n vertices has $O(n^{4/3})$ edges and that this bound is tight. In fact, this result generalizes the Szemerédi-Trotter theorem mentioned above. On the other hand, if we assume that the (non-bipartite) intersection graph associated with the set $\mathcal{S} \cup \mathcal{T}$, which contains the bipartite graph $\mathcal{B}(\mathcal{S}, \mathcal{T})$, is also $K_{2,2}$-free, then Theorem 1 implies that the number of edges drops to linear in n. In the examples where $\mathcal{B}(\mathcal{S}, \mathcal{T})$ has a superlinear number of edges, there must be many intersecting pairs of segments in $\mathcal{S}$ or in $\mathcal{T}$.

Remark 2. The key assumption in Lemma 1 is that there exists a $\frac{1}{r}$-cutting, whose size is sensitive to the number of intersecting d-tuples of objects. Under these circumstances, in terms of the smallest size of a $\frac{1}{r}$-cutting, one can give an upper bound on the number of edges of $K^d_{t,\ldots,t}$-free intersection hypergraphs with n vertices. For $d = 3$, we know some stronger bounds on the size of vertical decompositions of space induced by a set of triangles [5, 17], which imply the existence of $\frac{1}{r}$-cuttings of size $O(r^2\alpha(r) + \frac{mr^3}{n^3})$. Thus, in this case, we can deduce from Lemma 1 that every 3-uniform $K^3_{t,t,t}$-free intersection hypergraph of n triangles has $O(n^2\alpha(n))$ edges. It is an interesting open problem to establish nearly tight bounds on the maximum number of edges that a d-uniform $K^d_{t,\ldots,t}$-free intersection hypergraph induced by n semialgebraic sets in $\mathbb{R}^d$ can have. Lemma 1 does not apply in this case, because in the best currently known constructions of cuttings for semialgebraic sets, the exponent u is larger than d [1].

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