

Epsilon-Mnets: Hitting Geometric Set Systems with Subsets

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Abstract

The existence of Macbeath regions is a classical theorem in convex geometry [13], with recent applications in discrete and computational geometry. In this paper, we initiate the study of Macbeath regions in a combinatorial setting—and not only for the Lebesgue measure as is the case in the classical theorem—and establish near-optimal bounds for several basic geometric set systems.

1 Introduction

Given a convex body K in \mathbb{R}^d of unit volume, and a parameter $\epsilon > 0$, a classical theorem of Macbeath [13] from convex geometry implies the existence of disjoint convex bodies of K , each of volume $\Theta(\epsilon)$, called *Macbeath regions*, such that any half-space containing at least ϵ -th volume of K completely contains one of these convex bodies. Formally, consider the following theorem (as stated in [6]):

Theorem A (Macbeath regions). *Given a convex body $K \subset \mathbb{R}^d$ of unit volume, and a parameter $0 < \epsilon < 1/(2d)^{2d}$, there exists a set \mathcal{M} of $O\left(\frac{1}{\epsilon^{1-\frac{2}{d+1}}}\right)$ convex objects such that for any half-space h with $\text{vol}(h \cap K) \geq \epsilon$, there exists a $K_i \in \mathcal{M}$ such that $K_i \subset h \cap K$ and*

$$\text{vol}(K_i) \geq \frac{1}{(30d)^d} \cdot \epsilon.$$

Similar partitions of convex bodies was used by Edwald, Larmen and Rogers [9] for cap coverings, which were later further extended by Bárány and Larman [5]. They were also used for lower-bounds on range searching by Brönnimann, Chazelle and Pach [6]. Very recently, Macbeath regions were used in an elegant way by Arya, da Fonseca and Mount [3] for computing near-optimal Hausdorff approximations of polytopes. We refer the reader to Bárány [4] for a survey of these and several other applications of Macbeath regions.

Switching over to discrete and combinatorial geometry, a different structure— ϵ -nets—has been developed over the past three decades as a fundamental and powerful tool in computational geometry. Given a set system (X, \mathcal{R}) , and a parameter ϵ , an ϵ -net is a set $N \subseteq X$ such that $N \cap R \neq \emptyset$ for all $R \in \mathcal{R}$ with $|R| \geq \epsilon|X|$. A famous theorem of Haussler and Welzl [10] states the existence of ϵ -nets of size $O\left(\frac{d}{\epsilon} \log \frac{d}{\epsilon}\right)$ for (X, \mathcal{R}) , where d is the VC dimension of \mathcal{R} . This bound was later improved in [11] to an optimal bound of $(1 + o(1))\left(\frac{d}{\epsilon} \log \frac{1}{\epsilon}\right)$. By now ϵ -nets are an indispensable tool in combinatorics, geometry and algorithms (we refer the reader to the books [20, 16, 7, 17] for a small sampling of their constructions and applications).

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The starting point of our work is the observation that the two— ϵ -nets and Macbeath regions—are related. Indeed theorem A implies that for any convex body K in \mathbb{R}^d of volume V , it is possible to pick $O(\frac{1}{\epsilon})$ points in K (in fact, even less) which hit all half-spaces containing an ϵ -th fraction of the volume of K . However, the statement itself is much stronger than that: instead of just points, it states the existence of $O(\frac{1}{\epsilon})$ regions, each of volume $\Theta(\epsilon V)$, so that any half-space containing an ϵ -th fraction of the volume of K contains one of the regions completely. As we will prove in this paper, a strengthening of the ϵ -net statement is true for the counting measure for set systems induced by half-spaces in \mathbb{R}^3 : given any set P of points in \mathbb{R}^3 , there exist $O(\frac{1}{\epsilon})$ subsets of P , each of size $\Theta(\epsilon|P|)$, such that any half-space containing at least $\epsilon \cdot |P|$ points of P contains one of these regions completely. This raises the natural question: of the large number of results known for ϵ -nets for various geometric set systems, which can be optimally strengthened like the case above?

Geometric set systems can be categorized into two frequently studied types. Let \mathcal{O} be a family of geometric objects in \mathbb{R}^d —e.g., the family of all half-spaces, all balls and so on. We say that \mathcal{O} has union complexity $\varphi(\cdot)$ if the combinatorial complexity of the union of any r of the regions of \mathcal{O} is at most $r \cdot \varphi(r)$; we refer the reader to the survey [1] for bounds on the union complexity of many geometric objects. Given a set X of points in \mathbb{R}^d , we say that (X, \mathcal{R}) is a *primal set system* induced by \mathcal{O} if for each $R \in \mathcal{R}$, there exists an object $O \in \mathcal{O}$ such that $R = X \cap O$. On the other hand, given a finite set $\mathcal{S} \subseteq \mathcal{O}$ in \mathbb{R}^d , we say that $(\mathcal{S}, \mathcal{R})$ is a *dual set system* induced by \mathcal{S} if for each $R \in \mathcal{R}$, there exists a point $q \in \mathbb{R}^d$ contained in precisely the elements of R , i.e., $R = \{O \in \mathcal{S} \mid q \in O\}$.

In this paper we initiate a systematic study of the analogues of Macbeath regions—which we name ϵ -Mnets—for some commonly studied primal and dual geometric set-systems.

Definition (ϵ -Mnets). *Given a set system (X, \mathcal{R}) and a parameter $\epsilon > 0$, a collection $\mathcal{M} = \{X_1, \dots, X_t\}$ of subsets of X is an ϵ -Mnet for \mathcal{R} of size t if*

1. $|X_i| = \Omega(\epsilon \cdot |X|)$ for each $i = 1, \dots, t$ and,
2. for every $R \in \mathcal{R}$ with $|R| \geq \epsilon \cdot |X|$, there exists an index $j \in \{1, \dots, t\}$ such that $X_j \subseteq R$.

Furthermore, for any $\kappa \geq 2$, call \mathcal{M} a $\frac{1}{\kappa}$ -heavy ϵ -Mnet if each set in \mathcal{M} has size greater than $\frac{\epsilon|X|}{\kappa}$.

Our Results

Our first result establishes tight bounds for the sizes of ϵ -Mnets for the primal and dual set systems induced by axis-parallel rectangles in the plane. This already provides an example where ϵ -Mnets have larger sizes—by factors polynomial in $\frac{1}{\epsilon}$ —than ϵ -nets for the corresponding set systems. The proof of the following statement is in Section 2.

Theorem 1. *Let $\epsilon > 0$, $\kappa \geq 2$ be given parameters.*

- (a) **Dual set system.** *Given a set \mathcal{S} of axis-parallel rectangles in the plane, there exist $\frac{1}{2\kappa}$ -heavy ϵ -Mnets of size $O\left(\frac{4^\kappa}{\epsilon^{1+\frac{1}{\kappa}}}\right)$ for the dual set-system induced by \mathcal{S} .*

Furthermore, this is near-optimal: for any integer $n > 0$, there exists a set \mathcal{S} of n axis-parallel rectangles in \mathbb{R}^2 such that any $\frac{1}{\kappa}$ -heavy ϵ -Mnet for the dual set-system induced by \mathcal{S} has size $\Omega\left(\frac{1}{\epsilon^{1+\frac{1}{\kappa-1}}}\right)$.

- (b) **Primal set system.** *Given any set P of points in the plane, there exist ϵ -Mnets of size $O\left(\frac{1}{\epsilon} \log \frac{1}{\epsilon}\right)$ for the primal set-system induced by axis-parallel rectangles on P .*

Furthermore, this is near-optimal: for any integer $n > 0$, there exists a set P of n points in the plane such that any $\frac{1}{\kappa}$ -heavy ϵ -Mnet for the primal set-system induced by axis-parallel rectangles on P has size $\Omega\left(\frac{1}{\epsilon} \log_\kappa \frac{1}{\epsilon}\right)$.

