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Irréductibilité des objets combinatoires : probabilité asymptotique et interprétation

Irreducibility of combinatorial objects: asymptotic probability and interpretation

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THÈSE DE DOCTORAT

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Chapter 1

Introduction

1.1 Présentation du problème et de la stratégie

Plusieurs structures combinatoires admettent, au sens large, une notion d'irréductibilité. Par irréductibilité, nous entendons qu'une structure composée peut être décomposée en une collection de petits morceaux qui ne peuvent pas être décomposés davantage. En particulier, divers objets sur lesquels on peut canoniquement définir une topologie admettent une décomposition en composantes connexes. Par exemple, un graphe simple est une union disjointe de graphes connexes. On peut aussi penser à différents modèles de surfaces discrètes qui peuvent être connexes ou pas. L'irréductibilité apparait aussi lorsqu'on cherche à décomposer un objet en une séquence d'objets irréductibles. À titre d'exemple important, mentionnons les permutations qui peuvent être représentées comme des concaténations des permutations indécomposables (notez que les permutations peuvent également être décomposées en unions disjointes de cycles). Mentionnons aussi les tournois, c'est-à-dire, des graphes orientés obtenus en orientant chaque arête d'un graphe complet non orienté. La liste des structures irréductibles semble innombrable; pour les classifier, on peut utiliser une approche basée sur des partitions qui a été développée par Beissinger [5].

Cette thèse porte en grande partie sur la question suivante. Choisissons un objet aléatoire de taille n d'une classe combinatoire étiquetée qui admet une notion d'irréductibilité. Quelle est la probabilité que cet objet soit irréductible, lorsque sa taille tend vers l'infini ? Pour de nombreuses classes connues, cette probabilité tend vers 1. Par exemple, c'est le cas des graphes connexes [41], des tournois irréductibles [68] et des permutations indécomposables [21]. Dans d'autres cas, la probabilité limite est 0, comme pour les permutations constituées d'exactement un cycle ou pour les partitions d'ensembles constituées d'exactement une partie. Parfois, la probabilité limite d'irréductibilité est strictement comprise entre 0 et 1. Dans de nombreux cas, cette probabilité s'exprime en fonction de e. Par exemple, une forêt étiquetée aléatoire est connexe avec une probabilité limite de $1/\sqrt{e}$ [88], alors qu'un diagramme de cordes aléatoires est connexe avec une probabilité de $1/e$ [51, 95] et une permutation aléatoire est simple avec une probabilité de $1/e^2$ [2], respectivement. Cependant, on peut rencontrer des valeurs plus retorses, comme le nombre d'or $(\sqrt{5}-1)/2$ pour la probabilité limite qu'un poset de N-arbre aléatoire soit connexe [32, 92], ou $1/2,997\ldots = 0,3367\ldots$ pour la probabilité limite qu'une forêt aléatoire non étiquetée soit un arbre [79], voir aussi [10].

Au milieu des années 1960, Wright [99] s'est demandé comment distinguer ces limites. Il a considéré des classes des objets non étiquetés decomposables en composantes connexes. Les suites $(\mathfrak{a}_n)$ et $(\mathfrak{c}_n)$ comptant, respectivement, les objets de taille n et les objets connexes de taille n, satisfont la formule

$$\sum_{n=0}^{\infty} \mathfrak{a}_n z^n = \exp\left(\sum_{n=1}^{\infty} \mathfrak{c}_n z^n\right). \tag{1.1}$$

Wright a démontré [99, 100] que

$$\sum_{k=1}^{n-1} \mathfrak{a}_k \mathfrak{a}_{n-k} = o(\mathfrak{a}_n) \qquad \Leftrightarrow \qquad \sum_{k=1}^{n-1} \mathfrak{c}_k \mathfrak{c}_{n-k} = o(\mathfrak{c}_n)$$

et les deux conditions peuvent être utilisées comme nécessaires et suffisantes pour que la probabilité limite soit égale à 1, c'est-à-dire, que $\lim\limits_{n\to\infty} (\mathfrak{c}_n/\mathfrak{a}_n) = 1$. Il a aussi indiqué la condition nécessaire suivante : la fonction génératrice

$$A(z) = \sum_{n=0}^{\infty} \mathfrak{a}_n z^n$$

doit diverger pour tout z non nul, ce qui équivaut à $\lim\limits_{n\to\infty} (\mathfrak{a}_{n-1}/\mathfrak{a}_n) = 0$ dans le cas où $\mathfrak{a}_n \neq 0$. Des conditions similaires ont été obtenues dans le cas étiqueté (voir aussi [16]).

Au cours des décennies suivantes, plusieurs auteurs ont développé l'approche de Wright comme, par exemple, Bell, Bender, Cameron, Compton, Odlyzko et Richmond [6, 10, 16, 20], voir aussi [52]. La recherche a abouti à l'article de Bell, Bender, Cameron et Richmond [7] qui a prouvé le résultat suivant. Supposons que la limite $\rho = \lim\limits_{n\to\infty} (\mathfrak{c}_n/\mathfrak{a}_n)$ existe, alors sa valeur est liée au rayon de convergence R de la fonction génératrice

$$C(z) = \sum_{n=0}^{\infty} \mathfrak{c}_n z^n :$$

- $\rho = 1 \qquad \Leftrightarrow \qquad R = 0$,
- $\rho = 0 \qquad \Leftrightarrow \qquad R > 0$ et $C(R)$ diverge,
- $0 < \rho < 1 \qquad \Leftrightarrow \qquad R > 0$ et $C(R)$ converge.

De plus, ils ont étudié le cas où la limite n'existe pas, et ont décrit le comportement de $\limsup(\mathfrak{c}_n/\mathfrak{a}_n)$ et $\liminf(\mathfrak{c}_n/\mathfrak{a}_n)$ en fonction de R. Dans cette thèse, nous nous concentrons sur le premier cas : les suites $(\mathfrak{a}_n)$ et $(\mathfrak{c}_n)$ croissent rapidement et les fonctions génératrices $A(z)$ et $C(z)$ ont un rayon de convergence nul.

Nous ne nous intéresserons pas seulement à la probabilité limite. Un de nos objectifs est d'établir le développement asymptotique complet pour la probabilité qu'un objet aléatoire soit irréductible. Autrement dit, on souhaite avoir une expression du type

$$\frac{\mathfrak{c}_n}{\mathfrak{a}_n} = \rho\Big(1 + f_1(n) + f_2(n) + \ldots + f_r(n) + o\big(f_r(n)\big)\Big), \tag{1.2}$$

où $f_{k+1}(n) = o\big(f_k(n)\big)$ pour tout $k \in \mathbb{N}$, lorsque $n \to \infty$. D'une manière générale, on pourrait atteindre cet objectif à l'aide du principe d'inclusion-exclusion (nous verrons comment cela fonctionne pour différentes constructions combinatoires dans les sections 5.1, 6.1 et 7.1). Cependant, en pratique, cette méthode est peu maniable et principalement utilisée pour trouver les premiers termes. C'est par exemple le cas des permutations simples [2] et de certaines classes de surfaces connexes [15, 83]. Une autre méthode, bien que similaire dans son esprit, a été développée à la fin des années 1960 par

Wright qui utilisait les récurrences pour obtenir les asymptotiques. Par exemple, il a montré [102] que si les suites $(\mathfrak{c}_n)$ et $(\mathfrak{a}_n)$ satisfont la formule exponentielle (1.1), alors les conditions

$$\lim_{n\to\infty} \frac{\mathfrak{a}_{n-1}}{\mathfrak{a}_n} = 0 \qquad \text{et} \qquad \sum_{k=r}^{n-r} \mathfrak{a}_k \mathfrak{a}_{n-k} = O(\mathfrak{a}_{n-r}) \tag{1.3}$$

valables pour un entier r impliquent

$$\mathfrak{c}_n = \mathfrak{a}_n + \sum_{k=1}^{r-1} \alpha_k \mathfrak{a}_{n-k} + O(\mathfrak{a}_{n-r}), \tag{1.4}$$

et vice versa. Ici, la suite (α_n) est définie récursivement par $\alpha_0 = \mathfrak{a}_0 = 1$ et

$$\sum_{k=0}^{n} \alpha_k \mathfrak{a}_{n-k} = 0. \tag{1.5}$$

Cette méthode a permis à Wright d'obtenir des expressions asymptotiques pour les nombres de graphes connexes [102], de tournois irréductibles [103] et de graphes orientés fortement connexes [104]. Robinson a utilisé l'approche de Wright pour trouver des asymptotiques de graphes orientés acycliques [90].

Les nombreux résultats de Wright ont été repensés et généralisés en 1975 par Bender [9]. Par rapport aux travaux de Wright, l'avantage de l'approche de Bender est le passage des récurrences aux fonctions génératrices. En particulier, la suite (α_n) de la relation (1.5) a été déterminée en différenciant certaines séries entières. Comme nous le verrons plus tard, ce fait nous permet de comprendre la structure des asymptotiques et d'interpréter (α_n) de manière combinatoire. Nous discutons de l'approche de Bender en détail dans la section 2.2.

Il est important de souligner que la méthode utilisée par Bender [9] ne fonctionne bien que dans un certain nombre de cas. Selon la nature de la fonction génératrice, différentes techniques peuvent être nécessaires pour obtenir les asymptotiques souhaitées. Plus précisément, supposons qu'une fonction $F(z)$ soit analytique dans un certain voisinage de l'origine et qu'une série formelle $A(z)$ ait des coefficients croissant suffisamment vite (c'est-à-dire, satisfaisant les conditions (1.3) pour tout $r \in \mathbb{N}$). Alors la méthode de Bender convient pour établir le développement asymptotique complet des coefficients de la composition $F\big(A(z)\big)$. D'autre part, si nous cherchons les asymptotiques de $A\big(F(z)\big)$ ou de $A^{-1}(z)$, nous devrions utiliser une approche différente développée par Bender et Richmond [11] (ce sont les idées de [9] combinées avec la formule d'inversion de Lagrange). Un cas intéressant est étudié par Borinsky [13] : dans ses recherches, les deux fonctions impliquées dans la composition sont divergentes, mais leur taux de croissance n'est pas trop rapide. Plus précisément, Borinsky a établi des formules pour le comportement asymptotique du produit $F(z)G(z)$ et de la composition $F\big(G(z)\big)$, où les coefficients de $F(z)$ et $G(z)$ admettait un développement asymptotique de la forme

$$\alpha^{n+\beta}\,\Gamma(n+\beta)\left(d_0 + \frac{d_1}{(n+\beta-1)} + \frac{d_2}{(n+\beta-1)(n+\beta-2)} + \ldots\right)$$

pour certaines constantes $\alpha, \beta \in \mathbb{R}_{>0}$ et $d_k \in \mathbb{R}$.

D'autres méthodes sont à appliquer dans le cas où la fonction génératrice traitée a un rayon de convergence non nul. Selon les cas, on peut utiliser la méthode de Darboux (voir par exemple [22, 77, 98]) ou l'analyse des singularités [37] basée sur la méthode du point col et les articles de Hayman [48], Harris et Schoenfeld [47], Flajolet, Odlyzko et Richmond [36, 73, 75]. Pour en savoir plus, nous renvoyons le lecteur aux revues de la littérature de Bender [8] et Odlyzko [74] et aux livres de De Bruijn [25] et Flajolet et Sedgewick [38].

Ainsi, nous nous intéressons à l'asymptotique de la probabilité qu'un objet aléatoire d'un certain type soit irréductible. Dans notre recherche, nous poursuivons deux objectifs. Tout d'abord, nous aimerions obtenir le développement asymptotique complet de cette probabilité d'une manière commune, adaptée à différentes structures. En nous limitant aux classes d'objets dont les suites de comptage croissent assez rapidement, nous nous retrouvons dans une situation obéissant au théorème de Bender [9]. Le second objectif découle de l'observation que les coefficients du développement (1.2) sont entiers. Notamment, nous souhaitons avoir une interprétation combinatoire des constantes impliquées dans le développement asymptotique.

La stratégie pour atteindre les deux objectifs mentionnés ci-dessus consiste à forcer plusieurs techniques à interagir. Dans une première étape, nous interprétons les structures combinatoires en termes de fonctions génératrices. L'origine de cette idée importante est cachée dans la profondeur des siècles ; on peut voir des fonctions génératrices émerger sur la base des récurrences correspondantes dans les travaux de Bernoulli, Euler, Cayley, etc. Comme exemple d'une importance particulière, il convient de mentionner la contribution de Pólya [84, 85], dont les travaux ont été consacrés à l'énumération des objets sous l'action du groupe symétrique. Les approches contemporaines de l'analyse combinatoire, y compris la théorie de Pólya, peuvent être trouvées, par exemple, dans des livres d'auteurs tels que Comtet [22], Goulden et Jackson [42], Lando [55], Stanley [93, 94] et Wilf [98]. Notre premier choix est la méthode symbolique [38] qui est plutôt simple et assez accessible. Selon cette méthode, les structures composées sont obtenues à partir de structures simples à l'aide des constructions combinatoires comme : la séquence SEQ, le cycle CYC et l'ensemble SET. Le fait qu'un objet soit irréductible par rapport à l'une des trois constructions mentionnées s'interprète directement en termes de fonctions génératrices via des relations correspondantes.

Dès que nous avons des relations pour les fonctions génératrices, nous pouvons nous attaquer au premier objectif, c'est-à-dire, obtenir les développements asymptotiques. Comme nous l'avons déjà noté, notre méthode est basée sur le théorème de Bender qui nous permet d'établir un comportement asymptotique des coefficients de série formelle dans le cas où la série désirée est exprimée comme la composée d'une fonction analytique et de la série initiale. Le théorème de Bender est applicable pour les séries dont les coefficients croissent suffisamment vite. Ce fait nous limite à considérer des classes combinatoires étiquetées dont les suites de comptage satisfont (1.3) pour tout $r \in \mathbb{N}$. Notons que dans ce cas, pour la question de connexité (alias irréductibilité par rapport à la construction combinatoire SET), on peut garantir qu'une structure aléatoire est presque sûrement connexe [6, 7].

Le second objectif est plus délicat. À notre connaissance, le seul exemple de coefficients asymptotiques interprétés combinatoirement dans la littérature, dans l'esprit de notre travail, est le suivant. En 2005, Dixon [28] a montré que la probabilité t_n qu'une paire aléatoire de permutations à partir de S_n génère un groupe transitif pouvait être représentée par la série asymptotique

$$t_n = 1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \ldots + \frac{c_m}{n^m} + O\left(\frac{1}{n^{m+1}}\right). \tag{1.6}$$

Dixon a indiqué les premières valeurs des constantes c_k, de sorte que cette asymptotique avait la forme

$$t_n = 1 - \frac{1}{n} - \frac{1}{n^2} - \frac{4}{n^3} - \frac{23}{n^4} - \frac{171}{n^5} - \frac{1542}{n^6} - \ldots. \tag{1.7}$$

En prouvant les asymptotique ci-dessus, Dixon a utilisé le théorème de Bender et a établi que

$$t_n = 1 + \sum_{k=1}^{r-1} \frac{d_k}{(n)_k} + O\left(\frac{1}{n^r}\right) \tag{1.8}$$

pour tout entier strictement positif r, où $(n)_k = n(n-1)\dots(n-k+1)$ désignent les factorielles descendantes et les coefficients (d_k) sont définis par

$$\sum_{k=0}^{\infty} d_k z^k = \left(\sum_{k=0}^{\infty} k! z^k\right)^{-1}.$$

En 2009, Cori [23] a montré que $(-d_k)$ était la suite de comptage des permutations indécomposables.

Pour obtenir une interprétation combinatoire de ce type, nous avons besoin d'une technique qui relie les opérations sur les séries formelles et les structures combinatoires. Il y a plusieurs façons de le faire. Le moyen le plus naturel est d'utiliser à nouveau la méthode symbolique. Ainsi, nous décrivons d'abord l'irréductibilité des objets combinatoires en termes des fonctions génératrices à l'aide de la méthode symbolique. Cela nous permet de passer aux développements asymptotiques via le théorème de Bender. Finalement, après avoir obtenu les coefficients asymptotiques, nous appliquons encore fois la méthode symbolique, dans le sens inverse, pour leur donner une interprétation combinatoire.

Bien que la méthode symbolique soit claire et facile à utiliser, elle possède certaines limites. Dans certains cas, les coefficients obtenus ne peuvent pas être interprétés directement en termes de classes combinatoires. Par exemple, c'est le cas lorsque certains d'entre eux sont strictement négatifs. Pour surmonter cet obstacle, nous utilisons la théorie des espèces [12, 49]. On peut considérer cette théorie comme un langage différent qui pourrait servir le même objectif de traduction des décompositions de structures en opérations de fonctions génératrices et vice versa. Comparée à la méthode symbolique, la théorie des espèces est plus générale et puissante, bien qu'elle soit plus abstraite et plus difficile à mettre en œuvre. Un rôle clé dans la théorie des espèces est réservé à l'opération de substitution, qui embrasse toutes les constructions précédemment citées. Le principal avantage que nous utilisons activement est le concept d'espèces virtuelles. De la même façon que les entiers prolongent l'ensemble des nombres naturels, l'anneau des espèces virtuelles est l'extension du demi-anneau des espèces de structures. L'usage des espèces virtuelles nous permet de donner un sens combinatoire à plus d'expressions, y compris des suites à coefficients négatifs, et permet de passer à l'inverse sous substitution.

1.2 Contribution de la thèse

Les résultats de ce travail sont ou seront présentés dans les articles [30, 63, 64, 65]. Au cours de cette thèse, des résultats sur d'autres sujets ont également été obtenus, qui ne sont pas inclus ici afin de préserver l'unité du manuscrit. En particulier, dans un article commun avec Ivan Reshetnikov [71], nous fournissons des résultats sur des décompositions de fonctions définies sur des ensembles finis. Dans un autre article en collaboration avec Alexander Povolotsky [70], nous étudions l'analyse asymptotique des fonctions de corrélation non locales.

Cette thèse poursuit deux objectifs principaux. Le premier d'entre eux est de fournir une manière générale d'établir la probabilité asymptotique que de grands objets combinatoires soient connexes ou irréductibles dans un sens précis que nous définirons plus loin (voir les définitions 3.1.31, 3.1.39 et 3.1.46). Le second, en quelque sorte plus intéressant, est d'indiquer la signification combinatoire des coefficients impliqués dans les développements asymptotiques obtenus et d'en comprendre les raisons sous-jacentes. Pour donner au lecteur une idée du type de résultats qui nous intéresse, nous commençons par un exemple qui sera discuté en détail au chapitre 10.

Considérons le modèle Erdős–Rényi des graphes aléatoires $G(n, 1/2)$: pour tout entier positif n, on munit l'ensemble des graphes simples non orientés sur l'ensemble $[n] = \{1, \dots, n\}$ de la mesure de probabilité uniforme. Autrement dit, dans ce modèle, un graphe de taille n apparaît avec une probabilité $1/2^{\binom{n}{2}}$. La probabilité p_n qu'un graphe aléatoire soit connexe tend vers 1 lorsque n tend vers ∞. En 1959, Gilbert [41] a fourni une estimation plus précise montrant que

$$p_n = 1 - \frac{2n}{2^n} + O\left(\frac{n^2}{2^{3n/2}}\right).$$

En 1970, Wright [102] a calculé les quatre premiers termes du développement asymptotique de cette probabilité :

$$p_n = 1 - \binom{n}{1}\frac{1}{2^{n-1}} - 2\binom{n}{3}\frac{1}{2^{3n-6}} - 24\binom{n}{4}\frac{1}{2^{4n-10}} + O\left(\frac{n^5}{2^{5n}}\right).$$

La méthode de Wright peut être utilisée pour calculer plusieurs termes, pas à pas. Cependant, il ne donne ni la structure du développement asymptotique complet ni l'interprétation des coefficients $1, 2, 24, \ldots$.

Notre but est de fournir une telle structure : nous allons démontrer que le k-ième terme du développement asymptotique de p_n est de la forme

$$\mathfrak{it}_k \cdot 2^{k(k+1)/2} \cdot \binom{n}{k} \cdot \frac{1}{2^{kn}},$$

où $\mathfrak{it}_k$ compte le nombre de tournois étiquetés irréductibles de taille k (voir la définition 10.2.2). Nous exprimons ce résultat, ainsi que tous les autres résultats que nous verrons plus tard, à l'aide de deux notations. La première d'entre elles est le symbole $\approx$ pour le développement asymptotique, (voir la notation 2.2.1 pour la signification précise). La seconde est la probabilité asymptotique, $p_n = \mathbb{P}\big(G \text{ est connexe}\big)$, où G est un graphe aléatoire de taille n (pour le concept de *probabilité asymptotique*, voir la définition 3.1.2). En utilisant ces notations, on obtient le résultat annoncé sous la forme suivante.

Théorème 10.1.10. *La probabilité asymptotique qu'un graphe simple étiqueté aléatoire G avec n sommets soit connexe satisfait*

$$\mathbb{P}\big(G \textit{ est connexe}\big) \approx 1 - \sum_{k \geqslant 1} \mathfrak{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{kn}}, \tag{1.9}$$

où $\mathfrak{it}_k$ est le nombre de tournois irréductibles de taille k.

La preuve du théorème 10.1.10 est basée sur le théorème de Bender (théorème 2.2.5) et plutôt courte : nous exprimons la fonction génératrice exponentielle des graphes connexes en termes de fonction génératrice exponentielle des graphes, différencions cette expression pour obtenir l'asymptotique et notons que, d'une certaine manière, nous pouvons faire apparaitre la fonction génératrice exponentielle des tournois irréductibles dans le résultat. Cependant, le théorème 10.1.10 présente un nouveau résultat. L'explication de ce fait réside dans la relation entre les graphes et les tournois, cachée dans leurs structure. Bien que l'énumération des graphes connexes ait été donnée par Wright en 1970, il n'a pas fourni de signification combinatoire des coefficients. Le fait que $(\mathfrak{it}_k)$ soit la suite de comptage des tournois irréductibles est facile à obtenir empiriquement aujourd'hui, avec l'aide de l'encyclopédie en ligne des suites de nombres entiers [76] qui n'était pas disponible il y a cinquante ans. Néanmoins, même découvert, le lien entre graphes connexes et tournois irréductibles est plutôt non direct. En termes de classes combinatoires étiquetées, on peut décrire ce lien de la manière suivante. Compte tenu des classes combinatoires étiquetées $\mathcal{CG}$ des graphes connexes et $\mathcal{IT}$ des tournois irréductibles, respectivement, il existe un isomorphisme combinatoire des structures composées

$$\text{SET}(\mathcal{CG}) \cong \text{SEQ}(\mathcal{IT}). \tag{1.10}$$

Dans cette formule, l'utilisation de la construction SET reflète le fait qu'un graphe est l'union de ses composants connexes. De la même manière, l'utilisation de la construction SEQ signifie qu'un tournoi peut être représenté comme une séquence de tournois irréductibles. Ainsi, la relation (1.10) peut être interprétée comme un isomorphisme combinatoire des classes combinatoires étiquetées $\mathcal{G}$ des graphes et $\mathcal{T}$ des tournois, respectivement, ce qui signifie que leurs les suites de comptage sont les mêmes. Nous devons souligner ici que le concept d'*isomorphisme combinatoire* n'est rien d'autre que la coïncidence des nombres de structures de même taille (voir définition 3.1.11) : le nombre $\mathfrak{g}_n$ de graphes de taille n et le nombre $\mathfrak{t}_n$ de tournois de taille n sont tous deux égaux à

$$\mathfrak{g}_n = \mathfrak{t}_n = 2^{\binom{n}{2}}.$$

Alors que les deux tournois étiquetés de taille deux sont isomorphes (en échangeant les sommets), les deux graphes étiquetés de taille deux ne le sont pas (l'un est connexe, pas l'autre). En particulier, toute bijection entre $\mathcal{G}$ et $\mathcal{T}$ est un peu artificielle, car elle ne préserve pas les classes d'isomorphisme.

Maintenant, nous comprenons deux ingrédients de base se trouvant sous le théorème 10.1.10 : le théorème de Bender et l'interaction de deux structures différentes (graphes et tournois). Cela nous permet d'énoncer une généralisation pour les classes combinatoires arbitraires. Similaire à la relation (1.10), nous supposons qu'une classe combinatoire étiquetée $\mathcal{U}$ admet deux décompositions différentes:

$$\mathcal{U} = \text{SET}(\mathcal{V}) = \text{SEQ}(\mathcal{W})$$

(pour simplifier, on parle d'égalité, bien qu'en pratique on ait généralement un isomorphisme combinatoire, $\text{SET}(\mathcal{V}) \cong \text{SEQ}(\mathcal{W})$). Alors, sous certaines conditions, le développement asymptotique de la probabilité qu'un objet aléatoire appartenant à $\mathcal{U}$ est SET-irréductible peut être exprimée en fonction de la suite de comptage de $\mathcal{U}$, et les coefficients ont un sens combinatoire impliquant $\mathcal{W}$. Ici, grosso modo, la notion de SET-irréductibilité signifie qu'un objet de la classe $\mathcal{U}$ est connexe. Autrement dit, un objet de la classe $\mathcal{U}$ est SET-irréductible si et seulement s'il appartient à la classe $\mathcal{V}$ aussi. En ce qui concerne les conditions, la méthode fonctionne pour les classes combinatoires étiquetées dont les suites de comptage grossissent suffisamment vite (nous appelons ces classes *gargantuesques*, voir la définition 3.1.72). Plus précisément, le résultat est suivant.

Théorème 7.2.1. *Soit $\mathcal{U}$ une classe combinatoire étiquetée gargantuesque à coefficients strictement positifs, telle que $\mathcal{U} = \text{SET}(\mathcal{V})$ et $\mathcal{U} = \text{SEQ}(\mathcal{W})$ pour certaines classes combinatoires étiquetées $\mathcal{V}$ et $\mathcal{W}$. Supposons que $u \in \mathcal{U}$ soit un objet aléatoire de taille n. Alors*

$$\mathbb{P}(u \textit{ est } \text{SET-}\textit{irréductible}) \approx 1 - \sum_{k \geqslant 1} \mathfrak{w}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}, \tag{1.11}$$

où $(\mathfrak{u}_n)$ et $(\mathfrak{w}_n)$ sont les suites de comptage des classes $\mathcal{U}$ et $\mathcal{W}$, respectivement.

Le théorème 7.2.1 admet un certain nombre d'applications qui lient des structures de nature différente. Nous avons déjà vu ce genre de lien entre graphes connexes et tournois irréductibles (théorème 10.1.10). De manière similaire, on obtient des liens entre les multigraphes connexes et les multitournois irréductibles (proposition 10.1.18) et les (multi)digraphes faibles et les multitournois irréductibles (proposition 10.3.23). Dans tous les cas ci-dessus, la seconde structure apparaît sous forme de coefficients dans le développement asymptotique de la probabilité asymptotique pour la première structure.

Un autre exemple significatif est fourni par les surfaces à petits carreaux connexes et les permutations indécomposables. Le modèle des *surfaces à petits carreaux*, également appelé *origamis*, peut être obtenu comme suit [105]. Étant donné $n \in \mathbb{N}$, prenons une collection de carrés unitaires énumérés par des entiers de 1 à n et identifions leurs côtés par translation, de sorte que le résultat soit une surface de translation, c'est-à-dire une surface avec les notions globalement définies de nord, ouest, sud et est. Toutes les surfaces possibles qui peuvent être obtenues par la procédure décrite constituent la classe combinatoire étiquetée $\mathcal{O}$ des surfaces à petits carreaux. Le fait que toute surface est une collection de surfaces connexes peut être interprété par la relation

$$\mathcal{O} = \mathrm{SET}(\mathcal{CO}),$$

où $\mathcal{CO}$ désigne la sous-classe des surfaces à petits carreaux connexes. D'autre part, une identification de côtés de carreau dans le modèle ci-dessus correspond au couple de permutations $(\sigma_v, \sigma_h) \in S_n^2$ tel que le côté droit du k-ième carré est identifié au côté gauche du $\sigma_v(k)$-ème carré et le côté supérieur du k-ème carré est identifié au côté inférieur du $\sigma_h(k)$-ème carré. La classe combinatoire des paires de permutations de même taille est combinatoirement isomorphe à la classe $\mathcal{ML}(2)$ des paires d'ordres linéaires de même taille, ce qui signifie que leurs suites de comptage sont les mêmes : $(n!)^2$. La classe $\mathcal{ML}(2)$ peut être décomposée en une suite,

$$\mathcal{ML}(2) = \mathrm{SEQ}\big(\mathcal{IML}(2)\big),$$

où $\mathcal{ML}(2)$ est la classe combinatoire étiquetée des paires irréductibles d'ordres linéaires de même taille. Sa suite de comptage coïncide avec $n! \cdot \mathfrak{ip}_n$, où $\mathfrak{ip}_n$ compte les permutations indécomposables de taille n. Ainsi, l'isomorphisme combinatoire

$$\mathrm{SET}(\mathcal{CO}) \cong \mathrm{SEQ}\big(\mathcal{IML}(2)\big)$$

permet d'appliquer le théorème 7.2.1 qui nous donne le nouveau résultat suivant étroitement lié à la relation (1.8).

Théorème 11.1.8. *La probabilité asymptotique qu'une surface à petits carreaux S obtenue en collant n carreaux unitaires au hasard soit connexe satisfait*

$$\mathbb{P}\big(S \ \textit{est connexe}\big) \approx 1 - \sum_{k \geqslant 1} \frac{\mathfrak{ip}_k}{(n)_k}, \tag{1.12}$$

où $(n)_k = n(n-1)(n-2)\dots(n-k+1)$ sont les factorielles descendantes et $\mathfrak{ip}_k$ est le nombre de permutations indécomposables de taille k.

Notons qu'une surface à petits carreaux de taille n est connexe si et seulement si la paire de permutations correspondante génère un sous-groupe transitif de S_n. Ainsi, le développement asymptotique de la probabilité qu'une surface à petits carreaux aléatoire soit connexe peut être dérivé des résultats de Dixon [28] et Cori [23] mentionnés dans la section précédente. Cependant, l'apparition de permutations indécomposables dans ce contexte était plutôt surprenante et demandait à être expliquée. L'application de notre approche et du théorème 7.2.1 apporte un nouvel éclairage sur cette histoire. Nous divulguons le raisonnement sous-jacent en plaçant la structure et en montrant le lien entre les surfaces à petits carreaux et les permutations indécomposables dans un cadre général. Il ressort de notre approche que la décomposition (1.6) de t_n en asymptotique sur la base $1/(n)_k$ est plus naturelle que (1.7) sur la base $1/n^k$. Elle se généralise aussi plus facilement. L'une d'entre elles dévoile l'interaction entre les cartes combinatoires et les couplages parfaits indécomposables (théorème 11.2.4). Une autre donne les liens entre constellations et multipermutations indécomposables (théorème 11.3.6).

Après ce résultat sur l'asymptotique de la probabilité qu'un objet soit SET-irréductible (c'est-à-dire, connexe), nous pouvons étendre ce résultat aux constructions symboliques SEQ et CYC (théorèmes 5.2.1 et 6.2.1 respectivement). De plus, on peut vouloir obtenir des asymptotiques pour des constructions restreintes à un nombre fixe de composants irréductibles. Par exemple, pour tout $m \in \mathbb{N}$, la construction SEQ_m produit des séquences de m composants irréductibles. De la même manière, les constructions CYC_m et SET_m sont réservées pour les cycles et les ensembles de m composants irréductibles respectivement. Pour chaque construction ci-dessus, nous établissons la probabilité asymptotique qu'un objet aléatoire se décompose exactement en m composantes irréductibles. Notons que toutes les trois ont une interprétation naturelle en termes de principe d'inclusion-exclusion.

Pour donner une idée de ce à quoi ressemblent les résultats mentionnés ci-dessus, voici notre théorème général pour la construction SEQ_m.

Théorème 5.3.1. *Soit $\mathcal{U}$ une classe combinatoire étiquetée gargantuesque à coefficients strictement positifs, telle que $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ pour une classe combinatoire étiquetée $\mathcal{W}$. Supposons que $u \in \mathcal{U}$ soit un objet aléatoire de taille n. Alors pour tout $m \in \mathbb{N}$,*

$$\mathbb{P}(u \textit{ a } m \textit{ composants } \mathrm{SEQ}\textit{-irréductibles}) \approx \sum_{k\geqslant 0} m\left(\mathfrak{w}_k^{(m-1)} - 2\mathfrak{w}_k^{(m)} + \mathfrak{w}_k^{(m+1)}\right) \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}, \quad (1.13)$$

où $\left(\mathfrak{w}_k^{(m)}\right)$ est la suite de comptage de $\mathrm{SEQ}_m(\mathcal{W})$.

Ainsi, nous obtenons un outil qui nous permet d'obtenir des probabilités asymptotiques pour différentes structures ainsi qu'une interprétation combinatoire des coefficients impliqués. Tous les exemples déjà mentionnés dans cette section sont traitables, y compris les tournois irréductibles, les permutations indécomposables et les couplages parfaits indécomposables. Par exemple, nous montrons que les tournois irréductibles satisfont le théorème suivant.

Corollaire 10.2.12. *La probabilité asymptotique qu'un tournoi aléatoire étiqueté T avec n sommets soit irréductible satisfait*

$$\mathbb{P}\big(T \textit{ est irréductible}\big) \approx 1 - \sum_{k\geqslant 1} \left(2\mathfrak{it}_k - \mathfrak{it}_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{kn}}, \quad (1.14)$$

où $\mathfrak{it}_k$ et $\mathfrak{it}_k^{(2)}$ sont les nombres de tournois irréductible de taille k et de tournois de taille k composés par deux parties irréductibles respectivement.

Le corollaire 10.2.12 illustre une différence significative entre les développements asymptotiques pour les constructions SET et SEQ. Alors que, dans les conditions du théorème 7.2.1, les coefficients du développement asymptotique (1.11) sont toujours positifs, les coefficients de (1.13) pourraient être strictement négatifs même dans le cas $m = 1$. C'est d'ailleurs cas pour les tournois irréductibles, car

$$\left(2\mathfrak{it}_k - \mathfrak{it}_k^{(2)}\right) = 2,\ -2,\ 4,\ 32,\ 848,\ 38\,032,\ 3\,039\,136,\ 446\,043\,008,\ \dots\ .$$

En conséquence, il n'y a aucun espoir d'attendre que ces coefficients énumèrent les objets d'une classe combinatoire. D'autre part, il arrive souvent que des classes combinatoire étiquetées gargantuesques $\mathcal{U}$ et $\mathcal{V}$ satisfassent $\mathcal{U} = \mathrm{SET}(\mathcal{V})$, mais qu'il n'y ait pas de classe $\mathcal{W}$ telle que $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$, de sorte que le théorème 7.2.1 ne peut pas être appliqué directement. Nous aurions besoin d'un opérateur "anti-SEQ" qui produirait une classe combinatoire étiquetée $\mathcal{W}$ telle que $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$. Bien que la classe souhaitée $\mathcal{W}$ puisse ne pas exister, il existe encore des développements asymptotiques et leurs coefficients doivent être interprétés.

Toutes les observations ci-dessus sont la raison pour passer du langage des classes combinatoires au langage des espèces. Ce dernier nous permet, en considérant des espèces virtuelles, d'étendre le concept de suite de comptage à des suites dont les coefficients peuvent être strictement négatifs et de construire un opérateur "anti-SEQ". Plus précisément, les actions des opérateurs SEQ, CYC et SET se traduisent dans la théorie des espèces, à l'aide de compositions avec les espèces $\mathcal{L}$ des ordres totaux, $\mathcal{CP}$ des permutations cycliques et $\mathcal{E}$ des ensembles, respectivement :

$$\begin{array}{ccc} \mathrm{SEQ}(\mathcal{A}) & \rightsquigarrow & \mathcal{L} \circ \mathcal{A}, \\ \mathrm{CYC}(\mathcal{A}) & \rightsquigarrow & \mathcal{CP} \circ \mathcal{A}, \\ \mathrm{SET}(\mathcal{A}) & \rightsquigarrow & \mathcal{E} \circ \mathcal{A}. \end{array}$$

L'espèce $\mathcal{L}_+ = \mathcal{L} - \mathbf{1}$ admet un inverse pour la substitution, et son inverse $\mathcal{L}_+^{(-1)}$ est utilisée dans la généralisation du théorème 7.2.1 pour la théorie des espèces.

Théorème 8.3.4. *Soit $\mathcal{A}$ une espèce (pondérée) gargantuesque, telle que $\mathcal{A} = \mathcal{E} \circ \mathcal{B}$, où $\mathcal{B}$ est une espèce (pondéréé) quelconque. Soit $s \in \mathcal{A}$ une $\mathcal{A}$-structure sur $[n]$, où $n \in \mathbb{N}$. Alors*

$$\mathbb{P}(s \in \mathcal{B}) \approx 1 - \sum_{k \geqslant 1} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}, \tag{1.15}$$

où $\mathfrak{a}_n$ et $\mathfrak{c}_n$ sont les poids totaux sur $[n]$ des espèces $\mathcal{A}$ et $\mathcal{C} = \mathcal{E}^{-1} \circ \mathcal{B} \equiv \left(\mathbf{1} - \mathcal{L}_+^{(-1)}\right) \circ \mathcal{A}_+$ respectivement.

Un exemple important d'application du théorème 8.3.4 est le calcul de la probabilité asymptotique qu'un graphe aléatoire soit connexe dans le modèle Erdős–Réniy $G(n, p)$. Rappelons que, selon ce modèle, un graphe étiqueté G à k arêtes est choisi parmi l'ensemble de tous les graphes à n sommets avec probabilité

$$p^k q^{\binom{n}{2} - k},$$

où $q = 1 - p$. Dans le cas $p = q = 1/2$, nous avons la situation décrite dans le théorème 10.1.10. Dans le cas général, nous perdons le lien entre les graphes et les tournois. Néanmoins, nous pouvons décrire la structure combinatoire des coefficients du développement asymptotique de la probabilité qu'un graphe aléatoire soit connexe. Dans ce contexte plus général, le nombre de tournois irréductibles de taille k sera remplacé par un polynôme P_k évalué au point $p/(1-p)$. À cet effet, définissons le poids des graphes G à k arêtes par

$$w(G) = \rho^k, \qquad \text{où} \;\; \rho = \frac{p}{q}.$$

Par exemple, les poids des graphes dont la taille ne dépasse pas 3 sont indiqués sur la figure 1.1. Nous utilisons ces poids pour définir l'espèce pondérée $\mathcal{G}_w$ des graphes. Cette espèce peut être représentée comme la composition

$$\mathcal{G}_w = \mathcal{E} \circ \mathcal{CG}_w,$$

où $\mathcal{CG}_w$ est l'espèce pondérée des graphes connexes. En appliquant le théorème 8.3.4 à cette dernière relation, nous obtenons le résultat suivant.

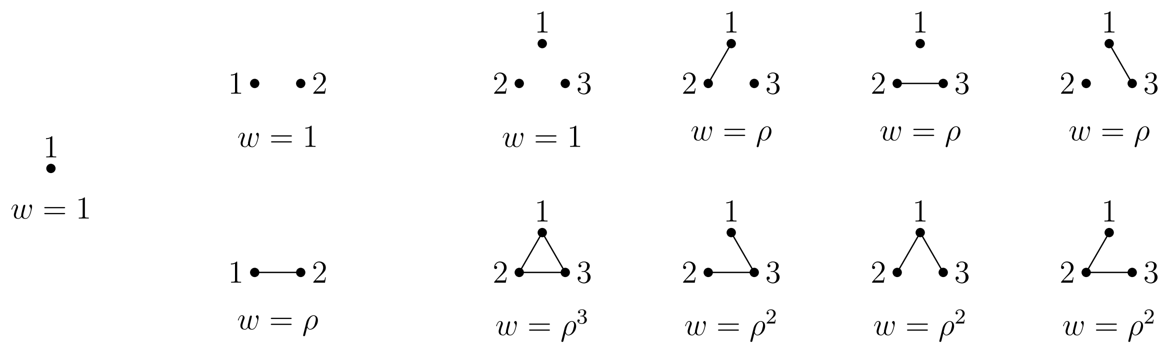


Figure 1.1: Poids des graphes étiquetés dont la taille ne dépasse pas 3.

Théorème 10.4.6. *Soit $p \in (0,1)$. La probabilité asymptotique qu'un graphe aléatoire G dans le modèle d'Erdős–Rényi $G(n,p)$ soit connexe satisfait*

$$\mathbb{P}(G \textit{ est connexe}) \approx 1 - \sum_{k \geqslant 1} P_k(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}}, \tag{1.16}$$

où $q = 1-p$, $\rho = p/q$ et

$$P_k(\rho) = \sum_{H \in \mathcal{G}_w[k]} (-1)^{\pi_0(H)-1} w(H)$$

est la somme des poids de tous les graphes $H \in \mathcal{G}_w[k]$ pris avec le signe $(-1)^{\pi_0(H)-1}$, où $\pi_0(H)$ est le nombre de composantes connexes du graphe H.

Indiquons explicitement les premiers termes du développement asymptotique de la probabilité qu'un graphe aléatoire du modèle d'Erdős–Rényi soit connexe. Pour cela, calculons $P_k(\rho)$ pour $k = 1, 2, 3$. Puisqu'il n'y a qu'un graphe de taille 1, qui est connexe et dont le poids est 1, on a

$$P_1(\rho) = 1.$$

Pour le deuxième polynôme, on a

$$P_2(\rho) = w\Big(\; 1 \bullet\!\!-\!\!\bullet 2 \; \Big) + w\Big(\; 1 \bullet \;\; \bullet 2 \; \Big) = \rho - 1.$$

De la même manière, comme on le voit sur la figure 1.1, il y a huit graphes de taille 3. Parmi eux, un est connexe et est de poids ρ^3, trois sont connexes et sont de poids ρ^2, trois ont deux composantes connexes et sont de poids ρ, et le dernier a trois composantes connexes et est de poids 1. Ainsi, on obtient

$$P_3(\rho) = \rho^3 + 3\rho^2 - 3\rho + 1.$$

Ce calcul nous donne les quatre premiers termes du développement asymptotique :

$$\mathbb{P}(G \text{ est connexe}) = 1 - \binom{n}{1} q^{n-1} - (\rho - 1)\binom{n}{2} q^{2n-3} - (\rho^3 + 3\rho^2 - 3\rho + 1)\binom{n}{3} q^{3n-6} + O(n^4 q^{4n}).$$

En particulier, on reconnait les deux premiers termes obtenus par Gilbert [41].

En plus de SEQ, les constructions combinatoires CYC, SET, SEQ_m, CYC_m et SET_m peuvent être traduites dans le langage de la théorie des espèces. Chacune d'elles correspond à une composition avec certaines espèces de structures. Ce fait nous permet d'énoncer des résultats asymptotiques généraux qui peuvent être considérés comme des analogues d'espèces du théorème 7.2.1 et du théorème 5.3.1 par exemple. Dans ces résultats, les coefficients impliqués dans les développements asymptotiques ont une signification combinatoire en tant que suites de comptage de certaines espèces virtuelles dépendant de structures initiales (théorèmes 8.1.1, 8.1.2, 8.2.1, 8.2.2 et 8.3.6).

Le dernier résultat à mentionner dans cette introduction concerne la probabilité asymptotique de graphes orientés fortement connexes. Le premier résultat pour cette probabilité a été obtenu par Wright [104]. Avec Sergey Dovgal, nous obtenons une simplification significative en imposant une structure aux coefficients.

Théorème 10.3.16. *La probabilité asymptotique qu'un graphe orienté étiqueté aléatoire D avec n sommets soit fortement connexe satisfait*

$$\mathbb{P}\big(D \text{ est fortement connexe}\big) \approx \sum_{k \geqslant 0} \mathfrak{ssd}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{n(n-1)/2}} \cdot \frac{\mathfrak{it}_{n-k}}{2^{nk}}, \tag{1.17}$$

où $\mathfrak{it}_k$ et $\mathfrak{ssd}_k$ sont respectivement les nombres de tournois irréductibles et de graphes orientés semi-forts de taille k.

La forme d'asymptotique (1.17) est assez simple, mais elle dépend du nombre de tournois irréductibles $\mathfrak{t}_{n-k}$. Dans le cas où nous voulons obtenir une décomposition par puissances de deux, nous pouvons encore utiliser l'asymptotique (1.14) qui nous amène au résultat suivant.