Our next result states the existence of small ϵ -Mnets for dual set systems as a function of the union complexity of the objects. Call a set \mathcal{S} of objects in \mathbb{R}^d *well-behaved* if for any subset $\mathcal{S}' \subseteq \mathcal{S}$ and any $Q \subseteq \mathbb{R}^d$, one can decompose the cells in the arrangement of \mathcal{S}' that intersect Q into cells of constant descriptive complexity, where the complexity of this decomposition is proportional to the total number of vertices in the cells that intersect Q ; we refer the reader to [8] for more details. The proof of the following statement is in Section 3.

Theorem 2. *Let \mathcal{R} be the dual set system induced by a set of well-behaved regions \mathcal{S} in \mathbb{R}^d with union complexity $\varphi(\cdot)$ and let $\epsilon > 0$ be a given parameter. Then there exists an ϵ -Mnet for \mathcal{R} of size $O(\frac{1}{\epsilon}\varphi(\frac{1}{\epsilon}))$.*

Interestingly, as $\varphi(m) = \Omega(m)$ for the dual set system induced by axis-parallel rectangles in the plane, Theorem 1 implies that the dependence of $\varphi(\cdot)$ in Theorem 2 cannot be reduced to, for example, $\log \varphi(\cdot)$, as is the case for ϵ -nets.

Our last result is to consider the primal case where the input is a set of points and the set system is defined by containment by geometric objects such as disks, lines, triangles and more generally, k -sided polygons in the plane. The proof of the following statement is in Section 4.

Theorem 3. *Let P be a set of n points, and $\epsilon > 0$ a given parameter. Then one can construct ϵ -Mnets of size:*

(a) $O(\frac{1}{\epsilon^{\lfloor d/2 \rfloor}})$ for the primal set system induced by half-spaces in \mathbb{R}^d , for $d \geq 2$.

Furthermore, this cannot be improved substantially: for any integers $d \geq 2$ and $n > 0$, there exists a set of n points in \mathbb{R}^d such that any ϵ -Mnet for the primal set system induced by half-spaces has size $\Omega(\frac{1}{\epsilon^{\lfloor \frac{d+1}{3} \rfloor}})$.

(b) $O(\frac{1}{\epsilon})$ for the primal set system induced by disks in the plane.

(c) $O(\frac{1}{\epsilon^3}(\log \frac{1}{\epsilon})^4)$ for the primal set system induced by triangles, and in general k -sided polygons in the plane (the constant in the asymptotic notation depends on k).

(d) $O(\frac{1}{\epsilon^2}(\log \frac{1}{\epsilon})^2)$ for the primal set system induced by lines, $O(\frac{1}{\epsilon^2}(\log \frac{1}{\epsilon})^3)$ for the one induced by cones, and $O(\frac{1}{\epsilon^2}(\log \frac{1}{\epsilon})^4)$ for the one induced by strips in the plane.

Furthermore, this is near-optimal: for any integer $n > 0$, there exists a set of n points in \mathbb{R}^2 such that any ϵ -Mnet for the primal set system induced by lines or cones or strips has size $\Omega(\frac{1}{\epsilon^2})$.

(e) $O(\frac{1}{\epsilon})$ for the primal set system induced by axis-parallel rectangles in \mathbb{R}^2 , all intersecting the y -axis.

Theorem 3 implies that near-linear bounds for ϵ -Mnets are not possible for even simple primal set-systems such as those induced by lines in the plane. This contrasts sharply with ϵ -net bounds for geometric set systems, which are near-linear for any set system with constant VC dimension.

2 Proof of Theorem 1

The following lemma, of independent interest, gives insight for studying ϵ -Mnets for both the primal and dual set systems induced by axis-parallel rectangles in the plane.

Lemma 2.1. *For any integers $r, d \geq 3$, consider the grid $G = \{0, \dots, r-1\}^d$ in \mathbb{R}^d consisting of r^d points. Then there exists a bijective mapping $\pi : G \mapsto \mathbb{R}^2$ such that the primal set system on G induced by axis-parallel lines can be realized by the primal set system induced by axis-parallel rectangles in \mathbb{R}^2 on the set $\{\pi(p), p \in G\}$.*

Proof. Let $[r]$ represent the set $\{0, \dots, r-1\}$. For any $i \in \{1, \dots, d\}$ and integers $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_d \in [r]$, consider the set of points

$$S_i(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_d) = \left\{ (a_1, \dots, a_{i-1}, t, a_{i+1}, \dots, a_d) : t \in [r] \right\}.$$

We call such a set a *line in direction i* . There are dr^{d-1} such lines, r^{d-1} in each of the d directions (along the axes) in \mathbb{R}^d .

We will show that there exists a mapping $\pi : G \mapsto \mathbb{R}^2$ such that for each line l in any direction, the inclusion-minimal axis-parallel rectangle containing the image, under $\pi(\cdot)$, of the points in l does not contain the image of any other point of G . Here is the mapping $\pi(\cdot)$ that we will use:

$$\pi((a_1, \dots, a_d)) = \sum_j a_j \vec{v}_j, \quad \text{where } \vec{v}_j = (r^j, r^{d+1-j}).$$

For any point $z \in G$, we will interpret $p = \pi(z)$ both as a vector and as a point, as suitable. When treating it as a vector, we will denote it by \vec{p} . For any $z' = (a_1, \dots, a_d) \in G$, let $\vec{V}_{<i}(z')$ denote the vector $\sum_{j < i} a_j \vec{v}_j$ and $\vec{V}_{>i}(z')$ denote the vector $\sum_{j > i} a_j \vec{v}_j$. Thus we can write $\pi(z') = \vec{V}_{<i}(z') + a_i \vec{v}_i + \vec{V}_{>i}(z')$.

Consider any line, say $l = S_i(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_d)$, and let R be the smallest rectangle containing the set of r mapped points of l in the plane, namely the set

$$f(l) = \left\{ \pi((a_1, \dots, a_{i-1}, t, a_{i+1}, \dots, a_d)) : t \in [r] \right\}.$$

Let $z_l = (a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_d)$ and $z_r = (a_1, \dots, a_{i-1}, r-1, a_{i+1}, \dots, a_d)$ be the two extreme points lying on l . As all the coordinates except the i -th one are the same for all points lying on l , the mapped point with the maximum x -coordinate is the one that maximizes $t \cdot r^i$, i.e., the point $\pi(z_r)$. Similarly, $\pi(z_r)$ has the maximum y -coordinate, and $\pi(z_l)$ has the minimum x - and y -coordinates. Furthermore, the width of R is defined by the difference in the x -coordinates of $\pi(z_r)$ and $\pi(z_l)$, and so it is precisely $(r-1)r^i$. Likewise, the height of R is $(r-1)r^{d+1-i}$.

It remains to show that for any other point, say $z = (b_1, \dots, b_d) \in G \setminus l$, $\pi(z)$ does not lie in R . Let $z' = (a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_d) \in G$ be the point lying on the line l with the same i -th coordinate as z . Let $p = \pi(z) = \vec{V}_{<i}(z) + b_i \vec{v}_i + \vec{V}_{>i}(z)$ and $q = \pi(z') = \vec{V}_{<i}(z') + b_i \vec{v}_i + \vec{V}_{>i}(z')$. Then

$$\vec{p} - \vec{q} = (\vec{V}_{<i}(z) - \vec{V}_{<i}(z')) + (\vec{V}_{>i}(z) - \vec{V}_{>i}(z')).$$

Since $\vec{p} \neq \vec{q}$, one of the above two summands must be non-zero. Without loss of generality assume that the second summand is non-zero. The other case is similar. As $\vec{V}_{>i}(z) - \vec{V}_{>i}(z') = \sum_{j > i} (b_j - a_j) \vec{v}_j$, it is a non-zero integral combination of the vectors v_j for $j > i$, and so its x -coordinate has magnitude at least r^{i+1} . On the other hand the x -coordinate of $(\vec{V}_{<i}(z) - \vec{V}_{<i}(z'))$ has magnitude at most $\sum_{1 \leq j < i} (r-1)r^j = r^i - r$. Therefore the difference in the x -coordinates between p and q is at least $r^{i+1} - (r^i - r)$, which is greater than the width of R . Hence, $p \notin R$. When $(\vec{V}_{<i}(z) - \vec{V}_{<i}(z')) \neq 0$, a similar argument holds for the y -coordinates of p and q , showing that the difference in their y -coordinates is larger than the height of R . \square

Case (a): Dual set system.

Lower-bound. We now show that for any integers $\kappa \geq 2$ and $n \geq 0$, there exists a set \mathcal{R} of n axis-parallel rectangles such that any $\frac{1}{\kappa}$ -heavy ϵ -Mnet for the dual set system induced by \mathcal{R} has size $\Omega\left(\frac{1}{\epsilon^{1+1/(\kappa-1)}}\right)$. Apply Lemma 2.1 with $d = \kappa$ and $r = \epsilon^{-\frac{1}{d-1}}$. Let G be the grid $[r]^d$ as before. We set $P = \{\pi(p) : p \in G\}$ and let \mathcal{R}' be the set of dr^{d-1} rectangles corresponding to the dr^{d-1} lines in G . Construct the required set \mathcal{R} by

replacing each rectangle of \mathcal{R}' with $\frac{\epsilon n}{d}$ copies. Note that $|\mathcal{R}| = \frac{\epsilon n}{d} \cdot dr^{d-1} = n$. Since each of the points in G is contained in d lines (one in each direction), each point of P is contained in d rectangles of \mathcal{R}' and consequently ϵn rectangles of \mathcal{R} . Since there is at most one line through two points in G , there are at most $\frac{\epsilon n}{d}$ rectangles of \mathcal{R} that contain any pair of points $p, q \in P$. Since for any $\frac{1}{\kappa}$ -heavy ϵ -Mnet \mathcal{M} , each $U \in \mathcal{M}$ has size greater than $\frac{\epsilon n}{\kappa}$, it must be that no set in \mathcal{M} can be contained in two sets $\mathcal{R}(p)$ and $\mathcal{R}(q)$ induced by two distinct points p and q in P . Therefore $|\mathcal{M}| \geq |P| = r^d = \epsilon^{-\frac{\kappa}{\kappa-1}} = \frac{1}{\epsilon^{1+\frac{1}{\kappa-1}}}$.

Upper-bound. We now establish an upper-bound for the dual set systems induced by axis-parallel rectangles in the plane.

Construct a hierarchical subdivision on \mathcal{S} , as follows. Let $k = \lceil \frac{1}{\epsilon^{1/\kappa}} \rceil$, and for $i = 0, \dots, \kappa$, set the parameters $n_i = \frac{n}{k^i}$, and $\epsilon_i = \epsilon(\frac{k}{2})^i$. At the 0-th level (here $i = 0$), let l_1^0, \dots, l_{k-1}^0 be a set of $k - 1$ vertical lines such that the number of rectangles of \mathcal{S} lying between two consecutive lines—call this region a ‘slab’—is at most $\frac{n_0}{k}$. Let \mathcal{S}_j^0 be the set of rectangles lying entirely in the j -th slab. For each index $j = 1, \dots, k - 1$, construct a $\frac{\epsilon_0}{4}$ -Mnet for all the rectangles of \mathcal{S} intersecting l_j^0 . Furthermore, construct an $\epsilon(\frac{k}{2})$ -Mnet for the rectangles in \mathcal{S}_j^0 , for each $j = 1, \dots, k - 1$ in the similar manner as above. The construction continues for κ steps: at the i -level, there are k^i total sub-problems, each sub-problem consists of at most $n_i = \frac{n}{k^i}$ rectangles and with $\epsilon_i = \epsilon(\frac{k}{2})^i$.

At the base case of the recursion, we use a direct $O(\frac{1}{\epsilon^2})$ -sized construction for the ϵ_κ -Mnet of the k^κ sub-problems at the last κ -level: for the sub-problem of computing a ϵ_κ -Mnet for a set of rectangles \mathcal{S}' where $|\mathcal{S}'| \leq n_\kappa$, construct a set L' of $\frac{8}{\epsilon_\kappa}$ vertical and $\frac{8}{\epsilon_\kappa}$ horizontal lines such that each vertical (resp. horizontal) slab induced by L' contains at most $\frac{\epsilon_\kappa |\mathcal{S}'|}{4}$ vertical (resp. horizontal) boundary edges of the rectangles in \mathcal{S}' . For each bounded cell c induced by L' , add to \mathcal{M} all the rectangles of \mathcal{S}' completely containing c , if their total number is at least $\frac{\epsilon_\kappa |\mathcal{S}'|}{2}$. Now take any point $q \in \mathbb{R}^2$ lying in at least $\epsilon_\kappa |\mathcal{S}'|$ rectangles of \mathcal{S}' and let c be the cell induced by L' containing q . At least one of the boundary edges of any rectangle R containing q but not containing c must lie in the vertical or horizontal slab induced by L' containing q . Thus there can be only $\frac{\epsilon_\kappa |\mathcal{S}'|}{2}$ such rectangles that contain q but not the cell c . The remaining at least $\frac{\epsilon_\kappa |\mathcal{S}'|}{2}$ rectangles that contain q must then all contain c , and so would form a set in \mathcal{M} of size at least $\frac{\epsilon_\kappa |\mathcal{S}'|}{2}$. Note that the total number of sets added to \mathcal{M} is $O(\frac{1}{\epsilon^2})$.

The next two claims conclude the proof by showing that all these Mnets together form an ϵ -Mnet \mathcal{M} for \mathcal{S} of the required size.

Claim 1. *Each set in \mathcal{M} has size $\Theta(\frac{\epsilon n}{2^\kappa})$. The size of \mathcal{M} is $O(\frac{4^\kappa}{\epsilon^{1+\frac{1}{\kappa}}})$.*

Proof. At the i -level there are k^i sub-problems, each of size at most $n_i = \frac{n}{k^i}$ with $\epsilon_i = \epsilon(\frac{k}{2})^i$. For each such sub-problem, we partition its set of at most n_i rectangles by $k - 1$ lines, and construct a $\frac{\epsilon_i}{4}$ -Mnet for the rectangles intersecting these $k - 1$ lines. Note that the set of rectangles intersecting any line, and clipped to one side of the line have linear union complexity [1] and by Theorem 2, there exists a $\frac{\epsilon_i}{4}$ -Mnet of size $O(\frac{1}{\epsilon_i})$. Hence the total size over all internal sub-problems is:

$$\sum_{i=0}^{\kappa} k^i \cdot (k - 1) \cdot O\left(\frac{1}{\epsilon_i}\right) \leq \sum_{i=0}^{\kappa} k^{i+1} \cdot O\left(\frac{2^i}{\epsilon k^i}\right) = \sum_{i=0}^{\kappa} O\left(\frac{2^i}{\epsilon^{1+\frac{1}{\kappa}}}\right) = O\left(\frac{2^\kappa}{\epsilon^{1+\frac{1}{\kappa}}}\right).$$

At the last level, after κ steps, we have k^κ sub-problems, each with at most $\frac{n}{k^\kappa}$ rectangles, and $\epsilon_\kappa = \epsilon(\frac{k}{2})^\kappa$. Now use a direct construction which constructs an ϵ -Mnet of size $O(\frac{1}{\epsilon^2})$, to get the total size of Mnet at the last step to be $O(\kappa^k \cdot \frac{1}{\epsilon^2}) = O(\frac{4^\kappa}{\epsilon^{2k^\kappa}}) = O(\frac{4^\kappa}{\epsilon})$.