Corollaire 10.3.18. *La probabilité asymptotique qu'un graphe orienté étiqueté aléatoire D avec n sommets soit fortement connexe satisfait*

$$\mathbb{P}\big(D \text{ est fortement connexe}\big) = \sum_{k=0}^{2r-1} \frac{2^{k(k+1)/2}}{2^{kn}} \sum_{\nu=0}^{[k/2]} \binom{n}{\nu, k-2\nu} \frac{\mathfrak{ssd}_\nu \beta_{k-2\nu}}{2^{\nu(k-\nu)}} + O\left(\frac{n^{2r}}{2^{2nr}}\right), \tag{1.18}$$

où $\mathfrak{ssd}_k$ est le nombre de graphes orienté semi-forts de taille k,

$$\beta_k = \begin{cases} 1, & \text{si } k = 0, \\ -2\mathfrak{it}_k + \mathfrak{it}_k^{(2)}, & \text{si } k \neq 0, \end{cases}$$

$\mathfrak{it}_k$ et $\mathfrak{it}_k^{(2)}$ sont les nombres de tournois irréductibles de taille k et de tournois de taille k composés de deux parties irréductibles respectivement.

La preuve du théorème 10.3.16 ne peut être déduite des théorèmes généraux mentionnés ci-dessus. Pour établir le résultat, nous appliquons directement le théorème de Bender, ainsi que des techniques supplémentaires. En particulier, nous utilisons une méthode d'énumération pour les graphes fortement connexes qui a été développée en 1973 par Robinson [90], et plus tard, indépendamment, par de Panafieu et Dovgal [29]. Nous pouvons voir que le développement asymptotique (1.17) a une forme moins simple que les développements précédentes. En particulier, il y a trois structures qui interagissent : les graphes fortement connexes, les graphes semi-forts et les tournois irréductibles. On peut espérer avoir une interprétation de la formule ci-dessus en termes de principe d'inclusion-exclusion. Malheureusement, nous n'en avons pas trouvé. Ce problème reste un défi pour les projets futurs.

1.3 Plan de la thèse

Ce manuscrit est structuré linéairement autour de parties et de chapitres dans un flux plutôt traditionnel qui passe des outils aux théorèmes généraux puis aux applications. Néanmoins, une lecture non linéaire à travers les chapitres est également possible. On peut le faire soit en considérant un type particulier d'objets qui apparaissent dans toutes ces sections, soit en se focalisant sur une construction particulière ou un formalisme particulier. Par exemple, si on s'intéresse aux asymptotique liées à la construction SEQ, alors il suffit de se limiter au chapitre 5 et à la section 8.1 de partie II ainsi qu'au chapitre 9 et à la section 10.2 de la partie III. De la même manière, si notre curiosité nous porte vers les tournois, nous pouvons aller directement aux sections 5.2, 5.3 et 10.2, en s'appuyant si besoin sur les outils correspondants de la partie I.

Décrivons la structure de la thèse. Dans la partie I, nous introduisons tous les outils nécessaires à notre recherche. Cette partie introductive est suffisamment illustrative et remplie d'exemples et de détails. Elle est destinée à être conviviale et, au delà ce ça, à servir aux étudiantes et étudiants qui découvrent le domaine pour se familiariser avec le matériel et les techniques. La majeure partie du contenu des chapitres 2–4 est bien connue, et le lecteur qui est familier peut ignorer les sujets correspondants sans perte de compréhension. La seule exception concerne le concept de séquences gargantuesques (ainsi que les classes combinatoires gargantuesques et les espèces gargantuesques qui en découlent), que nous définissons pour simplifier la présentation (voir les sections 2.1, 3.1.15, 3.2.6 et 4.2.4).

La partie II est consacrée aux résultats généraux. Dans les chapitres 5–7, nous travaillons avec la méthode symbolique. Premièrement, nous discutons, d'un point de vue combinatoire, les probabilités qu'un objet aléatoire soit irréductible en termes de décompositions par rapport aux constructions combinatoires SEQ, CYC et SET. Pour ces probabilités, nous établissons empiriquement les développements asymptotiques accompagnés de la signification combinatoire des coefficients impliqués. Ici, c'est le principe d'inclusion-exclusion qui joue le rôle clé pour obtenir les interprétations combinatoires. Deuxièmement, nous prouvons rigoureusement le résultat obtenu, via le théorème de Bender. Troisièmement, nous faisons de même pour les constructions restreintes SEQ_m, CYC_m et SET_m. Enfin, au chapitre 8, nous généralisons les résultats obtenus dans le cadre de la théorie des espèces.

Dans la partie III, nous discutons de nombreuses applications. Le chapitre 9 est consacré aux permutations et aux objets associés. Nous établissons les développements asymptotiques pour les probabilités qu'une permutation, un couplage parfait ou une multipermutation aléatoire soit indécomposable. Le chapitre 10 porte sur différents types de graphes. Nous montrons comment obtenir les probabilités qu'un graphe aléatoire soit connexe, qu'un tournoi aléatoire soit irréductible, qu'un graphe orienté aléatoire soit faiblement ou fortement connexe, etc. En particulier, nous le faisons pour le modèle d'Erdős–Rényi $G(n, p)$. Dans le chapitre 11, nous discutons de plusieurs modèles de surfaces. Nous étudions les surfaces à petits carreaux, les cartes combinatoires et les constellations et indiquons leurs connexions avec des permutations, des couplages parfaits et des multipermutations, respectivement. Pour tous les modèles mentionnés, nous fournissons des développements asymptotiques pour les probabilités qu'une surface aléatoire soit connexe ou consiste en un nombre fixe de composantes connexes.

Enfin, nous discutons de plusieurs directions possibles pour de futures recherches dans la partie IV.

Chapter 1

Introduction

1.1 Presentation of the problem and strategy

Various combinatorial structures admit, in a broad sense, a notion of irreducibility. Here, by irreducibility we mean that a compound structure can be decomposed into a collection of small pieces that cannot be decomposed further. As an example, various objects admit a decomposition into connected components. For instance, a simple graph is a disjoint union of connected ones, while directed graphs admit the notion of weak and strong connectedness. Also, one could refer to various models of discrete surfaces that can be connected or not. Another type of irreducibility is given by the decomposition of an object into a sequence of irreducible ones. As important examples, we must mention permutations, which can be represented as concatenations of indecomposable ones, and tournaments. Note that permutations can also be decomposed as disjoint unions of cycles. The list of irreducible structures looks countless; a unified approach to them based on partitions was developed by Beissinger [5].

We are interested in the following question. Pick a random object of size n from a labeled combinatorial class that admits a notion of irreducibility. What is the probability that this object is irreducible, as its size tends to infinity (say, what is the probability that a random graph with n vertices is connected, as $n \to \infty$)? For many known classes, this limiting probability tends to 1. For example, this is the case for connected graphs [41], irreducible tournaments [68] and indecomposable permutations [21]. In other cases, the limiting probability is 0, as for permutations consisting of exactly one cycle or for set partitions consisting of exactly one part. Sometimes, the limiting probability of irreducibility is strictly between 0 and 1. In many cases, it involves e. For instance, the limiting probability is $1/\sqrt{e}$ for a random labeled forest to be connected [88], while it is $1/e$ for a random chord diagram to be connected [51, 95] and $1/e^2$ for a random permutation to be simple [2], respectively. However, one could imagine more tricky values as well, as the golden ratio $(\sqrt{5}-1)/2$ for the limiting probability that a random N-tree poset is connected [32, 92], or $1/2.997\ldots = 0.3367\ldots$ for the limiting probability that a random unlabeled forest is a tree [79], see also [10].

Back in the mid-sixties, Wright [99] posed the question of how to distinguish between these cases. He limited himself to the connectedness case described by the exponential formula. In other words, he considered sequences $(\mathfrak{c}_n)$ and $(\mathfrak{a}_n)$ counting connected and all unlabeled object of some kind, respectively, such that

$$\sum_{n=0}^{\infty} \mathfrak{a}_n z^n = \exp\left(\sum_{n=1}^{\infty} \mathfrak{c}_n z^n\right). \tag{1.1}$$

Wright showed [99, 100] that

$$\sum_{k=1}^{n-1} \mathfrak{a}_k \mathfrak{a}_{n-k} = o(\mathfrak{a}_n) \qquad \Leftrightarrow \qquad \sum_{k=1}^{n-1} \mathfrak{c}_k \mathfrak{c}_{n-k} = o(\mathfrak{c}_n)$$

and both conditions can be used as necessary and sufficient for $\lim_{n\to\infty} (\mathfrak{c}_n/\mathfrak{a}_n) = 1$, i.e. for the limiting probability to be equal to 1. Also, he indicated the following necessary condition: the generating function

$$A(z) = \sum_{n=0}^{\infty} \mathfrak{a}_n z^n$$

must diverge for any non-zero z, which is equivalent to $\lim_{n\to\infty} (\mathfrak{a}_{n-1}/\mathfrak{a}_n) = 0$ in the case when $\mathfrak{a}_n \neq 0$. Similar conditions were obtained for the labeled case (see also [16]).

During the following decades, several authors developed Wright's approach as, for instance Bell, Bender, Cameron, Compton, Odlyzko and Richmond [6, 10, 16, 20], see also [52]. The research culminated in the paper of Bell, Bender, Cameron and Richmond [7] who proved the following result. Assuming that the limit $\rho = \lim_{n\to\infty} (\mathfrak{c}_n/\mathfrak{a}_n)$ exists, its value is in the following correspondence with the radius of convergence R of the generating function $C(z)$:

- $\rho = 1 \qquad \Leftrightarrow \qquad R = 0,$
- $\rho = 0 \qquad \Leftrightarrow \qquad R > 0$ and $C(R)$ diverge,
- $0 < \rho < 1 \qquad \Leftrightarrow \qquad R > 0$ and $C(R)$ converge.

Moreover, they studied the case when the limit does not exist, and described the behavior of $\limsup(\mathfrak{c}_n/\mathfrak{a}_n)$ and $\liminf(\mathfrak{c}_n/\mathfrak{a}_n)$ depending on the parameter R. In our investigation, we concentrate on the first case: the sequences $(\mathfrak{a}_n)$ and $(\mathfrak{c}_n)$ grow rapidly and the generating functions $A(z)$ and $C(z)$ have a radius of convergence $R = 0$.

Our interest is not limited to the limiting probability. One of our aims is to establish the full asymptotic expansion for the probability that a random object is irreducible. In other words, we wish to have an expression of the type

$$\frac{\mathfrak{c}_n}{\mathfrak{a}_n} = \rho\Big(1 + f_1(n) + f_2(n) + \ldots + f_r(n) + o\big(f_r(n)\big)\Big), \tag{1.2}$$

where $f_{k+1}(n) = o\big(f_k(n)\big)$ for every $k \in \mathbb{N}$, as $n \to \infty$. Generally speaking, one could reach this aim with the help of the inclusion-exclusion principle (we will see how this works for different combinatorial constructions in Sections 5.1, 6.1 and 7.1). However, in practice, this method is unwieldy and mainly used for finding the first few terms. For example, it is the case for simple permutations [2] and some classes of connected surfaces [15, 83]. Another, though similar in spirit, method was developed in the late sixties by Wright who used recurrences to get the asymptotics. For instance, he showed [102] that if sequences $(\mathfrak{c}_n)$ and $(\mathfrak{a}_n)$ satisfy exponential formula (1.1), then the conditions

$$\lim_{n\to\infty} \frac{\mathfrak{a}_{n-1}}{\mathfrak{a}_n} = 0 \qquad \text{and} \qquad \sum_{k=r}^{n-r} \mathfrak{a}_k \mathfrak{a}_{n-k} = O(\mathfrak{a}_{n-r}), \tag{1.3}$$

holding for some integer r, imply

$$\mathfrak{c}_n = \mathfrak{a}_n + \sum_{k=1}^{r-1} \alpha_k \mathfrak{a}_{n-k} + O(\mathfrak{a}_{n-r}), \tag{1.4}$$

and vice versa. Here, the sequence (α_n) is defined recursively by $\alpha_0 = \mathfrak{a}_0 = 1$ and

$$\sum_{k=0}^{n} \alpha_k \mathfrak{a}_{n-k} = 0. \tag{1.5}$$

This method allowed Wright to get asymptotic expressions for the number of connected graphs [102], irreducible tournaments [103] and strongly connected digraphs [104]. Also, Robinson used the approach of Wright for finding asymptotics of directed acyclic graphs [90].

The numerous results of Wright were rethought and generalized in 1975 by Bender [9]. Compared to the work of Wright, the advantage of Bender's approach is the transition from recurrences to generating functions. In particular, the sequence (α_n) from relation (1.5) was determined by differentiating some power series. As we will see later, this fact allows us to understand the structure of asymptotics and to interpret (α_n) combinatorially. We discuss Bender's approach in detail in Section 2.2.

It is important to emphasize that the method Bender used in [9] works well only in a number of cases. Depending on the nature of the generating function, different techniques may be required to obtain the desired asymptotics. More precisely, let a function $F(z)$ be analytic in some neighborhood of the origin and a formal power series $A(z)$ have coefficients growing sufficiently fast (i.e. satisfying conditions (1.3) for all $r \in \mathbb{N}$). Then Bender's method suits for establishing the full asymptotic expansion of the coefficients of the composition $F\big(A(z)\big)$. On the other hand, if we need to find the asymptotics of $A\big(F(z)\big)$ or $A^{-1}(z)$, we should use different approach developed by Bender and Richmond [11] (these are the ideas of [9] combined with the Lagrange inversion formula). An interesting case is studied by Borinsky [13]: in his research, both functions involved in the composition are divergent, but their rate of growth is not too fast. More precisely, Borinsky established formulas for the asymptotic behavior of the product $F(z)G(z)$ and composition $F\big(G(z)\big)$, where the coefficients of $F(z)$ and $G(z)$ admitted an asymptotic expansion of the form

$$\alpha^{n+\beta}\,\Gamma(n+\beta)\left(d_0 + \frac{d_1}{(n+\beta-1)} + \frac{d_2}{(n+\beta-1)(n+\beta-2)} + \ldots\right)$$

for some constants $\alpha, \beta \in \mathbb{R}_{>0}$ and $d_k \in \mathbb{R}$.

Other methods are to be applied in the case when the treated generating function has a nonzero radius of convergence. Depending on the situation, one can use Darboux's method (see, for example, [22, 77, 98]) or the singularity analysis [37] based on the saddle point method and the papers of Hayman [48], Harris and Schoenfeld [47], Flajolet, Odlyzko and Richmond [36, 73, 75]. To find out more about this, we refer the reader to the surveys of Bender [8], Odlyzko [74] and books of De Bruijn [25], Flajolet and Sedgewick [38].

So, we are interested in the asymptotics of the probability that a random object of a certain kind is irreducible. In our investigation, we pursue two goals. First, we would like to obtain the full asymptotic expansion of this probability in a common manner, suitable for different structures. Limiting counting sequences of the object classes to grow rapidly enough, we find ourselves in a situation obeying Bender's theorem [9]. Moreover, it turns out that the constants involved in expansion (1.2) in this case are integers. The second goal that we pursue follows from this observation. Namely, we wish to have a combinatorial interpretation of the constants involved in the asymptotic expansion.

The strategy for achieving the two above mentioned goals is in forcing several techniques to interplay. Our first step is to interpret combinatorial structures in terms of generating functions. The origin of this basic idea is hidden in the depth of centuries; one can see generating functions arising on the basis of corresponding recurrences in works of Bernoulli, Euler, Cayley etc. As an example of particular importance, we should mention the contribution of Pólya [84, 85], whose work was devoted to the enumeration of objects under group symmetry action. The contemporary approaches to combinatorial analysis, including Pólya theory, can be found, for instance, in books of such authors as Comtet [22], Goulden and Jackson [42], Lando [55], Stanley [93, 94] and Wilf [98]. Our first choice is the symbolic method [38] that is rather simple and clear enough. According to this method, compound structures are obtained from simpler ones using combinatorial constructions, such as sequence SEQ, cycle CYC and set SET. The fact that an object is irreducible with respect to one of the three mentioned constructions is directly interpreted in terms of generating functions via corresponding relations.

Once we have relations for generating functions, it becomes possible to proceed with the first goal, that is, with obtaining the asymptotic expansions. As we have just mentioned, our way is based on Bender's theorem which permits us to establish asymptotic behavior of formal power series coefficients in the case when the target power series is expressed in terms of the initial power series by means of an analytic function. Bender's theorem is applicable for series whose coefficients grow sufficiently fast. This fact limits us to considering labeled combinatorial classes whose counting sequences satisfy (1.3) for all $r \in \mathbb{N}$. Note that in this case, for the question of connectedness (aka irreducibility with respect to the combinatorial construction SET), one can guarantee that a random structure is almost surely connected [6, 7].

The second goal is more tricky. As far as we know, the only example of combinatorially interpreted asymptotic coefficients in the literature, in the spirit of our work, is the following. In 2005, Dixon [28] showed that the probability t_n that a random pair of permutations from S_n generates a transitive group could be represented as the asymptotic series

$$t_n = 1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \ldots + \frac{c_m}{n^m} + O\left(\frac{1}{n^{m+1}}\right). \tag{1.6}$$

Dixon indicated the first several values of constants c_k, so that this asymptotics had the form

$$t_n = 1 - \frac{1}{n} - \frac{1}{n^2} - \frac{4}{n^3} - \frac{23}{n^4} - \frac{171}{n^5} - \frac{1542}{n^6} - \ldots. \tag{1.7}$$

While proving the above asymptotics, Dixon used Bender's theorem and established that

$$t_n = 1 + \sum_{k=1}^{r-1} \frac{d_k}{(n)_k} + O\left(\frac{1}{n^r}\right) \tag{1.8}$$

for any positive integer r, where $(n)_k = n(n-1)\ldots(n-k+1)$ were the falling factorials and the coefficients (d_k) were defined by

$$\sum_{k=0}^{\infty} d_k z^k = \left(\sum_{k=0}^{\infty} k! z^k\right)^{-1}.$$

In 2009, Cori [23] showed that $(-d_k)$ was the counting sequence for indecomposable permutations.

To achieve such a combinatorial interpretation of coefficients, we need a technique that bridges between operations on power series and underlying combinatorial structures. There are several ways to do it. The most natural way is to use the symbolic method again. Thus, first we describe the irreducibility of combinatorial objects in terms of generating functions with the help of the symbolic method. This allows us to pass to asymptotic expansions via Bender's theorem. Having obtained the asymptotic coefficients, we apply the symbolic method once again, in the opposite direction. This allows to prescribe a structure for the asymptotic expansion and give the coefficients a combinatorial meaning.

Although the symbolic method is rather clear and easy to use, it does not always give satisfactory results. Sometimes the obtained coefficients cannot be interpreted in terms of combinatorial classes directly. For example, this is the case when some of them are negative (it is so, say, for irreducible tournaments). To cope with this obstacle, we use the theory of species [12, 49]. The reader can consider this theory as a different language that could serve the same aim of translating structure decompositions to generating function operations and vice versa. Compared to the symbolic method, the theory of species is more general, though it is more abstract and more difficult to implement. A key role in the theory of species is reserved for the substitution operation, that encompass all the previously discussed constructions. The main advantage that we actively use is the concept of virtual species. The same way as integers extend the set of natural numbers, the ring of virtual species is the extension of the semi-ring of species of structures. Using virtual species allows us to give combinatorial meaning to more expressions, including sequences with negative coefficients, and permits to pass to the inverse under substitution.

1.2 Contribution of the thesis

This work is based on the investigation, the results of which are presented in the papers [30, 63, 64, 65]. During the course of this PhD work, results on other topics were obtained as well, that are not included here in order to preserve the manuscript's unity. Thus, in a joint paper with Ivan Reshetnikov [71], we provide results on decompositions of functions defined on finite sets. In another, joint paper with Alexander Povolotsky [70], we study the asymptotic analysis of non-local correlation functions of the watermelon type.

The thesis pursue two main goals. The first of them is to provide a general way of establishing the asymptotic probability that large combinatorial objects are connected or irreducible in a precise sense that we will define later (see Definitions 3.1.31, 3.1.39 and 3.1.46). The second, somehow more interesting, is to indicate the combinatorial meaning of the coefficients involved in the obtained asymptotic expansions and to understand the underlying reasons. To give the reader an idea of what kind of results we are interested in, we start with an example that will be discussed in detail in Chapter 10.

Let us consider the Erdős–Rényi model of random graphs $G(n, 1/2)$: for each non-negative integer n, we endow the set of undirected simple graphs on the set $[n] = \{1, \ldots, n\}$ with the uniform probability. In other words, within this model, each graph of size n appears with probability $1/2^{\binom{n}{2}}$. The probability p_n that such a random graph is connected goes to 1 as n tends to ∞. In 1959, Gilbert [41] provided a more accurate estimation showing that

$$p_n = 1 - \frac{2n}{2^n} + O\left(\frac{n^2}{2^{3n/2}}\right).$$

In 1970, Wright [102] calculated the first four terms of the asymptotic expansion of this probability:

$$p_n = 1 - \binom{n}{1}\frac{1}{2^{n-1}} - 2\binom{n}{3}\frac{1}{2^{3n-6}} - 24\binom{n}{4}\frac{1}{2^{4n-10}} + O\left(\frac{n^5}{2^{5n}}\right).$$

Wright's method can be used to compute more terms, one by one. However, it gives neither the structure of the full asymptotic expansion nor the interpretation of the coefficients $1, 2, 24, \ldots$.

Our main goal is to provide such a structure: the k-th term of the asymptotic expansion of p_n is of the form

$$\mathfrak{it}_k \cdot 2^{k(k+1)/2} \cdot \binom{n}{k} \cdot \frac{1}{2^{kn}},$$

where $\mathfrak{it}_k$ counts the number of irreducible labeled tournaments of size k (see Definition 10.2.2). We express this result, as well as all the other results that we will see later, with the help of two notations. The first of them is the symbol $\approx$ for the asymptotic expansion (see Notation 2.2.1 for the precise meaning). The second is the asymptotic probability, $p_n = \mathbb{P}\big(G \text{ is connected}\big)$, where G is a random graph of size n (for the *asymptotic probability* concept, see Definition 3.1.2). Using these notations, we get the announced result in the following form.

Theorem 10.1.10. *The asymptotic probability that a random labeled simple graph G with n vertices is connected satisfies*

$$\mathbb{P}\big(G \textit{ is connected}\big) \approx 1 - \sum_{k \geqslant 1} \mathfrak{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{kn}}, \tag{1.9}$$

where $\mathfrak{it}_k$ is the number of irreducible tournaments of size k.

The proof of Theorem 10.1.10 is based on Bender's theorem (Theorem 2.2.5) and rather short: we express the exponential generating function of connected graphs in terms of the exponential generating function of graphs, differentiate this expression to obtain the asymptotics and notice that, somehow, the result involves the exponential generating function of irreducible tournaments. However, Theorem 10.1.10 presents a new result. The explanation of this fact lies in the relation between graphs and tournaments hidden in their structures. Although enumeration of connected graphs was given by Wright in 1970, he did not provide a combinatorial meaning of the coefficients. The fact that $(\mathfrak{it}_k)$ is the counting sequence of irreducible tournaments is easy to get empirically nowadays, with the help of the on-line encyclopedia of integer sequences [76], but it was concealed back in the seventies. Nevertheless, even when discovered, the link between connected graphs and irreducible tournaments is rather non-direct. In terms of labeled combinatorial classes, one can describe this link in the following way. Given the labeled combinatorial classes $\mathcal{CG}$ of connected graphs and $\mathcal{IT}$ of irreducible tournaments, respectively, there is a combinatorial isomorphism of compound structures

$$\mathrm{SET}(\mathcal{CG}) \cong \mathrm{SEQ}(\mathcal{IT}). \tag{1.10}$$

Here, the use of construction SET reflects the fact that any graph is the union of its connected components. In a similar way, using the construction SEQ means that any tournament can be represented as a sequence of irreducible ones. Thus, relation (1.10) can be interpreted as a combinatorial isomorphism of the labeled combinatorial classes $\mathcal{G}$ of graphs and $\mathcal{T}$ of tournaments, respectively, meaning that their counting sequences are the same. Here, we are obliged to emphasize that the *combinatorial isomorphism* concept is nothing more than the coincidence of the numbers of structures of the same size (see Definition 3.1.11): the number $\mathfrak{g}_n$ of graphs of size n and the number $\mathfrak{t}_n$ of tournaments of size n both are equal to

$$\mathfrak{g}_n = \mathfrak{t}_n = 2^{\binom{n}{2}}.$$

While two different labeled tournaments of size two are isomorphic by swapping the vertices, two different labeled graphs of size two are not. In particular, any bijection between $\mathcal{G}$ and $\mathcal{T}$ is kind of artificial, since it does not preserve isomorphism classes.

Now, we understand two basic ingredients lying under Theorem 10.1.10, namely, Bender's theorem and the interplay of two different structures (graphs and tournaments). This allows us to state a generalization for arbitrary combinatorial classes. Similar to relation (1.10), we suppose that a labeled combinatorial class $\mathcal{U}$ admits two different decompositions:

$$\mathcal{U} = \mathrm{SET}(\mathcal{V}) = \mathrm{SEQ}(\mathcal{W})$$

(for simplicity, we are talking about equality, although, in practice, one usually has a combinatorial isomorphism, $\mathrm{SET}(\mathcal{V}) \cong \mathrm{SEQ}(\mathcal{W})$). Then, under certain conditions, the asymptotic expansion for the probability that a random object from $\mathcal{U}$ is SET-irreducible can be expressed in terms of the counting sequence of $\mathcal{U}$, and the coefficients have precise combinatorial meaning. Here, roughly speaking, the notion of SET-irreducibility means that an object from the class $\mathcal{U}$ is connected. In other words, the object from the class $\mathcal{U}$ is SET-irreducible if and only if it belongs to the class $\mathcal{V}$ as well. As for the conditions, the method works for labeled combinatorial classes whose counting sequences grow sufficiently fast (we call such classes *gargantuan*, see Definition 3.1.72). More precisely, the new result that we establish is as follows.

Theorem 7.2.1. *Let $\mathcal{U}$ be a gargantuan labeled combinatorial class with positive coefficients, such that $\mathcal{U} = \mathrm{SET}(\mathcal{V})$ and $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ for some labeled combinatorial classes $\mathcal{V}$ and $\mathcal{W}$. Suppose that $u \in \mathcal{U}$ is a random object of size n. Then*

$$\mathbb{P}(u \text{ is SET-irreducible}) \approx 1 - \sum_{k \geqslant 1} \mathfrak{w}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}, \tag{1.11}$$

where $(\mathfrak{u}_n)$ and $(\mathfrak{w}_n)$ are the counting sequences of the classes $\mathcal{U}$ and $\mathcal{W}$, respectively.

Theorem 7.2.1 admits a number of applications that link structures of different nature. We have already seen this kind of link between connected graphs and irreducible tournaments (Theorem 10.1.10). In a similar way, we obtain links between connected multigraphs and irreducible multitournaments (Statement 10.1.18) and weak (multi)digraphs and irreducible multitournaments (Statement 10.3.23). In all the above cases, the second structure arises as coefficients in the asymptotic expansion of the asymptotic probability for the first structure.

Another significant example refers to connected square-tiled surfaces and indecomposable permutations. The model of *square-tiled surfaces*, also called *origamis*, can be obtained as follows [105]. Given $n \in \mathbb{N}$, take a collection of unit squares enumerated by integers from 1 to n and identify their sides by translation, so that the result is a translation surface, i.e. a surface with the globally defined notions of north, west, south and east. All the possible surfaces that can be obtained by the described procedure constitute the labeled combinatorial class $\mathcal{O}$ of square-tiled surfaces. The fact that any surface is a collection of connected ones can be interpreted by the relation

$$\mathcal{O} = \mathrm{SET}(\mathcal{CO}),$$

where $\mathcal{CO}$ denotes the subclass of connected square-tiled surfaces. On the other hand, each identification of square sides in the above model corresponds to the pair of permutations $(\sigma_v, \sigma_h) \in S_n^2$ such that the right side of the k-th square is identified to the left side of the $\sigma_v(k)$-th square and the top side of the k-th square is identified to the bottom side of the $\sigma_h(k)$-th square. The combinatorial class of pairs of permutations of the same size is combinatorially isomorphic to the class $\mathcal{ML}(2)$ of pairs of linear orders of the same size, meaning that their counting sequences are the same: $(n!)^2$. The class $\mathcal{ML}(2)$ can be decomposed as a sequence,

$$\mathcal{ML}(2) = \operatorname{SEQ}\big(\mathcal{IML}(2)\big),$$

where $\mathcal{ML}(2)$ is the labeled combinatorial class of irreducible pairs of linear orders of the same size. Its counting sequence coincide with $n! \cdot \mathfrak{ip}_n$, where $\mathfrak{ip}_n$ is the number of indecomposable permutations. Thus, the combinatorial isomorphism

$$\operatorname{SET}(\mathcal{CO}) \cong \operatorname{SEQ}\big(\mathcal{IML}(2)\big)$$

allows us to apply Theorem 7.2.1 which gives the following new result closely related to (1.8).

Theorem 11.1.8. *The asymptotic probability that a square-tiled surface S obtained by gluing n unit squares randomly is connected satisfies*

$$\mathbb{P}\big(S \text{ is connected}\big) \approx 1 - \sum_{k \geqslant 1} \frac{\mathfrak{ip}_k}{(n)_k}, \tag{1.12}$$

where $(n)_k = n(n-1)(n-2)\ldots(n-k+1)$ are the falling factorials and $\mathfrak{ip}_k$ is the number of indecomposable permutations of size k.

Note that a square-tiled surface of size n is connected if and only if the corresponding pair of permutations generates a transitive subgroup of S_n. Thus, the asymptotic expansion for the probability that a random square-tiled surface is connected can be derived from the results of Dixon [28] and Cori [23] mentioned in the previous section. However, the appearance of indecomposable permutations in this context was rather surprising and needed to be explained. Applying our approach and Theorem 7.2.1 sheds a new light on this story. We disclose the underlying reasoning by putting the structure and enclosing the link between square-tiled surfaces and indecomposable permutations in a general framework. It is seen from our approach that the decomposition (1.6) of t_n into asymptotics over the basis $1/(n)_k$ is more natural than (1.7) over the basis $1/n^k$. It also allows us to pass easily to the generalizations. One of them unveils the interplay between combinatorial maps and indecomposable perfect matchings (Theorem 11.2.4). Another one provides the links between constellations and indecomposable multipermutations (Theorem 11.3.6).

Once we have a result producing the asymptotic probability for SET-irreducible (aka connected) objects, it is reasonable to extend this result to other combinatorial structures, such as sequences SEQ and cycles CYC. It is possible, indeed, and we obtain asymptotic probabilities for SEQ-irreducible and CYC-irreducible objects (Theorem 5.2.1 and Theorem 6.2.1, respectively). Moreover, we get asymptotics for constructions restricted to the fixed number of irreducible components. For example, for any $m \in \mathbb{N}$, the construction SEQ_m produces sequences of m irreducible components. The same way, CYC_m and SET_m are reserved for cycles and sets of m irreducible components, respectively. For each of the above three constructions, we establish the asymptotic probability that a random object consists of exactly m irreducible components (Theorems 5.3.1, 6.3.1 and 7.3.1). Note that all of them have a natural interpretation in terms of inclusion-exclusion principle.

To give the reader an idea of what the above mentioned new results look like, here is our general theorem for the construction SEQ_m.

Theorem 5.3.1. *Let $\mathcal{U}$ be a gargantuan labeled combinatorial class with positive coefficients, such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ for some labeled combinatorial class $\mathcal{W}$. Suppose that $u \in \mathcal{U}$ is a random object of size n. Then for any $m \in \mathbb{N}$,*

$$\mathbb{P}(u \text{ has } m \text{ SEQ-irreducible components}) \approx \sum_{k\geqslant 0} m\Big(\mathfrak{w}_k^{(m-1)} - 2\mathfrak{w}_k^{(m)} + \mathfrak{w}_k^{(m+1)}\Big) \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}, \quad (1.13)$$

where $\big(\mathfrak{w}_k^{(m)}\big)$ is the counting sequence of $\mathrm{SEQ}_m(\mathcal{W})$.

Thus, we obtain a tool that permits us to get asymptotic probabilities for various structures together with combinatorial interpretation of the involved coefficients. All the examples already mentioned in this section are tractable, including irreducible tournaments, indecomposable permutations and indecomposable perfect matchings. For instance, we show that irreducible tournaments satisfy the following theorem.

Corollary 10.2.12. *The asymptotic probability that a random labeled tournament T with n vertices is irreducible satisfies*

$$\mathbb{P}\big(T \text{ is irreducible}\big) \approx 1 - \sum_{k\geqslant 1} \Big(2\mathfrak{it}_k - \mathfrak{it}_k^{(2)}\Big) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{kn}}, \quad (1.14)$$

where $(\mathfrak{it}_k)$ and $\big(\mathfrak{it}_k^{(2)}\big)$ are counting sequences for irreducible tournaments and tournaments with two irreducible parts, respectively.

Corollary 10.2.12 illustrates a significant difference between asymptotic expansions for probabilities for SET-irreducible and SEQ-irreducible objects. While, under conditions of Theorem 7.2.1, the coefficients in asymptotic expansion (1.11) are always non-negative, the coefficients in (1.13) could be negative even in the case $m = 1$, and it is actually the case for irreducible tournaments, since

$$\Big(2\mathfrak{it}_k - \mathfrak{it}_k^{(2)}\Big) = 2,\ -2,\ 4,\ 32,\ 848,\ 38\,032,\ 3\,039\,136,\ 446\,043\,008,\ \ldots\ .$$

As a consequence, there is no hope to expect that these coefficients (and even more so the coefficients involved in the asymptotic expansions for probabilities that an object has exactly m irreducible components, regardless of whether it is SET, SEQ or CYC-irreducible) enumerate objects of a combinatorial class. On the other hand, it often happens that gargantuan labeled combinatorial classes $\mathcal{U}$ and $\mathcal{V}$ satisfy $\mathcal{U} = \mathrm{SET}(\mathcal{V})$, but there is no class $\mathcal{W}$ such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$, and thus, Theorem 7.2.1 cannot be implemented directly. We would need an "anti-SEQ" operator constructing a labeled combinatorial class $\mathcal{W}$ such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$. Though a desired class $\mathcal{W}$, possibly, does not exist, there are still asymptotic expansions and their coefficients are to be interpreted.

All the above observations are the reason to proceed from the language of combinatorial classes to the language of species. The latter allows us, by considering virtual species, to extend the concept of counting sequence to sequences with negative coefficients and to create an "anti-SEQ" operator. Moreover, by considering weighted virtual species, we extend it to real numbers. More precisely, the actions of the operators SEQ, CYC and SET in the symbolic method translate in the theory of species to the composition with the species $\mathcal{L}$ of linear orders, $\mathcal{CP}$ of cyclic permutations and $\mathcal{E}$ of sets, respectively:

$$\begin{array}{ccc} \mathrm{SEQ}(\mathcal{A}) & \rightsquigarrow & \mathcal{L} \circ \mathcal{A}, \\ \mathrm{CYC}(\mathcal{A}) & \rightsquigarrow & \mathcal{CP} \circ \mathcal{A}, \\ \mathrm{SET}(\mathcal{A}) & \rightsquigarrow & \mathcal{E} \circ \mathcal{A}. \end{array}$$

The species $\mathcal{L}_+ = \mathcal{L} - \mathbf{1}$ admits an inverse under substitution, and we show that $\mathcal{L}_+^{(-1)}$ is involved in the generalization of Theorem 7.2.1 for the theory of species as follows.

Theorem 8.3.4. *Let $\mathcal{A}$ be a gargantuan (weighted) species of structures, such that $\mathcal{A} = \mathcal{E} \circ \mathcal{B}$ for some (weighted) species $\mathcal{B}$. Suppose that $s \in \mathcal{A}$ is a random $\mathcal{A}$-structure on $[n]$, where $n \in \mathbb{N}$. Then*

$$\mathbb{P}(s \in \mathcal{B}) \approx 1 - \sum_{k \geqslant 1} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}, \tag{1.15}$$

where $\mathfrak{a}_n$ and $\mathfrak{c}_n$ are the total weights on $[n]$ of the species $\mathcal{A}$ and $\mathcal{C} = \mathcal{E}^{-1} \circ \mathcal{B} \equiv \left(\mathbf{1} - \mathcal{L}_+^{(-1)}\right) \circ \mathcal{A}_+$, respectively.

An important example of the use of Theorem 8.3.4 is establishing the asymptotic probability that a random graph is connected in the Erdős–Rényi model $G(n,p)$. Recall that, according to this model, a labeled graph G with k edges is picked from the set of graphs with n vertices with probability

$$p^k q^{\binom{n}{2} - k},$$

where $q = 1 - p$. In the case $p = q = 1/2$, we have the situation described in Theorem 10.1.10. In the general case, we lose the connection between graphs and tournaments. However, we can still indicate the combinatorial structure for the coefficients of the asymptotic expansion of the probability that a random graph is connected. Namely, in this general case, the number of irreducible tournaments of size k will be replaced by a polynomial P_k evaluated at $p/(1-p)$. To this end, we can use the weight $w(G)$ of a graph G with k edges, defined as

$$w(G) = \rho^k, \qquad \text{where} \quad \rho = \frac{p}{q}.$$

For example, the weights of graphs whose size does not exceed 3 are indicated on Fig. 1.1. Defining the weight as above allows us to proceed to the weighted species $\mathcal{G}_w$ of graphs. This species can be represented as the composition

$$\mathcal{G}_w = \mathcal{E} \circ \mathcal{CG}_w,$$

where $\mathcal{CG}_w$ is the weighted species of connected graphs. Applying Theorem 8.3.4 to the latter relation, we obtain the following result.

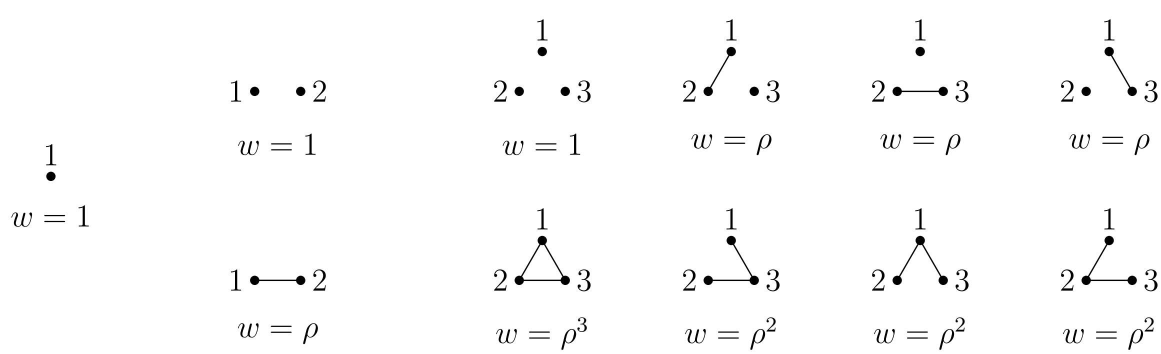


Figure 1.1: Weights of labeled graphs whose size does not exceed 3.

Theorem 10.4.6. *Let $p \in (0,1)$. The asymptotic probability that a random graph G in the Erdős–Rényi model $G(n,p)$ is connected satisfies*

$$\mathbb{P}(G \textit{ is connected}) \approx 1 - \sum_{k \geqslant 1} P_k(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}}, \tag{1.16}$$

where $q = 1 - p$, $\rho = p/q$ and

$$P_k(\rho) = \sum_{H \in \mathcal{G}_w[k]} (-1)^{\pi_0(H)-1} w(H)$$

is the sum of weights of graphs $H \in \mathcal{G}_w[k]$ taken with the sign $(-1)^{\pi_0(H)-1}$, where $\pi_0(H)$ is the number of connected components of the graph H.

Let us explicitly indicate the first several terms of the asymptotic expression for the probability that a random graph in the Erdős–Rényi model is connected. For this purpose, calculate $P_k(\rho)$ for $k = 1, 2, 3$. Since there is the only one graph of size 1 which is connected and whose weight is 1, we have

$$P_1(\rho) = 1.$$

For the second polynomial, one has

$$P_2(\rho) = w\left(\; 1 \bullet\!\!-\!\!\bullet 2 \; \right) + w\left(\; 1\bullet \;\; \bullet 2 \; \right) = \rho - 1.$$

The same way, as it is seen from Fig. 1.1, there are eight graphs of size 3. Among them, one is connected of weight ρ^3, three are connected of weight ρ^2, three have two connected components and weight ρ, and one has three connected components and weight 1. Thus, we obtain

$$P_3(\rho) = \rho^3 + 3\rho^2 - 3\rho + 1.$$

Hence,

$$\mathbb{P}(G \text{ is connected}) = 1 - \binom{n}{1} q^{n-1} - (\rho - 1)\binom{n}{2} q^{2n-3} - (\rho^3 + 3\rho^2 - 3\rho + 1)\binom{n}{3} q^{3n-6} + O(n^4 q^{4n}).$$

In particular, we see the first two terms obtained by Gilbert [41].

As well as SEQ, combinatorial constructions CYC, SET, SEQ_m, CYC_m and SET_m can be translated into the language of the theory of species. Each of them corresponds to a composition with some species of structures. This fact permits us to state general asymptotic results that can be viewed as species analogues of Theorem 7.2.1, Theorem 5.3.1 and others. In these results, the coefficients involved in the asymptotic expansions have a combinatorial meaning as counting sequences of some virtual species depending on initial structures (Theorem 8.1.1, Theorem 8.1.2, Theorem 8.2.1, Theorem 8.2.2, Theorem 8.3.6).

The last but not the least result to mention concerns the asymptotic probability of strongly connected directed graphs. The first result for this probability was obtained by Wright [104]. Together with Sergey Dovgal [30], we get a significant simplification by imposing a structure on the coefficients.

Theorem 10.3.16. *The asymptotic probability that a random labeled directed graph D with n vertices is strongly connected satisfies*

$$\mathbb{P}\big(D \text{ is strongly connected}\big) \approx \sum_{k\geqslant 0} \mathfrak{ssd}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{n(n-1)/2}} \cdot \frac{\mathfrak{it}_{n-k}}{2^{nk}}, \tag{1.17}$$

where $\mathfrak{it}_k$ and $\mathfrak{ssd}_k$ are the numbers of irreducible tournaments and semi-strong directed graphs of size k, respectively.

The form of asymptotics (1.17) is rather simple, but it depends on the number of irreducible tournaments $\mathfrak{t}_{n-k}$. In case we want to get a decomposition by powers of two, we can further use asymptotics (1.14) that leads us to the following result.

Corollary 10.3.18. *The asymptotic probability that a random labeled directed graph D with n vertices is strongly connected satisfies*

$$\mathbb{P}\big(D \text{ is strong}\big) = \sum_{k=0}^{2r-1} \frac{2^{k(k+1)/2}}{2^{kn}} \sum_{\nu=0}^{[k/2]} \binom{n}{\nu, k-2\nu} \frac{\mathfrak{ssd}_\nu \beta_{k-2\nu}}{2^{\nu(k-\nu)}} + O\left(\frac{n^{2r}}{2^{2nr}}\right), \tag{1.18}$$

where $\mathfrak{ssd}_k$ is the number of semi-strong digraphs of size k,

$$\beta_k = \begin{cases} 1, & \text{if } k = 0, \\ -2\mathfrak{it}_k + \mathfrak{it}_k^{(2)}, & \text{if } k \neq 0. \end{cases}$$

and $(\mathfrak{it}_k)$ and $\left(\mathfrak{it}_k^{(2)}\right)$ are counting sequences for irreducible tournaments and tournaments with two irreducible parts, respectively.

The proof of Theorem 10.3.16 cannot be deduced from the above mentioned general theorems. To establish the result, we apply Bender's theorem directly, together with some additional techniques. In particular, we use an enumeration method for strong digraphs that was developed in 1973 by Robinson [90], and later, independently, by de Panafieu and Dovgal [29]. We can see that the asymptotic expansion (1.17) has a more complex form than we have seen before. In particular, there are three structures that interplay: strong digraphs, semi-strong digraphs and irreducible tournaments. It looks like there should be an interpretation of the above formula in terms of inclusion-exclusion principle, but we did not find one. This problem remains a challenge for future projects.

1.3 Outline of the thesis

While the manuscript is structured linearly around parts and chapters in a rather traditional flow that moves from tools to general theorem and then to applications, a transversal reading across chapters is also possible. This is possible by either considering a particular kind of objects that appear in all of these sections, but also a particular construction or a particular formalism. For instance, if we are interested in the asymptotics related to the construction SEQ, then it is sufficient to limit ourselves to Chapter 5 and Section 8.1 from Part II along with Chapter 9 and Section 10.2 from Part III. The same way, if we are curious about tournaments, we can go directly to Sections 5.2, 5.3 and 10.2, with the support of corresponding tools from Part I if necessary.