At any level i , we construct a ϵ_i -Mnet on a set of at most $\frac{n}{k^i}$ rectangles. So each set in the constructed Mnet has size $\Omega(\epsilon_i \cdot \frac{n}{k^i}) = \Omega(\frac{\epsilon n}{2^i}) = \Omega(\frac{\epsilon n}{2^\kappa})$. \square

Claim 2. For each point $q \in \mathbb{R}^2$ lying in at least ϵn rectangles of \mathcal{S} , there exists a set $U \in \mathcal{M}$ such that q lies in all the rectangles of U .

Proof. Take a point q lying in at least ϵn rectangles of \mathcal{S} . At the 0-th level, say q lies in the vertical slab defined by lines l_j^0 and l_{j+1}^0 . If q is contained in at least $\frac{\epsilon n}{4}$ rectangles intersected by either l_j^0 or l_{j+1}^0 , say l_j^0 , then it is contained in at least $\frac{\epsilon n}{4}$ rectangles out of a total of at most n rectangles intersected by l_j^0 . So the $\frac{\epsilon}{4}$ -Mnet for l_j^0 will have a set U such that each rectangle in U contain q . Otherwise q is contained in at least $\frac{\epsilon n}{2} = \epsilon \left(\frac{k}{2}\right) \left(\frac{n}{k}\right) = \epsilon_1 n_1$ rectangles of the set \mathcal{S}_j^0 of size at most $n_1 = \frac{n_0}{k}$, and we proceed to this sub-problem.

In general, at the i -level, each sub-problem has at most $n_i = \frac{n}{k^i}$ rectangles, with $\epsilon_i = \epsilon \left(\frac{k}{2}\right)^i$. Then either q is contained in at least $\frac{\epsilon_i n_i}{4}$ rectangles intersecting one of the lines, and so will contain a set from the $\frac{\epsilon_i}{4}$ -Mnet constructed for each of the $k - 1$ vertical lines. Or q is contained in at least $\frac{\epsilon_i n_i}{2}$ rectangles out of a total of at most $n_{i+1} = \frac{n_i}{k}$ rectangles lying in one of the slabs defined by the $k - 1$ vertical lines. But as

$$\frac{\epsilon_i n_i}{2} = \frac{\epsilon}{2} \cdot \left(\frac{k}{2}\right)^i \cdot \frac{n}{k^i} = \epsilon \left(\frac{k}{2}\right)^{i+1} \frac{n}{k^{i+1}} = \epsilon_{i+1} n_{i+1},$$

q will be covered inductively by the ϵ_{i+1} -Mnet constructed for the n_{i+1} rectangles in one of the resulting sub-problems at level $i + 1$. \square

Case (b): Primal set system.

Lower-bound. We now show that for any integers $\kappa \geq 2$ and $n \geq 0$, there exists a set P of n points in \mathbb{R}^2 such that any $\frac{1}{\kappa}$ -heavy ϵ -Mnet of P for the primal set system induced by axis-parallel rectangles in the plane has size $\Omega\left(\frac{1}{\epsilon} \log_{\kappa} \frac{1}{\epsilon}\right)$. Apply Lemma 2.1 with $r = \kappa$, and with parameter d set with $r^{d-1} = \frac{1}{\epsilon}$. According to the lemma, there is a mapping π from the grid $G = \{0, \dots, r - 1\}^d$ to the plane so that for each subset $S \subset G$ of the grid obtained by intersecting G with an axis-parallel line, there exists an axis-parallel rectangle R in the plane such that $R \cap \pi(G) = \pi(S)$; i.e., R contains exactly the mapped points of S . There are $d r^{d-1} = \Theta\left(\frac{1}{\epsilon} \log_{\kappa} \frac{1}{\epsilon}\right)$ such subsets and let \mathcal{R} be the set of axis-parallel rectangles corresponding to these. Let P be the set of points obtained by replacing each point $p \in \pi(G)$ with $\frac{\epsilon n}{r}$ copies of p (note that P is not a multi-set; think of each copy of the same point in $\pi(G)$ as a distinct point). The number of points in P is $r^{d-1} \cdot \frac{\epsilon n}{r} = n$. Each rectangle in \mathcal{R} contains $r \cdot \frac{\epsilon n}{r} = \epsilon n$ points of P . Also, any pair of rectangles in \mathcal{R} share at most $\frac{\epsilon n}{r} = \epsilon \frac{n}{k}$ points of P . Thus no two rectangles in \mathcal{R} may share the same set $U \in \mathcal{M}$ of a $\frac{1}{k}$ -heavy ϵ -Mnet \mathcal{M} . Since each of them must contain some $U \in \mathcal{M}$, we have $|\mathcal{M}| \geq |\mathcal{R}|$ and the result follows.

Upper-bound. We now present a matching upper-bound for the primal set system induced by axis-parallel rectangles in the plane.

Assume $P = \{p_1, \dots, p_n\}$ are labeled in the order of increasing x -coordinates. Given P , construct a balanced binary subdivision of P with vertical lines: divide P by a vertical line into two equal-sized subsets P_0^1, P_1^1 , and then recursively divide each of these sets into two equal-sized subsets and so on for $\log \frac{1}{\epsilon}$ levels. At the i^{th} level of recursion, there are 2^i sets of size $\frac{n}{2^i}$.

Let P_j^i denote the j -th subset of P at level i , i.e.,

$$\text{For } 1 \leq i \leq \log \frac{1}{\epsilon}, \quad 0 \leq j < 2^i, \quad P_j^i = \left\{ p_{j \frac{n}{2^i} + 1}, \dots, p_{(j+1) \frac{n}{2^i}} \right\}.$$

For each set P_j^i , and for each of its two bounding lines, say lines l_0 and l_1 , construct a $2^{i-1} \epsilon$ -Mnet for the following primal set-system: the base set is P_j^i , and given a line $l \in \{l_0, l_1\}$, the sets are induced by axis-parallel rectangles intersecting the line l . Note that all points of P_j^i lie on the same side of l . Let \mathcal{M} be the union of all

these Mnets. Crucially, the primal set system induced by the set of axis-parallel rectangles on the same side of l admits an ϵ -Mnet of size $O(\frac{1}{\epsilon})$ by Theorem 3 (e).

We now prove that \mathcal{M} is an ϵ -Mnet of P , of size $O(\frac{1}{\epsilon} \log \frac{1}{\epsilon})$.

Claim 3. *Each set in \mathcal{M} has size $\Theta(\epsilon n)$, and size of \mathcal{M} is $O(\frac{1}{\epsilon} \log \frac{1}{\epsilon})$.*

Proof. The set P_j^i has size $\frac{n}{2^i}$, and so each set in a $(2^{i-1}\epsilon)$ -Mnet of P_j^i has size $\Omega(2^{i-1}\epsilon \cdot \frac{n}{2^i}) = \Omega(\epsilon n)$. Note that each $2^{i-1}\epsilon$ -Mnet has size $O(\frac{1}{2^{i-1}\epsilon})$, there are 2^i sets P_j^i at level i , and a total of $\log \frac{1}{\epsilon}$ levels. Hence the size of \mathcal{M} is $O(\frac{1}{2^i \epsilon} \cdot 2^i \cdot \log \frac{1}{\epsilon}) = O(\frac{1}{\epsilon} \log \frac{1}{\epsilon})$. \square

Claim 4. *Each axis-parallel rectangle containing at least ϵn points of P contains a set of \mathcal{M} .*

Proof. Let R be an axis-parallel rectangle containing at least ϵn points of P . Let i be the smallest index such that R intersects exactly one vertical line separating two sets P_j^i and P_{j+1}^i at level i . Say R intersects the line l separating P_j^i and P_{j+1}^i . Then R must contain at least $\frac{\epsilon n}{2}$ points from either P_j^i or P_{j+1}^i , say P_j^i . Let R' be the part of R on the side of l towards P_j^i . Thus R' must contain at least one set of the $2^{i-1}\epsilon$ -Mnet for P_j^i , as

$$|R \cap P_j^i| = |R' \cap P_j^i| \geq \frac{\epsilon n}{2} = 2^{i-1}\epsilon \cdot \frac{n}{2^i} = 2^{i-1}\epsilon \cdot |P_j^i|.$$

\square

3 Proof of Theorem 2

Given the input set \mathcal{S} of regions in \mathbb{R}^d , define the *depth* of any point $q \in \mathbb{R}^d$ with respect to \mathcal{S} to be the number of regions of \mathcal{S} containing q . The key tool used in the proof are *shallow cuttings*:

Theorem B ([15, 8]). *Given a set \mathcal{S} of n well-behaved regions in \mathbb{R}^d with union complexity $\varphi(\cdot)$ and two parameters $r, l > 0$, there exists a partition of \mathbb{R}^d into a set Ξ of interior-disjoint cells (of constant description complexity) such that*

1. *each cell of Ξ is intersected by the boundary of at most $\frac{n}{r}$ regions of \mathcal{S} , and*
2. *the number of cells in Ξ that contain points of depth less than l (with respect to \mathcal{S}) is $O\left(\left(\frac{rl}{n} + 1\right)^d \cdot \frac{n}{l} \cdot \varphi\left(\frac{n}{l}\right)\right)$.*

Such a partition Ξ is called a $(\frac{1}{r}, l)$ -shallow cutting of \mathcal{S} .

We will construct the required ϵ -Mnet \mathcal{M} as a union of $\log \frac{1}{\epsilon}$ collections \mathcal{M}_i , for $i = 0, \dots, \log \frac{1}{\epsilon}$. For a fixed index i , construct the sets in \mathcal{M}_i by setting $l_i = 2^{i+1}\epsilon n$, $r_i = \frac{1}{2^{i+1}\epsilon}$, and construct a $(\frac{1}{r_i}, l_i)$ -shallow cutting, denoted by Ξ_i , for \mathcal{S} . Call a cell $\Delta \in \Xi_i$ *shallow* if it contains points of depth less than l_i . For each $\Delta \in \Xi_i$, let $r(\Delta)$ be the set of regions in \mathcal{S} that completely contain Δ ; i.e., $S \in r(\Delta)$ if and only if $\Delta \subset S$. Now, for all shallow cells Δ with $r(\Delta) \geq \frac{\epsilon n}{2}$, add $r(\Delta)$ to \mathcal{M}_i .

We can trivially upper-bound $|\mathcal{M}_i|$ by the number of shallow cells of Ξ_i , i.e., cells containing a point of depth less than $l_i = 2^{i+1}\epsilon n$. Thus using Theorem B, we get

$$|\mathcal{M}_i| = O\left(\left(\frac{r_i \cdot 2^{i+1}\epsilon n}{n} + 1\right)^d \cdot \frac{n}{2^{i+1}\epsilon n} \cdot \varphi\left(\frac{n}{2^{i+1}\epsilon n}\right)\right) = O\left(4^d \cdot \frac{1}{2^i \epsilon} \cdot \varphi\left(\frac{1}{2^i \epsilon}\right)\right).$$

First we bound the size of $\mathcal{M} = \bigcup_i \mathcal{M}_i$:

$$|\mathcal{M}| \leq \sum_{i=0}^{\log \frac{1}{\epsilon}} |\mathcal{M}_i| = \sum_{i=0}^{\log \frac{1}{\epsilon}} O\left(4^d \cdot \frac{1}{2^i \epsilon} \cdot \varphi\left(\frac{1}{2^i \epsilon}\right)\right) = O\left(4^d \cdot \frac{1}{\epsilon} \cdot \varphi\left(\frac{1}{\epsilon}\right)\right) \sum_{i=0}^{\log \frac{1}{\epsilon}} \frac{1}{2^i} = O\left(4^d \cdot \frac{1}{\epsilon} \cdot \varphi\left(\frac{1}{\epsilon}\right)\right).$$

To see that sets in \mathcal{M} form the required ϵ -Mnet, let $p \in \mathbb{R}^d$ be any point contained in t regions of \mathcal{S} , where $t \geq \epsilon n$. Let i be the index such that $2^i \epsilon n \leq t < 2^{i+1} \epsilon n$. Let Δ_p be the shallow cell in the $(\frac{1}{r_i}, l_i)$ -shallow cutting that contains p . Recall that the $(\frac{1}{r_i}, l_i)$ -shallow cutting Ξ_i partitions \mathbb{R}^d into a set of cells such that each cell intersects the boundary of at most $\frac{n}{r_i} = 2^{i-1} \epsilon n$ objects in \mathcal{S} . Thus, of all the $t \geq 2^i \epsilon n$ regions containing p , the boundary of at most $2^{i-1} \epsilon n$ regions can intersect Δ_p . The remaining at least $2^{i-1} \epsilon n \geq \frac{\epsilon n}{2}$ regions of \mathcal{S} containing p must then completely contain Δ_p , and so are in the set $r(\Delta_p)$. Thus the set $r(\Delta_p)$ is added to \mathcal{M}_i , and we have $|r_i(\Delta)| \geq 2^{i-1} \epsilon n \geq \frac{\epsilon n}{2}$.

4 Proof of Theorem 3

(a). First we establish the upper-bound on the sizes of ϵ -Mnets for the primal set system induced by half-spaces in \mathbb{R}^d . For a point $p \in P$, let H_p be its dual hyperplane, and let $\mathcal{H} = \{H_p \mid p \in P\}$. Let \mathcal{H}^+ (resp. \mathcal{H}^-) be a set of upward-facing (resp. downward-facing) half-spaces defined by \mathcal{H} . Apply Theorem 2 to the dual set system induced by \mathcal{H}^+ (resp. \mathcal{H}^-) to get an ϵ -Mnet \mathcal{M}^+ (resp. \mathcal{M}^-), and let \mathcal{M} be the corresponding collection of sets for P corresponding to both \mathcal{M}^+ and \mathcal{M}^- . As \mathcal{M}^+ (resp. \mathcal{M}^-) is an ϵ -Mnet for \mathcal{H}^+ (resp. \mathcal{H}^-), for any point $q \in \mathbb{R}^d$ contained in at least ϵn half-spaces in \mathcal{H}^+ (resp. \mathcal{H}^-), there exists a set in \mathcal{M}^+ (resp. \mathcal{M}^-) of size $\Omega(\epsilon n)$, such that each half-space in this set contains q . Switching to the primal viewpoint, any upward-facing (resp. downward-facing) half-space H_q containing at least ϵn points of P , corresponds in the dual to a point q that is contained in at least ϵn downward-facing (resp. upward-facing) half-spaces in \mathcal{H}^+ (resp. \mathcal{H}^-). As \mathcal{M}^+ (resp. \mathcal{M}^-) is an ϵ -Mnet for \mathcal{H}^+ (resp. \mathcal{H}^-), it follows that \mathcal{M} is an ϵ -Mnet for the primal set system induced by half-spaces. To bound the size of \mathcal{M} obtained from Theorem 2, it suffices to note that for half-spaces, $r\varphi(r) = O(r^{\lfloor d/2 \rfloor})$ [1].