Let us describe the structure of the thesis. In Part I, we introduce all the tools necessary for our research. This introductory part is illustrative enough and filled with examples and details. It is intended to be reader-friendly and, even more, to serve for students who are new to the field to get acquainted with the material and techniques. Most of the material of Chapters 2–4 is well-known, and the reader familiar with it can skip the corresponding topics without loss of comprehension. The only exception concerns the concept of gargantuan sequences (as well as gargantuan combinatorial classes and gargantuan species based on it), which we define to simplify the presentation (see Sections 2.1, 3.1.15, 3.2.6 and 4.2.4).

Part II is devoted to the general results. In Chapters 5–7, we work with the symbolic method. First, we discuss, from a combinatorial point of view, the probabilities that a random object is irreducible in terms of decompositions with respect to combinatorial constructions SEQ, CYC and SET. For these probabilities, we establish empirically the asymptotic expansions accompanied by the combinatorial meaning of the involved coefficients. Here, it is the inclusion-exclusion principle that plays the key role for obtaining the combinatorial interpretations. Second, we prove the obtained result rigorously, via Bender's theorem. Third, we do the same for restricted constructions SEQ_m, CYC_m and SET_m. Finally, in Chapter 8, we generalize the obtained results for the theory of species.

In Part III, we discuss numerous applications. Chapter 9 is devoted to permutations and related objects. We establish the asymptotic expansions for the probabilities that a random permutation, perfect matching or multipermutation is indecomposable. Chapter 10 refers to different kinds of graphs. We show how to obtain the probabilities that a random graph is connected, a random tournament is irreducible, a random digraph is weakly or strongly connected etc. In particular, we do this for the Erdős–Rényi model $G(n, p)$. In Chapter 11, we discuss several model of surfaces. We study square-tiled surfaces, combinatorial maps and constellations and indicate their connections with permutations, perfect matchings and multipermutations, respectively. For all the mentioned models, we provide asymptotic expansions for the probabilities that a random surface is connected or consists of a fixed number of connected components.

We finish our work with Part IV, discussing possible directions for further research.

Part I

Tools

Chapter 2

Gargantuan sequences and Bender's theorem

In our investigation, we are interested in the asymptotic behavior of various combinatorial objects. The goal of this chapter is to introduce the reader to Bender's theorem [9] which is one of the key ingredients for studying this behavior. The conditions of Bender's theorem restrict the study to the combinatorial classes whose counting sequences grow sufficiently fast. In order to clarify the meaning of the words "sufficiently fast" and to simplify the presentation of the results of our research, we define *gargantuan sequences* and study their properties.

The chapter consists of two sections. Section 2.1 is devoted to gargantuan sequences and their properties. We define gargantuan and d-gargantuan sequences, establish several conditions for a sequence to be gargantuan and show how it works through numerous examples. This material is new, and represents the contribution of the author. In Section 2.2, we explain how to find the asymptotic behavior of composite gargantuan sequences with the help of Bender's theorem. We state different forms of Bender's theorem and provide numerous examples. This section is based on the paper of Bender [9], but the material is adapted for the purposes of our research. The presentation of the material of this chapter is self-contained and could serve as a student tutorial for those who wish to get acquainted with the topic.

2.1 Gargantuan sequences

Definition 2.1.1. We will call a sequence (a_n) *gargantuan*, if for any integer r the following two conditions hold:

$$\text{(i)} \quad \frac{a_{n-1}}{a_n} \to 0, \quad \text{as } n \to \infty; \qquad\qquad \text{(ii)} \quad \sum_{k=r}^{n-r} |a_k a_{n-k}| = O\big(a_{n-r}\big).$$

Example 2.1.2. The sequence $a_n = n!$ is gargantuan. Indeed, the first condition of Definition 2.1.1 obviously holds. To verify the second condition, note that $k!(n-k)! \geqslant (k+1)!(n-k-1)!$ as long as $k \leqslant (n-1)/2$. This observation implies that in the sum

$$\sum_{k=r}^{n-r} a_k a_{n-k} = \sum_{k=r}^{n-r} k!(n-k)!$$

the terms are decreasing for $k \leqslant (n-1)/2$ and increasing for $k \geqslant (n-1)/2$. Hence, we can replace all terms except the first and last by $(r+1)!(n-r-1)!$ without diminishing the sum:

$$\sum_{k=r}^{n-r} k!(n-k)! \leqslant 2r!(n-r)! + (n-2r+1)(r+1)!(n-r-1)! = O\big((n-r)!\big) = O(a_{n-r}).$$

Remark 2.1.3. Example 2.1.2 provides a way of checking whether a sequence is gargantuan. Similarly to $n!$, in many cases, a sequence (a_n) is positive and strongly increasing. For fixed integers n and $r < n/2$, define a finite sequence

$$x_k = a_k a_{n-k},$$

where $r \leqslant k \leqslant n-r$. Thus, the second condition of Definition 2.1.1 reads

$$\sum_{k=r}^{n-r} x_k = O\big(a_{n-r}\big).$$

Clearly, there is a symmetry $x_k = x_{n-k}$. Thus, if (x_k) is decreasing for $r \leqslant k < n/2$, then

$$\sum_{k=r}^{n-r} x_k \leqslant 2x_r + (n-2r-1)x_{r+1}.$$

Since $x_r = a_r a_{n-r}$, we have $x_r = O(a_{n-r})$. Hence, the condition (ii) of Definition 2.1.1 holds, as long as $nx_{r+1} = O(x_r)$. As a consequence, to verify that (a_n) is gargantuan, it is convenient to replace conditions (i) and (ii) of Definition 2.1.1 by two others that are stronger but easier to check. In a rigorous way, we state these conditions in Lemma 2.1.4.

Lemma 2.1.4. *If a sequence (a_n) satisfies the following two conditions*

(i)' $na_{n-1} = O(a_n)$,

(ii)' $x_k = |a_k a_{n-k}|$ *is decreasing for $k < n/2$ and for all but finitely many n,*

then (a_n) is gargantuan.

Proof. To check the first condition of Definition 2.1.1, it is sufficient to apply (i)'. To verify the second condition, note that the value of x_k is increasing for $k < n/2$ and decreasing otherwise. Hence, it follows from (i)' and (ii)' that

$$\sum_{k=r}^{n-r} |a_k a_{n-k}| = 2|a_r a_{n-r}| + \sum_{k=r+1}^{n-r-1} |a_k a_{n-k}| \leqslant 2|a_r a_{n-r}| + (n-2r-1)x_{r+1} = O(a_{n-r}).$$

□

Remark 2.1.5. Condition (i)' in Lemma 2.1.4 can be replaced by the weaker one:

$$n^{\alpha} a_{n-1} = O(a_n),$$

where $0 < \alpha < 1$. Indeed, let m be an integer such that $m\alpha \geqslant 1$. Then

$$na_{n-m} \leqslant n^{m\alpha} a_{n-m} = O\big(n^{(m-1)\alpha} a_{n-m+1}\big) = \ldots = O(a_n).$$

As a consequence, condition (ii) of Definition 2.1.1 holds, since

$$\sum_{k=r}^{n-r} a_k a_{n-k} \leqslant 2 \sum_{k=r}^{r+m-1} a_k a_{n-k} + \big(n-2r-2m-1\big) a_{r+m} a_{n-r-m} = O(a_{n-r}).$$

Remark 2.1.6. In order to check that (x_k) is decreasing for $k < n/2$, one may prove that the following equivalence takes place:

$$\frac{x_{k+1}}{x_k} \leqslant 1 \qquad \Leftrightarrow \qquad k+1 \leqslant n-k.$$

Example 2.1.7. Let us show that the sequence

$$a_n = \frac{2^{\binom{n}{2}}}{n!}$$

is gargantuan. In order to do so, apply Lemma 2.1.4. First of all, as $n \to \infty$, we have

$$\frac{na_{n-1}}{a_n} = n \cdot \frac{2^{\binom{n-1}{2}}}{(n-1)!} \cdot \frac{n!}{2^{\binom{n}{2}}} = \frac{n}{2^n} \to 0,$$

and hence, condition (i)' of Lemma 2.1.4 holds. Check condition (ii)', i.e. verify that the sequence

$$x_k = a_k a_{n-k}$$

is decreasing for $k < n/2$. For this goal, use Remark 2.1.6. In other words, consider

$$\frac{x_{k+1}}{x_k} = \frac{a_{k+1}a_{n-k-1}}{a_k a_{n-k}} = \frac{(n-k)}{(k+1)} \cdot \frac{2^{k+1}}{2^{n-k}}.$$

Since the function

$$f(x) = \frac{2^x}{x}$$

is increasing for large x, we have the following equivalences for large n:

$$\frac{x_{k+1}}{x_k} \leqslant 1 \quad \Leftrightarrow \quad \frac{2^{k+1}}{(k+1)} \leqslant \frac{2^{n-k}}{(n-k)} \quad \Leftrightarrow \quad k+1 \leqslant n-k.$$

As a consequence, (x_k) is decreasing for $k < n/2$, and, according to Lemma 2.1.4, (a_n) is gargantuan.

Lemma 2.1.8. *If (a_n) and (b_n) are two non-negative gargantuan sequences, then the sequence $(a_n b_n)$ is non-negative gargantuan as well.*

Proof. Let us check the conditions of Definition 2.1.1. The first condition trivially holds. The second condition reads

$$\sum_{k=r}^{n-r}(a_k b_k)(a_{n-k}b_{n-r}) \leqslant \left(\sum_{k=r}^{n-r} a_k a_{n-k}\right)\left(\sum_{k=r}^{n-r} b_k b_{n-k}\right) = O(a_{n-r})O(b_{n-r}) = O(a_{n-r}b_{n-r}).$$

The condition $a_n b_n \geqslant 0$ is trivial. □

Example 2.1.9. As we have seen in Examples 2.1.2 and 2.1.7, the sequences

$$a_n = n! \qquad \text{and} \qquad b_n = \frac{2^{\binom{n}{2}}}{n!}$$

are gargantuan. Hence, the sequences

$$c_n = a_n b_n = 2^{\binom{n}{2}}$$

and

$$d_n = c_n^2 = 2^{n(n-1)}$$

are gargantuan as well by Lemma 2.1.8.

Lemma 2.1.10. *If (a_n) is a gargantuan sequence and $b_n = b_0 c^n$, where $c \neq 0$, then the sequence $(a_n b_n)$ is gargantuan as well.*

Proof. Let us check the conditions of Definition 2.1.1. The first condition holds, since

$$\frac{a_{n-1}b_{n-1}}{a_n b_n} = \frac{1}{c} \cdot \frac{a_{n-1}}{a_n} \to 0,$$

as $n \to \infty$. The second condition reads

$$\sum_{k=r}^{n-r} \big|(a_k b_k)(a_{n-k} b_{n-r})\big| = b_0^2 |c|^n \sum_{k=r}^{n-r} |a_k a_{n-k}| = b_0^2 |c|^n \cdot O(a_{n-r}) = O(a_{n-r} b_{n-r}).$$

Therefore, $(a_n b_n)$ is gargantuan. □

Example 2.1.11. As we have seen in Example 2.1.9, the sequence

$$a_n = 2^{n(n-1)}$$

is gargantuan. Hence, the sequence

$$a_n \cdot 2^n = 2^{n^2}$$

is gargantuan as well by Lemma 2.1.10.

Lemma 2.1.12. *If (a_n) is a non-negative gargantuan sequence and $b_n = o(a_n)$, then, for any $K > 0$, the sequence $c_n = K a_n + b_n$ is gargantuan.*

Proof. Check the conditions of Definition 2.1.1. For the first condition, as $n \to \infty$, we have

$$\frac{c_{n-1}}{c_n} = \frac{K a_{n-1} + b_{n-1}}{K a_n + b_n} = \frac{a_{n-1}}{a_n} \left(1 + \frac{b_{n-1}}{K a_{n-1}}\right) \left(1 + \frac{b_n}{K a_n}\right)^{-1} \to 0.$$

To prove the second condition, note that there exists a constant $M > 0$ such that $|b_n| < M a_n$ for all $n \in \mathbb{N}$. Hence, $|c_n| \leqslant (K + M) a_n$ and we have

$$\sum_{k=r}^{n-r} |c_k c_{n-k}| \leqslant (K + M)^2 \sum_{k=r}^{n-r} a_k a_{n-k} = O(a_{n-r}) = O(c_{n-r}).$$

Since both conditions hold, the sequence (c_n) is gargantuan. □

Example 2.1.13. Let

$$a_n = 2^{n^2} \qquad \text{and} \qquad b_n = 2^n.$$

As we have seen in Example 2.1.11, the sequence (a_n) is gargantuan. Since $b_n = o(a_n)$, Lemma 2.1.12 implies that the sequence

$$a_n + b_n = 2^{n^2} + 2^n$$

is gargantuan as well.

Definition 2.1.14. Let $d \in \mathbb{N}$. We will call a sequence (a_n) *d-gargantuan*, if

$$a_n \neq 0, \qquad \Leftrightarrow \qquad n = dk \text{ for some } k \in \mathbb{Z}_{\geqslant 0}$$

and the subsequence a_{dk} is gargantuan.

Example 2.1.15. The sequence

$$a_n = \begin{cases} (n-1)!!, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

is 2-gargantuan. Indeed,

$$a_n \neq 0, \qquad \Leftrightarrow \qquad n = 2k \ \text{ for some } k \in \mathbb{Z}_{\geqslant 0},$$

meaning that the first condition of Definition 2.1.14 holds. Check that $a_{2n} = (2n-1)!!$ is gargantuan with the help of Lemma 2.1.4. The condition (i)' reads

$$na_{2n-2} = n(2n-3)!! = O\big((2n-1)!!\big) = O(a_{2n}).$$

As for the condition (ii)', consider the sequence

$$x_{2k} = a_{2k}a_{2n-2k}.$$

If $k \leqslant (n-1)/2$, then

$$\frac{x_{2k+2}}{x_{2k}} = \frac{(2k+1)!!(2n-2k-3)!!}{(2k-1)!!(2n-2k-1)!!} = \frac{2k+1}{2n-2k-1} \leqslant 1.$$

Hence, (x_{2k}) is decreasing for $k \leqslant (n-1)/2$. Thus, the conditions of Lemma 2.1.4 hold and (a_{2n}) is gargantuan.

2.2 Bender's theorem

In this section, we discuss our main tool for obtaining asymptotics. To simplify the presentation of the material, we widely use the following notation (which we have already seen in Section 1.2).

Notation 2.2.1. For a sequence (a_n) and an integer m, we write

$$a_n \approx \sum_{k \geqslant m} f_k(n),$$

if, for any integer $r \geqslant m+1$, one has

$$a_n = \sum_{k=m}^{r-1} f_k(n) + O\big(f_r(n)\big)$$

and the sequence (f_k) satisfies $f_{k+1}(n) = o\big(f_k(n)\big)$ for any integer k. In most of the cases, we have $m = 0$ or $m = 1$.

In order to give a motivation for Bender's theorem and to explain how one could figure it out, let us start with two examples.

Example 2.2.2. Let

$$A(z) = \sum_{n=1}^{\infty} a_n z^n$$

be a formal power series such that the sequence (a_n) is gargantuan. Suppose that we are interested in the asymptotic behavior of the coefficients of $B(z) = \big(A(z)\big)^2$. Clearly,

$$b_n = a_1 a_{n-1} + a_2 a_{n-2} + \ldots + a_{n-1} a_1,$$

and since (a_n) is gargantuan, for any fixed $r \in \mathbb{N}$ we have

$$b_n = 2a_1 a_{n-1} + 2a_2 a_{n-2} + \ldots + 2a_{r-1} a_{n-r+1} + O(a_{n-r}).$$

Note that in this example we can express $B(z)$ in terms of $A(z)$ as $B(z) = F\big(A(z)\big)$, where $F(y) = y^2$. It turns out that the coefficients of the asymptotics satisfy

$$b_n \approx \sum_{k \geqslant 0} c_k a_{n-k},$$

where

$$C(z) = 2A(z) = \left[\frac{\partial}{\partial y} F(y)\right]_{y=A(z)}.$$

Example 2.2.3. A natural generalization of Example 2.2.2 is provided by the function $F(y) = y^m$, where m is a positive integer. In this case, we consider a formal power series

$$A(z) = \sum_{n=1}^{\infty} a_n z^n,$$

whose sequence (a_n) is gargantuan, and we are interested in the asymptotic behavior of the formal power series

$$B(z) = F\big(A(z)\big) = \big(A(z)\big)^m.$$

Since we have

$$b_n = m a_1^{m-1} a_{n-m+1} + m(m-1) a_1^{m-2} a_2 a_{n-m} + \ldots,$$

one can guess that

$$b_n \approx \sum_{k \geqslant 0} c_k a_{n-k},$$

where

$$C(z) = \left[\frac{\partial}{\partial y} F(y)\right]_{y=A(z)} = m\big(A(z)\big)^{m-1}.$$

That is, indeed, the case. To verify this fact, we can use the following Lemma 2.2.4 proved by Bender in 1975 [9, Lemma 2].

Lemma 2.2.4. *Suppose that (a_n) is a gargantuan sequence such that $a_0 = 0$ and $a_n > 0$ for $n > 0$. For any fixed m and any vector $\mathbf{k} = (k_1, \ldots, k_m) \in \mathbb{Z}_{\geqslant 0}^m$, denote $a_{\mathbf{k}} = a_{k_1} \ldots a_{k_m}$. Then there exist constants $C > 0$ and $C_s > 0$ such that*

$$\sum_{K_{m,n}} a_{\mathbf{k}} \leqslant C^{m-1} a_{n-m+1}, \tag{2.1}$$

and for any fixed integer $s \in \mathbb{Z}_{\geqslant 0}$,

$$\sum_{K_{m,n,s}} a_{\mathbf{k}} \leqslant C_s^{m-1} a_{n-s}, \tag{2.2}$$

where the first sum runs over

$$K_{m,n} = \{\mathbf{k} = (k_1, \ldots, k_m) \mid k_1 + \ldots + k_m = n\},$$

while the second sum runs over

$$K_{m,n,s} = \{\mathbf{k} = (k_1, \ldots, k_m) \mid k_1 + \ldots + k_m = n,\ k_i \leqslant n - s\}.$$

Proof. Deduce (2.1) by induction on m. For $m = 1$ we have $a_n \leqslant a_n$, which is true. For $m = 2$ condition (2.1) reads

$$\sum_{j=1}^{n-1} a_j a_{n-j} \leqslant C a_{n-1},$$

which is true for sufficiently large $C \in \mathbb{R}$, since (a_n) is gargantuan. In general, we have

$$\begin{aligned}
\sum_{K_{m+1,n}} a_{\mathbf{k}} &= \sum_{j=1}^{n} a_j \sum_{K_{m,n-j}} a_{\mathbf{k}} \\
&\leqslant C^{m-1} \sum_{j=1}^{n} a_j a_{n-j-m+1} && \text{(by (2.1) for } m\text{)} \\
&\leqslant C^m a_{n-m} && \text{(by (2.1) for } m = 2\text{)}.
\end{aligned}$$

In a similar way, we establish (2.2). Clearly, it holds for $m = 1$. For $m = 2$, we have

$$\sum_{j=s}^{n-s} a_j a_{n-j} \leqslant C'_s a_{n-s}$$

for some C'_s, since (a_n) is gargantuan. In the general case, define

$$C_s = \max(C'_s, C + a_1 + \ldots + a_{s-1}).$$

Then, by induction on m,

$$\begin{aligned}
\sum_{K_{m+1,n,s}} a_{\mathbf{k}} &= \sum_{j=1}^{n-s} a_j \sum_{K_{m,n-j,s}} a_{\mathbf{k}} \\
&\leqslant C^{m-1} \sum_{j=s}^{n-s} a_j a_{n-j} + C_s^{m-1} \sum_{j=1}^{s-1} a_j a_{n-s} && \text{(by (2.1) and (2.2) for } m\text{)} \\
&\leqslant C^{m-1} C_s a_{n-s} + C_s^{m-1} (a_1 + \ldots + a_{s-1}) a_{n-s} && \text{(by (2.2) for } m = 2\text{)} \\
&\leqslant C_s a_{n-s}.
\end{aligned}$$

□

As a natural generalization of Example 2.2.3, one can obtain the following simple form of Bender's theorem [9, Theorem 2].

Theorem 2.2.5. *Consider the formal power series*

$$A(z) = \sum_{n=1}^{\infty} a_n z^n$$

and a function $F(y)$ which is analytic in some neighborhood of origin. Define

$$B(z) = \sum_{n=0}^{\infty} b_n z^n = F\big(A(z)\big) \qquad \text{and} \qquad C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial}{\partial y} F(y)\right]_{y=A(z)}.$$

Assume that the sequence (a_n) is gargantuan and $a_n \neq 0$ for any $n > 0$. Then

$$b_n \approx \sum_{k \geqslant 0} c_k a_{n-k}$$

and the sequence (b_n) is gargantuan.

Example 2.2.6. Let

$$A(z) = \sum_{n=1}^{\infty} a_n z^n$$

be a formal power series such that the sequence (a_n) is gargantuan. Suppose that we are interested in the asymptotic behavior of the coefficients of $B(z) = \exp(A(z)\big)$. Then, applying Theorem 2.2.5 for $F(y) = e^y$, we have

$$C(z) = \exp\big(A(z)\big) = B(z)$$

and

$$b_n \approx \sum_{k \geqslant 0} b_k a_{n-k}.$$

Example 2.2.7. Let

$$A(z) = \sum_{n=1}^{\infty} a_n z^n$$

be a formal power series such that the sequence (a_n) is gargantuan. If we study the asymptotic behavior of the coefficients of $B(z) = \log(1 + A(z)\big)$, then we need to apply Theorem 2.2.5 for the function $F(y) = \log(1 + y)$. In this case, we have

$$b_n \approx \sum_{k \geqslant 0} c_k a_{n-k},$$

where (c_k) are the coefficients of the formal power series

$$C(z) = \sum_{n=0}^{\infty} c_n z^n = \frac{1}{1 + A(z)}.$$

Let us also state more general versions of Bender's theorem [9, Theorem 2].

Theorem 2.2.8. *a) Consider the formal power series*

$$A(z) = \sum_{n=1}^{\infty} a_n z^n$$

and a function $F(x, y)$ *which is analytic in some neighborhood of* $(0; 0)$. *Define*

$$B(z) = \sum_{n=0}^{\infty} b_n z^n = F\bigl(z, A(z)\bigr) \qquad \text{and} \qquad C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial}{\partial y} F(x, y)\right]_{y=A(z)}.$$

Assume that the sequence (a_n) *is gargantuan and* $a_n \neq 0$ *for any* $n > 0$. *Then*

$$b_n \approx \sum_{k \geqslant 0} c_k a_{n-k}$$

and the sequence (b_n) *is gargantuan.*

b) Let

$$A(z) = \sum_{n=1}^{\infty} a_n z^n$$

be a formal power series such that the sequence (a_n) *is gargantuan and* $a_n \neq 0$ *for any* $n > 0$. *Suppose, that a function* $H(x, y)$ *converges in a neighborhood of the origin and, for some* $\lambda \in (0; 1)$, *the coefficients of the power series*

$$D(x) = \sum_{m=0}^{\infty} d_m z^m$$

satisfy $|d_n| = O(a_{[\lambda n]})$. *Then* $A(z)$ *and* $F(x, y) = H\bigl(D(x), y\bigr)$ *satisfy the conditions of Theorem 2.2.8 a).*

Chapter 3

The symbolic method

As we have already mentioned, we are interested in the asymptotic behavior of combinatorial objects. In the previous chapter, we have discussed the first tool necessary for studying asymptotics. The second ingredient of our investigation is a technique for counting combinatorial objects that allows us to translate the internal structure of the objects to formulas and vice versa. In this section, we introduce the symbolic method [4, 38] which is one of the simplest techniques of this kind.

The core principle of the symbolic method is to provide a translation scheme between combinatorial constructions and generating functions operations. Classes of combinatorial structures are created from simpler classes as from bricks, using elementary combinatorial constructions. The translation scheme allows us to express generating functions of compound combinatorial classes in terms of simpler ones. However, the converse is also true: if we have an expression that depends on the generating function of some combinatorial class, it is often possible to interpret this expression as a generating function of a more sophisticated combinatorial structure. The latter idea underlies the combinatorial interpretation of coefficients obtained using Bender's theorem.

Let us describe the structure of this chapter. Section 3.1 is devoted to labeled combinatorial structures and intended to be self-contained and filled with examples. Here, we mostly follow the book of Flajolet and Sedgewick [38], so that the reader familiar with it can skip the corresponding paragraphs without loss of understanding (see also [4]). In first twelve paragraphs, our only contribution is definitions of SEQ, CYC and SET-irreducible objects, which is done in order to simplify the presentation later. The last four paragraphs stand apart. In first of them, we define the Hadamard product of labeled combinatorial classes that happens to appear in different contexts later. Another one is devoted to graphic generating functions which is crucial for studying asymptotics of directed graphs (here, we follow papers of Robinson [90] and de Panafieu and Dovgal [29]). Finally, we end the exposition of labeled structures by defining gargantuan combinatorial classes and adapting Bender's theorem for them. The content of these two paragraphs is new and represents the author's contribution.

In Section 3.2, we outline unlabeled combinatorial structures. Here, we do not go deeply into details, limiting ourselves to an overview of concepts that differ from the labeled case, and some examples. The same way as in the first part, we follow [38], except the last paragraph, which is devoted to the adaptation of Bender's theorem to unlabeled combinatorial classes.

The last thing to mention before passing to the material is our notations. Usually, to designate a combinatorial class of some structures, we use the first letters of its name written in calligraphic in capital letters. For example, the combinatorial class of graphs is denoted by $\mathcal{G}$, while the combinatorial class of connected graphs is denoted by $\mathcal{CG}$. For designating the corresponding counting sequences, we use the same letters, but written in Gothic, say, $\mathfrak{cg}_n$ (the reason to use Gothic is to distinguish a single sequence denoted by several letters from the product of several items). Most of the notations are consistent with those of the book of Flajolet and Sedgewick [38]. Nevertheless, we allow ourselves to make some deviations. Thus, the combinatorial class of urns (or totally disconnected graphs) is denoted by $\mathcal{E}$ instead of $\mathcal{U}$. This is done to draw a parallel to a corresponding species of sets (see Chapter 4), and, at the same time, it is consistent with the fact that the exponential generating function of urns is the exponent e^z. The neutral combinatorial class is denoted by $\mathcal{N}$, rather than $\mathcal{E}$. Finally, the class of cyclic permutations is designated by $\mathcal{CP}$, which is consistent with our principle, but differs from the notation $\mathcal{C}$ of [38].

3.1 Labeled combinatorial structures

3.1.1 Combinatorial classes

Definition 3.1.1. A *combinatorial class* is a finite or countable set $\mathcal{A}$ equipped with a *size function*

$$|\cdot|\colon \mathcal{A} \to \mathbb{Z}_{\geqslant 0}$$

such that the number $\mathfrak{a}_n$ of elements of any given size n is finite. Each combinatorial class $\mathcal{A}$ is associated with its *counting sequence* $(\mathfrak{a}_n)$. For a given integer $n \in \mathbb{Z}_{\geqslant 0}$, we denote by $\mathcal{A}_n$ the subclass of $\mathcal{A}$ consisting of objects of size n.

Definition 3.1.2. Let $\mathcal{A}$ be a combinatorial class and $n \in \mathbb{Z}_{\geqslant 0}$. If $\mathfrak{a}_n \neq 0$, we suppose that the subclass $\mathcal{A}_n$ is endowed with the uniform probability. In other words, each object of size n appears with probability $1/\mathfrak{a}_n$. In the case when each object from $\mathcal{A}$ may satisfy some additional property Q, we write

$$\mathbb{P}\big(\text{a random object (from } \mathcal{A}\text{) satisfies } Q\big) \approx \sum_{k \geqslant m} f_k(n), \tag{3.1}$$

if

$$\mathbb{P}\big(\text{a random object from } \mathcal{A}_n \text{ satisfies } Q\big) \approx \sum_{k \geqslant m} f_k(n)$$

for some integer m and some sequence $\big(f_k\big)$ depending on n. We will call the right hand side of (3.1) *the asymptotic probability that a random object from $\mathcal{A}$ satisfies Q.*

Remark 3.1.3. In principle, one can consider more exquisite probability measures on $\mathcal{A}_n$ than uniform probability measures. However, for this purpose we will use the language of the theory of combinatorial species (see Remark 4.1.6 and Example 4.1.7).

Example 3.1.4. A *neutral* combinatorial class $\mathcal{N}$ consists of a single element of size 0. This object is often called *neutral* and denoted by ϵ.

Example 3.1.5. An *atomic* combinatorial class $\mathcal{Z}$ consists of a single element of size 1.

3.1.2 Labeled combinatorial classes

The rest of Section 3.1 is devoted to combinatorial classes that are referred to as *labeled* and comprised of labeled combinatorial objects. It is supposed that a *labeled object* of size n possesses n distinguishable atoms that bear labels from 1 to n (each atom bears its own label). For simplicity, the reader can think about labeled objects of size n as graphs whose set of vertices is $[n] = \{1, \ldots, n\}$.

Here, we use the word "graph" in broad sense. It should encompass many different structures, such as simple graphs, directed graphs, multigraphs, permutations, partitions, combinatorial maps, etc. In particular, we distinguish this broad sense from the one used in Example 3.1.14 below, where we refer to simple undirected graphs only.

In most of the cases, we deal with labeled combinatorial classes that satisfy the following additional *homogeneity property*: if a labeled object α belongs to a labeled combinatorial class $\mathcal{A}$, then any object α' obtained from α by permuting its labels belongs to $\mathcal{A}$ as well. However, in some cases, this natural property does not take place. What to do in such cases, we will discuss in Section 3.2.

Definition 3.1.6. The *exponential generating function* of the labeled combinatorial class $\mathcal{A}$ is the formal power series

$$A(z) = \sum_{n=0}^{\infty} \mathfrak{a}_n \frac{z^n}{n!}.$$

Example 3.1.7. The exponential generating functions of labeled combinatorial classes $\mathcal{N}$ and $\mathcal{Z}$ are

$$N(z) = 1 \qquad \text{and} \qquad Z(z) = z,$$

respectively.

Example 3.1.8. The labeled combinatorial class $\mathcal{E}$ of *totally disconnected graphs* comprises graphs with no edges. For each $n \in \mathbb{Z}_{\geqslant 0}$, there is a unique totally disconnected graph of size n (the order of labels does not matter, see Fig. 3.1). Hence, the exponential generating function of $\mathcal{E}$ is

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z.$$

The reader can think about the class $\mathcal{E}$ as the class of urns, where an urn of size n contains n different balls labeled by integers from 1 to n. Another useful point of view is to consider $\mathcal{E}$ as the class of sets of the form $[n]$, where $n \in \mathbb{Z}_{\geqslant 0}$.

Figure 3.1: Totally disconnected graph of size 4.

Example 3.1.9. Denote by $\mathcal{P}$ the labeled combinatorial class of *permutations*, that is, bijections of the form $[n] \to [n]$, where $n \in \mathbb{N}$. For a fixed integer n, the subclass $\mathcal{P}_n$ consists of permutations of size n, i.e. $\mathcal{P}_n = S_n$. The counting sequence of $\mathcal{P}$ is given by $\mathfrak{p}_n = n!$, and hence, its exponential generating function is

$$P(z) = \sum_{n=0}^{\infty} n! \frac{z^n}{n!} = \frac{1}{1-z}.$$

Given $n \in \mathbb{N}$, we can represent $\sigma \in S_n$ as a directed graph whose edges go from k to $\sigma(k)$, where $k \in [n]$ (see Fig. 3.2). This graph is a collection of cycles, possibly of size one (i.e. loops). Denote by $\mathcal{CP}$ the labeled combinatorial subclass of $\mathcal{P}$ consisting of *cyclic permutations*. Then each element in $\mathcal{CP}_n$ consists of a single cycle of maximal length. In particular, the counting sequence of $\mathcal{CP}$ is $\mathfrak{cp}_n = (n-1)!$ and its exponential generating function is

$$CP(z) = \sum_{n=0}^{\infty}(n-1)!\frac{z^n}{n!} = \log\frac{1}{1-z}.$$

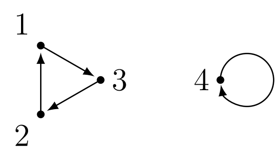


Figure 3.2: Labeled permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$ as a collection of cycles.

Example 3.1.10. Let us consider the labeled combinatorial class $\mathcal{L}$ of *linear orders*. A linear order $\prec = (l_1 \prec l_2 \prec \ldots \prec l_n)$ on the set $[n]$ can be represented as a linear directed graph whose edges go from l_{k+1} to l_k, where $k \in [n-1]$ (see Fig. 3.3). It is clear that the counting sequence of the class $\mathcal{L}$ satisfy $\mathfrak{l}_n = n!$. For any $n \in \mathbb{N}$, there is the following natural bijection:

$$\mathcal{L}_n \to S_n, \qquad (l_1 \prec l_2 \prec \ldots \prec l_n) \mapsto \begin{pmatrix} 1 & 2 & \ldots & n \\ l_1 & l_2 & \ldots & l_n \end{pmatrix}.$$

Figure 3.3: Linear order $(3 \prec 1 \prec 2 \prec 4)$ as a linear graph.

3.1.3 Combinatorially isomorphic classes

Definition 3.1.11. Two labeled combinatorial classes $\mathcal{A}$ and $\mathcal{B}$ are *combinatorially isomorphic* if their counting sequences are identical. In other words, there exists a bijection $\mathcal{A} \to \mathcal{B}$ which preserves size.

Notation 3.1.12. For combinatorially isomorphic combinatorial classes, we write $\mathcal{A} \cong \mathcal{B}$.

Example 3.1.13. It follows from Example 3.1.10, that labeled combinatorial classes $\mathcal{P}$ of permutations and $\mathcal{L}$ of linear orders are combinatorially isomorphic. However, this equality is somewhat artificial, because the underlying structures are different. All linear orders of the same size are isomorphic to each other, meaning that we can obtain one from another by changing labels of the underlying graphs. On the other hand, the isomorphism type of a permutation is determined by its decomposition into a product of cycles. In particular, even two permutations of size 2 are not isomorphic, since their decompositions are different: one of them is a cycle of size 2, and another is two cycles of size 1 each.

Example 3.1.14. Let $\mathcal{G}$ be the labeled combinatorial class of (simple undirected) *graphs*. The vertices of graphs of size n bear integer labels from 1 to n and any two vertices are either joined by an edge or not. Hence, the counting sequence $(\mathfrak{g}_n)$ of graphs satisfies

$$\mathfrak{g}_n = 2^{\binom{n}{2}}.$$

On the other hand, let $\mathcal{T}$ be the labeled combinatorial class of *tournaments*, i.e. complete directed graphs (each pair of vertices is joined by a directed edge). The same way as for the graphs, the vertices of a tournament of size n bear integer labels from 1 to n. Meanwhile, two vertices i and j are joined by an edge which goes either from i to j or in the opposite direction. Therefore, the counting sequence $(\mathfrak{t}_n)$ of tournaments satisfies

$$\mathfrak{t}_n = 2^{\binom{n}{2}},$$

meaning that $\mathfrak{t}_n = \mathfrak{g}_n$ and $\mathcal{T} \cong \mathcal{G}$. However, as well as in Example 3.1.13, we could call this equality artificial. Indeed, the two labeled tournaments of size 2 are isomorphic (by swapping the vertices), while the two labeled graphs are not.

Remark 3.1.15. Let $\mathcal{A}$ and $\mathcal{B}$ be two combinatorially isomorphic classes. It is natural to say that $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic*, if for any $n \in \mathbb{Z}_{\geqslant 0}$ there is a bijection $\mathcal{A}_n \to \mathcal{B}_n$ that is preserved under relabeling. We normally do not distinguish such classes and write simply $\mathcal{A} = \mathcal{B}$. The concepts of combinatorially isomorphic and isomorphic classes are much more developed within the framework of the theory of species (compare with Definitions 4.1.28 and 4.1.31).

3.1.4 Admissible constructions

Definition 3.1.16. An m-ary combinatorial construction Φ that associates a class $\mathcal{A}$ to a collection of classes $\mathcal{B}^{(1)}, \ldots, \mathcal{B}^{(m)}$ is *admissible* if and only if the counting sequence $(\mathfrak{a}_n)$ of $\mathcal{A}$ depends on the counting sequences $(\mathfrak{b}_n^{(1)}), \ldots, (\mathfrak{b}_n^{(m)})$ of $\mathcal{B}^{(1)}, \ldots, \mathcal{B}^{(m)}$ only.

Remark 3.1.17. The advantage of admissible constructions is that they can be directly translated into generating functions. In our investigation, we limit ourselves to considering unary and binary admissible combinatorial constructions (in other words, we have $m = 1$ or $m = 2$ in Definition 3.1.16).

3.1.5 Disjoint union

Definition 3.1.18. Suppose that labeled combinatorial classes $\mathcal{A}$ and $\mathcal{B}$ are disjoint: $\mathcal{A} \cap \mathcal{B} = \emptyset$. The *disjoint union* construction produces the class $\mathcal{C} = \mathcal{A} + \mathcal{B}$ with the consistent size function: for $\gamma \in \mathcal{C}$,

$$|\gamma|_{\mathcal{C}} = \begin{cases} |\gamma|_{\mathcal{A}}, & \text{if } \gamma \in \mathcal{A}, \\ |\gamma|_{\mathcal{B}}, & \text{if } \gamma \in \mathcal{B}. \end{cases}$$

In terms of counting sequences and exponential generating functions, one has

$$\mathfrak{c}_n = \mathfrak{a}_n + \mathfrak{b}_n \qquad \text{and} \qquad C(z) = A(z) + B(z).$$

Remark 3.1.19. The notation $\mathcal{A}+\mathcal{B}$ instead of $\mathcal{A}\cup\mathcal{B}$ is used to emphasize that the union of labeled combinatorial classes $\mathcal{A}$ and $\mathcal{B}$ is disjoint. The same way, we may write $\mathcal{B} = \mathcal{C} - \mathcal{A}$, meaning that $\mathcal{A} \subset \mathcal{C}$ and the labeled combinatorial class $\mathcal{B}$ consists of the elements of $\mathcal{C}$ that do not belong to $\mathcal{A}$.

Remark 3.1.20. If two labeled combinatorial classes $\mathcal{A}$ and $\mathcal{B}$ have non-empty intersection, then their union is not an admissible construction. Indeed, in the general case,

$$|\mathcal{A}_n \cup \mathcal{B}_n| = |\mathcal{A}_n| + |\mathcal{B}_n| - |\mathcal{A}_n \cap \mathcal{B}_n|,$$

meaning that the counting sequence of $\mathcal{A} \cup \mathcal{B}$ depends on the internal structures of initial classes, not only on their counting sequences. On the other hand, one can form the disjoint union construction for any pair of combinatorial classes $(\mathcal{A}, \mathcal{B})$ by taking

$$\mathcal{A} + \mathcal{B} = \big(\{1\} \times \mathcal{A}\big) \cup \big(\{2\} \times \mathcal{B}\big).$$

Here, technically, first we should rigorously define the labeled combinatorial classes $\{1\} \times \mathcal{A}$ and $\{2\} \times \mathcal{B}$. This can be done naturally: the second element in a pair $(1, \alpha) \in \{1\} \times \mathcal{A}$ inherit all the properties of $\alpha \in \mathcal{A}$ (such as the size, structure and labels), while the first element is considered as an additional label of the object.

3.1.6 Labeled product

Definition 3.1.21. Let $\mathcal{A}$ and $\mathcal{B}$ be two labeled combinatorial classes. The *labeled product* construction produces the labeled class $\mathcal{C} = \mathcal{A} \star \mathcal{B}$ obtained by considering all the ordered pairs from $\mathcal{A} \times \mathcal{B}$ and relabeling them in every possible order-consistent way. The size of an object $\gamma = (\alpha, \beta)$ is defined by

$$|\gamma|_{\mathcal{C}} = |\alpha|_{\mathcal{A}} + |\beta|_{\mathcal{B}}.$$

Remark 3.1.22. In contrast with Remark 3.1.20, the objects of the labeled product inherit the structures of both factors. The order-consistency condition means that if two atoms x and y of an object $\alpha \in \mathcal{A}$ or $\beta \in \mathcal{B}$ bear labels l_x and l_y such that $l_x < l_y$, then their labels l'_x and l'_y as atoms of $\gamma = (\alpha, \beta)$ possess the same relation: $l'_x < l'_y$. If objects $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ are of sizes k and $(n-k)$, respectively, then there are $\binom{n}{k}$ order-consistent relabelings of the pair $\gamma = (\alpha, \beta)$. Indeed, there are n atoms in γ, and hence, each of them bears a label from the set $[n]$. By choosing k labels for the atoms of the first element in the pair, we completely determine the labeling of γ due to the order-consistency condition. Thus, the counting sequence of the class $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ is given by the *binomial convolution formula*

$$\mathfrak{c}_n = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{a}_k \mathfrak{b}_{n-k}. \tag{3.2}$$

As a consequence, the exponential generating function of $\mathcal{C}$ is

$$C(z) = A(z) \cdot B(z).$$

Example 3.1.23. Let $\mathcal{A} = \mathcal{B} = \mathcal{E}$ be the labeled combinatorial class of sets of the form $[n]$, where $n \in \mathbb{Z}_{\geqslant 0}$. Then $\mathcal{A} \star \mathcal{B} = \mathcal{E}^2$ is the labeled combinatorial class of partitions of $[n]$ into two, possibly empty, parts (the order is important). Indeed, let us look at, say, $(\mathcal{A} \star \mathcal{B})_3$. There are four distinct objects of size 3 in the Cartesian product $\mathcal{A} \times \mathcal{B}$, namely,

$$(\{1,2,3\}, \emptyset), \qquad (\{1,2\}, \{1\}), \qquad (\{1\}, \{1,2\}), \qquad (\emptyset, \{1,2,3\})$$

The first and last pairs are already well-labeled (meaning that their elements bear labels 1, 2 and 3 with no repetition), so they are elements of $\mathcal{A} \star \mathcal{B}$ as well. For the pair $(\{1,2\}, \{1\})$, there are three different order-consistent relabelings:

$$(\{1,2\}, \{3\}), \qquad (\{1,3\}, \{2\}) \qquad \text{and} \qquad (\{2,3\}, \{1\}).$$

The same way, there are three order-consistent relabelings for the pair $(\{1\}, \{1,2\})$. Eventually, we have eight objects in $(\mathcal{A} \star \mathcal{B})_3$ that we can consider as partitions of the set $\{1,2,3\}$ into two ordered subsets. (see Fig. 3.4). In the general case, for an arbitrary $n \in \mathbb{Z}_{\geqslant 0}$, there are 2^n objects in $(\mathcal{A} \star \mathcal{B})_n$. Indeed, for a pair $\big([k], [n-k]\big)$, there are $\binom{n}{k}$ order-consistent relabelings, and

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Note that it is consistent with the fact that $A(z) \cdot B(z) = e^z \cdot e^z = e^{2z}$.

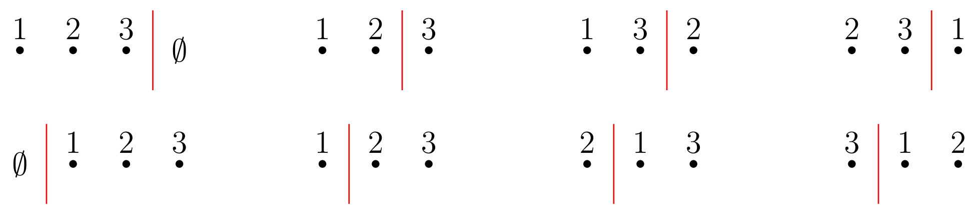


Figure 3.4: Objects of size 3 in $\mathcal{E}^2$.

Remark 3.1.24. Binomial convolution formula (3.2) is the reason why, dealing with labeled combinatorial classes, we need to work with exponential generating functions rather than ordinary ones of the form

$$\sum_{n=0}^{\infty} \mathfrak{a}_n z^n.$$

To understand how we could guess the form of an appropriate generating function, let us consider a generating function of the indefinite form

$$A(z) = \sum_{n=0}^{\infty} \mathfrak{a}_n \psi(n) z^n,$$

where ψ is a function to determine. In the general case, the product $C(z) = A(z) \cdot B(z)$ gives us the number $\mathfrak{c}_n$ of pairs of size n as

$$\mathfrak{c}_n = \frac{1}{\psi(n)} \sum_{k=0}^{n} \mathfrak{a}_k \mathfrak{b}_{n-k} \psi(k) \psi(n-k).$$

Since we wish $\mathfrak{c}_n$ to be consistent with formula (3.2), it is sufficient that

$$\frac{\psi(k)\psi(n-k)}{\psi(n)} = \binom{n}{k}.$$

In our case, we can put $\psi(n) = 1/n!$. Indeed, for $k = 1$, we have

$$\frac{\psi(1)\psi(n-1)}{\psi(n)} = n,$$

and hence,

$$\psi(n) = n\psi(1)\psi(n-1) = n!\big(\psi(1)\big)^n.$$

Putting $\psi(1) = 1$, one has

$$\psi(n) = \frac{1}{n!},$$

and direct calculations show that this function works well with binomial convolution (3.2).

3.1.7 Sequences

Definition 3.1.25. Let m be a non-negative integer and $\mathcal{A}$ be a labeled combinatorial class. In this case, the *m-sequence* construction SEQ_m produces all possible m-tuples of labeled combinatorial objects from $\mathcal{A}$:

$$\mathrm{SEQ}_m(\mathcal{A}) = \mathcal{A} \star \mathcal{A} \star \ldots \star \mathcal{A}.$$

Due to Definition 3.1.21, the size of an element $\beta = (\alpha_1, \ldots, \alpha_m)$ is defined by

$$|\beta| = |\alpha_1| + \ldots + |\alpha_m|.$$

Remark 3.1.26. The particular cases of m-sequences for $m = 0$ and $m = 1$ are

$$\mathrm{SEQ}_0(\mathcal{A}) = \mathcal{N} = \{\epsilon\} \qquad \text{and} \qquad \mathrm{SEQ}_1(\mathcal{A}) = \mathcal{A}.$$

Note that we do not distinguish classes $\mathcal{A}$ and $\mathrm{SEQ}_1(\mathcal{A})$, identifying objects $\alpha \in \mathcal{A}$ and sequences $(\alpha) \in \mathrm{SEQ}_1(\mathcal{A})$ comprising of one object (see also Remark 3.1.15).

Lemma 3.1.27. *If $\mathcal{A}$ and $\mathcal{B}$ are labeled combinatorial classes such that $\mathcal{B} = \mathrm{SEQ}_m(\mathcal{A})$, then their exponential generating functions satisfy*

$$B(z) = \big(A(z)\big)^m.$$

Proof. Follows directly from the binomial convolution rule (3.2) for the labeled product construction. □

Definition 3.1.28. Let $\mathcal{A}$ be a labeled combinatorial class that contains no object of size 0. The *sequence* construction generates the disjoint union of all the possible tuples of labeled combinatorial objects from $\mathcal{A}$:

$$\mathrm{SEQ}(\mathcal{A}) = \sum_{m=0}^{\infty} \mathrm{SEQ}_m(\mathcal{A}).$$

Lemma 3.1.29. *If $\mathcal{A}$ and $\mathcal{B}$ are labeled combinatorial classes such that $\mathcal{B} = \mathrm{SEQ}(\mathcal{A})$, then their exponential generating functions satisfy*

$$B(z) = \frac{1}{1 - A(z)}.$$

Proof. Follows directly from the definition of the sequence construction and Lemma 3.1.27, since

$$B(z) = \sum_{m=0}^{\infty} \big(A(z)\big)^m = \frac{1}{1 - A(z)}.$$

□

Remark 3.1.30. If a labeled combinatorial class $\mathcal{A}$ contains objects of size 0, then the sequence construction is not admissible. Indeed, for this case, even the number of objects of size 0 in $\mathrm{SEQ}(\mathcal{A})$ in not finite, since it is equal to

$$1 + \mathfrak{a}_0 + \mathfrak{a}_0^2 + \mathfrak{a}_0^3 + \ldots$$

and $\mathfrak{a}_0 \neq 0$. We will see in the next paragraphs that there are two more constructions with the same restriction, namely, cycles CYC and sets SET. To apply any of above-mentioned constructions to a class $\mathcal{A}$, we need to demand that $\mathfrak{a}_0 = 0$. Otherwise, the construction is not admissible.

Definition 3.1.31. Let $\mathcal{B} = \mathrm{SEQ}(\mathcal{A})$. Each object from $\mathrm{SEQ}_1(\mathcal{A}) \cap \mathcal{B}$ will be called SEQ-*irreducible*. Also, we will say that an object $b \in \mathrm{SEQ}_m(\mathcal{A}) \cap \mathcal{B}$ has exactly m SEQ-*irreducible components*.

Example 3.1.32. It follows directly from Example 3.1.10 that a linear order is a sequence of atoms. In other words,

$$\mathcal{L} = \mathrm{SEQ}(\mathcal{Z}).$$

Consequently, a linear order is SEQ-irreducible, if it consists of one atom, i.e. it is the trivial: (1).

Example 3.1.33. A tournament is *irreducible*, if for any partition $A \sqcup B$ of the set of its vertices there exist an edge from A to B and an edge from B to A (equivalently, a tournament is irreducible if and only if it is strongly connected, see [87, 91]). Denote by $\mathcal{IT}$ the labeled combinatorial class of irreducible tournaments. Then we have

$$\mathcal{T} = \mathrm{SEQ}(\mathcal{IT}).$$

Indeed, if a tournament T is not irreducible, then it consists of two nonempty subtournaments A and B, such that any edge between A and B is directed from A to B. Each of A and B is either irreducible or can be decomposed the same way further. Applying the induction, we obtain a decomposition of T into a sequence of irreducible subtournaments $T_1, \ldots, T_k$, such that for every pair $i < j$ all edges go from T_i to T_j (Fig. 3.5). This decomposition is unique, since the subtournaments T_i are the strongly connected components of the initial tournament T.

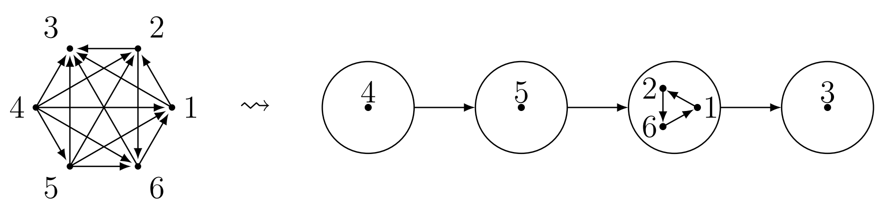


Figure 3.5: Decomposition of a tournament into a sequence of irreducible components.

3.1.8 Cycles

Definition 3.1.34. Let m be a non-negative integer and $\mathcal{A}$ be a labeled combinatorial class. In this case, the *m-cycle* construction CYC_m produces all possible m-cycles of labeled combinatorial objects from $\mathcal{A}$.

Remark 3.1.35. Cycles are tuples taken up to a circular shift. Rigorously, one can define m-cycle construction as

$$\mathrm{CYC}_m(\mathcal{A}) = \mathrm{SEQ}_m(\mathcal{A})/S,$$

where S is the equivalence relation between m-sequences such that

$$(\alpha_1, \ldots, \alpha_m) S (\alpha'_1, \ldots, \alpha'_m)$$

if and only if there exists a cyclic permutation τ such that $\alpha'_j = \alpha_{\tau(j)}$ for all j.

Lemma 3.1.36. *If $\mathcal{A}$ and $\mathcal{B}$ are labeled combinatorial classes such that $\mathcal{B} = \mathrm{CYC}_m(\mathcal{A})$, then their exponential generating functions satisfy*

$$B(z) = \frac{1}{m}\big(A(z)\big)^m.$$

Proof. Follows directly from Lemma 3.1.27 and the fact that each m-cycle corresponds to m different m-sequences. □

Definition 3.1.37. Let $\mathcal{A}$ be a labeled combinatorial class that contains no object of size 0. The *cycle* construction generates the disjoint union of all the possible cycles of labeled combinatorial objects from $\mathcal{A}$:

$$\mathrm{CYC}(\mathcal{A}) = \sum_{m=1}^{\infty} \mathrm{CYC}_m(\mathcal{A}).$$

Lemma 3.1.38. *If $\mathcal{A}$ and $\mathcal{B}$ are labeled combinatorial classes such that $\mathcal{B} = \mathrm{CYC}(\mathcal{A})$, then their exponential generating functions satisfy*

$$B(z) = \log \frac{1}{1 - A(z)}.$$

Proof. Follows directly from the definition of the cycle construction and Lemma 3.1.36, since

$$B(z) = \sum_{m=1}^{\infty} \frac{\big(A(z)\big)^m}{m} = \log \frac{1}{1 - A(z)}.$$

□

Definition 3.1.39. Let $\mathcal{B} = \mathrm{CYC}(\mathcal{A})$. Each object from $\mathrm{CYC}_1(\mathcal{A}) \cap \mathcal{B}$ will be called CYC-*irreducible*. Also, we will say that an object $b \in \mathrm{CYC}_m(\mathcal{A}) \cap \mathcal{B}$ has exactly m CYC-*irreducible components*.

Example 3.1.40. Consider the labeled combinatorial class $\mathcal{CP}$ of cyclic permutations introduced in Example 3.1.9. Since any cyclic permutation can be considered as a cycle of atoms, we have

$$\mathcal{CP} = \mathrm{CYC}(\mathcal{Z}).$$

3.1.9 Sets

Definition 3.1.41. Let m be a non-negative integer and $\mathcal{A}$ be a labeled combinatorial class. The *m-set* construction SET_m produces all possible m-sets of labeled combinatorial objects from $\mathcal{A}$.

Remark 3.1.42. Sets are tuples taken up to a permutation. In other words, m-set construction generates the labeled combinatorial class

$$\mathrm{SET}_m(\mathcal{A}) = \mathrm{SEQ}_m(\mathcal{A})/R,$$

where R is the equivalence relation between m-sequences such that

$$(\alpha_1, \ldots, \alpha_m) R (\alpha'_1, \ldots, \alpha'_m)$$

if and only if there exists a permutation σ such that $\alpha'_j = \alpha_{\sigma(j)}$ for all j.

Lemma 3.1.43. *If $\mathcal{A}$ and $\mathcal{B}$ are labeled combinatorial classes such that $\mathcal{B} = \mathrm{SET}_m(\mathcal{A})$, then their exponential generating functions satisfy*

$$B(z) = \frac{1}{m!}\big(A(z)\big)^m.$$

Proof. Follows directly from Lemma 3.1.27 and the fact that each m-set corresponds to $m!$ different m-sequences. □

Definition 3.1.44. Let $\mathcal{A}$ be a labeled combinatorial class that contains no object of size 0. The *set* construction applied to $\mathcal{A}$ generates a new labeled combinatorial class

$$\mathrm{SET}(\mathcal{A}) = \sum_{m=0}^{\infty} \mathrm{SET}_m(\mathcal{A}).$$

Lemma 3.1.45. *If $\mathcal{A}$ and $\mathcal{B}$ are labeled combinatorial classes such that $\mathcal{B} = \mathrm{SET}(\mathcal{A})$, then their exponential generating functions satisfy*

$$B(z) = \exp\big(A(z)\big).$$

Proof. Follows directly from the definition of the set construction and Lemma 3.1.43, since

$$B(z) = \sum_{m=0}^{\infty} \frac{\big(A(z)\big)^m}{m!} = \exp\big(A(z)\big).$$

□

Definition 3.1.46. Let $\mathcal{B} = \mathrm{SET}(\mathcal{A})$. Each object from $\mathrm{SET}_1(\mathcal{A}) \cap \mathcal{B}$ will be called SET-*irreducible*. Also, we will say that an object $b \in \mathrm{SET}_m(\mathcal{A}) \cap \mathcal{B}$ has exactly m SET-*irreducible components*.

Example 3.1.47. As we have seen in Example 3.1.9, any permutation is a collection of cyclic permutations. This fact can be represented by the relation

$$\mathcal{P} = \mathrm{SET}(\mathcal{CP}).$$

At the same time, it has been discussed in Example 3.1.40 that $\mathcal{CP} = \mathrm{CYC}(\mathcal{Z})$. As a consequence,

$$\mathcal{P} = \mathrm{SET}(\mathrm{CYC}(\mathcal{Z})).$$

The SET-irreducible components of a permutation are cycles, and hence, a permutation is SET-irreducible if and only if it consists of one cycle only.

Remark 3.1.48. It follows from Examples 3.1.13, 3.1.32 and 3.1.47 that

$$\mathrm{SEQ}(\mathcal{Z}) = \mathcal{L} \cong \mathcal{P} = \mathrm{SET}(\mathrm{CYC}(\mathcal{Z})).$$

This particular case reflects the general rule: for any labeled combinatorial class $\mathcal{A}$, the combinatorial constructions SEQ and SET(CYC) produce combinatorial isomorphic classes:

$$\mathrm{SEQ}(\mathcal{A}) \cong \mathrm{SET}(\mathrm{CYC}(\mathcal{A})).$$

Note that, in terms of the corresponding exponential generating functions, the above relation translates into

$$\frac{1}{1 - A(z)} = \exp\left(\log \frac{1}{1 - A(z)}\right).$$

Example 3.1.49. Let $\mathcal{G}$ and $\mathcal{CG}$ be the labeled combinatorial classes of graphs and connected graphs, respectively. Since any graph is an unordered collection of its connected components, we have

$$\mathcal{G} = \mathrm{SET}(\mathcal{CG}).$$

In terms of Definition 3.1.46, the SET-irreducible components of a graph are its connected components. In particular, a graph is SET-irreducible if and only if it is connected.

Example 3.1.50. Similarly to Example 3.1.49, we can consider the subclass $\mathcal{TR} \subset \mathcal{CG}$ of trees, that is, connected graphs without cycles. Applying the construction SET to this class,

$$\mathcal{F} = \mathrm{SET}(\mathcal{TR}),$$

we obtain the labeled combinatorial class $\mathcal{F}$ of forests, i.e. graphs without cycles.

3.1.10 Restricted constructions

Definition 3.1.51. Let $\mathcal{A}$ be a labeled combinatorial class and Ω be a predicate over the non-negative integers. In this case, we define $\mathrm{SEQ}_\Omega(\mathcal{A})$ to be the labeled combinatorial class of sequences whose number of elements satisfies Ω. In other words,

$$\mathrm{SEQ}_\Omega(\mathcal{A}) = \sum_{m \text{ satisfies } \Omega} \mathrm{SEQ}_m \mathcal{A}.$$

Example 3.1.52. If $\mathcal{A}$ is a labeled combinatorial class and k be a positive integer, then the notation

$$\mathrm{SEQ}_{>k}(\mathcal{A})$$

is reserved for the sequences whose number of components is larger than k. Similarly,

$$\mathrm{SEQ}_{\text{odd}}(\mathcal{A}) \qquad \text{and} \qquad \mathrm{SEQ}_{\text{even}}(\mathcal{A})$$

refer to sequences with odd and even number of components, respectively.

Remark 3.1.53. The same way, for a labeled combinatorial class $\mathcal{A}$ and a predicate Ω over the non-negative integers, we define

$$\mathrm{CYC}_\Omega(\mathcal{A}), \qquad \text{and} \qquad \mathrm{SET}_\Omega(\mathcal{A}).$$

3.1.11 Pointing

Definition 3.1.54. Let $\mathcal{A}$ be a labeled combinatorial class. The *pointing* construction generates a new labeled combinatorial class

$$\Theta\mathcal{A} = \sum_{n=0}^{\infty} \mathcal{A}_n \times \{1, \dots n\}.$$

In other words, any object of $\Theta\mathcal{A}$ is an object from $\mathcal{A}$ such that one of its labels is pointed.

Remark 3.1.55. The intuitive interpretation of pointing is that for any object $\alpha \in \mathcal{A}_n$ we point at one of its atoms. Since there are n possible choices to distinguish an atom of α, the counting sequence of the labeled combinatorial class $\mathcal{B} = \Theta\mathcal{A}$ satisfies

$$\mathfrak{b}_n = n \cdot \mathfrak{a}_n,$$

while its exponential generating function is determined by

$$B(z) = z\frac{d}{dz}A(z).$$

Example 3.1.56. If $\mathcal{A}$ is a labeled combinatorial class, then one has

$$\Theta\mathrm{CYC}(\mathcal{A}) \cong \Theta\mathcal{A} \star \mathrm{SEQ}(\mathcal{A}).$$

Indeed, let a pointed cycle consist of objects $\alpha_0, \ldots, \alpha_l \in \mathcal{A}$, where α_0 contain the pointed atom. Extracting α_0 from the cycle, we obtain the pair $\big(\alpha_0, (\alpha_1, \ldots, \alpha_l)\big)$, whose first element is a pointed object from $\mathcal{A}$ and second element belongs to $\mathrm{SEQ}(\mathcal{A})$ (Fig. 3.6). Another way to establish the desired combinatorial isomorphism is to verify the equality of the corresponding exponential generating functions,

$$z\frac{d}{dz}\ln\frac{1}{1-A(z)} = \left(z\frac{d}{dz}A(z)\right)\frac{1}{1-A(z)},$$

which can be verified by direct calculations.

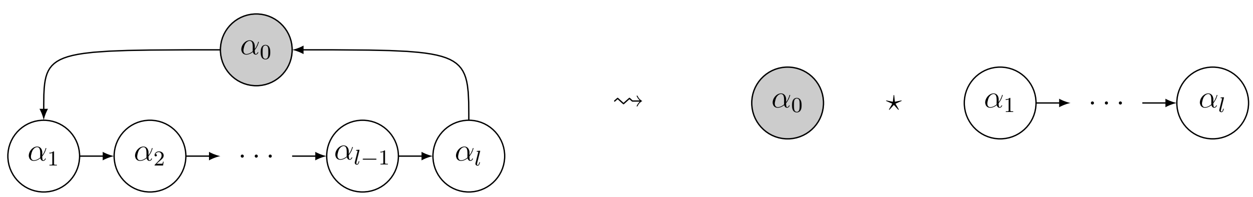


Figure 3.6: Bijection $\Theta\mathrm{CYC}(\mathcal{A}) \to \Theta\mathcal{A} \star \mathrm{SEQ}(\mathcal{A})$.

One more relation of such a type would be

$$\Theta\mathrm{CYC}_{>1}(\mathcal{A}) \cong \mathcal{A} \star \Theta\mathrm{CYC}(\mathcal{A}).$$

Indeed, let a pointed cycle consist of at least two objects $\alpha_0, \ldots, \alpha_l \in \mathcal{A}$, where α_0 contain the pointed atom. Now we extract α_1 from the cycle, so that the rest is the cycle $(\alpha_0, \alpha_2, \alpha_3, \ldots, \alpha_l)$ with the pointed element α_0 (Fig. 3.7). In terms of exponential generating functions, the isomorphism corresponds to the equality

$$z\frac{d}{dz}\left(\ln\frac{1}{1-A(z)} - A(z)\right) = \left(z\frac{d}{dz}A(z)\right)\frac{A(z)}{1-A(z)} = A(z)\left(z\frac{d}{dz}\ln\frac{1}{1-A(z)}\right).$$

For a generalization for $\Theta\mathrm{CYC}_{>m}(\mathcal{A})$, see Lemma 6.1.3.

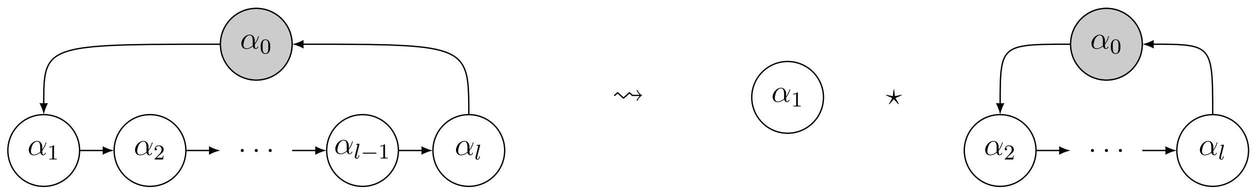


Figure 3.7: Bijection $\Theta\mathrm{CYC}_{>1}(\mathcal{A}) \to \mathcal{A} \star \Theta\mathrm{CYC}(\mathcal{A})$.

3.1.12 Substitution

Definition 3.1.57. Let $\mathcal{A}$ and $\mathcal{B}$ be two labeled combinatorial classes such that $\mathcal{B}$ contains no object of size 0. The *substitution* of $\mathcal{B}$ into $\mathcal{A}$ (or *composition* of $\mathcal{A}$ and $\mathcal{B}$) is a labeled combinatorial class defined by

$$\mathcal{A} \circ \mathcal{B} = \sum_{m=0}^{\infty} \mathcal{A}_m \times \mathrm{SET}_m(\mathcal{B}).$$

Remark 3.1.58. We can consider each element of $\mathcal{A} \circ \mathcal{B}$ as an object $\alpha \in \mathcal{A}$ whose atoms are replaced by arbitrary objects of $\mathcal{B}$. However, given an m-set $\beta \in \mathrm{SET}_m(\mathcal{B})$, it is *a priori* unclear what part of β should replace a node labeled by value $r \in \{1, \ldots, n\}$. One of possible methods of determining this part is the following. First, establish the "leader" of each part, that is, the value of its smallest label. Second, reorder the parts of β by the value of their "leader". Finally, substitute the part with "leader" of rank r to the node labeled by value r.

Remark 3.1.59. It follows directly from the definition of the substitution construction that if $\mathcal{C} = \mathcal{A} \circ \mathcal{B}$, then

$$C(z) = \sum_{m=0}^{\infty} \mathfrak{a}_m \cdot \frac{\big(B(z)\big)^m}{m!} = A\big(B(z)\big).$$

Example 3.1.60. The standard constructions SEQ, CYC and SET can be expressed in terms of substitution. Indeed, for any labeled combinatorial class $\mathcal{A}$ containing no object of size 0, one has

$$\mathrm{SEQ}(\mathcal{A}) \cong \mathcal{L} \circ \mathcal{A}, \qquad \mathrm{CYC}(\mathcal{A}) \cong \mathcal{CP} \circ \mathcal{A}, \qquad \mathrm{SET}(\mathcal{A}) \cong \mathcal{E} \circ \mathcal{A}.$$

The analogues of these representations will be widely used when applying theory of species (see also the introduction to Chapter 8).

3.1.13 Hadamard product

Definition 3.1.61. Let $A(z)$ and $B(z)$ be two exponential generating functions,

$$A(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \qquad \text{and} \qquad B(z) = \sum_{n=0}^{\infty} b_n \frac{z^n}{n!}.$$

Their *exponential Hadamard product* is the formal power series

$$A(z) \odot B(z) = \sum_{n=0}^{\infty} a_n b_n \frac{z^n}{n!}.$$

Definition 3.1.62. Let $\mathcal{A}$ and $\mathcal{B}$ be two labeled combinatorial classes. Define their *Hadamard product* as a new labeled combinatorial class $\mathcal{C} = \mathcal{A} \odot \mathcal{B}$ comprised by the pairs of the same size:

$$\mathcal{A} \odot \mathcal{B} = \{(\alpha, \beta) \in \mathcal{A} \times \mathcal{B} \mid |\alpha|_{\mathcal{A}} = |\beta|_{\mathcal{B}}\}.$$

The size function of an object $\gamma = (\alpha, \beta)$ equals to the size of its components:

$$|\gamma|_{\mathcal{C}} = |\alpha|_{\mathcal{A}} = |\beta|_{\mathcal{B}}.$$

As a consequence, the counting sequence $(\mathfrak{c}_n)$ of the Hadamard product $\mathcal{C} = \mathcal{A} \odot \mathcal{B}$ satisfy

$$\mathfrak{c}_n = \mathfrak{a}_n \cdot \mathfrak{b}_n,$$

and the following relation for exponential generating functions holds:

$$C(z) = A(z) \odot B(z).$$

Remark 3.1.63. The reader can think of an object $\gamma = (\alpha, \beta) \in \mathcal{A} \odot \mathcal{B}$ as a labeled graph whose edges bear two colors, say, red and blue. The set of red edges, together with all vertices, represent the graph α, while the set of blue edges give rise to the graph β.

Example 3.1.64. Some elements of the labeled combinatorial classes $\mathcal{G} \odot \mathcal{G}$ and $\mathcal{T} \odot \mathcal{T}$ are depicted on Fig. 3.8. We denote these classes by $\mathcal{MG}(2)$ and $\mathcal{MT}(2)$ and call the objects belonging to them *multigraphs* and *multitournaments* (of rank 2), respectively.

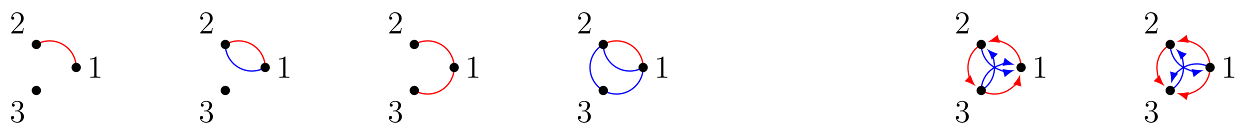


Figure 3.8: Some elements of the labeled combinatorial classes $\mathcal{G} \odot \mathcal{G}$ and $\mathcal{T} \odot \mathcal{T}$.

Multigraphs possess a naturally defined notion of connectivity, as well as multitournaments have a notion of irreducibility. Similarly to Examples 3.1.49 and 3.1.33, we have

$$\mathcal{MG}(2) = \mathrm{SET}\big(\mathcal{CMG}(2)\big) \qquad \text{and} \qquad \mathcal{MT}(2) = \mathrm{SEQ}\big(\mathcal{IMT}(2)\big), \tag{3.3}$$

where $\mathcal{CMG}(2)$ and $\mathcal{IMT}(2)$ are the labeled combinatorial classes of connected multigraphs and irreducible multitournaments, respectively.

Example 3.1.65. Let $\mathcal{ML}(2) = \mathcal{L} \odot \mathcal{L}$ be the labeled combinatorial class of pairs of linear orders of the same size. This class admits a natural SEQ-decomposition. Indeed, if we consider an object $L \in \mathcal{ML}(2)$ as a multitournament, then the desired decomposition can be seen as a particular case of the second identity of (3.3) (see Fig. 3.9).

We can state the condition of irreducibility of L directly (compare with the condition for a tournament to be irreducible, see Example 3.1.33). Let

$$L = \Big((l_1 \prec_1 \ldots \prec_1 l_n), (l'_1 \prec_2 \ldots \prec_2 l'_n)\Big)$$

be an object of size n. In this case, L is reducible, if there exists a partition $[n] = A \sqcup B$ such that every pair of elements $a \in A$ and $b \in B$ satisfy

$$a \prec_1 b \qquad \text{and} \qquad a \prec_2 b.$$

Otherwise, L is referred to as irreducible.

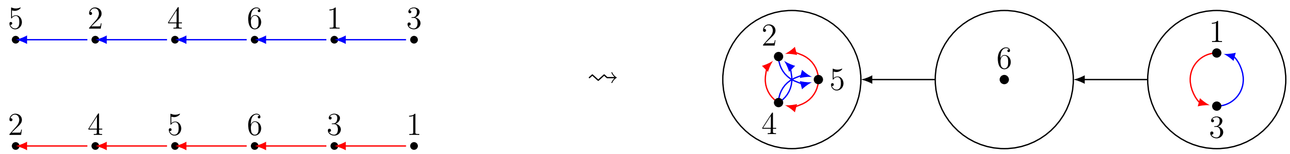


Figure 3.9: Decomposition of a pair of linear orders into a sequence of irreducibles.

3.1.14 Graphic generating functions

Definition 3.1.66. Let $\mathcal{A}$ be a labeled combinatorial class. Its *graphic generating function* is

$$\widehat{A}(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!\, 2^{\binom{n}{2}}}.$$

Notation 3.1.67. We denote by Δ the operator converting exponential generating function of a labeled combinatorial class $\mathcal{A}$ into its graphic generating function:

$$\Delta A(z) = \widehat{A}(z).$$

Remark 3.1.68. In the literature, graphic generating functions are also called *special generating function* [90] or *chromatic generating function* [94].

Remark 3.1.69. Graphic generating functions are widely used for different kinds of directed graphs. They are based on the idea of *strong connectivity*: a directed graph is strongly connected if for any two vertices there is a directed path from the first vertex to the second one. Each directed graph can be decomposed into a collection of strongly connected components. This leads us to the convolution rule

$$\mathfrak{c}_n = \sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)} \mathfrak{a}_k \mathfrak{b}_{n-k} \tag{3.4}$$

called the *arrow product*. In contrast with binomial convolution rule (3.2), there is a supplementary factor $2^{k(n-k)}$ corresponding to possible edges from one strongly connected component to another.

We could deduce the form of the generating function consistent with arrow product as follows. Consider a generating function of the indefinite form

$$A(z) = \sum_{n=0}^{\infty} \mathfrak{a}_n \psi(n) z^n,$$

where ψ is a function to determine. The necessary condition to have the relation

$$C(z) = A(z) \cdot B(z)$$

is that

$$\mathfrak{c}_n \psi(n) = \sum_{k=0}^{n} \mathfrak{a}_k \mathfrak{b}_{n-k} \psi(k) \psi(n-k).$$

In our case, this means that

$$\frac{\psi(k)\psi(n-k)}{\psi(n)} = \binom{n}{k} 2^{k(n-k)}. \tag{3.5}$$

In particular, for $k = 1$,

$$\psi(n) = \frac{\psi(1)\psi(n-1)}{n2^{n-1}} = \frac{\big(\psi(1)\big)^n}{n!2^{\binom{n}{2}}}.$$

Putting $\psi(1) = 1$, we have

$$\psi(n) = \frac{1}{n!2^{\binom{n}{2}}},$$

and direct calculations show that (3.5) holds indeed.

Remark 3.1.70. Sometimes it is convenient to interpret the action of the operator Δ^{-1} as Hadamard multiplication by the exponential generating function $G(z)$ of the labeled combinatorial class $\mathcal{G}$ of graphs. In other words, for any labeled combinatorial class $\mathcal{A}$,

$$\Delta^{-1}\widehat{A}(z) = \widehat{A}(z) \odot G(z) = A(z).$$

Example 3.1.71. Let $\mathcal{D}$ be the labeled combinatorial class of directed graphs. In a directed graph, for each ordered pair of different vertices (i, j), there are two possibilities: they are either joined by a directed edge $\overrightarrow{ij}$ or not. Hence, the number of directed graphs of size n is $\mathfrak{d}_n = 2^{n(n-1)}$ and the exponential generating function of the class $\mathcal{D}$ is

$$D(z) = \sum_{n=0}^{\infty} 2^{n(n-1)} \frac{z^n}{n!}.$$

We can see from this relation that

$$D(z) = G(z) \odot G(z) \qquad \text{and} \qquad \widehat{D}(z) = G(z).$$

In particular, there is a combinatorial isomorphism

$$\mathcal{D} \cong \mathcal{G} \odot \mathcal{G} \cong \mathcal{T} \odot \mathcal{T}.$$

One can establish a bijection between $\mathcal{D}_n$ and $(\mathcal{G} \odot \mathcal{G})_n$ the following way. Take a directed graph from $\mathcal{D}_n$. Replacing each directed edge $\overrightarrow{ij}$ by a red edge in the case when $i < j$, and by a blue edge otherwise, we get an element of $(\mathcal{G} \odot \mathcal{G})_n$ (Fig. 3.10). The similar way, we obtain a bijection between $\mathcal{T}_n$ and $\mathcal{G}_n$ by replacing each directed edge $\overrightarrow{ij}$ by an edge in the case when $i < j$ (and just erasing it otherwise).

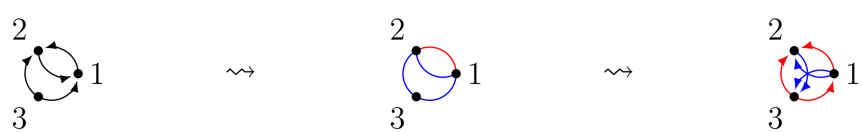


Figure 3.10: Bijections between $\mathcal{D}$, $\mathcal{G} \odot \mathcal{G}$ and $\mathcal{T} \odot \mathcal{T}$.

3.1.15 Gargantuan labeled combinatorial classes

Definition 3.1.72. We will call a labeled combinatorial class $\mathcal{A}$ *gargantuan*, if the sequence $(\mathfrak{a}_n/n!)$ is gargantuan. In other words, $\mathcal{A}$ is *gargantuan*, if for any integer r the following two conditions hold:

$$\text{(i)} \quad n \cdot \frac{\mathfrak{a}_{n-1}}{\mathfrak{a}_n} \to 0 \text{ as } n \to \infty; \qquad \text{(ii)} \quad \sum_{k=r}^{n-r} \binom{n}{k} \mathfrak{a}_k \mathfrak{a}_{n-k} = O\big(n^r \mathfrak{a}_{n-r}\big).$$

Example 3.1.73. As we have seen in Example 3.1.14, counting sequences for labeled graphs and labeled tournaments are the same:

$$\mathfrak{g}_n = \mathfrak{t}_n = 2^{\binom{n}{2}}.$$

On the other hand, it has been shown in Example 2.1.7 that the sequence

$$\frac{\mathfrak{g}_n}{n!} = \frac{\mathfrak{t}_n}{n!} = \frac{2^{\binom{n}{2}}}{n!}$$

is gargantuan. Hence, labeled combinatorial classes $\mathcal{G}$ and $\mathcal{T}$ are gargantuan.

Example 3.1.74. The labeled combinatorial class $\mathcal{TR}$ of trees is not gargantuan. Indeed, it is well-known that the counting sequence of the class $\mathcal{TR}$ satisfies

$$\mathfrak{tr}_n = n^{n-2}$$

(for various proofs, see [18], [66] or [86]). Therefore,

$$\frac{\mathfrak{tr}_{n-1}}{(n-1)!} \cdot \frac{n!}{\mathfrak{tr}_n} = \left(\frac{n-1}{n}\right)^{n-3} \to \frac{1}{e} \neq 0,$$

as $n \to \infty$. Hence, the first condition of Definition 2.1.1 does not hold for the sequence $\big(\mathfrak{tr}_n/n!\big)$, meaning that it is not gargantuan. As a consequence, by Definition 3.1.72, the class $\mathcal{TR}$ is not gargantuan either.

Example 3.1.75. The sequence

$$n \cdot \frac{\mathfrak{p}_{n-1}}{\mathfrak{p}_n} = 1$$

does not tend to 0 as n tends to infinity. Hence, the labeled combinatorial class $\mathcal{P}$ of permutations is not gargantuan.

Replacing ordinary power series by exponential generating functions, we obtain the following adaptation of Bender's lemma and theorem for labeled combinatorial classes whose counting sequences are gargantuan.

Lemma 3.1.76 (Bender's lemma for labeled combinatorial classes)**.** *Let $\mathcal{A}$ be a gargantuan labeled combinatorial class such that $\mathfrak{a}_0 = 0$ and $\mathfrak{a}_n > 0$ for $n > 0$. Then there exist constants $C > 0$ and $C_s > 0$ such that*

$$\sum_{K_{m,n}} \binom{n}{k_1, \ldots, k_m} \mathfrak{a}_{k_1} \ldots \mathfrak{a}_{k_m} \leqslant C^{m-1} n^{m-1} \mathfrak{a}_{n-m+1}, \tag{3.6}$$

and for any fixed integer $s \in \mathbb{Z}_{\geqslant 0}$,

$$\sum_{K_{m,n,s}} \binom{n}{k_1, \ldots, k_m} \mathfrak{a}_{k_1} \ldots \mathfrak{a}_{k_m} \leqslant C_s^{m-1} n^s \mathfrak{a}_{n-s}, \tag{3.7}$$

where the first sum runs over

$$K_{m,n} = \{(k_1, \ldots, k_m) \mid k_1 + \ldots + k_m = n\}$$

and the second sum runs over

$$K_{m,n,s} = \{(k_1, \ldots, k_m) \mid k_1 + \ldots + k_m = n,\ k_i \leqslant n - s\}.$$

Proof. Since $\mathcal{A}$ is gargantuan, we can apply Lemma 2.2.4 to the sequence $\big(\mathfrak{a}_n/n!\big)$. Due to (2.1), there exists a constant C such that

$$\sum_{K_{m,n}} \frac{\mathfrak{a}_{k_1}}{k_1!} \cdots \frac{\mathfrak{a}_{k_m}}{k_m!} \leqslant C^{m-1} \frac{\mathfrak{a}_{n-m+1}}{(n-m+1)!},$$

while (2.2) gives us

$$\sum_{K_{m,n,s}} \frac{\mathfrak{a}_{k_1}}{k_1!} \cdots \frac{\mathfrak{a}_{k_m}}{k_m!} \leqslant C_s^{m-1} \mathfrak{a}_{n-s}(n-s)!$$

for a family of constants C_s, $s \in \mathbb{Z}_{\geqslant 0}$. Now, to obtain (3.6) and (3.7), it is sufficient to multiply these two inequalities by $n!$ and note that for any non-negative integer s holds

$$\frac{n!}{(n-s)!} \leqslant n^s.$$

□

Definition 3.1.77. Let $d \in \mathbb{N}$. We will call a labeled combinatorial class $\mathcal{A}$ *d-gargantuan*, if the sequence $(\mathfrak{a}_n/n!)$ is d-gargantuan.

Lemma 3.1.78. *Let $d \in \mathbb{N}$ and $\mathcal{A}$ be a labeled combinatorial class.*
a) The class $\mathcal{A}$ is d-gargantuan if and only if for any integer r the following two conditions hold:

$$\text{(i)} \quad n^d \cdot \frac{\mathfrak{a}_{d(n-1)}}{\mathfrak{a}_{dn}} \to 0 \ \text{ as } \ n \to \infty; \qquad\qquad \text{(ii)} \quad \sum_{k=r}^{n-r} \binom{dn}{dk} \mathfrak{a}_{dk}\mathfrak{a}_{d(n-k)} = O\big(n^{dr}\mathfrak{a}_{d(n-r)}\big).$$

b) The class $\mathcal{A}$ is d-gargantuan if for any integer r the following two conditions hold:

$$\text{(i)'} \quad n^{d+1}a_{d(n-1)} = O(a_{dn}), \quad \text{as } \ n \to \infty,$$
$$\text{(ii)'} \quad x_k = \binom{dn}{dk} |a_{dk}a_{d(n-k)}| \ \text{is decreasing for } k < n/2,$$

Proof. a) By definition 2.1.14 it means, that the sequence

$$b_n = \frac{\mathfrak{a}_{dn}}{(dn)!}$$

is gargantuan. The first condition of (b_n) to be gargantuan reads

$$\frac{b_{n-1}}{b_n} = \frac{(dn)!}{(dn-d)!} \cdot \frac{\mathfrak{a}_{d(n-1)}}{\mathfrak{a}_{dn}} \to 0,$$

which is equivalent to condition (i). For the second condition, note that

$$\sum_{k=r}^{n-r} b_k b_{n-k} = \frac{1}{(dn)!} \sum_{k=r}^{n-r} \binom{dn}{dk} \mathfrak{a}_{dk} \mathfrak{a}_{d(n-k)}.$$

Hence, the identity

$$\sum_{k=r}^{n-r} b_k b_{n-k} = O(b_{n-r})$$

is equivalent to condition (ii).

b) Suppose that conditions (i)' and (ii)' hold. Then (i) holds as well. To verify (ii), note that

$$(n-2r-1)\binom{dn}{d(r+1)} = O\left(n^{d+1}\binom{dn}{dr}\right).$$

Hence, we have

$$\begin{aligned}
\sum_{k=r}^{n-r} \binom{dn}{dk} \mathfrak{a}_{dk} \mathfrak{a}_{d(n-k)} &= 2\binom{dn}{dr} \mathfrak{a}_{dr} \mathfrak{a}_{d(n-r)} + \sum_{k=r+1}^{n-r-1} \binom{dn}{dk} \mathfrak{a}_{dk} \mathfrak{a}_{d(n-k)} \\
&\leqslant 2\binom{dn}{dr} \mathfrak{a}_{dr} \mathfrak{a}_{d(n-r)} + (n-2r-1)\binom{dn}{d(r+1)} \mathfrak{a}_{d(r+1)} \mathfrak{a}_{d(n-r-1)} \\
&= O\big(n^{dr} \mathfrak{a}_{d(n-r)}\big).
\end{aligned}$$

□

3.1.16 Bender's theorem for labeled combinatorial classes

Theorem 3.1.79 (Bender's theorem for labeled combinatorial classes). *a) Let $\mathcal{A}$ be a gargantuan labeled combinatorial class such that $\mathfrak{a}_0 = 0$ and $\mathfrak{a}_n \neq 0$ for $n > 0$. Suppose that $\mathcal{B}$ is a labeled combinatorial class whose exponential generating function has the form*

$$B(z) = F\big(A(z)\big),$$

where $F(y)$ is a function analytic in some neighborhood of origin. Then the class $\mathcal{B}$ is gargantuan and

$$\mathfrak{b}_n \approx \sum_{k\geqslant 0} \binom{n}{k} c_k \mathfrak{a}_{n-k},$$

where (c_k) are the coefficients of exponential power series

$$C(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!} = \left[\frac{\partial}{\partial y} F(y)\right]_{y=A(z)}.$$

b) Let $\mathcal{A}$ be a d-gargantuan labeled combinatorial class such that $\mathfrak{a}_0 = 0$ and $\mathfrak{a}_{dn} \neq 0$ for $n > 0$. Suppose that $\mathcal{B}$ is a labeled combinatorial class whose exponential generating function has the form

$$B(z) = F\big(A(z)\big),$$

where $F(y)$ is a function analytic in some neighborhood of origin. Then the class $\mathcal{B}$ is d-gargantuan and

$$\mathfrak{b}_{dn} \approx \sum_{k\geqslant 0} \binom{dn}{dk} c_{dk} \mathfrak{a}_{d(n-k)},$$

where (c_{dk}) are the coefficients of exponential power series

$$C(z) = \sum_{n=0}^{\infty} c_{dn} \frac{z^{dn}}{(dn)!} = \left[\frac{\partial}{\partial y} F(y)\right]_{y=A(z)}.$$

Proof. Since a gargantuan labeled combinatorial class is d-gargantuan for $d = 1$, it is sufficient to proof Theorem 3.1.79 for d-gargantuan classes. To this aim, in the statement of Theorem 2.2.5, consider a replacement

$$a_n \mapsto \frac{\mathfrak{a}_{dn}}{(dn)!}.$$

Consistently, the product $a_k b_{n-k}$ is replaced by

$$a_k b_{n-k} \mapsto \frac{\mathfrak{a}_{dk} \mathfrak{b}_{d(n-k)}}{(dk)!(dn-dk)!} = \frac{1}{(dn)!} \binom{dn}{dk} \mathfrak{a}_{dk} \mathfrak{b}_{d(n-k)}.$$

As a consequence, the asymptotic expansion takes a form

$$\frac{\mathfrak{b}_{dn}}{(dn)!} \approx \frac{1}{(dn)!} \sum_{k\geqslant 0} \binom{dn}{dk} c_{dk} \mathfrak{a}_{d(n-k)}.$$

Multiplying both sides by $(nd)!$, we get the desired result. □

Remark 3.1.80. The reader may wonder why we need two different parts of Theorem 3.1.79. Indeed, instead of d-gargantuan labeled combinatorial class $\mathcal{A}$, it would be natural to consider the gargantuan labeled combinatorial class $\mathcal{U}$ whose counting sequence satisfy $\mathfrak{u}_n = \mathfrak{a}_{dn}$. It turns out, however, that we cannot proceed this way. The reason is hidden in the nature of labeled objects, namely, in the binomial convolution rule (3.2). Making a substitution $n \to dn$, we must change this rule as well in the following way:

$$\binom{n}{k} \to \binom{dn}{dk}.$$

Indeed, it is so, since the gargantuan sequence $\mathfrak{a}_n/n!$ is to be replaced by $\mathfrak{a}_{dn}/(dn)!$ which satisfies the conditions of Lemma 3.1.78, see the proof of Lemma 3.1.78.