For the lower-bound for ϵ -Mnets for the primal set system induced by half-spaces in \mathbb{R}^d , we first prove the following more general theorem.

Theorem 4. *Given a real parameter $\epsilon > 0$, integer $n > 1$ and two constants δ and k , there exists a set P of n points in the plane, and a set \mathcal{D} of $\Omega(\frac{1}{\epsilon^{\delta+1}})$ curves, each of degree at most δ , such that a) each curve contains ϵn points of P and b) no two curves in \mathcal{D} have more than $\frac{\epsilon n}{k}$ points of P in common. In particular, any $\frac{1}{k}$ -heavy ϵ -Mnet for the primal set system on P induced by curves of degree at most δ has size $\Omega(\frac{1}{\epsilon^{\delta+1}})$ (the constants in the asymptotic notation depend on k and δ).*

Proof. Denote by G the set of $\frac{\delta k}{\epsilon}$ grid points in $\{0, \dots, \delta k - 1\} \times \{0, \dots, \lceil \frac{1}{\epsilon} \rceil - 1\}$. The set of curves in \mathcal{D} will be all univariate functions in x of the form

$$y = \sum_{i=0}^{\delta} a_i \cdot x^i, \quad \text{where each } a_i \in \left\{0, 1, \dots, \left\lceil \frac{1}{\epsilon(\delta+1)(\delta k)^i} \right\rceil - 1\right\}.$$

Clearly we have

$$|\mathcal{D}| = \prod_{i=0}^{\delta} \frac{1}{\epsilon(\delta+1)(\delta k)^i} = \Omega\left(\frac{1}{\epsilon^{\delta+1}(\delta k)^{\Theta(\delta^2)}}\right) = \Omega\left(\frac{1}{\epsilon^{\delta+1}}\right).$$

Since for each value of $x \in \{0, \dots, \delta k - 1\}$, the corresponding value of y for each of the curves in \mathcal{D} lies in $\{0, \dots, \lceil \frac{1}{\epsilon} \rceil - 1\}$, each of the curves of \mathcal{D} contain precisely δk points of G . Furthermore, as these curves have degree at most δ , no two intersect in more than δ points of G .

Let P be the set of n points obtained by replacing each point of G with $\frac{\epsilon n}{\delta k}$ copies to get a set of n points in the plane. Now each curve in \mathcal{D} contains $\delta k \cdot \frac{\epsilon n}{\delta k} = \epsilon n$ points of P and every pair of curves have less than $d \cdot \frac{\epsilon n}{\delta k} = \frac{\epsilon n}{k}$ points of P in common.

Finally observe that any $\frac{1}{k}$ -heavy ϵ -Mnet \mathcal{M} for the primal set system on P induced by \mathcal{D} must consist of at least $|\mathcal{D}|$ sets: each curve $D \in \mathcal{D}$ must completely contain a set $R \in \mathcal{M}$ of size at least $\frac{\epsilon n}{k}$, and furthermore R cannot be contained in any other curve $D' \in \mathcal{D}$, as any two curves of \mathcal{D} have less than $\frac{\epsilon n}{k}$ points of P in common. \square

Now we show the desired lower-bound for ϵ -Mnets for the primal set system induced by half-spaces in \mathbb{R}^d .

Corollary 4.1. *For any $\epsilon > 0$ and integers n and d , there exists a set P of n points in \mathbb{R}^d such that any ϵ -Mnet for the primal set system on P induced by half-spaces has size $\Omega\left(\frac{1}{\epsilon^{\lceil \frac{d+1}{3} \rceil}}\right)$.*

Proof. First assume that $\frac{d-2}{3}$ is an integer, and apply Theorem 4 with $\delta = \frac{d-2}{3}$ and $k = 2$ to get a set P of n points in \mathbb{R}^2 and a set \mathcal{D} of curves such that any ϵ -Mnet for the primal set system induced by \mathcal{D} on P has size $\Omega(\frac{1}{\epsilon^{\delta+1}})$. We now use Veronese maps [17] to map the incidences between points and curves in \mathcal{D} to incidences between points and half-spaces in \mathbb{R}^d . More precisely, consider the map:

$$\pi : p = (p_x, p_y) \in \mathbb{R}^2 \longrightarrow (x, x^2, \dots, x^{2\delta}, y, yx, \dots, yx^\delta, y^2) \in \mathbb{R}^d.$$

We claim that for any curve $D \in \mathcal{D}$, say defined by the equation $y = \sum_{i=0}^{\delta} a_i \cdot x^i$, there exists a half-space H_D in \mathbb{R}^d such that the set of points of P contained in D is precisely the set of points of $\pi(P)$ contained in H_D . The required half-space can be constructed as follows:

$$\begin{aligned} p \in D \quad \text{if and only if} \quad & \left(y - \sum_{i=0}^{\delta} a_i \cdot x^i \right) = 0 \\ & \left(y - \sum_{i=0}^{\delta} a_i \cdot x^i \right)^2 \leq 0 \\ & \left(a'_1 x + a'_2 x^2 + \dots + a'_{2\delta} x^{2\delta} \right) + \left(-2y \cdot (a_0 x^0 + \dots + a_\delta x^\delta) \right) + y^2 \leq a'_0 \end{aligned}$$

for constants $a'_0, \dots, a'_{2\delta}$ depending on a_0, \dots, a_δ . Labeling the coordinates in $\mathbb{R}^{3\delta+2}$ with $x_1, \dots, x_{3\delta+2}$, the required half-space H_D is then

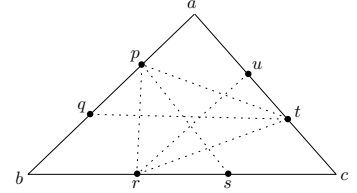
$$H_D : a'_1 \cdot x_1 + \dots + a'_{2\delta} \cdot x_{2\delta} + (-2a_0) \cdot x_{2\delta+1} + \dots + (-2a_\delta) x_{3\delta+1} + x_{3\delta+2} \leq a'_0,$$

containing precisely the points that lie on the curve $D \in \mathcal{D}$. This now implies a lower-bound of $\Omega(\frac{1}{\epsilon^{\delta+1}}) = \Omega(\frac{1}{\epsilon^{(d+1)/3}})$ for the ϵ -Mnet for the primal set system induced by half-spaces in \mathbb{R}^d . Finally, the lower-bound follows for any value of d by applying the bound for the largest $d' \leq d$ with integer value of $\frac{d'-2}{3}$. \square

(b). By Veronese maps, points P and disks D can be lifted to half-spaces H in \mathbb{R}^3 such that each point is lifted to a point in \mathbb{R}^3 and each disk is lifted to a half-space in \mathbb{R}^3 in such a way that their incidences are preserved. Now the required upper-bound follows from applying the bound in part (a) for half-spaces in \mathbb{R}^3 to the lifted point set of P .

(c). As a k -sided polygon can be partitioned into k triangles, one of which must contain at least $\frac{\epsilon n}{k}$ points,

an $\frac{\epsilon}{k}$ -Mnet with respect to triangles is an ϵ -Mnet with respect to k -sided polygons. Thus from now on we restrict ourselves to the primal set system induced by triangles in the plane.