Example 3.1.81. Let $\mathcal{A}$ be a gargantuan labeled combinatorial class such that $\mathfrak{a}_0 = 0$ and $\mathfrak{a}_n \neq 0$ for $n > 0$. Apply Theorem 3.1.79 for the class $\mathcal{A}$ and the function $F(y) = e^y$. In this case,

$$B(z) = F\big(A(z)\big) = \exp\big(A(z)\big)$$

can be interpreted as the exponential generating function for $\mathcal{B} = \mathrm{SET}(\mathcal{A})$. Since

$$C(z) = \left[\frac{\partial}{\partial y} F(y)\right]_{y=A(z)} = B(z),$$

Theorem 3.1.79 implies that $\mathcal{B} = \mathrm{SET}(\mathcal{A})$ is gargantuan and

$$\mathfrak{b}_n \approx \sum_{k\geqslant 0} \binom{n}{k} \mathfrak{b}_k \mathfrak{a}_{n-k}.$$

Similar statements can be obtained for other combinatorial constructions as well.

3.2 Unlabeled combinatorial structures

In this section, we discuss unlabeled combinatorial structures. We would like to warn the reader that the use of the word "unlabeled" here is very relative. In fact, there are two different cases that can be treated with the help of methods described below. First of all, we mean "truly unlabeled" structures, which can be considered as labeled ones with erased labels, so that objects are taken up to isomorphism. A typical example to be mentioned here is unlabeled graphs. Another case refers to objects that possess labels but cannot be treated by the approach of Section 3.1. This is inherent to labeled combinatorial classes that are not stable under relabeling of the atoms, for instance, indecomposable permutations.

Let us mention the differences between the unlabeled and labeled universes. In the unlabeled case, the "standard" Cartesian product is used instead of labeled product, which leads to employing ordinary generating functions instead of exponential ones. This also leads to some changes in defining combinatorial constructions. In particular, the set construction splits into two. This becomes possible, since the components of unlabeled combinatorial objects are indistinguishable. Depending on whether repetitions are allowed or not, we get two different constructions: MSET and PSET.

Nevertheless, apart from the above changes, the concept of the symbolic method is the same for the unlabeled and labeled cases. Therefore, in this section, we allow ourselves to outline the material, without going into details. As well as in the previous section, we mostly follow the book of Flajolet and Sedgewick [38], which we recommend to the reader who wants to learn the topic more deeply (see also [4] for rather short introduction). The only exception is the last paragraph devoted to gargantuan unlabeled combinatorial classes and Bender's theorem.

3.2.1 Unlabeled combinatorial classes

By an *unlabeled combinatorial class* we mean a combinatorial class in the sense of Definition 3.1.1 equipped with the ordinary generating function defined below.

Definition 3.2.1. The *ordinary generating function* of an unlabeled combinatorial class $\mathcal{A}$ is the formal power series

$$A(z) = \sum_{n=0}^{\infty} \mathfrak{a}_n z^n.$$

Example 3.2.2. The ordinary generating functions of unlabeled combinatorial classes $\mathcal{N}$ and $\mathcal{Z}$ are

$$N(z) = 1 \qquad \text{and} \qquad Z(z) = z,$$

respectively. It is seen that they coincide with the exponential generating functions of the corresponding labeled combinatorial classes (see Example 3.1.7).

Example 3.2.3. Consider two different symbols, say, letters a and b. A *binary word* is a sequence of elements taken from the set $\{a, b\}$ comprising these two symbols. The number of elements in the sequence is the *size of the word*. We denote by $\mathcal{BW}$ the unlabeled combinatorial class of binary words:

$$\mathcal{BW} = \{\epsilon, a, b, aa, ab, ba, bb, \ldots\}.$$

Its counting sequence is $\mathfrak{bw}_n = 2^n$, and hence, its ordinary generating function is

$$BW(z) = \sum_{n=0}^{\infty} 2^n z^n = \frac{1}{1-2z}.$$

Example 3.2.4. Let p be a prime number and $\mathbb{F}_p$ be the finite field of integers taken modulo p. Denote by $\mathcal{PO}(p)$ the unlabeled combinatorial class of *monic polynomials* over $\mathbb{F}_p$. In this case, the counting sequence of $\mathcal{PO}(p)$ is $\mathfrak{po}_n(p) = p^n$. In particular, for $p = 2$,

$$\mathfrak{po}_n(2) = 2^n = \mathfrak{bw}_n.$$

Similarly to labeled combinatorial classes, the coincidence of counting sequences allows us to say that the corresponding unlabeled combinatorial classes are combinatorially isomorphic. In our case,

$$\mathcal{PO}(2) \cong \mathcal{BW}.$$

Example 3.2.5. We can formally consider the combinatorial class $\mathcal{P}$ of *permutations* as an unlabeled combinatorial class. To emphasize that we deal with the unlabeled case, we will designate the corresponding substances with the lower index "u". In this case (compare with Example 3.1.9), the size of a permutation $[n] \to [n]$ is n, the counting sequence of ${}_u\mathcal{P}$ is still given by

$${}_u\mathfrak{p}_n = \mathfrak{p}_n = n!,$$

but its ordinary generating function is

$${}_uP(z) = \sum_{n=0}^{\infty} n! z^n.$$

The reader may find this approach artificial, since the labels are somehow incorporated into the definition of permutations as bijections $[n] \to [n]$. However, we will see in Chapter 9 that this approach can lead to some non-trivial results.

3.2.2 Cartesian product

Definition 3.2.6. Let $\mathcal{A}$ and $\mathcal{B}$ be two unlabeled combinatorial classes. The *Cartesian product* construction generates the class

$$\mathcal{C} = \mathcal{A} \times \mathcal{B} = \{\gamma = (\alpha, \beta) \mid \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}$$

with the size of an element $\gamma = (\alpha, \beta)$ defined by

$$|\gamma|_{\mathcal{C}} = |\alpha|_{\mathcal{A}} + |\beta|_{\mathcal{B}}.$$

In terms of counting sequences and ordinary generating functions, one has

$$\mathfrak{c}_n = \sum_{k=0}^{n} \mathfrak{a}_k \mathfrak{b}_{n-k} \qquad \text{and} \qquad C(z) = A(z) \cdot B(z).$$

Example 3.2.7. Let us consider the unlabeled combinatorial atomic class $\mathcal{Z}$ which consists of a single element of size 1. We can interpret the disjoint union $\mathcal{Z} + \mathcal{Z}$ as a binary alphabet $\{a, b\}$. Put

$$\mathcal{A} = \mathcal{B} = \mathcal{Z} + \mathcal{Z}.$$

Then the Cartesian product

$$\mathcal{A} \times \mathcal{B} = (\mathcal{Z} + \mathcal{Z})^2$$

represents the unlabeled combinatorial class of binary words of length two: $\{aa, ab, ba, bb\}$.

3.2.3 Sequences

Definition 3.2.8. Let m be a non-negative integer and $\mathcal{A}$ be an unlabeled combinatorial class. In this case, the *m-sequence* construction SEQ_m produces all possible m-tuples of objects from $\mathcal{A}$:

$$\mathrm{SEQ}_m(\mathcal{A}) = \mathcal{A} \times \mathcal{A} \times \ldots \times \mathcal{A}.$$

The size of an element $\beta = (\alpha_1, \ldots, \alpha_m)$ is defined by

$$|\beta| = |\alpha_1| + \ldots + |\alpha_m|.$$

Definition 3.2.9. Let $\mathcal{A}$ be an unlabeled combinatorial class that contains no object of size 0. The *sequence* construction generates the disjoint union of all the possible tuples of unlabeled combinatorial objects from $\mathcal{A}$, so that

$$\mathrm{SEQ}(\mathcal{A}) = \sum_{m=0}^{\infty} \mathrm{SEQ}_m(\mathcal{A}) = \{\epsilon\} + \mathcal{A} + (\mathcal{A} \times \mathcal{A}) + (\mathcal{A} \times \mathcal{A} \times \mathcal{A}) + \ldots.$$

Remark 3.2.10. Let $m \in \mathbb{Z}_{\geqslant 0}$ and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be unlabeled combinatorial classes satisfying

$$\mathcal{B} = \mathrm{SEQ}(\mathcal{A}) \qquad \text{and} \qquad \mathcal{C} = \mathrm{SEQ}_m(\mathcal{A}),$$

As well as in the labeled case, each object from $\mathcal{A} \cap \mathcal{B}$ will be called SEQ-*irreducible*, and we will say that each object from $\mathcal{C} \cap \mathcal{B}$ has exactly m SEQ-*irreducible components*. The generating function operations for constructions SEQ and SEQ_m are the same for labeled and unlabeled cases:

$$B(z) = \frac{1}{1 - A(z)} \qquad \text{and} \qquad C(z) = \big(A(z)\big)^m$$

(see also [38, Theorems I.1 and I.3]).

Example 3.2.11. For any $m \in \mathbb{Z}_{\geqslant 0}$, the unlabeled combinatorial class $\mathrm{SEQ}_m(\mathcal{Z})$ comprises the unique object of size m that can be interpreted as the corresponding integer. Hence, the unlabeled combinatorial class $\mathrm{SEQ}(\mathcal{Z})$ describes the set of non-negative integers. The only SEQ-irreducible integer is 1, while any other positive integer can be represented as a "sum" of SEQ-irreducible components.

Example 3.2.12. Given $m \in \mathbb{Z}_{\geqslant 0}$, the unlabeled combinatorial class $\mathrm{SEQ}_m(\mathcal{Z} + \mathcal{Z})$ consists of binary words of length n (the particular case $m = 2$ was presented in Example 3.2.7). The union of all these classes, according to Definition 3.2.9, is the unlabeled combinatorial class

$$\mathcal{BW} = \mathrm{SEQ}(\mathcal{Z} + \mathcal{Z})$$

of binary words. In this example, a word SEQ-irreducible if and only if it is a letter, meaning that its size equals 1. All other words, except the empty word, can be decomposed into a "sum" of SEQ-irreducible components. Note that the order of components is important.

Example 3.2.13. A permutation $\sigma \colon [n] \to [n]$ is *decomposable*, if there exists a positive integer $k < n$ such that $\sigma\big([k]\big) = [k]$. Otherwise, σ is said to be *indecomposable*. Given a permutation $\sigma \in S_n$, we can indicate the maximal partition of $[n]$ into consecutive intervals,

$$[n] = I_1 + \ldots + I_m,$$

such that $\sigma\big(I_k\big) = I_k$ for each $k \in [m]$. For any k, the restriction $\sigma|_{I_k}$ can be interpreted as an indecomposable permutation $\big[|I_k|\big] \to \big[|I_k|\big]$ by the corresponding order-consisting relabeling. Hence, any permutation can be represented as a concatenation of indecomposable parts in a unique way, meaning that we have the relation

$$_u\mathcal{P} = \mathrm{SEQ}(_u\mathcal{IP})$$

for the unlabeled combinatorial classes $_u\mathcal{P}$ of permutations and $_u\mathcal{IP}$ of indecomposable permutations, respectively.

Note that this result is very different from what we have seen for the labeled case. If we consider labeled combinatorial classes $\mathcal{P}$ and $\mathcal{IP}$, then we have $\mathcal{P} \cong \mathrm{SEQ}(\mathcal{Z})$, see Remark 3.1.48. At the same time, the class $\mathcal{IP}$ is not invariant under relabeling. Indeed, let us consider a permutation $\tau \in S_n$ as a graph whose set of vertices $[n]$ lie on the number line and whose edges are arcs joining k with $\tau(k)$, for all $k \in [n]$. From this point of view, τ is decomposable, if there exists a vertical line $x = x_0$ separating one component of the graph from others. For example, the permutation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

is decomposable with the separating line $x = 3.5$ (see the left part of Fig. 3.11).

Now, let us see what happens when relabeling. When we permute labels, the natural order is disrupted, and to restore it, we need to reorder the vertices of the graph. The restoration process does not change the cyclic structure of the graph which is determined by $\sigma\tau\sigma^{-1}$, where σ is the permutation of labels. However, the relative position of the graph components on the plane may change. In particular, a decomposable permutation may become indecomposable after relabeling, and vice versa (see the central and right parts of Fig. 3.11).

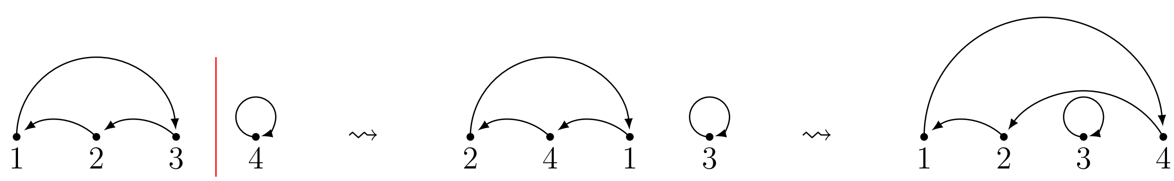


Figure 3.11: Permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$ under relabeling $(1, 2, 3, 4) \mapsto (2, 4, 1, 3)$.

3.2.4 Cycles

Definition 3.2.14. Let m be a non-negative integer and $\mathcal{A}$ be an unlabeled combinatorial class. The *m-cycle* construction CYC_m produces all possible m-cycles of unlabeled combinatorial objects from $\mathcal{A}$. If $\mathcal{A}$ contains no object of size 0, then the *cycle* construction CYC generates the disjoint union of all the possible cycles of unlabeled combinatorial objects from $\mathcal{A}$:

$$\mathrm{CYC}(\mathcal{A}) = \sum_{m=0}^{\infty} \mathrm{CYC}_m(\mathcal{A}).$$

Remark 3.2.15. Let $m \in \mathbb{Z}_{\geqslant 0}$ and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be unlabeled combinatorial classes satisfying

$$\mathcal{B} = \mathrm{CYC}(\mathcal{A}) \qquad \text{and} \qquad \mathcal{C} = \mathrm{CYC}_m(\mathcal{A}),$$

The same way as in the labeled case, each object from $\mathcal{A} \cap \mathcal{B}$ will be called CYC-*irreducible*, and we will say that each object from $\mathcal{C} \cap \mathcal{B}$ has exactly m CYC-*irreducible components*. Contrary to the labeled case, the generating functions operations for constructions CYC_m and CYC have more sophisticated form and need additional techniques to be established. One can prove that the corresponding relations are:

$$B(z) = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log \frac{1}{1 - A(z^k)}. \qquad \text{and} \qquad C(z) = [u^m] \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log \frac{1}{1 - u^k A(z^k)}$$

For the proof and further information, we refer interested readers to [38, Theorems I.1 and I.3].

3.2.5 Sets

Definition 3.2.16. Let m be a non-negative integer and $\mathcal{A}$ be an unlabeled combinatorial class. In this case, the *m-multiset* construction MSET_m produces all possible m-multisets of unlabeled combinatorial objects from $\mathcal{A}$, while the *m-powerset* construction PSET_m produces the unlabeled combinatorial class $\mathrm{PSET}_m(\mathcal{A}) \subset \mathrm{MSET}_m(\mathcal{A})$ such that involved multisets have no repetitions.

Definition 3.2.17. Let $\mathcal{A}$ be an unlabeled combinatorial class that contains no object of size 0. The *multiset* and the *powerset* constructions applied to $\mathcal{A}$ generate new unlabeled combinatorial classes

$$\mathrm{MSET}(\mathcal{A}) = \sum_{m=0}^{\infty} \mathrm{MSET}_m(\mathcal{A}) \qquad \text{and} \qquad \mathrm{PSET}(\mathcal{A}) = \sum_{m=0}^{\infty} \mathrm{PSET}_m(\mathcal{A}),$$

respectively.

Remark 3.2.18. Regarding the generating functions operations, the behavior of unlabeled set constructions is also more complicated than the one of their labeled analogues. One can prove that if unlabeled combinatorial classes $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ satisfy

$$\mathcal{B} = \mathrm{MSET}(\mathcal{A}) \qquad \text{and} \qquad \mathcal{C} = \mathrm{MSET}_m(\mathcal{A}),$$

then the following relations hold for their ordinary generating functions:

$$B(z) = \prod_{n=1}^{\infty} \frac{1}{(1 - z^n)^{\mathfrak{a}_n}} = \exp\left(\sum_{k=1}^{\infty} \frac{A(z^k)}{k} \right).$$

and

$$C(z) = [u^m] \exp\left(\sum_{k=1}^{\infty} \frac{u^k}{k} A(z^k) \right)$$

On the other hand, if

$$\mathcal{B} = \mathrm{PSET}(\mathcal{A}) \qquad \text{and} \qquad \mathcal{C} = \mathrm{PSET}_m(\mathcal{A}),$$

then

$$B(z) = \prod_{n=1}^{\infty} (1 + z^n)^{\mathfrak{a}_n} = \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1} A(z^k)}{k} \right).$$

and

$$C(z) = [u^m] \exp\left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{u^k}{k} A(z^k) \right)$$

In both cases, each object from $\mathcal{A} \cap \mathcal{B}$ will be called SET-*irreducible*, and we will say that each object from $\mathcal{C} \cap \mathcal{B}$ has exactly m SET-*irreducible components*. For the proof of the above relations and further information, we refer interested readers to [38, Theorems I.1 and I.3].

Example 3.2.19. Let ${}_u\mathcal{G}$ and ${}_u\mathcal{CG}$ be unlabeled combinatorial classes of graphs and connected graphs, respectively. In this case, they satisfy

$${}_u\mathcal{G} = \mathrm{MSET}({}_u\mathcal{CG}).$$

Similarly, the unlabeled combinatorial classes ${}_u\mathcal{T}$ of tournaments and ${}_u\mathcal{IT}$ of irreducible tournaments, respectively, satisfy

$${}_u\mathcal{T} = \mathrm{SEQ}({}_u\mathcal{IT})$$

(compare with Examples 3.1.33 and 3.1.49). However, in contrast with the labeled case,

$${}_u\mathcal{G} \not\cong {}_u\mathcal{T},$$

since their counting sequences are different. Indeed, there are two unlabeled graphs of size 2, but the unlabeled tournament of size 2 is unique.

3.2.6 Gargantuan unlabeled combinatorial classes

Definition 3.2.20. Let $\mathcal{A}$ be an unlabeled combinatorial class. We will call $\mathcal{A}$ *gargantuan* (respectively, *d-gargantuan*), if its counting sequence (a_n) is gargantuan (respectively, d-gargantuan).

Example 3.2.21. As we have seen in Example 2.1.2, the sequence $\mathfrak{p}_n = n!$ is gargantuan. Hence, the unlabeled combinatorial class $\mathcal{P}$ of permutations is gargantuan.

Theorem 3.2.22 (Bender's theorem for unlabeled combinatorial classes)**.** *a) Let $\mathcal{A}$ be a gargantuan unlabeled combinatorial class such that $\mathfrak{a}_0 = 0$ and $\mathfrak{a}_n \neq 0$ for $n > 0$. Suppose that $\mathcal{B}$ is an unlabeled combinatorial class whose ordinary generating function has the form*

$$B(z) = F\big(A(z)\big),$$

where $F(y)$ is a function analytic in some neighborhood of origin. Then the class $\mathcal{B}$ is gargantuan and

$$\mathfrak{b}_n \approx \sum_{k \geqslant 0} c_k \mathfrak{a}_{n-k},$$

where (c_k) are the coefficients of ordinary power series

$$C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial}{\partial y} F(y)\right]_{y=A(z)}.$$

b) Let $d \in \mathbb{N}$ and $\mathcal{A}$ be a d-gargantuan unlabeled combinatorial class such that $\mathfrak{a}_0 = 0$ and $\mathfrak{a}_{dn} \neq 0$ for $n > 0$. Suppose that $\mathcal{B}$ is an unlabeled combinatorial class whose ordinary generating function has the form

$$B(z) = F\big(A(z)\big),$$

where $F(y)$ is a function analytic in some neighborhood of origin. Then the class $\mathcal{B}$ is d-gargantuan and

$$\mathfrak{b}_{dn} \approx \sum_{k \geqslant 0} c_{dk} \mathfrak{a}_{d(n-k)},$$

where (c_{dk}) *are the coefficients of ordinary power series*

$$C(z) = \sum_{n=0}^{\infty} c_{dn} z^{dn} = \left[\frac{\partial}{\partial y} F(y)\right]_{y=A(z)}.$$

Proof. The proof follows from Theorem 2.2.5 the same way as the proof of Theorem 3.1.79. □

Remark 3.2.23. In the unlabeled case, we could consider d-gargantuan unlabeled combinatorial class $\mathcal{A}$ as a gargantuan class $\mathcal{U}$ comprised with the same objects, but with a different size function:

$$|\alpha|_{\mathcal{U}} = \frac{|\alpha|_{\mathcal{A}}}{d}.$$

The counting sequence and the ordinary generating function of the class $\mathcal{U}$ are

$$\mathfrak{u}_n = \mathfrak{a}_{dn} \qquad \text{and} \qquad U(z) = A\big(z^{1/d}\big),$$

respectively. Using this construction, one can see that the statement of Theorem 3.2.22 b) follows from the part a). Thus, one could completely abandon the concept of d-gargantuan unlabeled combinatorial classes and use the gargantuan ones only. However, to emphasize the analogy with the labeled case, we prefer to state Theorem 3.2.22 in full generality. We will see the role of this analogy in Section 9.2, discussing asymptotic behavior of indecomposable perfect matchings.

Chapter 4

The theory of species

This chapter is devoted to the theory of species. As well as the symbolic method, the theory of species allows us to translate the internal structure of combinatorial objects to generating series operations and vice versa. However, it is more powerful and can be considered as a generalization of the symbolic method (in particular, the labeled and unlabeled structures are united in a common framework). The analysis of compound structures within species theory is purely algebraic and proceeds in terms of transformations and combinations of simpler structures. At a deep level, the theory of species is based on category theory that provides a language for its concepts. The high level of abstraction gives the theory of species its power, but, at the same time, it makes this theory more difficult for comprehension and for initial familiarity with.

Regarding our investigation, it is important to emphasize the following. Passing from the symbolic method to the theory of species, we replace combinatorial constructions by substitutions. Consequently, to move from an arbitrary structure to an irreducible one within the theory of species, we must be able to inverse corresponding substitutional laws. This can be done with the help of the concept of virtual species that has no analogues in the language of the symbolic method. The closest analogy of the construction of virtual species from species of structures is the one of $\mathbb{Z}$ from $\mathbb{N}$. Defining the ring of virtual species fills up the absence of a combinatorial operation of subtraction (as an inverse to disjoint union). Furthermore, having a combinatorial form of subtraction allows us to give a combinatorial meaning to the multiplicative inverse and the inverse under substitution. Thus, integer sequences that had no combinatorial interpretation within the framework of the symbolic method acquire a new meaning in the theory of species.

The presentation of this chapter is intended to be reader-friendly and suitable for students who take the first steps in mastering the basics of the theory of species. In Section 4.1, we discuss species of structures, including weighted species, associated series and operations on species. Section 4.2 is devoted to virtual species and its applications for multiplicative and substitutional inverses. To make the material clearer, its presentation is provided with a number of examples and remarks. However, we omit some details and proofs, otherwise the volume of the chapter would exceed all conceivable limits. Most of the time, we follow the book of Bergeron, Labelle and Leroux [12], that the reader is invited to consult in case of need to dive deeper. The only exception is the last paragraph where we define gargantuan species and adapt Bender's theorem for them. This paragraph is entirely the contribution of the author.

Concerning notations, we also follow [12]. There are, however, some differences. We denote by $\mathcal{Z}$ (instead of $\mathcal{X}$) the species of singletons to make clearer the parallel with the atomic combinatorial class in the symbolic method. The same way, we denote by $\mathcal{P}$ and $\mathcal{CP}$ species of permutations and cyclic permutations, while in [12] they are designated by $\mathcal{S}$ and $\mathcal{C}$, respectively. Almost every-where, the nature of structures is reflected in notations written in first letters of their names. A small exception is the species $\mathcal{E}$ of sets that came from the French word "ensemble" meaning "set".

4.1 The theory of species of structures

4.1.1 Species of structures

Definition 4.1.1. A *species of structures* is a rule F such that

1. for each finite set U, F produces a finite set $F[U]$ (whose elements are *F-structures on U*);
2. for each bijection $\sigma\colon U \to V$, F produces a *transport function* (of F-structures along σ)

$$F[\sigma]\colon F[U] \to F[V],$$

satisfying the following properties:

(a) for all bijections $\sigma\colon U \to V$ and $\tau\colon V \to W$,

$$F[\tau \circ \sigma] = F[\tau] \circ F[\sigma];$$

(b) for the identity map $\mathrm{Id}_U\colon U \to U$,

$$F[\mathrm{Id}_U] = \mathrm{Id}_{F[U]}.$$

Notation 4.1.2. For simplicity, we use the following notations.

- $\sigma \cdot s$ is used to designate $F[\sigma](s)$, where $s \in F[U]$ is an F-structure on the set U.
- $F[n]$ is used to designate the set $F\big[\{1,\ldots,n\}\big]$, where $n \in \mathbb{N}$. Elements of $F[n]$ are *labeled F-structures* of order n.

Example 4.1.3. Let us describe the most commonly used species of structures.

- The *empty species* $\mathbf{0}$, defined by
$$\mathbf{0}[U] := \emptyset$$
for every finite set U, and equipped with the trivial transport function.
- The *characteristic of the empty set*, species $\mathbf{1}$, defined by
$$\mathbf{1}[U] = \begin{cases} \{U\}, & \text{if } U = \emptyset, \\ \emptyset, & \text{if } U \neq \emptyset \end{cases}$$
and equipped with the trivial transport function.
- The *characteristic of singletons*, species $\mathcal{Z}$, defined by
$$\mathcal{Z}[U] = \begin{cases} \{U\}, & \text{if } |U| = 1, \\ \emptyset, & \text{if } |U| \neq 1 \end{cases}$$
and equipped with the trivial transport function.

- The *species of sets* $\mathcal{E}$, defined by
$$\mathcal{E}[U] = \{U\}$$
for every finite set U, and equipped with the trivial transport function.

- The species of *linear orders* $\mathcal{L}$, such that for any finite set U, the rule $\mathcal{L}$ produces the set $\mathcal{L}[U]$ of linear orders on U, and the transport function is defined in a natural way.

- The species of *permutations* $\mathcal{P}$, defined by
$$\mathcal{P}[U] = \{\psi_U : U \to U \mid \psi_U \text{ is bijective}\}$$
and equipped with the transport function
$$\mathcal{P}[U] \to \mathcal{P}[V], \qquad \psi_U \mapsto \psi_V = \sigma\psi_U\sigma^{-1},$$
along the bijection $\sigma : U \to V$ of finite sets U and V.

- The species of *cyclic permutations* $\mathcal{CP}$, defined by
$$\mathcal{CP}[U] = \{\psi_U \in \mathcal{P}[U] \mid \psi_U \text{ is a cycle}\},$$
and equipped with the same transport function as $\mathcal{P}$ (which is well-defined for $\mathcal{CP}$, since conjugations preserve the cyclic type of permutations).

4.1.2 Weighted species

Let $\mathbb{A}$ be a ring of formal power series (in an arbitrary number of variables $t_1, t_2, \ldots$) over a ring $\mathbb{K}$. In our investigation, $\mathbb{K} = \mathbb{Z}$ or $\mathbb{K} = \mathbb{R}$, but in principle, it could be $\mathbb{C}$ as well. An $\mathbb{A}$-*weighted set* is a pair (A, w), where A is a (finite or countable) set and w is a function
$$w : A \to \mathbb{A}$$
whose value $w(a) \in \mathbb{A}$ is the *weight* of the element $a \in A$. The set A is *summable*, if for any monomial $\kappa = (t_1^{n_1} t_2^{n_2} \ldots)$ the set
$$W_\mu = \big\{a \in A \mid [\kappa]w(a) \neq 0\big\}$$
is finite (here, $[\kappa]w(a)$ designates the operation of extracting the coefficient of κ in the formal power series $w(a)$). The *total weight* of the weighted set A is the element
$$|A|_w = \sum_{a \in A} w(a).$$
In particular, for any monomial κ,
$$[\kappa]|A|_w = \sum_{a \in A} [\kappa]w(a).$$

Definition 4.1.4. Let $\mathbb{A}$ be a ring of formal power series (in an arbitrary number of variables) over a ring $\mathbb{K}$. An $\mathbb{A}$-*weighted species* is a rule F such that

1. for each finite set U, the rule F produces a finite or summable $\mathbb{A}$-weighted set $\big(F[U], w_U\big)$;

2. for each bijection $\sigma\colon U \to V$, the rule F produces a transport function
$$F[\sigma]\colon \big(F[U], w_U\big) \to \big(F[V], w_V\big),$$
satisfying the following properties:

   (a) $F[\sigma]$ preserves the weights:
$$w_U = w_V \circ F[\sigma];$$

   (b) for all bijections $\sigma\colon U \to V$ and $\tau\colon V \to W$,
$$F[\tau \circ \sigma] = F[\tau] \circ F[\sigma];$$

   (c) for the identity map $\mathrm{Id}_U\colon U \to U$,
$$F[\mathrm{Id}_U] = \mathrm{Id}_{F[U]}.$$

Notation 4.1.5. To emphasize that a species F is weighted, we write $F = F_w$.

Remark 4.1.6. In the case when $\mathbb{A} \subset \mathbb{R}$ and all weights are positive, there is a natural family of probability measures that is assigned to an $\mathbb{A}$-weighted species F_w. Namely, given an integer $n \in \mathbb{Z}_{\geqslant 0}$, we endow $F_w[n]$ with the discrete probability by normalizing the weight of a structure by the total weight
$$\big|F_w[n]\big|_w = \sum_{s \in F_w} |w(s)|.$$
In other words, a random structure $s \in F_w[n]$ appears with probability
$$\mathbb{P}(s) = \frac{w(s)}{\big|F_w[n]\big|_w}.$$

Example 4.1.7. Consider the species $\mathcal{G}$ of graphs and an integer $n \in \mathbb{Z}_{\geqslant 0}$. Fix a positive real number ρ and define the weight of a graph $G \in \mathcal{G}$ to be
$$w(G) = \rho^{|E(G)|},$$
where $|E(G)|$ is the number of edges in G. Then the naturally assigned probability measure on $\mathcal{G}[n]$ takes the value
$$\mathbb{P}(G) = \frac{\rho^{|E(G)|}}{(1+\rho)^{\binom{n}{2}}}$$
on the graph G. Since there exist the unique pair $(p, q) \in \mathbb{R}^2_{>0}$ such that
$$\frac{p}{q} = \rho \qquad \text{and} \qquad p + q = 1,$$
we can rewrite the above probability as
$$\mathbb{P}(G) = p^{|E(G)|} q^{\binom{n}{2} - |E(G)|}.$$
The latter form has the following interpretation: to obtain a random graph, we can pick each edge independently with probability p. This leads us to the Erdős–Rényi model [41] (see also [33]).

4.1.3 Isomorphism of structures

Definition 4.1.8. Let F be a species of structures and let U and V be two finite sets of the same cardinality. Structures $s_u \in F[U]$ and $s_v \in F[V]$ are *isomorphic*, if there exists a bijection $\sigma\colon U \to V$ such that

$$s_v = \sigma \cdot s_u.$$

Remark 4.1.9. Since the transport function $F[\sigma]$ is a bijection, for any structure $s_v \in F[V]$ one can find a structure $s_u \in F[U]$ such that s_v and s_u are isomorphic. As a consequence, for studying properties of species of structures up to isomorphisms, it is sufficient to consider sets of the form $[n]$, where n is a non-negative integer.

Example 4.1.10. Consider the species of permutations $\mathcal{P}$. Since the transport function along σ is a conjugation by σ, two permutation are isomorphic if and only if they have the same cycle type.

4.1.4 Exponential generating series

Definition 4.1.11. The *exponential generating series* of a species of structures F is the formal power series

$$F(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!},$$

where $f_n = \big|F[n]\big|$ is the number of labeled F-structures of order n.

Remark 4.1.12. For weighted species $F = F_w$, we need to take the formal power series

$$F_w(z) = \sum_{n=0}^{\infty} \big|F[n]\big|_w \frac{z^n}{n!},$$

where $\big|F[n]\big|_w$ is the total weight of the set of labeled F_w-structures on $[n]$.

Remark 4.1.13. Following the book of Bergeron, Labelle and Leroux [12], we use the expression "exponential generating series" when talking about species, while the expression "exponential generating function" is associated to the same concept attributed to labeled combinatorial classes.

Example 4.1.14. Let us list the exponential generating series of species of structures mentioned in Example 4.1.3. In all the cases, the reader can verify the identities by direct enumeration.

- For the *empty species* $\mathbf{0}$, we have

$$\mathbf{0}(z) = 0.$$

- For the species $\mathbf{1}$ of empty set, we have

$$\mathbf{1}(z) = 1.$$

- For the species $\mathcal{Z}$ of singletons, we have

$$\mathcal{Z}(z) = z.$$

- For the species $\mathcal{E}$ of sets, we have

$$\mathcal{E}(z) = e^z.$$

- For the species $\mathcal{L}$ of linear orders, we have

$$\mathcal{L}(z) = \frac{1}{1-z}.$$

- For the species $\mathcal{P}$ of permutations, we have

$$\mathcal{P}(z) = \frac{1}{1-z}.$$

- For the species $\mathcal{CP}$ of cyclic permutations, we have

$$\mathcal{CP}(z) = \log \frac{1}{1-z}.$$

4.1.5 Type generating series

Definition 4.1.15. Let F be a species of structures and $n \in \mathbb{Z}_{\geqslant 0}$. Structures $s, t \in F[n]$ are *isomorphic* or have the same *isomorphism type*, if there is an automorphism $\pi\colon [n] \to [n]$ such that

$$t = \pi \cdot s.$$

Isomorphism types of F-structures of order n (or *unlabeled F-structures* of order n) are equivalence classes modulo $\sim$ of F-structures on $[n]$, where the equivalence relation $\sim$ is defined by

$$s \sim t \qquad \Leftrightarrow \qquad s \text{ and } t \text{ are isomorphic.}$$

Definition 4.1.16. The *type generating series* of a species of structures F is the formal power series

$$\widetilde{F}(z) = \sum_{n=0}^{\infty} \widetilde{f}_n z^n,$$

where $\widetilde{f}_n$ is the number of unlabeled F-structures of order n.

Remark 4.1.17. In the case of weighted structures, Definition 4.1.4 guarantees that isomorphic structures have the same weight. Hence, the weight of an isomorphism type is well-defined, and we can define the type generating series of F_w by

$$\widetilde{F}_w(z) = \sum_{n=0}^{\infty} |F[n]/\sim|_w z^n,$$

where $\sim$ is the above discussed isomorphism relation and $|F[n]/\sim|_w$ is the total weight of unlabeled F_w-structures on $[n]$.

Remark 4.1.18. In contrast with exponential generating series, type generating series are "ordinary", there is no factorial at the denominator. This fact emphasize a parallel between species and combinatorial classes. In each of these two cases, we use exponential series for labeled objects and ordinary series for unlabeled objects, respectively. The reason, as we have discussed in Remark 3.1.24, is the binomial convolution rule for multiplication of labeled structures. In particular, as we will see in Lemma 4.1.44, this choice of series is consistent with the operation of product of species.

Example 4.1.19. Let us list the type generating series of species of structures mentioned in Examples 4.1.3 and 4.1.14.

- For the *empty species* $\mathbf{0}$, we have
$$\widetilde{\mathbf{0}}(z) = 0.$$

- For the species $\mathbf{1}$ of empty set, we have
$$\widetilde{\mathbf{1}}(z) = 1.$$

- For the species $\mathcal{Z}$ of singletons, we have
$$\widetilde{\mathcal{Z}}(z) = z.$$

- For the species $\mathcal{E}$ of sets, we have
$$\widetilde{\mathcal{E}}(z) = \frac{1}{1-z}.$$

- For the species $\mathcal{L}$ of linear orders, we have
$$\widetilde{\mathcal{L}}(z) = \frac{1}{1-z}.$$

- For the species $\mathcal{P}$ of permutations, we have
$$\widetilde{\mathcal{P}}(z) = \prod_{k=1}^{\infty} \frac{1}{1-z^k}.$$

- For the species $\mathcal{CP}$ of cyclic permutations, we have
$$\widetilde{\mathcal{CP}}(z) = \frac{z}{1-z}.$$

For all the above cases except cyclic permutations, the reader can verify the identities by direct enumeration. For evaluating the type generating series of cyclic permutations, the additional techniques is needed, see Example 4.1.54.

4.1.6 Cycle index series

Definition 4.1.20. Let $n \in \mathbb{N}$ and $\sigma \in S_n$. The *cycle type* of σ is the sequence (σ_k), where σ_k is the number of cycles of length k in the decomposition of σ into disjoint cycles.

Remark 4.1.21. Since $\sigma_k = 0$ for any $k > n$, we can write the cycle type of σ in the vector form $(\sigma_1, \dots, \sigma_n)$.

Notation 4.1.22. For a bijection $\sigma \colon U \to U$, we denote by
$$\operatorname{Fix} \sigma = \{u \in U \mid \sigma(u) = u\}$$
the set of fixed points of σ.

Definition 4.1.23. Let F be a species of structures. Its *cycle index series* is the formal power series in an infinite number of variables $z_1, z_2, z_3 \ldots$ of the form

$$Z_F(z_1, z_2, z_3, \ldots) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} |\mathrm{Fix}\, F[\sigma]|\, z_1^{\sigma_1} z_2^{\sigma_2} z_3^{\sigma_3} \ldots,$$

where $F[\sigma]$ is the transport function $F[n] \to F[n]$.

Remark 4.1.24. The cycle type of $F[\sigma]$,

$$\Big(\big(F[\sigma]\big)_1, \big(F[\sigma]\big)_2, \ldots \Big),$$

depends on the cycle type $(\sigma_1, \sigma_2, \ldots)$ of the permutation σ only. As a consequence, in Definition 4.1.23, all permutations of the same cycle type make the same contribution. Given $n \in \mathbb{N}$ and a cycle type $(k_1, \ldots, k_n)$, where $k_1 + 2k_2 + \ldots + nk_n = n$, the number of permutations of size n with this cycle type is equal to

$$\frac{n!}{1^{k_1} k_1! \ldots n^{k_n} k_n!}.$$

Denote by $\mathrm{fix}\, F[k_1, \ldots, k_n]$ the number of F-structures on $[n]$ which are fixed under the action of any permutation of cycle type $(k_1, \ldots, k_n)$. Then we obtain the following expression for the cycle index series of the species F:

$$Z_F(z_1, z_2, z_3, \ldots) = \sum_{n=0}^{\infty} \sum_{C_n} \mathrm{fix}\, F[k_1, \ldots, k_n] \frac{n!}{1^{k_1} k_1! \ldots n^{k_n} k_n!}, \tag{4.1}$$

where $C_n = \big\{ (k_1, \ldots, k_n) \mid k_1 + 2k_2 + \ldots + nk_n = n \big\}$.

Remark 4.1.25. For weighted species $F = F_w$, the cycle index series takes the form

$$Z_{F_w}(z_1, z_2, z_3 \ldots) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} |\mathrm{Fix}\, F[\sigma]|_w\, z_1^{\sigma_1} z_2^{\sigma_2} z_3^{\sigma_3} \ldots,$$

where $|\mathrm{Fix}\, F[\sigma]|_w$ is the total weight of the set of F_w-structures on $[n]$ that are fixed under the transport function $F[\sigma]$.

Lemma 4.1.26. *For any species of structures F, one has*

$$F(z) = Z_F(z, 0, 0, \ldots)$$

and

$$\widetilde{F}(z) = Z_F(z, z^2, z^3, \ldots).$$

Proof. For the proof, see [12, Theorem 1.2.8]. □

Example 4.1.27. Let us list the cycle index series of species of structures mentioned in Examples 4.1.3, 4.1.14 and 4.1.19.

- For the *empty species* $\mathbf{0}$, we have

$$Z_{\mathbf{0}}(z_1, z_2, \ldots) = 0.$$

- For the species **1** of empty set, we have

$$Z_{\mathbf{1}}(z_1, z_2, \ldots) = 1.$$

- For the species $\mathcal{Z}$ of singletons, we have

$$Z_{\mathcal{Z}}(z_1, z_2, \ldots) = z_1.$$

- For the species $\mathcal{E}$ of sets, we have

$$Z_{\mathcal{E}}(z_1, z_2, \ldots) = \exp\left(\sum_{k=1}^{\infty} \frac{z_k}{k}\right).$$

- For the species $\mathcal{L}$ of linear orders, we have

$$Z_{\mathcal{L}}(z_1, z_2, \ldots) = \frac{1}{1 - z_1}.$$

- For the species $\mathcal{P}$ of permutations, we have

$$Z_{\mathcal{P}}(z_1, z_2, \ldots) = \prod_{k=1}^{\infty} \frac{1}{1 - z_k}.$$

- For the species $\mathcal{CP}$ of cyclic permutations, we have

$$Z_{\mathcal{CP}}(z_1, z_2, \ldots) = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log \frac{1}{1 - z_k},$$

where ϕ is Euler's totient function. For all the above cases except cyclic permutations, the reader can verify the identities by direct enumeration. For evaluating the cycle index series of cyclic permutations, additional techniques are needed, see Example 4.1.54.

4.1.7 Equipotent and isomorphic species

Definition 4.1.28. Two species of structures F and G are *equipotent* if for any finite set U there is a bijection

$$\alpha_U \colon F[U] \to G[U].$$

Notation 4.1.29. For equipotent species F and G, we write $F \equiv G$.

Remark 4.1.30. If species of structures F and G are equipotent, this immediately implies that their exponential generating series coincide,

$$F(z) = G(z),$$

since $|F[n]| = |G[n]|$ for any $n \in \mathbb{Z}_{\geqslant 0}$. However, the fact that $F \equiv G$ does not mean equality of associated cycle index series and, moreover, equality of associated type generating series. In the general case,

$$\begin{aligned} \widetilde{F}(z) &\neq \widetilde{G}(z), \\ Z_F(z_1, z_2, \ldots) &\neq Z_G(z_1, z_2, \ldots). \end{aligned}$$

The concept of equipotent species is a direct analogue of the concept of the combinatorial isomorphism of labeled combinatorial classes.

Definition 4.1.31. Two species of structures F and G are *isomorphic* if for any finite set U there is a bijection $\alpha_U : F[U] \to G[U]$ such that for any F-structure $s \in F[U]$ and any bijection $\sigma : U \to V$ we have

$$\sigma \cdot \alpha_U(s) = \alpha_V(\sigma \cdot s).$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} F[U] & \xrightarrow{\alpha_U} & G[U] \\ {\scriptstyle F[\sigma]}\big\downarrow & & \big\downarrow{\scriptstyle G[\sigma]} \\ F[V] & \xrightarrow{\alpha_V} & G[V] \end{array}$$

Notation 4.1.32. For isomorphic species F and G, we write $F \simeq G$.

Remark 4.1.33. The concept of isomorphism is in the agreement with the series associated to species of structures, meaning that if $F \simeq G$, then

$$\begin{aligned} F(z) &= G(z), \\ \widetilde{F}(z) &= \widetilde{G}(z), \\ Z_F(z_1, z_2, \ldots) &= Z_G(z_1, z_2, \ldots). \end{aligned}$$

Example 4.1.34. An appropriate example of species of structures that are equipotent but not isomorphic is served by permutations $\mathcal{P}$ and linear orders $\mathcal{L}$ (compare with Example 3.1.13). Indeed, any two linear orders are isomorphic, while two permutations are isomorphic if and only if they have the same cycle type, see Example 4.1.10. This fact is also reflected in the behavior of the associated series. As we have seen in Examples 4.1.14 and 4.1.19, the exponential generating series of $\mathcal{P}$ and $\mathcal{L}$ are equal,

$$\mathcal{P}(z) = \frac{1}{1-z} = \mathcal{L}(z),$$

but their type generating series do not coincide,

$$\widetilde{\mathcal{P}}(z) = \prod_{k=1}^{\infty} \frac{1}{1-z^k} \neq \frac{1}{1-z} = \widetilde{\mathcal{L}}(z).$$

Example 4.1.35. Another example of equipotent but non-isomorphic species is species of graphs $\mathcal{G}$ and tournaments $\mathcal{T}$ (compare with Example 3.1.14). Indeed,

$$\mathcal{G}(z) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{z^n}{n!} = \mathcal{T}(z),$$

so that there is a bijection $\mathcal{G}[U] \to \mathcal{T}[U]$ for any finite set U. At the same time, the number of unlabeled structures of order 2 is different: there are two unlabeled graphs with two vertices (connected and disconnected), but the unlabeled tournament with two vertices is unique. Hence,

$$\widetilde{\mathcal{G}}(z) \neq \widetilde{\mathcal{T}}(z).$$

Remark 4.1.36. For weighted species, Definitions 4.1.28 and 4.1.31 of equipotent and isomorphic species remain the same, but with the additional restriction that all bijections α_U have to preserve weights.

4.1.8 Sum of species of structures

Definition 4.1.37. Let F and G be two species of structures. The *sum* $F+G$ is a new species of structures that,

1. for each finite set U, produces a disjoint union
$$(F+G)[U] = F[U] + G[U];$$

2. for each structure $s \in (F+G)[U]$ and each bijection $\sigma\colon U \to V$, produces the transport function
$$(F+G)[\sigma](s) = \begin{cases} F[\sigma](s), & \text{if } s \in F[U], \\ G[\sigma](s), & \text{if } s \in G[U]. \end{cases}$$

Remark 4.1.38. In the case when $F[U] \cap G[U] \neq \emptyset$, we may proceed the same way as we have passed for the disjoint union of combinatorial classes (see Remark 3.1.20). Namely, construct distinct non-intersecting copies of the sets $F[U]$ and $G[U]$, say,

$$\{1\} \times F[U] \qquad \text{and} \qquad \{2\} \times G[U].$$

Then it is sufficient to set

$$(F+G)[U] = \big(\{1\} \times F[U]\big) \cup \big(\{2\} \times G[U]\big).$$

Moreover, Definition 4.1.37 can be generalized onto summable families of species of structures, where $(F_i)_{i \in I}$ is *summable* if for any finite set U, there is a finite number of indices $i \in I$ such that $F_i[U] \neq \emptyset$. For the summable family $(F_i)_{i \in I}$ of species,

$$\left(\sum_{i \in I} F_i\right)[U] = \bigcup_{i \in I} \{i\} \times F_i[U]$$

and

$$\left(\sum_{i \in I} F_i\right)[\sigma](s,i) = \big(F_i[\sigma](s), i\big),$$

where $\sigma\colon U \to V$ is a bijection and

$$(s,i) \in \left(\sum_{i \in I} F_i\right)[U].$$

Lemma 4.1.39. *For any two species of structures F and G, the associated series of the species of structures $F+G$ satisfy*

$$\begin{aligned} (F+G)(z) &= F(z) + G(z), \\ (\widetilde{F+G})(z) &= \widetilde{F}(z) + \widetilde{G}(z), \\ Z_{F+G}(z_1, z_2, \ldots) &= Z_F(z_1, z_2, \ldots) + Z_G(z_1, z_2, \ldots). \end{aligned}$$

Proof. For the proof, see [12, Proposition 1.3.3]. □

Remark 4.1.40. In the case of weighted species F_w and G_v, the weight of a sum structure is defined by

$$\begin{cases} w(s), & \text{if } s \in F[U], \\ v(s), & \text{if } s \in G[U]. \end{cases}$$

This permits to extend Lemma 4.1.39 onto series of weighted species as well [12, Proposition 2.3.11].

Example 4.1.41. Let F be a species of structures and $n \in \mathbb{N}$. Denote the sum of n copies of F by nF. For this sum, we have

$$\begin{aligned} (nF)(z) &= nF(z), \\ \widetilde{(nF)}(z) &= n\widetilde{F}(z), \\ Z_{nF}(z_1, z_2, \ldots) &= nZ_F(z_1, z_2, \ldots). \end{aligned}$$

In particular, if $F = \mathbf{1}$, then the species $n\mathbf{1}$ possesses n structures on the empty set and no structure on any other set. This fact can be interpreted as an embedding of the semi-ring of natural numbers in the semi-ring of species.

Example 4.1.42. Any species of structures F can be canonically represented as a sum of a summable family of species. Indeed, define a countable family $(F_n)_{n\geqslant 0}$ of species F *restricted to cardinality* n by

$$F_n[U] = \begin{cases} F[U], & \text{if } |U| = n, \\ \emptyset, & \text{if } |U| \neq n, \end{cases}$$

with the natural transport functions. Then the following *canonical decomposition* takes place:

$$F = F_0 + F_1 + F_2 + F_3 + \ldots.$$

For instance, the species $\mathcal{E}$ of sets is decomposed into

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \ldots,$$

where $\mathcal{E}_0 = \mathbf{1}$, $\mathcal{E}_1 = \mathcal{Z}$ and for any $n \in \mathbb{Z}_{\geqslant 0}$ holds $\big|\mathcal{E}[n]\big| = 1$. This fact is reflected in formulas for associated series that turn into the following identities:

$$\begin{aligned} e^z &= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots, \\ \frac{1}{1-z} &= 1 + z + z^2 + \ldots, \\ \exp\left(\sum_{k=1}^{\infty} \frac{z_k}{k}\right) &= \sum_{n=1}^{\infty} \sum_{C_n} \frac{z_1^{k_1} \ldots z_n^{k_n}}{1^{k_1} k_1! \ldots n^{k_n} k_n!}, \end{aligned}$$

where $C_n = \big\{(k_1, \ldots, k_n) \mid k_1 + 2k_2 + \ldots + nk_n = n\big\}$.

4.1.9 Product of species of structures

Definition 4.1.43. Let F and G be two species of structures. The *product* $F \cdot G$ is a new species of structures such that any $F \cdot G$-structure on a finite set U is an unordered pair $s = (f, g)$, where f is an F-structure on a subset $U_F \subset U$ and g is a G-structure on a complement $U_G = U \setminus U_F$. In other words,

1. for each finite set U, the product $F \cdot G$ produces a disjoint union of Cartesian products

$$(F \cdot G)[U] = \sum_{\substack{U_F \cup U_G = U \\ U_F \cap U_G = \emptyset}} F[U_F] \times G[U_G];$$

2. for each structure $s = (f, g) \in (F \cdot G)[U]$ and each bijection $\sigma\colon U \to V$, the product $F \cdot G$ produces the transport function

$$(F \cdot G)[\sigma](s) = \Big(F\big[\sigma|_{U_F}\big](f), G\big[\sigma|_{U_G}\big](g) \Big).$$

Lemma 4.1.44. *For any two species of structures F and G, the associated series of the species of structures $F \cdot G$ satisfy*

$$\begin{aligned} (F \cdot G)(z) &= F(z) \cdot G(z), \\ \widetilde{(F \cdot G)}(z) &= \widetilde{F}(z) \cdot \widetilde{G}(z), \\ Z_{F \cdot G}(z_1, z_2, \ldots) &= Z_F(z_1, z_2, \ldots) \cdot Z_G(z_1, z_2, \ldots). \end{aligned}$$

Proof. For the proof, see [12, Proposition 1.3.8]. □

Remark 4.1.45. For weighted species F_w and G_v, the weight of a product structure $s = (f, g)$ is defined by $w(f)v(g)$. With this precision, Lemma 4.1.44 can be extended onto series of weighted species as well [12, Proposition 2.3.11].

Example 4.1.46. Consider the species $\mathcal{L}$ of linear orders. Notice, that its restriction $\mathcal{L}_2$ to cardinality 2 is isomorphic to $\mathcal{Z} \cdot \mathcal{Z}$. More generally, for species $\mathcal{L}$ restricted to cardinality $n \in \mathbb{N}$, one has $\mathcal{L}_n = \mathcal{Z}^n$. This leads to the following canonical decomposition of $\mathcal{L}$:

$$\mathcal{L} = \mathbf{1} + \mathcal{Z} + \mathcal{Z}^2 + \mathcal{Z}^3 + \ldots.$$

Example 4.1.47. Let $\mathcal{DE}$ be the species of *derangements*, that is, permutations without fixed points. Then we have the relation

$$\mathcal{P} = \mathcal{DE} \cdot \mathcal{E},$$

since any permutation can be represented as a product of a derangement and a set of fixed points. This relation leads us, with the help of Lemma 4.1.44, to the expressions for the series associated to $\mathcal{DE}$, namely,

$$\begin{aligned} \mathcal{DE}(z) &= \frac{e^{-z}}{1 - z}, \\ \widetilde{\mathcal{DE}}(z) &= \prod_{n=2}^{\infty} \frac{1}{1 - z^k}, \\ Z_{\mathcal{DE}}(z_1, z_2, \ldots) &= \prod_{k=1}^{\infty} \frac{\exp\big(- z_k/k \big)}{1 - z_k}. \end{aligned}$$

Remark 4.1.48. The operations of addition and multiplication of species of structures are commutative and associative up to isomorphism. Also, they possess as a neutral element the empty species $\mathbf{0}$ and the species $\mathbf{1}$ of empty set, respectively. Moreover, multiplication distributes over addition and possesses the species $\mathbf{0}$ as absorbing element. Thus, the family of species taken up to isomorphisms constitute a semi-ring.

Notation 4.1.49. We denote by Spe the semi-ring of species.

4.1.10 Substitution of species of structures

Definition 4.1.50. Let F and G be two species of structures such that $G[\emptyset] = \emptyset$. The *composite* $F \circ G$ (or *the substitution of* G *in* F) is a new species of structures such that any $F \circ G$-structure on a finite set U is a triplet $s = \big(\pi, \varphi, (\gamma_p)_{p\in\pi}\big)$, where π is a partition of U, φ is an F-structure on the set of classes of π and γ_p is a G-structure on p. In other words,

1. for each finite set U, the composite $F \circ G$ produces a disjoint union of Cartesian products
$$(F \circ G)[U] = \sum_{\pi \text{ partition of } U} F[\pi] \times \prod_{p\in\pi} G[p];$$

2. for each structure $s = (\pi, \varphi, \gamma) \in (F \circ G)[U]$ and each bijection $\sigma \colon U \to V$, the composite $F \circ G$ produces the transport function
$$(F \circ G)[\sigma](s) = \big(\bar{\pi}, \bar{\varphi}, (\bar{\gamma}_{\bar{p}})_{\bar{p}\in\bar{\pi}}\big),$$

where

- $\bar{\pi}$ is the partition of V obtained by transport of π along σ,
- $\bar{p} = \sigma(p) \in \bar{\pi}$ and σ induce the bijection $\bar{\sigma}$ between sets of classes of π and $\bar{\pi}$,
- the structure $\bar{\varphi}$ is obtained from the structure φ by F-transport along $\bar{\sigma}$,
- the structure $\bar{\gamma}_{\bar{p}}$ is obtained from the structure γ_p by G-transport along $\sigma|_p$.

Remark 4.1.51. The operation of substitution is associative up to isomorphism. The species $\mathcal{Z}$ is a neutral element and $\mathbf{0}$ and $\mathbf{1}$ are absorbing elements, so that for any species F, we have
$$F \circ \mathcal{Z} = F,$$
and, if additionally $F[\emptyset] = \emptyset$, then
$$\mathcal{Z} \circ F = F$$
and
$$\mathbf{0} \circ F = \mathbf{0} \qquad \text{and} \qquad \mathbf{1} \circ F = \mathbf{1}.$$
Note also, that the condition $F[\emptyset] = \emptyset$ is equivalent to $F \circ \mathbf{0} = \mathbf{0}$.

Lemma 4.1.52. *For any two species of structures F and G such that $G[\emptyset] = \emptyset$, the associated series of the species of structures $F \circ G$ satisfy*
$$\begin{aligned} (F \circ G)(z) &= F\big(G(z)\big), \\ \widetilde{(F \circ G)}(z) &= Z_F\big(\widetilde{G}(z), \widetilde{G}(z^2), \widetilde{G}(z^3), \ldots\big), \\ Z_{F\circ G}(z_1, z_2, \ldots) &= Z_F\big(Z_G(z_1, z_2, \ldots), Z_G(z_2, z_4, \ldots), \ldots\big). \end{aligned}$$

Proof. For the proof, see [12, Theorem 1.4.2]. □

Remark 4.1.53. In the case of weighted species F_w and G_v, the weight of a composite structure $s = \big(\pi, \varphi, (\gamma_p)_{p\in\pi}\big)$ is defined by

$$w(f)\prod_{p\in\pi} v(\gamma_p).$$

For weighted species, the relation between exponential generating series in Lemma 4.1.52 remains unchanged. As for cycle index series, we have

$$Z_{F_w\circ G_v}(z_1, z_2, \dots) = Z_{F_w}\big(Z_{G_v}(z_1, z_2, \dots), Z_{G_{v^2}}(z_2, z_4, \dots), \dots\big),$$

where the weighting v in the series $Z_{G_{v^k}}(z_k, z_{2k}, z_{3k}, \dots)$ (in the ring $\mathbb{A}$) is raised to the power k (see [12, Proposition 2.3.11]).

Example 4.1.54. Since any permutation can be viewed as a collection of cycles, we have the following representation:

$$\mathcal{P} = \mathcal{E} \circ \mathcal{CP}.$$

In terms of exponential generating series, this leads immediately to

$$\mathcal{CP}(z) = \log\frac{1}{1-z}.$$

For cycle index series, one has

$$\prod_{k=1}^{\infty}\frac{1}{1-z_k} = \exp\sum_{k=1}^{\infty}\frac{1}{k}Z_{\mathcal{CP}}(z_k, z_{2k}, \dots),$$

which, after taking the logarithm and using properties of Möbius function, gives

$$Z_{\mathcal{CP}}(z_1, z_2, \dots) = \sum_{k=1}^{\infty}\frac{\phi(k)}{k}\log\frac{1}{1-z_k}.$$

In particular, by Lemma 4.1.26,

$$\widetilde{\mathcal{CP}}(z) = \sum_{k=1}^{\infty}\frac{\phi(k)}{k}\log\frac{1}{1-z^k} = \frac{z}{1-z}.$$

Note that

$$\widetilde{\mathcal{P}}(z) = \prod_{k=1}^{\infty}\frac{1}{1-z^k} \neq \frac{1-z}{1-2z} = \widetilde{\mathcal{E}}\big(\widetilde{\mathcal{CP}}(z)\big).$$

Thus, this example shows us that in the general case, the relation

$$\widetilde{F\circ G}(z) = \widetilde{F}\big(\widetilde{G}(z)\big)$$

does not hold.

4.1.11 Derivative of species of structures

Definition 4.1.55. Let F be a species of structures. The *derivative* F' is a new species of structures such that any F'-structure on a finite set U is an F-structure on $U^+ = U \cup \{*_U\}$, where $*_U \notin U$. In other words,

1. for each finite set U, the derivative F' produces the set
$$F'[U] = F[U^+];$$
2. for each structure $s \in F'[U]$ and each bijection $\sigma\colon U \to V$, the derivative F' produces the transport function
$$F'[\sigma](s) = F[\sigma^+](s),$$
where $\sigma^+\colon U^+ \to V^+$ is such the bijection that
$$\sigma^+(u) = \begin{cases} \sigma(u), & \text{if } u \in U, \\ *_V, & \text{if } u = *_U. \end{cases}$$

Remark 4.1.56. The concept of derivation is consistent with other operations on species. Namely, for all species F and G, one has

$$\begin{aligned} (F+G)' &= F' + G', \\ (F \cdot G)' &= F' \cdot G + F \cdot G' \end{aligned}$$

and, if additionally $G[\emptyset] = \emptyset$,

$$(F \circ G)' = (F' \circ G) \cdot G'$$

(see [12, Remark 1.4.10]).

Lemma 4.1.57. *For any species of structures F, the associated series of the species of structures F' satisfy*

$$\begin{aligned} F'(z) &= \frac{d}{dz} F(z), \\ \widetilde{F}'(z) &= \left(\frac{\partial}{\partial z_1} Z_F \right)(z, z^2, z^3 \ldots), \\ Z_{F'}(z_1, z_2, z_3 \ldots) &= \left(\frac{\partial}{\partial z_1} Z_F \right)(z_1, z_2, z_3, \ldots). \end{aligned}$$

Proof. For the proof, see [12, Proposition 1.4.8]. □

Remark 4.1.58. For weighted species, the weight of an F'-structure s is simply $w(s)$ and the statement of Lemma 4.1.57 remains unchanged.

Example 4.1.59. Let us consider the derivative of the species $\mathcal{E}$ of sets. By Definition 4.1.55, an $\mathcal{E}'$-structure on a finite set U is a set on $U \cup \{*_U\}$. Removing $*_U$, we obtain a set U again. Thus,

$$\mathcal{E}' = \mathcal{E}.$$

Note that for species $\mathcal{E}_n$ restricted to cardinality $n \in \mathbb{N}$, the derivative produces the same species but restricted to cardinality $n-1$:

$$\mathcal{E}_n' = \mathcal{E}_{n-1}.$$

Example 4.1.60. Let us consider the derivative of the species $\mathcal{CP}$ of cyclic permutations. By Definition 4.1.55, a $\mathcal{CP}'$-structure on a finite set U is a $\mathcal{CP}$-structure on $U \cup \{*_U\}$. The latter can be identified with a linear order on U by forgetting $*_U$, and hence,

$$\mathcal{CP}' = \mathcal{L}.$$

This relation gives one more way to obtain the exponential generating series $\mathcal{CP}(z)$ by integrating

$$\mathcal{CP}'(z) = \mathcal{L}(z) = \frac{1}{1-z}.$$

As for restrictions to a given cardinality $n \in \mathbb{N}$, one has

$$\mathcal{CP}'_n = \mathcal{L}_{n-1}.$$

Example 4.1.61. Let us proceed with the derivative of the species $\mathcal{L}$ of linear orders. In this case, a $\mathcal{L}'$-structure on a finite set U is a linear order on $U \cup \{*_U\}$. Removing $*_U$, we obtain a pair of linear orders on a partition of U into two subsets. Thus,

$$\mathcal{L}' = \mathcal{L}^2.$$

This species identity translates into the following well-known identity of series:

$$\frac{d}{dz}\left(\frac{1}{1-z}\right) = \left(\frac{1}{1-z}\right)^2.$$

Taking a look at the species $\mathcal{L}_n$ of cardinality $n \in \mathbb{N}$, one notice that the following relation takes place:

$$\mathcal{L}'_n = \mathcal{L}_0 \cdot \mathcal{L}_{n-1} + \mathcal{L}_1 \cdot \mathcal{L}_{n-2} + \ldots + \mathcal{L}_{n-1} \cdot \mathcal{L}_0.$$

Since $\mathcal{L}_n = \mathcal{Z}^n$, this identity can be rewritten in a pretty familiar form

$$\mathcal{L}'_n = (\mathcal{Z}^n)' = n\mathcal{Z}^{n-1}.$$

4.1.12 Pointing in a species of structures

Definition 4.1.62. Let F be a species of structures. The species $F^\bullet$ produces *pointed F-structures* of the form $s = (f, u)$, where f is an F-structure on a finite set U and $u \in U$ is a distinguished element. In other words,

1. for each finite set U, the rule $F^\bullet$ produces the set

$$F^\bullet[U] = F[U] \times U;$$

2. for each structure $s = (f, u) \in F^\bullet[U]$ and each bijection $\sigma \colon U \to V$, the rule $F^\bullet$ produces the transport function

$$F^\bullet[\sigma](s) = \big(F[\sigma](f), \sigma(u)\big).$$

Remark 4.1.63. The operation of pointing is related to derivation by the equation

$$F^\bullet = \mathcal{Z} \cdot F'.$$

Lemma 4.1.64. *For any species of structures F, the associated series of the species of structures F' satisfy*

$$\begin{aligned} F^{\bullet}(z) &= z\frac{d}{dz}F(z), \\ \widetilde{F}^{\bullet}(z) &= z\left(\frac{\partial}{\partial z_1}Z_F\right)(z, z^2, z^3, \ldots), \\ Z_{F^{\bullet}}(z_1, z_2, z_3, \ldots) &= z_1\left(\frac{\partial}{\partial z_1}Z_F\right)(z_1, z_2, z_3, \ldots). \end{aligned}$$

Proof. For the proof, see [12, Proposition 2.1.2]. □

Remark 4.1.65. For weighted species, the weight of an $F^{\bullet}$-structure $s = (f, u)$ is $w(f)$ and the statement of Lemma 4.1.57 remains valid.

Example 4.1.66. Let us consider the species $\mathcal{CP}$ of cyclic permutations. When we distinguish an element of a cycle, it can be interpreted as a beginning of the row of elements determined by this cycle. In other words, by distinguishing an element in a cycle, we get a linear order on these elements (compare with Example 3.1.56). Hence, for any positive integer n,

$$\mathcal{CP}^{\bullet}_n = \mathcal{L}_n \qquad \text{and} \qquad \mathcal{CP}^{\bullet} = \mathcal{L} - \mathbf{1}.$$

4.1.13 Cartesian product of species of structures

Definition 4.1.67. Let F and G be two species of structures. The *Cartesian product* $F \times G$ is a new species of structures such that any $F \times G$-structure on a finite set U is a pair $s = (f, g)$, where $f \in F[U]$ and $g \in G[U]$. In other words,

1. for each finite set U, the Cartesian product $F \times G$ produces the set

$$(F \times G)[U] = F[U] \times G[U];$$

2. for each structure $s = (f, g) \in (F \times G)[U]$ and each bijection $\sigma\colon U \to V$, the Cartesian product $F \times G$ produces the transport function

$$(F \times G)[\sigma](s) = \big(F[\sigma](f), G[\sigma](g)\big).$$

Remark 4.1.68. Cartesian product of species possesses associativity and commutativity. The species $\mathcal{E}$ serves as a neutral element for the Cartesian product, that is, for any species F, we have

$$\mathcal{E} \times F = F \times \mathcal{E} = F.$$

Moreover, for any $n \in \mathbb{N}$, the restriction to the cardinality n gives

$$\mathcal{E}_n \times F = F \times \mathcal{E}_n = F_n.$$

Besides, there are the following distributivity laws: for any species F, G and H, we have

$$F \times (G + H) = F \times G + F \times H$$

and

$$(F \times G)^{\bullet} = F^{\bullet} \times G = F \times G^{\bullet}.$$

Definition 4.1.69. The *Hadamard product* of two index series

$$f(z_1, z_2, \ldots) = \sum_{n=0}^{\infty} \sum_{C_n} f_{k_1,\ldots,k_n} \frac{z_1^{k_1} \ldots z_n^{k_n}}{1^{k_1} k_1! \ldots n^{k_n} k_n!}$$

and

$$g(z_1, z_2, \ldots) = \sum_{n=0}^{\infty} \sum_{C_n} g_{k_1,\ldots,k_n} \frac{z_1^{k_1} \ldots z_n^{k_n}}{1^{k_1} k_1! \ldots n^{k_n} k_n!}$$

is defined by

$$(f \odot g)(z_1, z_2, \ldots) = \sum_{n=0}^{\infty} \sum_{C_n} f_{k_1,\ldots,k_n} g_{k_1,\ldots,k_n} \frac{z_1^{k_1} \ldots z_n^{k_n}}{1^{k_1} k_1! \ldots n^{k_n} k_n!},$$

where $C_n = \{(k_1, \ldots, k_n) \mid k_1 + 2k_2 + \ldots + nk_n = n\}$.

Lemma 4.1.70. *For any two species of structures F and G, the associated series of the species of structures $F \times G$ satisfy*

$$\begin{aligned} (F \times G)(z) &= F(z) \odot G(z), \\ (\widetilde{F \times G})(z) &= Z_{F \times G}(z, z^2, \ldots), \\ Z_{F\times G}(z_1, z_2, \ldots) &= Z_F(z_1, z_2, \ldots) \odot Z_G(z_1, z_2, \ldots). \end{aligned}$$

Proof. For the proof, see [12, Proposition 2.1.7]. □

Remark 4.1.71. For weighted species F_w and G_v, the weight of the Cartesian product structure $s = (f, g)$ is defined by $w(f)v(g)$. With this precision, Lemma 4.1.70 can be extended onto series of weighted species as well (see also [12, Proposition 2.3.11]).

Example 4.1.72. We have seen in Example 4.1.34, that species $\mathcal{L}$ of linear orders and $\mathcal{P}$ of permutations are equipotent, but not isomorphic. However, it turns out that there is an isomorphism

$$\mathcal{L} \times \mathcal{L} \simeq \mathcal{L} \times \mathcal{P}.$$

Indeed, for any $n \in \mathbb{N}$ and for any set U of cardinality n, the pair of orders

$$(u_1 \prec_1 u_2 \prec_1 \ldots \prec_1 u_n) \qquad \text{and} \qquad (u_{i_1} \prec_2 u_{i_2} \prec_2 \ldots \prec_2 u_{i_n})$$

naturally corresponds to the pair of linear order and permutation

$$(u_1 \prec u_2 \prec \ldots \prec u_n) \qquad \text{and} \qquad \begin{pmatrix} 1 & 2 & \ldots & n \\ i_1 & i_2 & \ldots & i_n \end{pmatrix},$$

respectively, and vice versa. The same way, for any $m \in \mathbb{N}$,

$$\mathcal{L} \times \underbrace{\mathcal{L} \times \ldots \times \mathcal{L}}_{m} \equiv \mathcal{L} \times \underbrace{\mathcal{P} \times \ldots \times \mathcal{P}}_{m}.$$

4.2 Virtual species

Recall that we denote by Spe the semi-ring of species. In this section, we discuss its extension to the ring of virtual species Virt. The construction of this extension is similar to the construction of $\mathbb{Z}$ from $\mathbb{N}$ and serves the same goal, namely, to define an operation of subtraction. With the help of this operation, we construct multiplicative and substitutional inverses and provide combinatorial interpretations.

4.2.1 Definition of virtual species, operations and series

Definition 4.2.1. A *virtual species* is an element of the quotient set

$$\mathrm{Virt} = (\mathrm{Spe} \times \mathrm{Spe})/\sim,$$

where the equivalence relation $\sim$ is defined by

$$(F, G) \sim (H, K) \qquad \Leftrightarrow \qquad F + K \simeq G + H,$$

$F, G, H, K \in \mathrm{Spe}$.

Notation 4.2.2. We write $F - G$ to designate the class of (F, G) modulo $\sim$, where $F, G \in \mathrm{Spe}$. The pair (F, G) is a *representative* of the class $F - G$.

Definition 4.2.3. Let $F, G, H, K \in \mathrm{Spe}$, so that $\Phi = F - G$ and $\Psi = H - K$ are two virtual species. The operations are extended from Spe to Virt the following way:

- the *sum* of Φ and Ψ is
$$\Phi + \Psi = (F + H) - (G + K),$$
- the *product* of Φ and Ψ is
$$\Phi \cdot \Psi = (F \cdot H + G \cdot K) - (F \cdot K + G \cdot H),$$
- the *derivative* of Φ is
$$\Phi' = F' - G',$$
the *pointing* of Φ is
$$\Phi^{\bullet} = F^{\bullet} - G^{\bullet},$$
- the *Cartesian product* of Φ and Ψ is
$$\Phi \times \Psi = (F \times H + G \times K) - (F \times K + G \times H).$$

Remark 4.2.4. The set Virt of virtues species is a commutative ring under the operations of addition and multiplication. Species

$$\mathbf{0} = (\mathbf{0} - \mathbf{0}) \qquad \text{and} \qquad \mathbf{1} = (\mathbf{1} - \mathbf{0})$$

serve as zero and one of this ring, respectively. All the usual properties of the operations mentioned in Definition 4.2.3, such as $(\Phi + \Psi)' = \Phi' + \Psi'$, remain valid.

Definition 4.2.5. Let $\Phi = F - G$ be a virtual species, where $F, G \in \mathrm{Spe}$. There are the following series associated to Φ:

- the *exponential generating series* is

$$\Phi(z) = F(z) - G(z),$$

- the *type generating series* is

$$\widetilde{\Phi}(z) = \widetilde{F}(z) - \widetilde{G}(z),$$

- the *cycle index series* is

$$Z_\Phi(z_1, z_2, \ldots) = Z_F(z_1, z_2, \ldots) - Z_G(z_1, z_2, \ldots).$$

Remark 4.2.6. The behavior of the series associated to virtual species $\Phi + \Psi$ and $\Phi \cdot \Psi$ is consistent with operations and independent of the choice of representatives $\Phi = F - G$ and $\Psi = H - K$. In other words, these series satisfy the following natural properties:

$$\begin{aligned} (\Phi + \Psi)(z) &= \Phi(z) + \Psi(z), \\ \widetilde{\Phi + \Psi}(z) &= \widetilde{\Phi}(z) + \widetilde{\Psi}(z), \\ Z_{\Phi+\Psi}(z_1, z_2, \ldots) &= Z_\Phi(z_1, z_2, \ldots) + Z_\Psi(z_1, z_2, \ldots) \end{aligned}$$

and

$$\begin{aligned} (\Phi \cdot \Psi)(z) &= \Phi(z) \cdot \Psi(z), \\ \widetilde{\Phi \cdot \Psi}(z) &= \widetilde{\Phi}(z) \cdot \widetilde{\Psi}(z), \\ Z_{\Phi\cdot\Psi}(z_1, z_2, \ldots) &= Z_\Phi(z_1, z_2, \ldots) \cdot Z_\Psi(z_1, z_2, \ldots). \end{aligned}$$

Definition 4.2.7. A virtual species Φ is written in *reduced form* $\Phi = F - G$, if the species F and G are *unrelated*, meaning that there is no isomorphic subspecies in F and G apart from the empty species. Species F and G are the *positive* and *negative* parts of Φ, respectively.

Lemma 4.2.8. *Every virtual species Φ can be written in reduced form $\Phi = \Phi^+ - \Phi^-$ in a unique way.*

Proof. For the proof, see [12, Proposition 2.5.7]. □

Remark 4.2.9. As well as species of structures, any virtual species of structures Φ can be canonically decomposed into the sum of a summable family $(\Phi_n)_{n\geqslant 0}$ with respect to cardinality. Moreover, if $\Phi = F - G$, then there is a decomposition of the form

$$\Phi = \sum_{n=0}^{\infty} \left(\Phi_n^+ - \Phi_n^-\right),$$

where for every $n \in \mathbb{Z}_{\geqslant 0}$, we have $\Phi_n^+ = (F_n - G_n)^+$ and $\Phi_n^- = (F_n - G_n)^-$.

4.2.2 Multiplicative inverse

Lemma 4.2.10. *If F is a species of structures satisfying $F_0 = \mathbf{1}$ (meaning that there exists a unique F-structure on the empty set), then the virtual species*

$$F^{-1} = \sum_{k=0}^{\infty} (-1)^k (F_+)^k,$$

where $F_+ = F - \mathbf{1}$, is the multiplicative inverse of F, meaning that

$$F \cdot F^{-1} = F^{-1} \cdot F = \mathbf{1}.$$

Proof. For the proof, see [12, Exercise 2.5.7]. □

Remark 4.2.11. Lemma 4.2.10 remains valid for a virtual species Φ satisfying $\Phi_0 = \mathbf{1}$. Moreover, for any integers $m, n \in \mathbb{Z}$, one has

$$\Phi^m \cdot \Phi^n = \Phi^{m+n},$$

where $\Phi^n = (F - G)^n$ is defined by

$$\Phi^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} F^{n-k} \cdot G^k.$$

Example 4.2.12. Consider the species $\mathcal{L}$ of linear orders. Since, according to Example 4.1.46,

$$\mathcal{L} = \mathbf{1} + \mathcal{Z} + \mathcal{Z}^2 + \mathcal{Z}^3 + \dots,$$

direct calculation shows that the multiplicative inverse $\mathcal{L}^{-1}$ has the following simple form:

$$\mathcal{L}^{-1} = \mathbf{1} - \mathcal{Z}.$$

The associated series are

$$\begin{aligned} \mathcal{L}^{-1}(z) &= 1 - z, \\ \widetilde{\mathcal{L}^{-1}}(z) &= 1 - z, \\ Z_{\mathcal{L}^{-1}}(z_1, z_2, \dots) &= 1 - z_1. \end{aligned}$$

Example 4.2.13. Consider the species $\mathcal{E}$ of sets. Since $\mathcal{E}_0 = \mathbf{1}$, by Lemma 4.2.10 there exists a multiplicative inverse

$$\mathcal{E}^{-1} = (\mathbf{1} + \mathcal{E}_+)^{-1} = \sum_{k=0}^{\infty} (-1)^k (\mathcal{E}_+)^k.$$

The distribution law for the product over the sum gives us

$$\begin{aligned} \mathcal{E}^{-1}(z) &= e^{-z}, \\ \widetilde{\mathcal{E}^{-1}}(z) &= 1 - z, \\ Z_{\mathcal{E}^{-1}}(z_1, z_2, \dots) &= \exp\left(- \sum_{k=1}^{\infty} \frac{z^k}{k} \right). \end{aligned}$$

As for reduced form $\mathcal{E}^{-1} = (\mathcal{E}^{-1})^+ - (\mathcal{E}^{-1})^-$, we have

$$(\mathcal{E}^{-1})^+ = \mathbf{1} + (\mathcal{E}_+)^2 + (\mathcal{E}_+)^4 + \dots$$

and

$$(\mathcal{E}^{-1})^- = \mathcal{E}_+ + (\mathcal{E}_+)^3 + (\mathcal{E}_+)^5 + \dots.$$

4.2.3 Substitution of virtual species and its inverse

Definition 4.2.14. Let $\Phi = F - G$ and $\Psi = H - K$ be two virtual species, where $F, G, H, K \in \mathrm{Spe}$. Suppose that $\Psi_0 = \mathbf{0}$. The *substitution* $\Phi \circ \Psi$ is a virtual species

$$\Phi \circ \Psi = \Phi(\mathcal{X} + \mathcal{Y}) \times \big(\mathcal{E}(\mathcal{X}) \cdot \mathcal{E}^{-1}(\mathcal{Y})\big) \,|_{\mathcal{X}=H, \mathcal{Y}=K},$$

where $\mathcal{X}$ and $\mathcal{Y}$ are two different species of singletons.

Remark 4.2.15. The reason for such a complicated form of Definition 4.2.14 comes from an extension of the theory of species to the multisort context. Without diving into this theory deeply, let us say that the idea of this generalization is to consider structures constructed on multisets, that is, sets consisting of several sorts of elements. One can distinguish different sorts of elements by considering different species of singletons, say, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, ... Then the sum, say, $\mathcal{X}+\mathcal{Y}$, represents singletons of two different kinds, and hence, if F is a (one-sort) species, then the substitution $F(\mathcal{X}+\mathcal{Y})$ is a two-sort species. Moreover, we can set the proportion of elements of different sorts and, for $m, n \in \mathbb{N}$, have

$$F(m\mathcal{X}+n\mathcal{Y}) = F(\mathcal{X}+\mathcal{Y}) \times \big(\mathcal{E}^m(\mathcal{X}) \cdot \mathcal{E}^n(\mathcal{Y})\big). \tag{4.2}$$

Now, for virtual species, it is natural to generalize (4.2) as

$$F(\mathcal{X}-\mathcal{Y}) = \Phi(\mathcal{X}+\mathcal{Y}) \times G\big(\mathcal{E}(\mathcal{X}) \cdot \mathcal{E}^{-1}(\mathcal{Y})\big),$$

so that

$$F(\Psi) = F(H-K) = \Phi(\mathcal{X}+\mathcal{Y}) \times G\big(\mathcal{E}(\mathcal{X}) \cdot \mathcal{E}^{-1}(\mathcal{Y})\big) \,|_{\mathcal{X}=H,\, \mathcal{Y}=K},$$

and extend it by linearity:

$$(F-G) \circ \Psi = F \circ \Psi - G \circ \Psi.$$

For details, we refer interested reader to [12, Section 2.4].

Remark 4.2.16. All the usual properties of the substitution remain valid for virtual species, including those involving associated series series:

$$\begin{aligned} (\Phi \circ \Psi)(z) &= \Phi\big(\Psi(z)\big), \\ \widetilde{(\Phi \circ \Psi)}(z) &= Z_\Phi\big(\widetilde{\Psi}(z), \widetilde{\Psi}(z^2), \dots\big), \\ Z_{\Phi\circ\Psi}(z_1, z_2, \dots) &= Z_\Phi\big(Z_\Psi(z_1, z_2, \dots), Z_\Psi(z_2, z_4, \dots), \dots\big). \end{aligned}$$

Lemma 4.2.17. *If Ψ is a virtual species of structures satisfying $\Psi_0 = \mathbf{0}$ and $\Psi_1 = \mathcal{Z}$, then there exists a unique virtual species $\Psi^{(-1)}$ that is the inverse of Ψ under substitution, i.e.*

$$\Psi \circ \Psi^{(-1)} = \Psi^{(-1)} \circ \Psi = \mathcal{Z}.$$

Proof. For the proof, see [12, Proposition 2.5.19]. □

Remark 4.2.18. The explicit form of the substitutional inverse $\Psi^{(-1)}$ is

$$\Psi^{(-1)} = \sum_{k=0}^{\infty} (-1)^k \Delta_\Psi^k(\mathcal{Z}),$$

where the linear operator $\Delta_\Psi \colon \text{Virt} \to \text{Virt}$ is defined by

$$\Delta_\Psi(\Phi) = \Phi \circ \Psi - \Phi.$$

In particular,

$$\Psi^{(-1)} = \mathcal{Z} - (\Psi - \mathcal{Z}) + (\Psi \circ \Psi - 2\Psi + \mathcal{Z}) - (\Psi \circ \Psi \circ \Psi - 3\Psi \circ \Psi + 3\Psi - \mathcal{Z}) + \dots.$$

Example 4.2.19. Let Ψ be a virtual species of structures such that $\Psi_0 = \mathbf{0}$ and $\Psi_1 = \mathcal{Z}$. Taking a derivative of the identity

$$\Psi^{(-1)} \circ \Psi = \mathcal{Z}$$

leads us to the relation

$$\big((\Psi^{(-1)})' \circ \Psi\big) \cdot \Psi' = \mathbf{1}.$$

As a consequence, we obtain the following expression for the derivative of the inverse under substitution:

$$(\Psi^{(-1)})' = (\Psi')^{-1} \circ \Psi^{(-1)}.$$

4.2.4 Gargantuan species and Bender's theorem

Definition 4.2.20. Let F be a (virtual, weighted) species of structures. We call F *gargantuan* if the sequence $\big(|F[n]|/n!\big)$ is gargantuan.

Theorem 4.2.21 (Bender's theorem for species)**.** *Let $\mathcal{A}$, $\mathcal{B}$ and F be three (virtual, weighted) species of structures such that*

- *the species $\mathcal{A}$ is gargantuan,*
- *the total weight $\mathfrak{a}_n = |\mathcal{A}[n]|$ of the species $\mathcal{A}$ on the set $[n]$ is nonzero for each $n \in \mathbb{N}$,*
- *the exponential generating series $F(z)$ is analytic in some neighborhood of the origin,*
- *$\mathcal{B} = F(\mathcal{A}_+)$, where $\mathcal{A}_+ = \mathcal{A} - \mathcal{A}_0$.*

Then the species $\mathcal{B}$ is gargantuan and the total weight $\mathfrak{b}_n = |\mathcal{B}[n]|$ satisfies

$$\mathfrak{b}_n \approx \sum_{k \geqslant 0} \binom{n}{k} \mathfrak{c}_k \mathfrak{a}_{n-k},$$

where $\mathfrak{c}_k$ is the total weight of the species $\mathcal{C} = F' \circ \mathcal{A}_+$ on the set $[k]$.

Proof. It is sufficient to apply Theorem 2.2.5 to the formal power series $\mathcal{A}(z)$ and analytic (in some neighborhood of origin) function $F(y)$. □

Remark 4.2.22. The same way as it has been done for combinatorial classes, one can extend Definition 4.2.20 and Theorem 4.2.21 to d-gargantuan species, $d \in \mathbb{N}$.

Part II

General results

Preliminaries and the strategy

The second part of the presented thesis is devoted to establishing the asymptotic expansions for the probabilities that a random object is SEQ, CYC or SET-irreducible. In this preliminary section, we discuss the strategy that allows us to obtain all the desired results. To make the presentation less abstract, we take SEQ-irreducibles as a model example.

Let $\mathcal{U}$ and $\mathcal{W}$ be two labeled combinatorial classes, such that

$$\mathcal{U} = \mathrm{SEQ}(\mathcal{W}).$$

Pick a random object of size n from $\mathcal{U}$ (i.e. pick a random object from $\mathcal{U}_n$). What is the probability that this object is SEQ-irreducible, as $n \to \infty$? In other words, what is the asymptotic probability that the sequence u consists of one element only?

The strategy to obtain the asymptotic expansion for this probability is as follows. First, express the asymptotic behavior of $\mathfrak{w}_n$ in terms of $(\mathfrak{u}_k)$ with the help of Bender's theorem for labeled combinatorial classes. For this part of work, we use the relations between exponential generating functions of $\mathcal{U}$ and $\mathcal{W}$:

$$W(z) = 1 - \frac{1}{U(z)}.$$

Second, divide the obtained expression by $\mathfrak{u}_n$. Thus, on the left hand side, we get the fraction $\mathfrak{w}_n/\mathfrak{u}_n$ whose meaning is exactly the probability that a random object from $\mathcal{U}$ belongs to $\mathcal{W}$, while the right hand side stands for the desired asymptotics in terms of $\mathfrak{u}_n$.

The same way, for a fixed positive integer m, we may state a question about the probability that a random labeled object $u \in \mathcal{U}$ belongs to the labeled combinatorial class

$$\mathcal{W}^{(m)} = \mathrm{SEQ}_m(\mathcal{W}).$$

To answer that questions, we implement the same method. As we will see, the results involve coefficients of corresponding exponential generating functions

$$W^{(s)}(z) = \sum_{n=1}^{\infty} \mathfrak{w}_n^{(s)} \frac{z^n}{n!} = \big(W(z)\big)^s$$

for $s \in \{m-1, m, m+1\}$.

Note what part of the work is common for the mentioned cases. Applying Bender's theorem for labeled combinatorial classes (i.e. Theorem 3.1.79), we need to indicate two input structures: a gargantuan class $\mathcal{A}$ and a function $F(y)$ which is analytic near the origin. While the taken function varies from case to case, the gargantuan class is always the same: $\mathcal{U}$ minus a single element of size zero whose exponential generating series is $U(z) - 1$. We denote this class by $\tilde{\mathcal{U}}$.

The structure of this part is the following. In Chapter 5 we discuss decompositions with respect to the construction SEQ. We establish asymptotic expansions for the probabilities that a random object is SEQ-irreducible or consists of exactly m SEQ-irreducible components. We show that the involved coefficients have a combinatorial meaning and can be explained with the help of the inclusion-exclusion principle. Also, we provide analogous results for the unlabeled case.

Chapters 6 and 7 are devoted to decompositions with respect to the constructions CYC and SET, respectively. We establish the corresponding asymptotics and interpret their coefficients in terms of the inclusion-exclusion principle. Note that for these two constructions, the same simple method does not work for the unlabeled case. We discuss what we could do with that in Part IV.

Finally, in Chapter 8, we generalize the obtained results for species of structures and get the interpretation of coefficients in terms of virtual species.

Chapter 5

SEQ-irreducibles

The goal of this chapter is to establish asymptotic expansions for the probabilities concerning the construction SEQ, as well as to interpret combinatorially the involved coefficients. In Section 5.1, we discuss different approaches to this problem and obtain the desired results in a less formal way. In Sections 5.2 and 5.3 we provide short proofs of the obtained results, while Sections 5.4 and 5.5 are devoted to generalizations.

Throughout the chapter, we consider two combinatorial classes $\mathcal{U}$ and $\mathcal{W}$ such that

$$\mathcal{U} = \mathrm{SEQ}(\mathcal{W}). \tag{5.1}$$

In Sections 5.1–5.3 these classes are gargantuan and labeled, while in Sections 5.4 and 5.5 they are d-gargantuan labeled and gargantuan unlabeled, respectively. In all the cases, by Lemma 3.1.29 and Remark 3.2.10, formula (5.1) translates into the following relation of the exponential generating functions:

$$W(z) = 1 - \frac{1}{U(z)}. \tag{5.2}$$

Also, we consider a family of labeled combinatorial classes

$$\mathcal{W}^{(m)} = \mathrm{SEQ}_m(\mathcal{W}), \tag{5.3}$$

together with their counting sequences $\left(\mathfrak{w}_n^{(m)}\right)$, where $m \in \mathbb{N}$. According to Lemma 3.1.29 and Remark 3.2.10, the corresponding generating functions have the following form:

$$W^{(m)}(z) = \big(W(z)\big)^m. \tag{5.4}$$

In particular, for $m = 1$, we have $W^{(m)}(z) = W(z)$.

5.1 SEQ-irreducibles and the inclusion-exclusion principle

Let $\mathcal{U}$ and $\mathcal{W}$ be two labeled combinatorial classes such that $\mathcal{U}$ is gargantuan and relation (5.1) holds, that is,

$$\mathcal{U} = \mathrm{SEQ}(\mathcal{W}).$$

Let us suppose that the behavior of $\mathcal{U}$ is known. Say, there is an explicit expression for its counting sequence $(\mathfrak{u}_n)$. Having the asymptotic expression for the counting sequence $(\mathfrak{w}_n)$, we can obtain the asymptotic probability that a random element of $\mathcal{U}$ is SEQ-irreducible by dividing it by $\mathfrak{u}_n$. Hence, our goal is to establish the behavior of $\mathcal{W}$. Let us have a look at how this can be done.