Consider any triangle T in the plane that contains ϵn points of P . By moving the sides of the triangle we can ensure that each side of T contains at least two points of P and this can be done in such a way that no point outside T enters the interior of P . Some points in the interior of T may have moved to its boundary and some point outside T may also have moved to the boundary. Since at most 6 points may be on the boundary of T , due to P being in general position, the interior of T still contains at least $\frac{\epsilon n}{2}$ points, assuming $\epsilon n \geq 12$ (observe that for $\epsilon n < 12$, the collection of singletons of P is an ϵ -Mnet of size $O(\frac{1}{\epsilon})$). Thus we can further restrict ourselves to the interior of triangles each of whose sides contain at least two points. The figure above shows a triangle with each side containing two points of P . The points q and r could be identical, they could both be equal at the corner b of the triangle. Similarly s and t could be at c and u and p could be at a . Observe that the triangles aqt , bsp , cur and prt cover the triangle T and therefore one of them must contain at least $\frac{\epsilon n}{4}$ points of P . Each of these triangles are of the following type: at least two of the corners are in P and all sides contain at least two points of P . We call such triangles *anchored* triangles. Thus we can again restrict ourselves to the problem of anchored triangles in the plane containing ϵn points of P .

Let \mathcal{O} be the set of all anchored triangles for P . Let $\mathcal{O}' = \{\Delta_1, \dots, \Delta_t\}$ be a maximal set of t triangles from \mathcal{O} such that $|\Delta_i \cap P| = \epsilon n$ and $|\Delta_i \cap \Delta_j \cap P| \leq \frac{\epsilon n}{2}$.

Lemma 4.1. $|\mathcal{O}'| \leq 2 \cdot f_{\mathcal{O}}(\frac{\epsilon}{2} \cdot \log \frac{1}{\epsilon}, 2c \log \frac{1}{\epsilon})$, where $f_{\mathcal{O}}(m, l)$ is the maximum number of subsets of size at most l in the primal set system induced by objects in \mathcal{O} on any subset of m points of P , and c is some fixed constant.

Proof. Pick each point of P independently at random with probability $p = \frac{c}{2\epsilon n} \cdot \log \frac{1}{\epsilon}$ to get a random sample S .

First, observe that with probability greater than $\frac{1}{2}$, the sets $\Delta_i \cap S$, $i = 1 \dots t$, are distinct and $|S| \leq \frac{c}{\epsilon} \cdot \log \frac{1}{\epsilon}$: consider the range space (P, \mathcal{R}') , where $\mathcal{R}' = \{(\Delta_i \setminus \Delta_j) \cap P \mid \forall 1 \leq i < j \leq t\}$. From the definition of \mathcal{O}' , each set in \mathcal{R}' has size at least $\epsilon n - \frac{\epsilon n}{2} = \Theta(\epsilon n)$. We now use the fact that ranges induced by polygons with k sides have VC dimension at most $2k + 1$ [17]; it is easy to see that \mathcal{R}' is a subset of the ranges induced by polygons (or union of polygons) with at most 9 sides, and so the VC dimension of \mathcal{R}' is at most 19. Then by the Haussler-Welzl theorem [10], for $c > 19 \cdot 4$, with probability greater than $\frac{3}{4}$, S is an ϵ -net for (P, \mathcal{R}') . Now observe that if $\Delta_i \cap S = \Delta_j \cap S$, then the set $(\Delta_i \setminus \Delta_j) \cap S$ is empty, a contradiction to the fact that S is an ϵ -net for \mathcal{R}' . From standard concentration estimates from Chernoff bounds, it follows that $|S| \geq \frac{c}{\epsilon} \cdot \log \frac{1}{\epsilon}$ with probability less than $\frac{1}{4}$.

For each $\Delta_i \in \mathcal{O}'$, let X_i be the random variable which is 1 if $|\Delta_i \cap S| \geq 2c \log \frac{1}{\epsilon}$, and 0 otherwise. For a fixed i , by linearity of expectation, we have $E[|\Delta_i \cap S|] = \frac{c}{2} \cdot \log \frac{1}{\epsilon}$. By Markov's inequality applied to each X_i ,

$$\Pr[X_i = 1] = \Pr[|\Delta_i \cap S| \geq 2c \cdot \log \frac{1}{\epsilon}] = \Pr[|\Delta_i \cap S| \geq 4 \cdot E[|\Delta_i \cap S|]] \leq \frac{1}{4}.$$

Hence $E[Y] = E[\sum X_i] \leq \frac{t}{4}$, and by Markov's inequality applied to Y , we get that $\Pr[\sum X_i \geq \frac{t}{2}] \leq \frac{1}{4}$.

We can conclude that there exists a subset S of size $\frac{c}{\epsilon} \log \frac{1}{\epsilon}$ such that $\Delta_i \cap S$ are distinct for all objects in \mathcal{O}' , and for at least $\frac{|\mathcal{O}'|}{2}$ of the objects in \mathcal{O}' , we have $|\Delta_i \cap S| \leq 2c \log \frac{1}{\epsilon}$. Therefore we can get the required bound on the size of \mathcal{O}' :

$$\frac{|\mathcal{O}'|}{2} \leq f(|S|, l) = f\left(\frac{c}{\epsilon} \log \frac{1}{\epsilon}, 2c \log \frac{1}{\epsilon}\right).$$

□

Remark: After the appearance of the conference version of this paper, the statement of Lemma 4.1 has been formalized as the *shallow packing lemma*. We refer the reader to [18] for details and recent history.

We will need the following theorem from [14].

Theorem C (Simplicial partition theorem). *Given a set P of n points in \mathbb{R}^d , and an integer parameter $t > 0$, there exists a partition of P into t sets, each of size $\Theta(\frac{n}{t})$, such that any hyperplane intersects the convex-hull of at most $O(t^{1-1/d})$ sets of the partition.*

Take this set \mathcal{O}' of maximal objects, each containing ϵn points of P , and every pair of objects in \mathcal{O}' intersecting in less than $\frac{\epsilon n}{2}$ points. For each object $\Delta_i \in \mathcal{O}'$, do the following: apply the simplicial partition theorem to $\Delta_i \cap P$ with the parameter t , set to a large enough constant, to get a partition of $\Delta_i \cap P$ into t sets of size $\Theta(\frac{|\Delta_i \cap P|}{t})$. Add each of these $t = O(1)$ sets to the ϵ -Mnet \mathcal{M} for P .

Claim 5. \mathcal{M} is an ϵ -Mnet for the primal set-system induced by \mathcal{O} , of size $O\left(f_{\mathcal{O}}\left(\frac{c}{\epsilon} \cdot \log \frac{1}{\epsilon}, 2c \log \frac{1}{\epsilon}\right)\right)$.

Proof. First note that each set added to \mathcal{M} had size $\Theta(\frac{|\Delta_i \cap P|}{t}) = \Theta(\epsilon n)$, and the number of such sets is $O(|\mathcal{O}'| \cdot t) = O(|\mathcal{O}'|)$. It remains to show that any object containing ϵn points of P contains one set of \mathcal{M} . Take any triangle Δ containing ϵn points of P (any triangle containing greater than ϵn points can always be shrunk to a triangle containing fewer points). By the maximality of \mathcal{O}' , there exists $\Delta_i \in \mathcal{O}'$ such that $|\Delta \cap \Delta_i| \geq \frac{\epsilon n}{2}$. Furthermore, of all the sets in the simplicial partition of Δ_i , each edge of $\partial\Delta$ can intersect only $O(\sqrt{t})$ sets; so in total the three bounding segments of Δ can intersect at most $O(3\sqrt{t})$ sets. Each of these sets has $O(\frac{|\Delta_i \cap P|}{t})$ points. So these sets can contribute at most $O(3\sqrt{t} \cdot \frac{|\Delta_i \cap P|}{t})$ points of Δ_i to Δ . Setting t to be a large-enough constant (say, $t = 38$), this is less than $\frac{\epsilon n}{2}$. Therefore Δ must contain a point in Δ_i which lies in a partition for Δ_i not intersecting $\partial\Delta$, i.e., the partition lies completely inside Δ . \square

Finally, when \mathcal{O} is a set of anchored triangles in the plane, a routine application of the Clarkson-Shor method [17] implies that $f_{\mathcal{O}}(n, l) = O(n^3 \cdot l)$. Then Lemma 5 implies the existence of ϵ -Mnets for the primal set system induced by \mathcal{O} of size $O\left(\left(\frac{c}{\epsilon} \log \frac{1}{\epsilon}\right)^3 \cdot 2c \log \frac{1}{\epsilon}\right) = O\left(\frac{1}{\epsilon^3} (\log \frac{1}{\epsilon})^4\right)$.