5.1.1 Recurrence relations

The first naive idea is to pass from (5.1) to the recurrence relation. We can rewrite the initial relation as follows:

$$\mathcal{U} = \mathrm{SEQ}(\mathcal{W}) = \sum_{m=0}^{\infty} \mathrm{SEQ}_m(\mathcal{W}) = \{\epsilon\} + \sum_{m=1}^{\infty} \mathrm{SEQ}_m(\mathcal{W}) = \{\epsilon\} + \mathcal{W} \star \mathcal{U}.$$

Hence, for any positive integer n,

$$\mathfrak{u}_n = \sum_{k=1}^{n} \binom{n}{k} \mathfrak{w}_k \mathfrak{u}_{n-k}.$$

Taking into account that $\mathfrak{u}_0 = 1$, this gives us the following result.

Lemma 5.1.1. *If $\mathcal{U}$ and $\mathcal{W}$ are labeled combinatorial classes such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$, then for any positive integer n,*

$$\mathfrak{w}_n = \mathfrak{u}_n - \sum_{k=1}^{n-1} \binom{n}{k} \mathfrak{w}_k \mathfrak{u}_{n-k}. \tag{5.5}$$

Corollary 5.1.2. *Let $\mathcal{U}$ and $\mathcal{W}$ be labeled combinatorial classes such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$. Suppose that $u \in \mathcal{U}$ is a random object of size n. Then*

$$\mathbb{P}\big(u \textit{ is } \mathrm{SEQ}\textit{-irreducible}\big) = 1 - \sum_{k=1}^{n-1} \binom{n}{k} \mathfrak{w}_k \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}. \tag{5.6}$$

The results of Lemma 5.1.1 and Corollary 5.1.2 possess two advantages: they are exact and valid for all labeled combinatorial classes, not just gargantuan ones. However, we cannot accept the obtained results as satisfactory enough. In the general case, all the summands in (5.6) may make a significant contribution, and so we cannot even be sure that the leading term of the asymptotic expansion is 1. For gargantuan classes, formula (5.5) shows that $\mathfrak{w}_n \sim \mathfrak{u}_n$, and hence,

$$\mathbb{P}\big(u \text{ is SEQ-irreducible}\big) = 1 + O\left(\frac{\mathfrak{u}_{n-1}}{\mathfrak{u}_n}\right),$$

but we cannot get further precision directly, since the summands $n\mathfrak{w}_1\mathfrak{u}_{n-1}$ and $n\mathfrak{w}_{n-1}\mathfrak{u}_1$ are commensurate. It is possible, however, to obtain more terms recursively. Namely, replacing, in the first approximation, $\mathfrak{w}_{n-1}$ by $\mathfrak{u}_{n-1}$, we have two terms of the expansion:

$$\mathfrak{w}_n = \mathfrak{u}_n - n(\mathfrak{w}_1 + \mathfrak{u}_1)\mathfrak{u}_{n-1} + O(\mathfrak{u}_{n-2}).$$

Using this more accurate approximation, we can obtain the third term etc.

5.1.2 Using the underlying structure

Let us try a different approach now. In order to understand what are the first terms of the asymptotic expansion of $\mathfrak{w}_n$ for the class $\mathcal{U}$, let us focus on the structure of the objects included in this class. If $\mathcal{U}$ is gargantuan, then the main contribution is made by objects that have a large SEQ-irreducible part, and so we have the following relation:

$$\mathfrak{u}_n = \mathfrak{w}_n + \binom{n}{1}\lambda_1\mathfrak{w}_{n-1} + \binom{n}{2}\lambda_2\mathfrak{w}_{n-2} + \binom{n}{3}\lambda_3\mathfrak{w}_{n-3} + \ldots.$$

Here, the constants λ_k are determined by the possible structure of the object. Say, if a sequence consists of two elements such that one of them is of size $n-1$, then the second is of size 1. There are two options for the mutual arrangement of elements, hence,

$$\lambda_1 = 2\mathfrak{w}_1.$$

The same way, if one of the elements of the sequence is of size $n-2$, then we have additionally either an element of size 2 or two elements of size 1. Counting possible arrangements, we have

$$\lambda_2 = 2\mathfrak{w}_2 + 6\mathfrak{w}_1^2,$$

and so on. This reasoning leads us to the following result.

Lemma 5.1.3. *If $\mathcal{U}$ and $\mathcal{W}$ are labeled combinatorial classes such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$, then for any positive integer n,*

$$\mathfrak{u}_n = \sum_{C_n} \frac{n!}{(1!)^{c_1} \cdot \ldots \cdot (n!)^{c_n}} \cdot \frac{(c_1 + \ldots + c_n)!}{c_1! \cdot \ldots \cdot c_n!} \cdot \left(\mathfrak{w}_1^{c_1} \cdot \ldots \cdot \mathfrak{w}_n^{c_n}\right), \tag{5.7}$$

where $C_n = \{(c_1, \ldots, c_n) \mid c_1 + 2c_2 + \ldots + nc_n = n\}$.

Proof. Let us enumerate all possible structures in $\mathcal{U}_n$ for any positive integer n. Each object in $\mathcal{U}_n$ is a sequence whose elements sizes are determined by the n-sequence $(c_1, \ldots, c_n)$ such that c_k is the number of elements of size k (and hence, $c_1 + 2c_2 + \ldots + nc_n = n$). Let us find the number of objects determined by the same n-sequence. Given an n-sequence $(c_1, \ldots, c_n)$, we need to choose

1. the order of elements sizes:
$$\frac{(c_1 + \ldots + c_n)!}{c_1! \cdot \ldots \cdot c_n!} \text{ choices;}$$
2. labels for each element:
$$\frac{n!}{(1!)^{c_1} \cdot \ldots \cdot (n!)^{c_n}} \text{ choices;}$$
3. elements themselves:
$$\mathfrak{w}_1^{c_1} \cdot \ldots \cdot \mathfrak{w}_n^{c_n} \text{ choices.}$$

Adding up, we get the desired expression (5.7).

Note that the same result can be obtained formally from the expression for the exponential generating function of $\mathcal{U}$ as a geometric progression:

$$U(z) = 1 + W(z) + W^2(z) + W^3(z) + \ldots.$$

□

Lemma 5.1.3 works well, if we wish to obtain the asymptotic behavior of $\mathcal{U}$ in terms of gargantuan class $\mathcal{W}$. However, for the inverse task, this lemma is not good enough. The same way, as with the help of Lemma 5.1.1, we could get the asymptotic expansion recursively, term by term. The result would be the same though written in slightly different form. For instance, the first two terms of the asymptotics are

$$\mathfrak{w}_n = \mathfrak{u}_n - 2n\mathfrak{w}_1\mathfrak{u}_{n-1} + O(\mathfrak{u}_{n-2}),$$

which is the same as we have seen above, since $\mathfrak{u}_1 = \mathfrak{w}_1$.

5.1.3 The inclusion-exclusion principle

We have seen two ways presented in Lemmas 5.1.1 and 5.1.3 that lead us to the desired asymptotics. As we have mentioned, they both do not permit to have all terms at once, meaning that we need a recursive procedure to get one term after another. Nevertheless, the second proof of Lemma 5.1.3 suggests a possible solution for this obstacle.

Lemma 5.1.4. *If $\mathcal{U}$ and $\mathcal{W}$ are labeled combinatorial classes such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$, then for any positive integer n,*

$$\mathfrak{w}_n = \sum_{C_n} (-1)^{(c_1+\ldots+c_n)-1} \cdot \frac{n!}{(1!)^{c_1} \cdot \ldots \cdot (n!)^{c_n}} \cdot \frac{(c_1+\ldots+c_n)!}{c_1! \cdot \ldots \cdot c_n!} \cdot \left(\mathfrak{u}_1^{c_1} \cdot \ldots \cdot \mathfrak{u}_n^{c_n}\right), \tag{5.8}$$

where $C_n = \{(c_1, \ldots, c_n) \mid c_1 + 2c_2 + \ldots + nc_n = n\}$.

Proof. With the help of relation (5.2), express the exponential generating function of $\mathcal{W}$ as follows:

$$W(z) = \tilde{U}(z) - \tilde{U}^2(z) + \tilde{U}^3(z) - \ldots,$$

where $\tilde{U}(z) = U(z) - 1$. Now, the desired formula (5.8) can be derived from this relation directly. □

For gargantuan classes, Lemma 5.1.4 immediately provides as many terms as we want. For example, the first three terms of the asymptotic probability that a random object $u \in \mathcal{U}$ from a gargantuan labeled combinatorial class $\mathcal{U}$ is SEQ-irreducible are

$$\mathbb{P}\big(u \text{ is SEQ-irreducible}\big) = 1 - 2\mathfrak{u}_1 \binom{n}{1} \frac{\mathfrak{u}_{n-1}}{\mathfrak{u}_n} - \left(2\mathfrak{u}_2 - 6\mathfrak{u}_1^2\right) \binom{n}{2} \frac{\mathfrak{u}_{n-2}}{\mathfrak{u}_n} + O\left(\frac{\mathfrak{u}_{n-3}}{\mathfrak{u}_n}\right).$$

However, in this form the meaning of the involved coefficients is somehow hidden. To reach out this meaning, use the inclusion-exclusion principle and rewrite the coefficients in terms of $\mathfrak{w}_k$. The idea is to enumerate $\mathfrak{u}_n$ objects from $\mathcal{U}_n$ according to their structure. Roughly speaking, any object is either SEQ-irreducible or consists of at least two SEQ-irreducible components. In the latter case, first, enumerate those objects whose first component is of size less than $n/2$. Second, list objects with the last component of size less than $n/2$. Finally, we need to deduce objects counted twice, since both their first and last components are smaller than $n/2$. In total, there are the following four cases schematically depicted on Fig. 5.1.

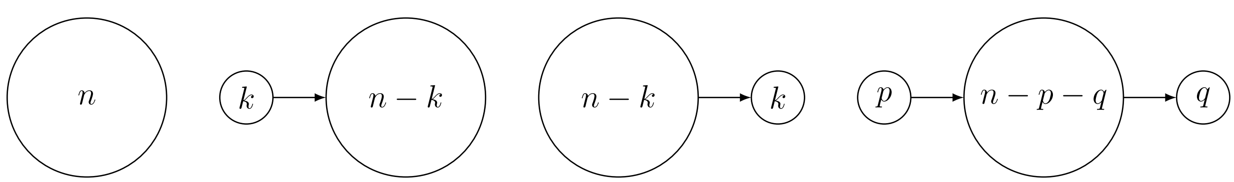


Figure 5.1: Schema for counting objects in $\mathcal{U}_n$.

1. There are $\mathfrak{w}_n$ SEQ-irreducible objects of size n.

2. For each $k \leqslant n/2$, there are $\binom{n}{k}\mathfrak{w}_k\mathfrak{u}_{n-k}$ objects decomposed into a sequence of length at least two whose first element is of size k.

3. For each $k \leqslant n/2$, there are $\binom{n}{k}\mathfrak{w}_k\mathfrak{u}_{n-k}$ objects decomposed into a sequence of length at least two whose last element is of size k.

4. For each pair $p, q \leqslant n/2$, there are $\binom{n}{p,q}\mathfrak{w}_p\mathfrak{w}_q\mathfrak{u}_{n-p-q}$ objects decomposed into a sequence of length at least two whose first and last elements are of size p and q, respectively.

Summarizing, by the inclusion-exclusion principle, we obtain the following result.

Lemma 5.1.5. *If $\mathcal{U}$ and $\mathcal{W}$ are labeled combinatorial classes such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$, then for any positive integer n,*

$$\mathfrak{u}_n = \mathfrak{w}_n + 2\sum_{k=1}^{[n/2]}\binom{n}{k}\mathfrak{w}_k\mathfrak{u}_{n-k} - \sum_{p=1}^{[n/2]}\sum_{q=1}^{[n/2]}\binom{n}{p,q}\mathfrak{w}_p\mathfrak{w}_q\mathfrak{u}_{n-p-q}. \tag{5.9}$$

Corollary 5.1.6. *Let $\mathcal{U}$ and $\mathcal{W}$ be labeled combinatorial classes such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$. Suppose that $u \in \mathcal{U}$ is a random object of size n. Then*

$$\mathbb{P}\big(u \text{ is SEQ-irreducible}\big) = 1 - 2\sum_{k=1}^{[n/2]}\binom{n}{k}\mathfrak{w}_k\frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n} + \sum_{p=1}^{[n/2]}\sum_{q=1}^{[n/2]}\binom{n}{p,q}\mathfrak{w}_p\mathfrak{w}_q\frac{\mathfrak{u}_{n-p-q}}{\mathfrak{u}_n}. \tag{5.10}$$

Lemma 5.1.5, as well as Lemma 5.1.4, is valid for any labeled combinatorial classes $\mathcal{U}$ and $\mathcal{W}$ such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$. In the case when $\mathcal{U}$ is gargantuan, we obtain the asymptotic expansion together with the combinatorial interpretation of the coefficients. Indeed, let $r \in \mathbb{N}$ and n is large enough (it is sufficient to have $n \geqslant 2r$). Then, since $\mathfrak{w}_n \leqslant \mathfrak{u}_n$ for any $n \in \mathbb{N}$, we have

$$2\sum_{k=1}^{[n/2]}\binom{n}{k}\mathfrak{w}_k\mathfrak{u}_{n-k} \leqslant 2\sum_{k=1}^{r-1}\binom{n}{k}\mathfrak{w}_k\mathfrak{u}_{n-k} + \sum_{k=r}^{n-r}\binom{n}{k}\mathfrak{u}_k\mathfrak{u}_{n-k} = 2\sum_{k=1}^{r-1}\binom{n}{k}\mathfrak{w}_k\mathfrak{u}_{n-k} + O(n^r\mathfrak{u}_{n-r}).$$

The same way, as

$$\mathfrak{w}_k^{(2)} = \sum_{p=1}^{k-1}\binom{k}{p}\mathfrak{w}_p\mathfrak{w}_{k-p},$$

one has the following estimation:

$$\sum_{p=1}^{[n/2]}\sum_{q=1}^{[n/2]}\binom{n}{p,q}\mathfrak{w}_p\mathfrak{w}_q\mathfrak{u}_{n-p-q} \leqslant \sum_{k=1}^{r-1}\binom{n}{k}\mathfrak{w}_k^{(2)}\mathfrak{u}_{n-k} + \sum_{r\leqslant p+q\leqslant n}\binom{n}{p,q}\mathfrak{u}_p\mathfrak{u}_q\mathfrak{u}_{n-p-q}.$$

According to Lemma 3.1.76,

$$\sum_{r\leqslant p+q\leqslant n}\binom{n}{p,q}\mathfrak{u}_p\mathfrak{u}_q\mathfrak{u}_{n-p-q} = O(n^r\mathfrak{u}_{n-r}).$$

Hence, for any $r \in \mathbb{N}$,

$$\mathfrak{u}_n = \mathfrak{w}_n + \sum_{k=1}^{r-1}\big(2\mathfrak{w}_k - \mathfrak{w}_k^{(2)}\big)\binom{n}{k}\mathfrak{u}_{n-k} + O(n^r\mathfrak{u}_{n-r}).$$

Rearranging the summands and dividing by $\mathfrak{u}_n$, we get the asymptotic probability

$$\mathbb{P}\big(u \text{ is SEQ-irreducible}\big) \approx 1 - \sum_{k\geqslant 1}\big(2\mathfrak{w}_k - \mathfrak{w}_k^{(2)}\big)\binom{n}{k}\frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

We will give a short proof of this relation in Section 5.2 (see Theorem 5.2.1). The meaning of the coefficient $\big(2\mathfrak{w}_k - \mathfrak{w}_k^{(2)}\big)$ is illustrated by Fig. 5.1: it is SEQ-irreducible objects of size k counted twice minus ordered pairs of SEQ-irreducible objects of total size k.

5.1.4 Several SEQ-irreducible components

The aim of this section is to generalize the statement of Lemma 5.1.5. We start with enumerating objects consisting of two SEQ-irreducible parts. On the one hand, the number of such objects is $\mathfrak{w}_n^{(2)}$. On the other hand, let us apply the inclusion-exclusion principle to count them in a different way. The idea is to enumerate objects with at least two SEQ-irreducible components with respect to the sizes and then to deduce those with at least three components.

1. First, count objects whose first component has size $k \leqslant n/2$. There are $\binom{n}{k}\mathfrak{w}_k\mathfrak{u}_{n-k}$ of them.
2. Second, deduce all $\binom{n}{k,p}\mathfrak{w}_k\mathfrak{w}_p\mathfrak{u}_{n-k-p}$ objects consisting of at least three components such that the first and second components are of sizes $k \leqslant n/2$ and $p \leqslant [(n-k)/2]$, respectively.
3. The same way, deduce all $\binom{n}{k,p}\mathfrak{w}_k\mathfrak{w}_p\mathfrak{u}_{n-k-p}$ objects consisting of at least three components such that the first and last components are of sizes $k \leqslant n/2$ and $p \leqslant [(n-k)/2]$, respectively.
4. Finally, we need to add $\binom{n}{k,p,q}\mathfrak{w}_k\mathfrak{w}_p\mathfrak{w}_q\mathfrak{u}_{n-k-p-q}$ objects consisting of at least three components such that the first, second and last components are of sizes $k \leqslant n/2$, $p \leqslant [(n-k)/2]$ and $q \leqslant [(n-k)/2]$, respectively.

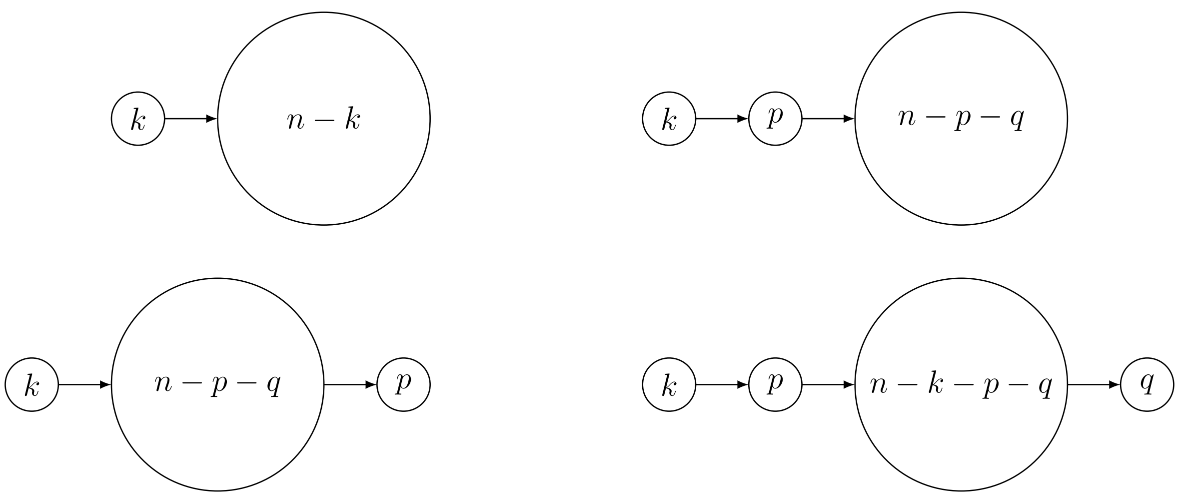


Figure 5.2: Schema for counting objects with two SEQ-irreducible components.

Schematically, we depict the structures of discussed objects on Fig. 5.2. Adding them up, we obtain the number of objects consisting of two SEQ-irreducible components whose first component is of the size $k \leqslant n/2$.

The same way, we get the number of objects with two SEQ-irreducible components, whose second component is of the size $k \leqslant n/2$. Note that in the case when n is even, objects consisting of two components of the same size are counted twice. Hence, we need to deduce them from the total sum. Taking this observation into account, we obtain the following result.

Lemma 5.1.7. *If $\mathcal{U}$ and $\mathcal{W}$ are labeled combinatorial classes such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$, then for any positive integer n,*

$$\mathfrak{w}_n^{(2)} = 2\sum_{k=1}^{[n/2]} \mathfrak{w}_k \binom{n}{k} \left(\mathfrak{u}_{n-k} - 2\sum_{p=1}^{[(n-k)/2]} \binom{n-k}{p} \mathfrak{w}_p \mathfrak{u}_{n-k-p} + \sum_{p,q=1}^{[(n-k)/2]} \binom{n-k}{p,q} \mathfrak{w}_p \mathfrak{w}_q \mathfrak{u}_{n-k-p-q} \right) - \Lambda,$$

where

$$\Lambda = \begin{cases} \binom{n}{n/2}\mathfrak{w}_{n/2}^2, & \text{if } n \text{ is even},\\ 0, & \text{if } n \text{ is odd}. \end{cases}$$

In the case when $\mathcal{U}$ is gargantuan, we can pass from Lemma 5.1.7 to the asymptotic expansion in a similar way that we have done for Lemma 5.1.5. Namely, for given $r \in \mathbb{N}$, we suppose that n is large enough, and hence, leaving only terms with factors $\mathfrak{u}_n, \mathfrak{u}_{n-1}, \ldots, \mathfrak{u}_{n-r+1}$, we have

$$\mathfrak{w}_n^{(2)} = 2\sum_{k=1}^{r-1}\left(\mathfrak{w}_k - 2\mathfrak{w}_k^{(2)} + \mathfrak{w}_k^{(3)}\right)\binom{n}{k}\mathfrak{u}_{n-k} + O(n^r\mathfrak{u}_{n-r}).$$

This lead us to the following asymptotic expansion for the probability that a random object $u \in \mathcal{U}$ consists of two SEQ-irreducible components:

$$\mathbb{P}(u \text{ has two SEQ-irreducible components}) \approx 2\sum_{k\geqslant 0}\left(\mathfrak{w}_k - 2\mathfrak{w}_k^{(2)} + \mathfrak{w}_k^{(3)}\right)\cdot\binom{n}{k}\cdot\frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

The reasoning leading to the approximation above can be generalize to objects consisting of the arbitrary number m of SEQ-irreducible components. We will prove in Section 5.3 that the following relation takes place:

$$\mathbb{P}(u \text{ has } m \text{ SEQ-irreducible components}) \approx m\sum_{k\geqslant 0}\left(\mathfrak{w}_k^{(m-1)} - 2\mathfrak{w}_k^{(m)} + \mathfrak{w}_k^{(m+1)}\right)\cdot\binom{n}{k}\cdot\frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

5.2 Asymptotic probability of SEQ-irreducible objects

Theorem 5.2.1 (SEQ asymptotics)**.** *Let $\mathcal{U}$ be a gargantuan labeled combinatorial class with positive coefficients, such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ for some labeled combinatorial class $\mathcal{W}$. Suppose that $u \in \mathcal{U}$ is a random object of size n. Then*

$$\mathbb{P}(u \text{ is SEQ-irreducible}) \approx 1 - \sum_{k\geqslant 1}\left(2\mathfrak{w}_k - \mathfrak{w}_k^{(2)}\right)\cdot\binom{n}{k}\cdot\frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}. \tag{5.11}$$

Proof. Let us apply Theorem 3.1.79 a) to the gargantuan labeled combinatorial class $\tilde{\mathcal{U}} = \mathcal{U} - \{\epsilon\}$ and the function

$$F(y) = 1 - \frac{1}{1+y}.$$

According to (5.2),

$$F\big(U(z) - 1\big) = 1 - \frac{1}{U(z)} = W(z),$$

while, due to (5.4),

$$C(z) = \frac{1}{\big(U(z)\big)^2} = \big(1 - W(z)\big)^2 = 2W(z) - W^{(2)}(z).$$

Together with the statement of Theorem 3.1.79 a), this implies

$$\mathfrak{w}_n \approx \sum_{k\geqslant 0}\binom{n}{k}c_k\mathfrak{u}_{n-k} = \mathfrak{u}_n - \sum_{k\geqslant 1}\binom{n}{k}\left(2\mathfrak{w}_k - \mathfrak{w}_k^{(2)}\right)\mathfrak{u}_{n-k},$$

and it is sufficient to divide both sides by $\mathfrak{u}_n$ to finish the proof. $\square$

Example 5.2.2. As we have seen in Example 3.1.33, any tournament can be decomposed as a sequence of irreducible tournaments,

$$\mathcal{T} = \mathrm{SEQ}(\mathcal{IT}).$$

Since the labeled combinatorial class $\mathcal{T}$ is gargantuan (see Example 3.1.73), we can apply Theorem 5.2.1 that gives us

$$\mathbb{P}\big(\text{tournament is irreducible}\big) \approx 1 - \sum_{k\geqslant 1}\Big(2\mathfrak{it}_k - \mathfrak{it}_k^{(2)}\Big)\cdot\binom{n}{k}\cdot\frac{2^{k(k+1)/2}}{2^{nk}},$$

where the counting sequence of irreducible tournaments (A054946 in the OEIS [76]) is

$$(\mathfrak{it}_k) = 1,\ 0,\ 2,\ 24,\ 544,\ 22\,320,\ 1\,677\,488,\ 236\,522\,496,\ 64\,026\,088\,576,\ \ldots$$

and the counting sequences of tournaments consisting of two irreducible parts is

$$\big(\mathfrak{it}_k^{(2)}\big) = 0,\ 2,\ 0,\ 16,\ 240,\ 6\,608,\ 315\,840,\ 27\,001\,984,\ 4\,268\,194\,560,\ \ldots$$

In particular, as it was shown by Wright [103],

$$\mathbb{P}\big(\text{tournament is irreducible}\big) = 1 - \binom{n}{1}\cdot 2^{2-n} + \binom{n}{2}\cdot 2^{4-2n} - \binom{n}{3}\cdot 2^{8-3n} - \binom{n}{4}\cdot 2^{15-4n} + O\big(n^5\cdot 2^{-5n}\big).$$

Remark 5.2.3. Theorem 5.2.1 provides us an indirect method of showing that a labeled combinatorial class $\mathcal{U}$ is not gargantuan. Indeed, let $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ and $u \in \mathcal{U}$ be a random object of size n. Suppose that

$$\lim_{n\to\infty} \mathbb{P}\big(u \text{ is SEQ-irreducible}\big) \neq 1.$$

Then, in accordance with Theorem 5.2.1, $\mathcal{U}$ is not gargantuan.

Example 5.2.4. Consider the labeled combinatorial class $\mathcal{L}$ of linear orders. As we have seen in Example 3.1.32,

$$\mathcal{L} = \mathrm{SEQ}(\mathcal{Z}).$$

Since for $n > 1$ any linear order $\prec\, \in \mathcal{L}_n$ is SEQ-reducible in terms of this decomposition, we have

$$\mathbb{P}(\prec \text{ is SEQ-irreducible}) = 0.$$

Hence, by Remark 5.2.3, $\mathcal{L}$ is not gargantuan. Note that this fact is consistent with Example 3.1.75.

5.3 Objects with several SEQ-irreducible components

Theorem 5.3.1 (SEQ_m asymptotics)**.** *Let $\mathcal{U}$ be a gargantuan labeled combinatorial class with positive coefficients, such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ for some labeled combinatorial class $\mathcal{W}$. Suppose that $u \in \mathcal{U}$ is a random object of size n. Then for any $m \in \mathbb{N}$,*

$$\mathbb{P}(u \textit{ has } m \text{ SEQ-}\textit{irreducible components}) \approx \sum_{k\geqslant 0} \beta_k^{(m)}\cdot\binom{n}{k}\cdot\frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}, \tag{5.12}$$

where

$$\beta_k^{(m)} = m\Big(\mathfrak{w}_k^{(m-1)} - 2\mathfrak{w}_k^{(m)} + \mathfrak{w}_k^{(m+1)}\Big).$$

Proof. Apply Theorem 3.1.79 a) to the gargantuan class $\tilde{\mathcal{U}} = \mathcal{U} - \{\epsilon\}$ and the function

$$F_m(y) = \left(1 - \frac{1}{1+y}\right)^{m+1}.$$

According to (5.2) and (5.4),

$$F_m\big(U(z) - 1\big) = \left(1 - \frac{1}{U(z)}\right)^{m} = W^{(m)}(z),$$

while

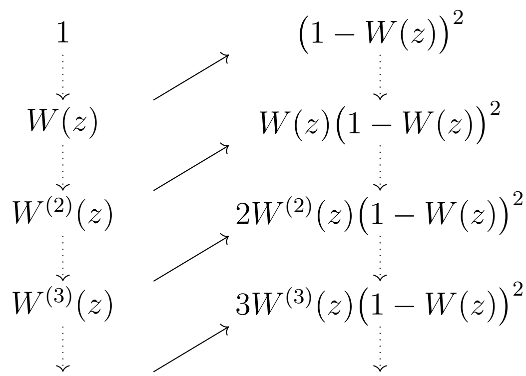
$$\begin{aligned} C_m(z) &= \left(1 - \frac{1}{U(z)}\right)^{m-1} \frac{m}{\big(U(z)\big)^2} \\ &= m\big(W(z)\big)^{m-1}\big(1 - W(z)\big)^2 \\ &= m\Big(W^{(m-1)}(z) - 2W^{(m)}(z) + W^{(m+1)}(z)\Big). \end{aligned}$$

Hence, the statement of Theorem 3.1.79 a) gives us

$$\mathfrak{w}_n^{(m)} \approx \sum_{k\geqslant 0} \binom{n}{k} c_k \mathfrak{u}_{n-k} = \sum_{k\geqslant 0} \binom{n}{k} \beta_k^{(m)} \mathfrak{u}_{n-k}.$$

To finish the proof, divide the obtained formula by $\mathfrak{u}_n$. □

Remark 5.3.2. Defining $W^{(0)}(z) = \big(W(z)\big)^0 = 1$, we can see that Theorem 5.2.1 is the particular case of Theorem 5.3.1 for $m = 1$. The coefficients of asymptotic probabilities fit into the general scheme depicted below. Here, in the left column, there are exponential generating functions corresponding to the family of labeled combinatorial classes $\mathcal{W}^{(m)}$, $m \in \mathbb{Z}_{\geqslant 0}$, while the right column represents the same functions multiplied by $\big(1 - W(z)\big)^2$. By Theorem 5.3.1, the coefficients involved into the asymptotic expansion of $\mathcal{W}^{(m)}$ constitute the exponential generating function $m\big(W(z)\big)^{m-1}\big(1-W(z)\big)^2$. This fact is reflected by solid arrows pointing from the left column to the right one.



Remark 5.3.3. Every labeled combinatorial object $u \in \mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ has a fixed number m of SEQ-irreducible components. This trivial fact implies the identity

$$\mathfrak{u}_n = \sum_{m=1}^{\infty} \mathfrak{w}_n^{(m)}$$

for any positive integer n. As a consequence, in the proof of Theorem 5.3.1 the sum of the series $C_m(z)$ over all $m \geqslant 1$ should be equal to 1. We can check it directly as follows:

$$\sum_{m=1}^{\infty} C_m(z) = \sum_{m=1}^{\infty} m\Big(W^{(m-1)}(z) - 2W^{(m)}(z) + W^{(m+1)}(z)\Big) = W^{(0)}(z) = 1.$$

Corollary 5.3.4. *Let $\mathcal{U}$ be a gargantuan labeled combinatorial class with positive coefficients, such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ for some labeled combinatorial class $\mathcal{W}$. Suppose that $u \in \mathcal{U}$ is a random object of size n. Then for any $m \in \mathbb{N}$,*

$$\mathbb{P}(u \text{ has } m \text{ SEQ-irreducible components}) = m \cdot (n)_{m-1} \cdot \frac{\mathfrak{u}_1^{m-1}\mathfrak{u}_{n-m+1}}{\mathfrak{u}_n} + O\left(n^m \cdot \frac{\mathfrak{u}_{n-m}}{\mathfrak{u}_n}\right), \qquad (5.13)$$

where $(n)_m = n(n-1)(n-2)\dots(n-m+1)$ are the falling factorials.

Proof. According to Theorem 5.3.1, we need to look at the leading term of the exponential generating function $m\Big(W^{(m-1)}(z) - 2W^{(m)}(z) + W^{(m+1)}(z)\Big)$. Taking into account that $\mathfrak{w}_1 = \mathfrak{u}_1$, we obtain

$$m\Big(W^{(m-1)}(z) - 2W^{(m)}(z) + W^{(m+1)}(z)\Big) = m! \cdot \mathfrak{u}_1^{m-1} \cdot \frac{z^{m-1}}{(m-1)!} + \dots$$

As a consequence, the first non-zero coefficient in asymptotic probability (5.12) turns out to be

$$m! \cdot \mathfrak{u}_1^{m-1}\left(\binom{n}{m-1} \cdot \frac{\mathfrak{u}_{n-m+1}}{\mathfrak{u}_n}\right) = m \cdot (n)_{m-1} \cdot \frac{\mathfrak{u}_1^{m-1}\mathfrak{u}_{n-m+1}}{\mathfrak{u}_n}.$$

□

Example 5.3.5. Apply Theorem 5.3.1 for tournaments (it is possible, since $\mathcal{T} = \mathrm{SEQ}(\mathcal{IT})$ and $\mathcal{T}$ is gargantuan, see Example 5.2.2). In this case, we have the family of labeled combinatorial classes $\mathcal{IT}^{(m)}$ of tournaments consisting of m irreducible parts, where m runs over all positive integers. The counting sequences of $\mathcal{IT}^{(m)}$ for $m \leqslant 5$ are listed in Table 5.1.

n	1	2	3	4	5	6	7	8	9	
$(\mathfrak{it}_n)$	1	0	2	24	544	22 320	1 677 488	236 522 496	64 026 088 576	…
$(\mathfrak{it}_n^{(2)})$	0	2	0	16	240	6 608	315 840	27 001 984	4 268 194 560	…
$(\mathfrak{it}_n^{(3)})$	0	0	6	0	120	2 160	70 224	3 830 400	366 729 600	…
$(\mathfrak{it}_n^{(4)})$	0	0	0	24	0	960	20 160	758 016	46 448 640	…
$(\mathfrak{it}_n^{(5)})$	0	0	0	0	120	0	8 400	201 600	8 628 480	…

Table 5.1: Counting sequences $\left(\mathfrak{it}_n^{(m)}\right)$ for $m \leqslant 5$.

According to Theorem 5.3.1, the asymptotic probability, that a random tournament of size n has exactly m irreducible parts, reads

$$\mathbb{P}\big(\text{tournament has } m \text{ irreducible parts}\big) \approx \sum_{k \geqslant 0} \beta_k^{(m)} \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}},$$

where $\beta_k^{(m)} = m\left(\mathfrak{it}_k^{(m-1)} - 2\mathfrak{it}_k^{(m)} + \mathfrak{it}_k^{(m+1)}\right)$. Sequences $\left(\beta_k^{(m)}\right)$ for $m \leqslant 5$ are listed in Table 5.2.

k	0	1	2	3	4	5	6	7	8	
$\left(\beta_k^{(1)}\right)$	1	−2	2	−4	−32	−848	−38 032	−3 039 136	−446 043 008	...
$\left(\beta_k^{(2)}\right)$	0	2	−8	16	−16	368	22 528	2 232 064	372 697 856	...
$\left(\beta_k^{(3)}\right)$	0	0	6	−36	120	0	9 744	586 656	60 297 600	...
$\left(\beta_k^{(4)}\right)$	0	0	0	24	−192	960	960	153 216	10 063 872	...
$\left(\beta_k^{(5)}\right)$	0	0	0	0	120	−1 200	8 400	16 800	2 177 280	...

Table 5.2: Asymptotic coefficients for tournaments.

In particular, according to Corollary 5.3.4, the leading term of this asymptotic probability is

$$\mathbb{P}\big(\text{tournament has } m \text{ irreducible parts}\big) = m \cdot (n)_{m-1} \cdot \frac{2^{m(m-1)/2}}{2^{(m-1)n}} + O\left(\frac{n^m}{2^{nm}}\right).$$

Note that in Table 5.2, the sum over any column is zero (see Remark 5.3.3).

5.4 Asymptotics for d-gargantuan classes

The goal of this section is to generalize the results of Sections 5.2 and 5.3 to d-gargantuan labeled combinatorial classes for any $d \in \mathbb{N}$.

Theorem 5.4.1. *Let d be a positive integer and $\mathcal{U}$ be a d-gargantuan labeled combinatorial class, such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ for some labeled combinatorial class $\mathcal{W}$. Suppose that $u \in \mathcal{U}$ is a random object of size dn. Then*

$$\mathbb{P}(u \textit{ is } \mathrm{SEQ}\textit{-irreducible}) \approx 1 - \sum_{k\geqslant 1} \left(2\mathfrak{w}_{dk} - \mathfrak{w}_{dk}^{(2)}\right) \cdot \binom{dn}{dk} \cdot \frac{\mathfrak{u}_{d(n-k)}}{\mathfrak{u}_{dn}}. \tag{5.14}$$

Proof. It is sufficient to apply Theorem 3.1.79 b) to the d-gargantuan labeled combinatorial class $\tilde{\mathcal{U}} = \mathcal{U} - \{\epsilon\}$ and the function

$$F(y) = 1 - \frac{1}{1+y}.$$

□

Theorem 5.4.2. *Let d be a positive integer and $\mathcal{U}$ be a d-gargantuan labeled combinatorial class, such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ for some labeled combinatorial class $\mathcal{W}$. Suppose that $u \in \mathcal{U}$ is a random object of size dn. Then for any $m \in \mathbb{N}$,*

$$\mathbb{P}(u \textit{ has } m \textit{ } \mathrm{SEQ}\textit{-irreducible components}) \approx \sum_{k\geqslant 0} \beta_{dk}^{(m)} \cdot \binom{dn}{dk} \cdot \frac{\mathfrak{u}_{d(n-k)}}{\mathfrak{u}_{dn}}, \tag{5.15}$$

where

$$\beta_{dk}^{(m)} = m\left(\mathfrak{w}_{dk}^{(m-1)} - 2\mathfrak{w}_{dk}^{(m)} + \mathfrak{w}_{dk}^{(m+1)}\right).$$

In particular, the leading term of (5.15) *is*

$$\mathbb{P}(u \textit{ has } m \textit{ } \mathrm{SEQ}\textit{-irreducible components}) =$$
$$= m \cdot \frac{(dn)!}{(d!)^{m-1}(d(n-m+1))!} \cdot \frac{u_d^{m-1} u_{d(n-m+1)}}{u_{dn}} + O\left(n^{dm} \cdot \frac{u_{d(n-m)}}{u_{dn}}\right). \tag{5.16}$$

Proof. To establish (5.15), it is sufficient to apply Theorem 3.1.79 b) to the d-gargantuan labeled combinatorial class $\tilde{\mathcal{U}} = \mathcal{U} - \{\epsilon\}$ and the function

$$F_m(y) = \left(1 - \frac{1}{1+y}\right)^{m+1}.$$

Let us evaluate the leading term of asymptotic expansion (5.15). To this end, we need to look at the exponential generating function

$$W^{(m-1)}(z) - 2W^{(m)}(z) + W^{(m+1)}(z)$$

which determines its behavior. Among three summands, it is sufficient to look at the behavior of the smallest one, namely, $W^{(m-1)}(z)$. Since the first non-zero coefficient in the exponential generating functions $W(z)$ is equal to $\mathfrak{w}_d = \mathfrak{u}_d$, we have

$$\begin{aligned} W^{(m-1)}(z) &= \big(W(z)\big)^{m-1} \\ &= u_d^{m-1} \cdot \frac{z^{d(m-1)}}{(d!)^{m-1}} + \dots \\ &= \left(\frac{\big(d(m-1)\big)!}{(d!)^{m-1}} \cdot u_d^{m-1}\right) \cdot \frac{z^{d(m-1)}}{\big(d(m-1)\big)!} + \dots . \end{aligned}$$

Thus, using formula (5.15), we obtain the first non-zero coefficient in the asymptotic probability that a random object has exactly m SEQ-irreducible components which is

$$\left(\binom{dn}{d(m-1)} \cdot \frac{u_{d(n-m+1)}}{u_{dn}}\right) \cdot m \cdot \left(\frac{\big(d(m-1)\big)!}{(d!)^{m-1}} \cdot u_d^{m-1}\right).$$

Regrouping factors, we obtain exactly (5.16). □

5.5 Asymptotics for unlabeled classes

The goal of this section is to generalize the results of Sections 5.2 and 5.3 to unlabeled combinatorial classes. Since a gargantuan class is d-gargantuan for $d = 1$, we state theorems for d-gargantuan unlabeled combinatorial classes directly.

Theorem 5.5.1. *Let $d \in \mathbb{N}$ and $\mathcal{U}$ be a d-gargantuan unlabeled combinatorial class with positive coefficients, such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ for some unlabeled combinatorial class $\mathcal{W}$. Suppose that $u \in \mathcal{U}$ is a random object of size dn. Then*

$$\mathbb{P}(u \textit{ is } \mathrm{SEQ}\textit{-irreducible}) \approx 1 - \sum_{k \geqslant 1} \left(2\mathfrak{w}_{dk} - \mathfrak{w}_{dk}^{(2)}\right) \cdot \frac{\mathfrak{u}_{d(n-k)}}{\mathfrak{u}_{dn}}. \tag{5.17}$$

Proof. Apply Theorem 3.2.22 to the d-gargantuan unlabeled combinatorial class $\tilde{\mathcal{U}} = \mathcal{U} - \{\epsilon\}$ and the function

$$F(y) = 1 - \frac{1}{1+y}.$$

□

Theorem 5.5.2. *Let $d \in \mathbb{N}$ and $\mathcal{U}$ be a d-gargantuan unlabeled combinatorial class with positive coefficients, such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ for some unlabeled combinatorial class $\mathcal{W}$. Suppose that $u \in \mathcal{U}$ is a random object of size dn. Then for any $m \in \mathbb{N}$,*

$$\mathbb{P}(u \text{ has } m \text{ SEQ-irreducible components}) \approx \sum_{k \geqslant 0} m\Big(\mathfrak{w}_{dk}^{(m-1)} - 2\mathfrak{w}_{dk}^{(m)} + \mathfrak{w}_{dk}^{(m+1)}\Big) \cdot \frac{\mathfrak{u}_{d(n-k)}}{\mathfrak{u}_{dn}}. \tag{5.18}$$

In particular, the leading term of (5.18) *is*

$$\mathbb{P}(u \text{ has } m \text{ SEQ-irreducible components}) = m \cdot \frac{u_d^{m-1} u_{d(n-m+1)}}{u_{dn}} + O\left(\frac{u_{d(n-m)}}{u_{dn}}\right). \tag{5.19}$$

Proof. Apply Theorem 3.2.22 to the d-gargantuan unlabeled combinatorial class $\tilde{\mathcal{U}} = \mathcal{U} - \{\epsilon\}$ and the function

$$F_m(y) = \left(1 - \frac{1}{1+y}\right)^{m+1}.$$

The formula for the leading term follows from the fact that $W^{(m)}(z) = \mathfrak{u}_d^m z^{dm} + \ldots$. □

Example 5.5.3. As we have seen in Example 3.2.13, any permutation can be decomposed as a sequence of indecomposable permutations. We can formally consider the combinatorial classes ${}_u\mathcal{P}$ of permutations and ${}_u\mathcal{IP}$ of indecomposable permutations as unlabeled, meaning that they satisfy

$${}_u\mathcal{P} = \mathrm{SEQ}({}_u\mathcal{IP}).$$

Also, we consider the family of unlabeled combinatorial classes ${}_u\mathcal{IP}^{(m)}$ of permutations consisting of m indecomposable parts, where m runs over all positive integers. The counting sequences $\big(\mathfrak{ip}_n^{(m)}\big)$ of ${}_u\mathcal{IP}^{(m)}$ for $m \leqslant 5$ are listed in Table 5.3.

n	1	2	3	4	5	6	7	8	9	10	11	
$(\mathfrak{ip}_n)$	1	1	3	13	71	461	3 447	29 093	273 343	2 829 325	31 998 903	...
$(\mathfrak{ip}_n^{(2)})$	0	1	2	7	32	177	1 142	8 411	69 692	642 581	6 534 978	...
$(\mathfrak{ip}_n^{(3)})$	0	0	1	3	12	58	327	2 109	15 366	125 316	1 135 329	...
$(\mathfrak{ip}_n^{(4)})$	0	0	0	1	4	18	92	531	3 440	24 892	200 344	...
$(\mathfrak{ip}_n^{(5)})$	0	0	0	0	1	5	25	135	800	5 226	37 690	...

Table 5.3: Counting sequences $\big(\mathfrak{ip}_n^{(m)}\big)$ for $m \leqslant 5$.