(d). The upper-bounds for the primal set systems induced by lines, strips, cones in the plane again follow from Lemma 5. The function $f(n, l)$ correspondingly denotes the number of subsets of size l induced by the objects of the appropriate type (lines, strips, cones). For lines, $f(n, l) = O(n^2)$ implies the existence of ϵ -Mnets of size $O(\frac{1}{\epsilon^2} (\log \frac{1}{\epsilon})^2)$; for strips $f(n, l) = O(n^2 \cdot l)$ implies the existence of ϵ -Mnets of size $O(\frac{1}{\epsilon^2} (\log \frac{1}{\epsilon})^3)$; and for cones, $f(n, l) = O(n^2 \cdot l^2)$ implies the existence of ϵ -Mnets of size $O(\frac{1}{\epsilon^2} (\log \frac{1}{\epsilon})^4)$.

The lower-bound for the primal set system induced by lines, strips and cones in the plane follows from Theorem 4 by setting $\delta = 1$:

Corollary 4.2. *For any $\epsilon > 0$ and integer n , there exists a set P of n points in the plane such that any ϵ -Mnet for the primal set system on P induced by lines must have size $\Omega(\frac{1}{\epsilon^2})$.*

As the set system induced by lines is a special case for the ones induced by strips and cones, this implies the same lower-bound for the primal set system induced by strips and cones in the plane.

(e). As each rectangle contains ϵn points of P and intersects the y -axis, for each rectangle R , take the portion of the rectangle on the side of the y -axis that contains at least $\frac{\epsilon n}{2}$ points. We can construct $\frac{\epsilon}{2}$ -Mnets for the two sides of the y -axis separately and return the union of the two Mnets. Now for the primal set system induced by axis-parallel rectangles with one vertical edge lying on the y -axis, we have $f(n, l) = O(n)$ [22]. Now Lemma 5 implies that one can construct $\frac{\epsilon}{2}$ -Mnets of size $O(\frac{1}{\epsilon})$.

5 Conclusion and future work

We conclude our study by observing that the above series of results— with proofs that use different techniques— indicate an intriguing relation between the sizes of ϵ -nets and the sizes of ϵ -Mnets. In all cases, they obey the following pattern: if there exist ϵ -nets of size $O(\frac{1}{\epsilon}f(\frac{1}{\epsilon}))$ for some primal or dual set system, then the size of ϵ -Mnets for the same set system is $O(\frac{1}{\epsilon}c^f(\frac{1}{\epsilon}))$, where c is some constant. For example, for all spaces known to have linear-sized ϵ -nets (which is optimal), our proofs establish the existence of linear-sized ϵ -Mnets (which is optimal). For the primal set system induced by axis-parallel rectangles in the plane, ϵ -nets have size $O(\frac{1}{\epsilon} \log \log \frac{1}{\epsilon})$ (shown to be optimal) [2, 21]; our results show the existence of ϵ -Mnets of size $O(\frac{1}{\epsilon} \log \frac{1}{\epsilon})$ (which we show to be optimal). For the primal set system induced by half-spaces in \mathbb{R}^d , ϵ -nets have size $O(\frac{d}{\epsilon} \log \frac{1}{\epsilon})$ (shown to be optimal [12]); our results establish the existence of ϵ -Mnets for this set system of size $O(\frac{1}{\epsilon^{(d+1)/3}})$. Similarly, for the remaining set systems for which there exist ϵ -nets of size $O(\frac{1}{\epsilon} \log \frac{1}{\epsilon})$, we show the existence of ϵ -Mnets of size $O(\frac{1}{\epsilon^c})$. It would be interesting to see if there is any connection with the (still) open problem of finding the right bound on the size of ϵ -nets for the primal set system induced by lines in the plane.

References

- [1] P. K. Agarwal, J. Pach, and M. Sharir. State of the union (of geometric objects): A review. In J. Goodman, J. Pach, and R. Pollack, editors, *Computational Geometry: Twenty Years Later*, pages 9–48. AMS, 2008.
- [2] B. Aronov, E. Ezra, and M. Sharir. Small-size ϵ -nets for axis-parallel rectangles and boxes. *SIAM Journal on Computing*, 39(7):3248–3282, 2010.
- [3] S. Arya, G. D. da Fonseca, and D. Mount. Optimal area-sensitive bounds for polytope approximation. In *Proceedings of the International Symposium on Computational Geometry (SoCG)*, pages 363–372, 2012.
- [4] I. Bárány. Random polytopes, convex bodies, and approximation. In *Stochastic Geometry*, pages 77–118. Springer, 2007.
- [5] I. Bárány and D. G. Larman. Convex bodies, economic cap coverings, random polytopes. *Mathematika*, 35:274–291, 1988.
- [6] H. Brönnimann, B. Chazelle, and J. Pach. How hard is half-space range searching. *Discrete & Computational Geometry*, 10:143–155, 1993.
- [7] B. Chazelle. *The Discrepancy Method: Randomness and Complexity*. Cambridge University Press, 2000.
- [8] C. Chekuri, K. L. Clarkson, and S. Har-Peled. On the set multi-cover problem in geometric settings. In *Proceedings of the International Symposium on Computational Geometry (SoCG)*, pages 341–350, 2009.
- [9] G. Ewald, D. G. Larman, and C. A. Rogers. The directions of the line segments and of the r -dimensional balls on the boundary of a convex body in euclidean space. *Mathematika*, 17:1–20, 1970.
- [10] D. Haussler and E. Welzl. Epsilon-nets and simplex range queries. *Discrete Comput. Geom.*, 2:127–151, 1987.
- [11] J. Komlós, J. Pach, and G. Woeginger. Almost tight bounds for epsilon nets. *Discrete & Computational Geometry*, pages 163–173, 1992.
- [12] A. Kupavskii, N. H. Mustafa, and J. Pach. New lower bounds for epsilon-nets. In *Proceedings of the International Symposium on Computational Geometry (SoCG)*, 2016.

- [13] A. M. Macbeath. A theorem on non-homogeneous lattices. *Annals of Math*, 56:269–293, 1952.
- [14] J. Matoušek. Efficient partition trees. *Discrete & Computational Geometry*, 8:315–334, 1992.
- [15] J. Matoušek. Reporting points in halfspaces. *Computational Geometry*, 2(3):169 – 186, 1992.
- [16] J. Matoušek. *Geometric Discrepancy: An Illustrated Guide*. Springer, 1999.
- [17] J. Matoušek. *Lectures in Discrete Geometry*. Springer, 2002.
- [18] N. H. Mustafa. A simple proof of the shallow packing lemma. *Discrete & Computational Geometry*, 55(3):739–743, 2016.
- [19] N. H. Mustafa and S. Ray. Near-optimal generalisations of a theorem of Macbeath. In *Proceedings of the International Symposium on Theoretical Aspects of Computer Science (STACS)*, pages 578–589, 2014.
- [20] J. Pach and P. K. Agarwal. *Combinatorial Geometry*. John Wiley & Sons, 1995.
- [21] J. Pach and G. Tardos. Tight lower bounds for the size of epsilon-nets. *Journal of the AMS*, 26:645–658, 2013.
- [22] E. Pyrga and S. Ray. New existence proofs epsilon-nets. In *Proceedings of the International Symposium on Computational Geometry (SoCG)*, pages 199–207, 2008.