Since the unlabeled combinatorial class ${}_u\mathcal{P}$ is gargantuan (see Example 3.2.21), we can apply Theorem 5.5.1. Taking into account that

$$\frac{\mathfrak{p}_{n-k}}{\mathfrak{p}_n} = \frac{1}{(n)_k},$$

where $(n)_k = n(n-1)\ldots(n-k+1)$ are the falling factorials, the theorem gives us

$$\mathbb{P}\big(\text{permutation is indecomposable}\big) \approx 1 - \sum_{k \geqslant 1} \frac{2\mathfrak{ip}_k - \mathfrak{ip}_k^{(2)}}{(n)_k}.$$

In particular, as it was shown by Comtet [21],

$$\mathbb{P}\big(\text{permutation is indecomposable}\big) =$$
$$= 1 - \frac{2}{n} - \frac{1}{(n)_2} - \frac{4}{(n)_3} - \frac{19}{(n)_4} - \frac{110}{(n)_5} - \frac{745}{(n)_6} - \frac{5752}{(n)_7} - \frac{49775}{(n)_8} - \frac{476994}{(n)_9} - \frac{5016069}{(n)_{10}} + O\left(\frac{1}{n^{11}}\right).$$

The same way, we can apply Theorem 5.5.2 and, for every $m \in \mathbb{N}$, obtain

$$\mathbb{P}\big(\text{permutation has } m \text{ indecomposable parts}\big) \approx \sum_{k \geqslant 1} \frac{\beta_k^{(m)}}{(n)_k}.$$

where $\beta_k^{(m)} = m\Big(\mathfrak{ip}_k^{(m-1)} - 2\mathfrak{ip}_k^{(m)} + \mathfrak{ip}_k^{(m+1)}\Big)$. Sequences $\big(\beta_k^{(m)}\big)$ for $m \leqslant 5$ are listed in Table 5.4.

n	0	1	2	3	4	5	6	7	8	9	
$\big(\beta_n^{(1)}\big)$	1	−2	−1	−4	−19	−110	−745	−5 752	−49 775	−476 994	...
$\big(\beta_n^{(2)}\big)$	0	2	−2	0	4	38	330	2 980	28 760	298 650	...
$\big(\beta_n^{(3)}\big)$	0	0	3	0	6	36	237	1 740	14 172	127 200	...
$\big(\beta_n^{(4)}\big)$	0	0	0	4	4	20	108	672	4 728	37 144	...
$\big(\beta_n^{(5)}\big)$	0	0	0	0	5	10	45	240	1 470	10 140	...

Table 5.4: Asymptotic coefficients for permutations.

In particular,

$$\mathbb{P}\big(\text{permutation has } m \text{ indecomposable parts}\big) = \frac{m}{(n)_{m-1}} + O\left(\frac{1}{n^m}\right).$$

Chapter 6

CYC-irreducibles

In this chapter, we study the asymptotic probabilities that a random object is CYC-irreducible or consists of a fixed number of cycles. Section 6.1 is devoted to less formal discussion of the topic. We follow rather intuitive way to get the desired expansions for gargantuan labeled classes and explain combinatorially the obtained coefficients. In Sections 6.2 and 6.3 we prove the empirically obtained results more formally, with the help of Bender's theorem. Section 6.4 is devoted to the generalization for d-gargantuan labeled combinatorial classes. Unlike the previous chapter, we do not mention unlabeled combinatorial classes whose asymptotic behavior cannot be treated by Theorem 3.1.79. What is the problem with unlabeled classes, and what could be the approach to solving it, we will discuss in part IV.

Throughout the chapter, we use the following notations. We designate by $\mathcal{V}$ and $\mathcal{W}$ two labeled combinatorial classes, such that

$$\mathcal{V} = \mathrm{CYC}(\mathcal{W}), \tag{6.1}$$

meaning that their exponential generating functions are connected by the relation

$$V(z) = \ln \frac{1}{1 - W(z)} \qquad \Leftrightarrow \qquad W(z) = 1 - e^{-V(z)}. \tag{6.2}$$

Also, we consider labeled combinatorial classes

$$\mathcal{W}^{[m]} = \mathrm{CYC}_m(\mathcal{W}) \tag{6.3}$$

and their exponential generating functions

$$W^{[m]}(z) = \sum_{n=1}^{\infty} \mathfrak{w}_n^{[m]} \frac{z^n}{n!} = \frac{\big(W(z)\big)^m}{m}, \tag{6.4}$$

where m runs over the set of positive integers and $\big(\mathfrak{w}_n^{[m]}\big)$ is the counting sequences of the class $\mathcal{W}^{[m]}$. Note that in the particular case $m = 1$, we have $\mathcal{W}^{[1]} = \mathcal{W}$ and, consequently, $W^{[1]}(z) = W(z)$.

6.1 CYC-irreducibles and the inclusion-exclusion principle

Let us start this chapter with the combinatorial reasoning and the underlying inclusion-exclusion principle. Here, the basic ideas are the same as in Section 5.1. That is why we allow ourselves to make a presentation more concise. We suppose that the labeled combinatorial classes $\mathcal{V}$ and $\mathcal{W}$

satisfy (6.1) and that the behavior of $\mathcal{V}$ is known. Our goal is to express the behavior of $\mathcal{W}$ in terms of $\mathcal{V}$ and establish asymptotic expansions in the case when $\mathcal{V}$ is gargantuan.

The first idea is that, for gargantuan class $\mathcal{V}$, the main contribution in the asymptotic behavior is made by objects that have a large CYC-irreducible part. Accordingly, we can approximate $\mathfrak{v}_n$ by the sum over $\mathfrak{w}_k$, where $k = n, n-1, n-2, \ldots$. The coefficients rely on the structure of objects constructed with the help of CYC-operation. For instance, the first three terms of the above decomposition are

$$\mathfrak{v}_n = \mathfrak{w}_n + \mathfrak{w}_1 \binom{n}{1} \mathfrak{w}_{n-1} + (\mathfrak{w}_2 + 2\mathfrak{w}_1) \binom{n}{2} \mathfrak{w}_{n-2} + O\big(n^3 \mathfrak{w}_{n-3}\big).$$

Similarly to Lemmas 5.1.3 and 5.1.4, we obtain the following result.

Lemma 6.1.1. *If $\mathcal{V}$ and $\mathcal{W}$ are labeled combinatorial classes such that $\mathcal{V} = \mathrm{CYC}(\mathcal{W})$, then for any positive integer n,*

$$\mathfrak{v}_n = \sum_{C_n} \frac{n!}{(1!)^{c_1} \cdot \ldots \cdot (n!)^{c_n}} \cdot \frac{(c_1 + \ldots + c_n - 1)!}{c_1! \cdot \ldots \cdot c_n!} \cdot \big(\mathfrak{w}_1^{c_1} \cdot \ldots \cdot \mathfrak{w}_n^{c_n}\big) \tag{6.5}$$

and

$$\mathfrak{w}_n = \sum_{C_n} (-1)^{(c_1+\ldots+c_n)-1} \cdot \frac{n!}{(1!)^{c_1} \cdot \ldots \cdot (n!)^{c_n}} \cdot \frac{1}{c_1! \cdot \ldots \cdot c_n!} \cdot \big(\mathfrak{v}_1^{c_1} \cdot \ldots \cdot \mathfrak{v}_n^{c_n}\big), \tag{6.6}$$

where $C_n = \{(c_1, \ldots, c_n) \mid c_1 + 2c_2 + \ldots + nc_n = n\}$.

Proof. The first formula comes from the expression for the exponential generating function of $\mathcal{V}$ in terms of the exponential generating function of $\mathcal{W}$ (see (6.2)):

$$V(z) = 1 + W(z) + \frac{W^2(z)}{2} + \frac{W^3(z)}{3} + \ldots$$

The second formula can be derived from the inverse relation

$$W(z) = V(z) - \frac{V^2(z)}{2!} + \frac{V^3(z)}{3!} - \ldots.$$

□

Remark 6.1.2. As well as (5.1.3), relation (6.5) admits a purely combinatorial explanation coming from the enumeration of all possible structures in $\mathcal{V}_n$, $n \in \mathbb{N}$. Indeed, any cycle in $\mathcal{V}_n$ determines the n-sequence $(c_1, \ldots, c_n)$ such that c_k is the number of elements of size k in this cycle. The number of objects with the same n-sequence $(c_1, \ldots, c_n)$ is determined by the choice of

1. the order of elements sizes:
$$\frac{(c_1 + \ldots + c_n - 1)!}{c_1! \cdot \ldots \cdot c_n!} \quad \text{choices;}$$
2. labels for each element:
$$\frac{n!}{(1!)^{c_1} \cdot \ldots \cdot (n!)^{c_n}} \quad \text{choices;}$$
3. elements themselves:
$$\mathfrak{w}_1^{c_1} \cdot \ldots \cdot \mathfrak{w}_n^{c_n} \quad \text{choices.}$$

Adding up, we obtain the desired expression (6.5).

Relations in the statement of Lemma 6.1.1 hold for any labeled combinatorial classes, not just gargantuan. However, if $\mathcal{V}$ is gargantuan, then formula (6.6) provides the asymptotic expansion with as many terms as we want. For instance, we can get the following first three terms of the asymptotic probability that a random object $v \in \mathcal{V}$ is CYC-irreducible:

$$\mathbb{P}\big(v \text{ is CYC-irreducible}\big) = 1 - \mathfrak{v}_1 \binom{n}{1} \frac{\mathfrak{v}_{n-1}}{\mathfrak{v}_n} - \big(\mathfrak{v}_2 - \mathfrak{v}_1^2\big) \binom{n}{2} \frac{\mathfrak{v}_{n-2}}{\mathfrak{v}_n} + O\left(\frac{\mathfrak{v}_{n-3}}{\mathfrak{v}_n}\right).$$

In order to interpret combinatorially the involved coefficients, we rewrite them in terms of $\mathfrak{w}_k$. For this goal, we need to come back to the asymptotic expansion for $\mathfrak{v}_n$ rewritten in a different form. Also, we need the following additional observation.

Lemma 6.1.3. *If $\mathcal{W}$ is a labeled combinatorial class and m is a positive integer, then*

$$\Theta\mathrm{CYC}_{>m}(\mathcal{W}) \cong \mathrm{SEQ}_m(\mathcal{W}) \star \Theta\mathrm{CYC}(\mathcal{W}).$$

Proof. Assume that a pointed cycle consist of $l+1$ objects $\omega_0, \ldots, \omega_l \in \mathcal{W}$, where $l \geqslant m$ and ω_0 contain the pointed atom. Extract the subsequence $(\omega_1, \ldots, \omega_m)$ from the cycle. Thus, the rest is equivalent to the cycle $(\omega_0, \omega_{m+1}, \omega_{m+2}, \ldots, \omega_l)$ with the pointed atom belonging to ω_0 (Fig. 6.1). Clearly, we obtain a bijection. □

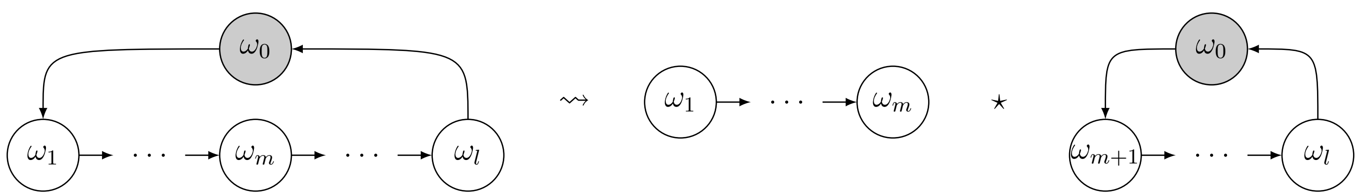


Figure 6.1: Mapping $\Theta\mathrm{CYC}_{>m}(\mathcal{W}) \to \mathrm{SEQ}_m(\mathcal{W}) \star \Theta\mathrm{CYC}(\mathcal{W})$.

Remark 6.1.4. Lemma 6.1.3 can be verified directly with the help of exponential generating functions. Indeed, the exponential generating function of $\mathrm{SEQ}_m(\mathcal{W}) \star \Theta\mathrm{CYC}(\mathcal{W})$ is

$$\big(W(z)\big)^m \left(z\frac{d}{dz} \ln \frac{1}{1-W(z)}\right) = \left(z\frac{d}{dz}W(z)\right) \frac{\big(W(z)\big)^m}{1-W(z)}.$$

On the other hand, the exponential generating function of $\Theta\mathrm{CYC}_{>m}(\mathcal{W})$ turns out to be

$$z\frac{d}{dz}\left(\ln\frac{1}{1-W(z)} - \sum_{k=1}^{m} \frac{\big(W(z)\big)^k}{k}\right) = \left(z\frac{d}{dz}W(z)\right)\frac{1}{1-W(z)} - \left(z\frac{d}{dz}W(z)\right)\sum_{k=1}^{m}\big(W(z)\big)^{k-1}$$
$$= \left(z\frac{d}{dz}W(z)\right)\frac{\big(W(z)\big)^m}{1-W(z)}.$$

We have seen another combinatorial isomorphism of this kind in Example 3.1.56,

$$\Theta\mathrm{CYC}(\mathcal{W}) \cong \Theta\mathcal{W} \star \mathrm{SEQ}(\mathcal{W}),$$

meaning that a cycle with a distinguished element is equivalent to the pair consisting of this element and a sequence. In our case, we are interested in the asymptotic behavior of $\mathfrak{v}_n$, and the significant contribution is made by large CYC-irreducible components which play the role of distinguished elements. More precisely, let us say that we wish to obtain first r terms of the asymptotic expansion, where r is an arbitrary positive integer. Then for sufficiently large n (say, $n > 2r$) each object $v \in \mathcal{V}_n$ either has a CYC-irreducible component of size at least $(n-r)$ or makes a negligible contribution.

Now, to enumerate CYC-irreducible objects, we deduce objects with at least two components from all the objects. According to Lemma 6.1.3, an object with at least two components, one of which is large, corresponds to the pair of the CYC-irreducible element and the object with large component. Hence, we have the approximation

$$\mathfrak{w}_n \approx \mathfrak{u}_n - \sum_{k\geqslant 1} \mathfrak{w}_k \binom{n}{k} \mathfrak{v}_{n-k}$$

leading to the asymptotic probability for a random object $v \in \mathcal{V}$ to be CYC-irreducible:

$$\mathbb{P}(v \text{ is CYC-irreducible}) \approx 1 - \sum_{k\geqslant 1} \mathfrak{w}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{v}_{n-k}}{\mathfrak{v}_n}.$$

Note that, on the contrary to the case of SEQ-irreducibles, we do not provide the exact formula for $\mathfrak{w}_n$ that could be an analogue of (5.9). We do not need to go deeply into the precise behavior of negligible "tails", so we "kill" symmetries for simplifying the computations.

In the same way, we can obtain the asymptotic expansion for the probability that a random object $v \in \mathcal{V}$ has several CYC-irreducible components. Here, we use the observation that

$$\mathrm{CYC}_{\geqslant m}(\mathcal{W}) = \mathrm{CYC}_m(\mathcal{W}) + \mathrm{CYC}_{>m}(\mathcal{W}).$$

Together with Lemma 6.1.3, this allows us to obtain the following asymptotics:

$$\mathbb{P}(v \text{ has } m \text{ CYC-irreducible components}) \approx \sum_{k\geqslant 0} \left(\mathfrak{w}_k^{(m-1)} - \mathfrak{w}_k^{(m)}\right) \cdot \binom{n}{k} \cdot \frac{\mathfrak{v}_{n-k}}{\mathfrak{v}_n},$$

where $\left(\mathfrak{w}_k^{(m)}\right)$ is the counting sequence of the labeled combinatorial class

$$\mathcal{W}^{(m)} = \mathrm{SEQ}_m \mathcal{W}.$$

The reader will find the short proof of the above relation in Sections 6.2 and 6.3.

6.2 Asymptotic probability of CYC-irreducible objects

Theorem 6.2.1 (CYC asymptotics)**.** *Let $\mathcal{V}$ be a gargantuan labeled combinatorial class with positive coefficients, such that $\mathcal{V} = \mathrm{CYC}(\mathcal{W})$ for some labeled combinatorial class $\mathcal{W}$. Suppose that $v \in \mathcal{V}$ is a random object of size n. Then*

$$\mathbb{P}(v \textit{ is CYC-irreducible}) \approx 1 - \sum_{k\geqslant 1} \mathfrak{w}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{v}_{n-k}}{\mathfrak{v}_n}. \tag{6.7}$$

Proof. Let us apply Theorem 3.1.79 a) to the gargantuan labeled combinatorial class $\mathcal{V}$ and the function

$$F(y) = 1 - e^{-y}.$$

According to (6.2),

$$F\big(V(z)\big) = 1 - e^{-V(z)} = W(z),$$

while, due to (6.4),

$$C(z) = e^{-V(z)} = 1 - W(z).$$

Together with the statement of Theorem 3.1.79, this implies

$$\mathfrak{w}_n \approx \sum_{k\geqslant 0} \binom{n}{k} c_k \mathfrak{v}_{n-k} = \mathfrak{v}_n - \sum_{k\geqslant 1} \binom{n}{k} \mathfrak{w}_k \mathfrak{v}_{n-k}.$$

Since $\mathfrak{w}_n/\mathfrak{v}_n$ is the probability that v is CYC-irreducible, it is sufficient to divide both sides by $\mathfrak{v}_n$ to finish the proof. □

6.3 Objects with several CYC-irreducible components

Theorem 6.3.1 (CYC$_m$ asymptotics)**.** *Let $\mathcal{V}$ be a gargantuan labeled combinatorial class with positive coefficients, such that $\mathcal{V} = \mathrm{CYC}(\mathcal{W})$ for some labeled combinatorial class $\mathcal{W}$. Suppose that $v \in \mathcal{V}$ is a random object of size n. Then for any $m \in \mathbb{N}$,*

$$\mathbb{P}(v \text{ has } m \text{ CYC-irreducible components}) \approx \sum_{k\geqslant 0} \left(\mathfrak{w}_k^{(m-1)} - \mathfrak{w}_k^{(m)}\right) \cdot \binom{n}{k} \cdot \frac{\mathfrak{v}_{n-k}}{\mathfrak{v}_n}. \tag{6.8}$$

where $\big(\mathfrak{w}_k^{(m)}\big)$ is the counting sequence of the labeled combinatorial class $\mathcal{W}^{(m)} = \mathrm{SEQ}_m(\mathcal{W})$.

Proof. Apply Theorem 3.1.79 a) to the gargantuan class $\mathcal{V}$ and the function

$$F_m(y) = \frac{\big(1 - e^{-y}\big)^m}{m}.$$

According to (6.2) and (6.4),

$$F_m\big(V(z)\big) = \frac{\big(1 - e^{-V(z)}\big)^m}{m} = \frac{\big(W(z)\big)^m}{m} = W^{[m]}(z),$$

while

$$\begin{aligned} C_m(z) &= \big(1 - e^{-V(z)}\big)^{m-1} e^{-V(z)} \\ &= \big(W(z)\big)^{m-1}\big(1 - W(z)\big) \\ &= W^{(m-1)}(z) - W^{(m)}(z). \end{aligned}$$

Hence, the statement of Theorem 3.1.79 gives us

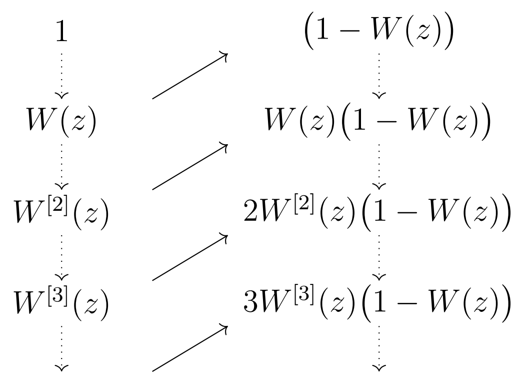
$$\mathfrak{w}_n^{[m]} \approx \sum_{k\geqslant 0} \binom{n}{k} c_k \mathfrak{v}_{n-k} = \sum_{k\geqslant 0} \binom{n}{k} \left(\mathfrak{w}_k^{(m-1)} - \mathfrak{w}_k^{(m)}\right) \mathfrak{v}_{n-k}.$$

To finish the proof, divide the obtained formula by $\mathfrak{v}_n$. □

Remark 6.3.2. Since $W^{(m)}(z) = mW^{[m]}(z)$ for every positive integer m, we can rewrite formula (6.8) in terms of counting sequences $\big(\mathfrak{w}_k^{[m-1]}\big)$ and $\big(\mathfrak{w}_k^{[m]}\big)$:

$$\mathbb{P}(v \text{ has } m \text{ CYC-irreducible components}) \approx \sum_{k\geqslant 0} \Big((m-1)\mathfrak{w}_k^{[m-1]} - m\mathfrak{w}_k^{[m]}\Big) \cdot \binom{n}{k} \cdot \frac{\mathfrak{v}_{n-k}}{\mathfrak{v}_n}.$$

Remark 6.3.3. Defining $W^{[0]}(z) = \big(W(z)\big)^0 = 1$, we can see that Theorem 6.2.1 is the particular case of Theorem 6.3.1 for $m = 1$. The coefficients of asymptotic probabilities fit into the general scheme depicted below. The left column corresponds to the exponential generating functions of labeled combinatorial classes $\mathcal{W}^{[m]}$, $m \in \mathbb{Z}_{\geqslant 0}$. The right column represents the same functions multiplied by $m\big(1 - W(z)\big)$. According to Theorem 6.3.1, the coefficients involved into the asymptotic expansion of $\mathcal{W}^{[m]}$ constitute the exponential generating function $W^{(m-1)}(z)\big(1 - W(z)\big)$. This fact is reflected by solid arrows pointing from the left column to the right one.



Corollary 6.3.4. *Let $\mathcal{V}$ be a gargantuan labeled combinatorial class with positive coefficients, such that $\mathcal{V} = \mathrm{CYC}(\mathcal{W})$ for some labeled combinatorial class $\mathcal{W}$. Suppose that $v \in \mathcal{U}$ is a random object of size n. Then*

$$\mathbb{P}(v \textit{ has } m \text{ CYC-}\textit{irreducible components}) = \binom{n}{m-1} \cdot \frac{\mathfrak{v}_1^{m-1}\mathfrak{v}_{n-m+1}}{\mathfrak{v}_n} + O\left(n^m \cdot \frac{\mathfrak{v}_{n-m}}{\mathfrak{v}_n}\right). \qquad (6.9)$$

Proof. According to Theorem 6.3.1, we need to look at the leading term of the exponential generating function $\big(W(z)\big)^{m-1} - \big(W(z)\big)^m$. Since $\mathfrak{w}_1 = \mathfrak{v}_1$, we have

$$\big(W(z)\big)^{m-1} - \big(W(z)\big)^m = \mathfrak{v}_1^{m-1} \cdot \frac{z^{m-1}}{(m-1)!} + \ldots$$

As a consequence, the first non-zero coefficient in asymptotic probability (6.8) turns out to be

$$\mathfrak{v}_1^{m-1}\left(\binom{n}{m-1} \cdot \frac{\mathfrak{v}_{n-m+1}}{\mathfrak{v}_n}\right) = \binom{n}{m-1} \cdot \frac{\mathfrak{v}_1^{m-1}\mathfrak{v}_{n-m+1}}{\mathfrak{v}_n}.$$

□

Remark 6.3.5. Every labeled combinatorial object $v \in \mathcal{V} = \mathrm{CYC}(\mathcal{W})$ has a fixed number m of CYC-irreducible components. This fact implies the natural identity

$$\mathfrak{v}_n = \sum_{m=1}^{\infty} \mathfrak{w}_n^{[m]}$$

for any positive integer n. As a consequence, in the proof of Theorem 6.3.1 the sum of the series $C_m(z)$ over all $m \geqslant 1$ should be equal to 1. We can check it directly:

$$\sum_{m=1}^{\infty} C_m(z) = \sum_{m=1}^{\infty} \Big(\big(W(z)\big)^{m-1} - \big(W(z)\big)^{m} \Big) = \big(W(z)\big)^{0} = 1.$$

6.4 Asymptotics for d-gargantuan classes

The aim of this section is to generalize the results of Sections 6.2 and 6.3 to d-gargantuan labeled combinatorial classes, $d \in \mathbb{N}$.

Theorem 6.4.1. *Let d be a positive integer and $\mathcal{V}$ be a d-gargantuan labeled combinatorial class, such that $\mathcal{V} = \mathrm{CYC}(\mathcal{W})$ for some labeled combinatorial class $\mathcal{W}$. Suppose that $v \in \mathcal{V}$ is a random object of size dn. Then*

$$\mathbb{P}(v \text{ is } \mathrm{CYC}\text{-irreducible}) \approx 1 - \sum_{k \geqslant 1} \mathfrak{w}_{dk} \cdot \binom{dn}{dk} \cdot \frac{\mathfrak{v}_{d(n-k)}}{\mathfrak{v}_{dn}}. \tag{6.10}$$

Proof. It is sufficient to apply Theorem 3.1.79 b) to the d-gargantuan labeled combinatorial class $\mathcal{V}$ and the function

$$F(y) = 1 - e^{-y}.$$

□

Theorem 6.4.2. *Let d be a positive integer and $\mathcal{V}$ be a d-gargantuan labeled combinatorial class, such that $\mathcal{V} = \mathrm{CYC}(\mathcal{W})$ for some labeled combinatorial class $\mathcal{W}$. Suppose that $v \in \mathcal{V}$ is a random object of size dn. Then for any $m \in \mathbb{N}$,*

$$\mathbb{P}(v \text{ has } m \text{ } \mathrm{CYC}\text{-irreducible components}) \approx \sum_{k \geqslant 0} \Big(\mathfrak{w}_{dk}^{(m-1)} - \mathfrak{w}_{dk}^{(m)} \Big) \cdot \binom{dn}{dk} \cdot \frac{\mathfrak{v}_{d(n-k)}}{\mathfrak{v}_{dn}}, \tag{6.11}$$

where $\big(\mathfrak{w}_k^{(m)}\big)$ is the counting sequence of the labeled combinatorial class $\mathcal{W}^{(m)} = \mathrm{SEQ}_m(\mathcal{W})$. In particular, the leading term of (6.11) is

$$\mathbb{P}(v \text{ has } m \text{ } \mathrm{CYC}\text{-irreducible components}) =$$
$$= \frac{(dn)!}{(d!)^{m-1}(d(n-m+1))!} \cdot \frac{v_d^{m-1} v_{d(n-m+1)}}{v_{dn}} + O\left(n^{dm} \cdot \frac{v_{d(n-m)}}{v_{dn}} \right). \tag{6.12}$$

Proof. To establish (6.11), it is sufficient to apply Theorem 3.1.79 b) to the d-gargantuan labeled combinatorial class $\mathcal{V}$ and the function

$$F_m(y) = \frac{(1 - e^{-y})^m}{m}.$$

In accordance with (6.11), the leading term of the desired asymptotics is determined by the exponential generating function

$$W^{(m-1)}(z) - W^{(m)}(z).$$

Since the first non-zero coefficient in the exponential generating functions $W(z)$ is equal to $\mathfrak{w}_d = \mathfrak{v}_d$, we have

$$\begin{aligned} W^{(m-1)}(z) - W^{(m)}(z) &= \big(W(z)\big)^{m-1}\big(1 - W(z)\big) \\ &= v_d^{m-1} \cdot \frac{z^{d(m-1)}}{(d!)^{m-1}} + \dots \\ &= \left(\frac{\big(d(m-1)\big)!}{(d!)^{m-1}} \cdot v_d^{m-1} \right) \cdot \frac{z^{d(m-1)}}{\big(d(m-1)\big)!} + \dots . \end{aligned}$$

Hence, formula (6.11) implies that the first non-zero coefficient in the asymptotic probability that a random object has exactly m CYC-irreducible components is

$$\left(\binom{dn}{d(m-1)} \cdot \frac{v_{d(n-m+1)}}{v_{dn}} \right) \cdot \left(\frac{\big(d(m-1)\big)!}{(d!)^{m-1}} \cdot v_d^{m-1} \right) = \frac{(dn)!}{(d!)^{m-1}(d(n-m+1))!} \cdot \frac{v_d^{m-1} v_{d(n-m+1)}}{v_{dn}}.$$

□

Chapter 7

SET-irreducibles

This chapter is devoted to asymptotic expansions of the probabilities concerning SET-decompositions of gargantuan labeled combinatorial classes. As in the previous two chapters, the first section (Section 7.1) constitutes of rather informal discussion. With the help of the inclusion-exclusion principle, we establish exact formulas that lead to desired approximations together with the combinatorial interpretation of the involved coefficients. In Sections 7.2 and 7.3, we use Bender's theorem to prove rigorously the obtained results. Section 7.4 is related to the generalization for d-gargantuan labeled combinatorial classes. Finally, some partial results related to unlabeled combinatorial classes are discussed in Section 7.5.

Let us describe the notations that we use in this chapter. We designate by $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ three labeled combinatorial classes, such that[1]

$$\mathcal{U} = \mathrm{SET}(\mathcal{V}) \qquad \text{and} \qquad \mathcal{U} = \mathrm{SEQ}(\mathcal{W}). \tag{7.1}$$

Consequently, their exponential generating functions satisfy the following relations:

$$U(z) = \exp\big(V(z)\big) \qquad \text{and} \qquad U(z) = \frac{1}{1 - W(z)}. \tag{7.2}$$

Also, we study the family of labeled combinatorial classes

$$\mathcal{V}^{\{m\}} = \mathrm{SET}_m(\mathcal{V}) \tag{7.3}$$

parametrized by a positive integer m. The corresponding exponential generating functions are

$$V^{\{m\}}(z) = \sum_{n=1}^{\infty} \mathfrak{v}_n^{\{m\}} \frac{z^n}{n!} = \frac{\big(V(z)\big)^m}{m!}, \tag{7.4}$$

where $\big(\mathfrak{v}_n^{\{m\}}\big)$ is the counting sequences of the class $\mathcal{V}^{\{m\}}$. In particular, $\mathcal{V}^{\{1\}} = \mathcal{V}$ and $V^{\{1\}}(z) = V(z)$.

7.1 SET-irreducibles and the inclusion-exclusion principle

The same way as we have it done in the previous chapters, we start with some exact formulas that can serve for establishing asymptotics for gargantuan labeled combinatorial classes.

[1]To be rigorous, we should have written $\mathcal{U} = \mathrm{SET}(\mathcal{V}) \cong \mathrm{SEQ}(\mathcal{W})$, but for simplicity, we use the sign of equality.

Lemma 7.1.1. *If $\mathcal{U}$ and $\mathcal{V}$ are labeled combinatorial classes such that $\mathcal{U} = \mathrm{SET}(\mathcal{V})$, then for any positive integer n,*

$$\mathfrak{u}_n = \sum_{C_n} \frac{n!}{(1!)^{c_1} \cdot \ldots \cdot (n!)^{c_n}} \cdot \frac{1}{c_1! \cdot \ldots \cdot c_n!} \cdot \left(\mathfrak{v}_1^{c_1} \cdot \ldots \cdot \mathfrak{v}_n^{c_n}\right) \tag{7.5}$$

and

$$\mathfrak{v}_n = \sum_{C_n} (-1)^{(c_1+\ldots+c_n)-1} \cdot \frac{n!}{(1!)^{c_1} \cdot \ldots \cdot (n!)^{c_n}} \cdot \frac{(c_1 + \ldots + c_n - 1)!}{c_1! \cdot \ldots \cdot c_n!} \cdot \left(\mathfrak{u}_1^{c_1} \cdot \ldots \cdot \mathfrak{u}_n^{c_n}\right), \tag{7.6}$$

where $C_n = \{(c_1, \ldots, c_n) \mid c_1 + 2c_2 + \ldots + nc_n = n\}$.

Proof. Both formulas come from the relation $U(z) = \exp\big(V(z)\big)$. To obtain the first formula, it is sufficient to use the decomposition of the exponent into the series:

$$U(z) = 1 + V(z) + \frac{V^2(z)}{2!} + \frac{V^3(z)}{3!} + \ldots$$

and then to pass to the summation over partitions. The second formula can be derived in a similar way from the inverse relation

$$V(z) = \tilde{U}(z) - \frac{\tilde{U}^2(z)}{2} + \frac{\tilde{U}^3(z)}{3} - \ldots,$$

where $\tilde{U}(z) = U(z) - 1$. □

Remark 7.1.2. As well as (5.1.3) and (6.5), relation (7.5) can be explained with the help of pure combinatorial reasoning. Indeed, for each $n \in \mathbb{N}$, any set in $\mathcal{U}_n$ determines the n-sequence $(c_1, \ldots, c_n)$ such that c_k is the number of elements of size k in this set. The number of objects with the same n-sequence $(c_1, \ldots, c_n)$ is determined by the choice of

1. labels for each element:
$$\frac{n!}{(1!)^{c_1} \cdot \ldots \cdot (n!)^{c_n}} \quad \text{choices;}$$
2. elements themselves:
$$\mathfrak{v}_1^{c_1} \cdot \ldots \cdot \mathfrak{v}_n^{c_n} \quad \text{choices.}$$

Note that we need to divide the product of the above numbers by $c_1! \cdot \ldots \cdot c_n!$, because the order of elements in the set is not important. Taking the sum over C_n, we obtain the desired expression (6.5).

In the case when $\mathcal{U}$ is gargantuan, one can use (7.6) to obtain the asymptotic expansion. Indeed, in this case $\mathfrak{u}_k = o(\mathfrak{u}_{k+1})$ for any $k \in \mathbb{N}$. Therefore, fixing an arbitrary $r \in \mathbb{N}$, we can consider the linear combination of $\mathfrak{v}_k$, where $k = n, (n-1), (n-2), \ldots, (n-r)$, as an approximation of $\mathfrak{u}_n$ for large n. For example, for $r = 2$, we have

$$\mathfrak{v}_n = \mathfrak{u}_n - \mathfrak{u}_1 \binom{n}{1} \mathfrak{u}_{n-1} - (\mathfrak{u}_2 - 2\mathfrak{u}_1^2) \binom{n}{2} \mathfrak{u}_{n-2} - O\big(n^3 \mathfrak{u}_{n-3}\big).$$

Yet, a combinatorial interpretation of the coefficients is not clear for this formula. In order to understand the meaning of the involved values, we need to do some preliminary work.

Let us fix a large positive integer n. For each $k \leqslant n$ and for every partition $\lambda = (\lambda_1, \ldots, \lambda_{l(\lambda)})$ of the integer k, define $B_{\lambda,n}$ to be the number of labeled objects in $\mathcal{U}_n$ containing SET-irreducible components of sizes $\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}$ with respect to the labels. For example, if $n = 3$, $k = 2$ and $\lambda = (1, 1)$, then the only possible structure of the object consists of three SET-irreducible components of size 1, so that the total amount of such objects is $\mathfrak{v}_1^3$. On the other hand, each object enters into $B_{(1,1),3}$ exactly three times because there are three possible ways to choose two components out of three. Thus, $B_{(1,1),3} = 3\mathfrak{v}_1^3$.

More generally, a partition λ is determined by the tuple $(c_1, \ldots, c_k)$ of non-negative integers satisfying $c_1 + 2c_2 + \ldots + kc_k = k = |\lambda|$. In terms of this tuple, the number $B_{\lambda,n}$ is determined by the following sequence of choices.

1. Choose k vertices among given n labeled vertices:
$$\binom{n}{k} \text{ choices.}$$
2. Distribute these k vertices into $l(\lambda) = c_1+\ldots+c_k$ groups, such that c_k groups contain k elements:
$$\frac{k!}{(1!)^{c_1} \cdot \ldots \cdot (k!)^{c_k}} \cdot \frac{1}{c_1! \cdot \ldots \cdot c_k!} \text{ choices.}$$
3. For every group, choose a SET-irreducible labeled object of appropriate size:
$$\mathfrak{v}_1^{c_1} \cdot \ldots \cdot \mathfrak{v}_k^{c_k} \text{ choices.}$$
4. For the remaining $(n-k)$ vertices, choose a labeled object of size $(n-k)$:
$$\mathfrak{u}_{n-k} \text{ choices.}$$

Thus, $B_{\lambda,n}$ satisfy the following relation:
$$B_{\lambda,n} = \binom{n}{k} \cdot \frac{k!}{(1!)^{c_1} \cdot \ldots \cdot (k!)^{c_k}} \cdot \frac{\mathfrak{v}_1^{c_1} \cdot \ldots \cdot \mathfrak{v}_k^{c_k}}{c_1! \cdot \ldots \cdot c_k!} \cdot \mathfrak{u}_{n-k}.$$
As a consequence, the number of labeled objects of size n in $\mathcal{U}$, due to the inclusion-exclusion principle, is provided with the formula
$$\mathfrak{u}_n = \sum_{\lambda\colon |\lambda| \leqslant n} (-1)^{l(\lambda)-1} B_{\lambda,n}.$$
Explicitly, we have the following result.

Lemma 7.1.3. *If $\mathcal{U}$ and $\mathcal{V}$ are labeled combinatorial classes such that $\mathcal{U} = \mathrm{SET}(\mathcal{V})$, then for any positive integer n,*
$$\mathfrak{u}_n = \sum_{k=1}^{n} \sum_{C_k} (-1)^{(c_1+\ldots+c_k)-1} \cdot \binom{n}{k} \cdot \frac{k!}{(1!)^{c_1} \cdot \ldots \cdot (k!)^{c_k}} \cdot \frac{\mathfrak{v}_1^{c_1} \cdot \ldots \cdot \mathfrak{v}_k^{c_k}}{c_1! \cdot \ldots \cdot c_k!} \cdot \mathfrak{u}_{n-k}, \tag{7.7}$$
where $C_k = \{(c_1, \ldots, c_k) \mid c_1 + 2c_2 + \ldots + kc_k = k\}$.

Let us factor out of the second sum in (7.7) the factors depending on n, and denote by $\mathfrak{w}_k$ the rest. In other words, we define $\mathfrak{w}_k$ by the relation:

$$\mathfrak{w}_k = \sum_{C_k} (-1)^{(c_1+\ldots+c_k)-1} \cdot \frac{k!}{(1!)^{c_1} \cdot \ldots \cdot (k!)^{c_k}} \cdot \frac{\mathfrak{v}_1^{c_1} \cdot \ldots \cdot \mathfrak{v}_k^{c_k}}{c_1! \cdot \ldots \cdot c_k!}. \tag{7.8}$$

With this notation, we can rewrite expression (7.7) as

$$\mathfrak{u}_n = \sum_{k=1}^{n} \mathfrak{w}_k \cdot \binom{n}{k} \cdot \mathfrak{u}_{n-k}.$$

Note that, in terms of exponential generating functions, we have

$$U(z) = 1 + U(z) \cdot W(z) \qquad \Leftrightarrow \qquad W(z) = 1 - \frac{1}{U(z)}$$

where

$$W(z) = \sum_{n=0}^{\infty} \mathfrak{w}_n \cdot \frac{z^n}{n!}.$$

This observation means that if there is a combinatorial class $\mathcal{W}$ such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$, then $(\mathfrak{w}_n)$ is the counting sequence of $\mathcal{W}$.

Now, let us come back to the asymptotics of $\mathfrak{v}_n$ in the case when $\mathcal{U}$ is gargantuan. The main idea is the same as in Sections 5.1 and 6.1: the contribution into the asymptotic expansion is made by objects that have a large SET-irreducible component. To be precise, fix a positive integer r and assume that n is large enough (say, $n > 2r$). Then any object that makes a contribution into first r terms of the asymptotics is either SET-irreducible or has a component of size greater then $(n-r)$. The number of SET-irreducible objects is $\mathfrak{v}_n$. On the other hand, by the inclusion-exclusion principle, the asymptotics of the number of objects containing a sufficiently large component is determined by the summands corresponding to $k < r$ in (7.7). This fact implies that

$$\mathfrak{u}_n = \mathfrak{v}_n + \sum_{k\geqslant 1} \mathfrak{w}_k \cdot \binom{n}{k} \cdot \mathfrak{u}_{n-k} + O(n^r u_{n-r}).$$

Rearranging the terms and dividing by $\mathfrak{u}_n$, we get the asymptotic expansion for the probability that a random object $u \in \mathcal{U}$ is SET-irreducible:

$$\mathbb{P}(u \text{ is SET-irreducible}) \approx 1 - \sum_{k\geqslant 1} \mathfrak{w}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

We provide a short proof of this informally obtained approximation in the next section.

7.2 Asymptotic probability of SET-irreducible objects

Theorem 7.2.1 (SET asymptotics). *Let $\mathcal{U}$ be a gargantuan labeled combinatorial class with positive coefficients, such that $\mathcal{U} = \mathrm{SET}(\mathcal{V})$ and $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ for some labeled combinatorial classes $\mathcal{V}$ and $\mathcal{W}$. Suppose that $u \in \mathcal{U}$ is a random object of size n. Then*

$$\mathbb{P}(u \textit{ is SET-irreducible}) \approx 1 - \sum_{k\geqslant 1} \mathfrak{w}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}. \tag{7.9}$$

Proof. Apply Theorem 3.1.79 a) to the gargantuan labeled combinatorial class $\tilde{\mathcal{U}} = \mathcal{U} - \{\epsilon\}$ and the function

$$F(y) = \ln(1 + y).$$

Then, according to (7.2),

$$F\big(U(z) - 1\big) = \ln\big(U(z)\big) = V(z),$$

while

$$C(z) = \frac{1}{U(z)} = 1 - W(z).$$

Hence, the statement of Theorem 3.1.79 a) gives us

$$\mathfrak{v}_n \approx \sum_{k\geqslant 0} \binom{n}{k} c_k \mathfrak{u}_{n-k} = \mathfrak{u}_n - \sum_{k\geqslant 1} \binom{n}{k} \mathfrak{w}_k \mathfrak{u}_{n-k}.$$

Dividing by $\mathfrak{u}_n$, we obtain desired formula (7.9). □

Example 7.2.2. As we have discussed in Examples 3.1.14, 3.1.33 and 3.1.49, the labeled combinatorial classes of graphs and tournaments are combinatorially equivalent,

$$\mathcal{G} \cong \mathcal{T},$$

while any graph is a collection of connected graphs,

$$\mathcal{G} = \mathrm{SET}(\mathcal{CG}),$$

and any tournament can be represented as a sequence of irreducible subtournaments,

$$\mathcal{T} = \mathrm{SEQ}(\mathcal{IT}).$$

Together with the fact that the class $\mathcal{G}$ is gargantuan (see Example 3.1.73), this means that we can apply Theorem 5.2.1 to this class. Since a graph is SET-irreducible if and only if it is connected, Theorem 5.2.1 implies that

$$\mathbb{P}\big(\text{graph is connected}\big) \approx 1 - \sum_{k\geqslant 1} \mathfrak{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}},$$

where the counting sequence of irreducible tournaments (A054946 in the OEIS [76]) is

$$(\mathfrak{it}_k) = 1,\ 0,\ 2,\ 24,\ 544,\ 22\,320,\ 1\,677\,488,\ 236\,522\,496,\ 64\,026\,088\,576,\ \ldots$$

In particular, as it was shown by Wright [102],

$$\mathbb{P}\big(\text{graph is connected}\big) = 1 - \binom{n}{1} \cdot 2^{1-n} - 2 \cdot \binom{n}{3} \cdot 2^{6-3n} - 24 \cdot \binom{n}{4} \cdot 2^{10-4n} + O\big(n^5 \cdot 2^{-5n}\big).$$

Remark 7.2.3. Theorem 7.2.1 can serve as an indirect method of showing that a labeled combinatorial class $\mathcal{U}$ is not gargantuan. Indeed, let $\mathcal{U} = \mathrm{SET}(\mathcal{V})$ and $u \in \mathcal{U}$ be a random object of size n. Suppose that

$$\lim_{n\to\infty} \mathbb{P}(u \text{ is SET-irreducible}) \neq 1.$$

Then Theorem 7.2.1 implies that $\mathcal{U}$ is not gargantuan.

Example 7.2.4. Consider the labeled combinatorial class $\mathcal{F}$ of forests. As we have seen in Example 3.1.50, any forest is a collection of trees, meaning that

$$\mathcal{F} = \mathrm{SET}(\mathcal{TR}),$$

where $\mathcal{TR}$ is the labeled combinatorial class of trees. It is well-known [18] that the counting sequence of trees is $\mathfrak{tr}_n = n^{n-2}$. Hence, for large n,

$$\frac{\mathfrak{tr}_{n+1}}{\mathfrak{tr}_n} = (n+1)\left(1+\frac{1}{n}\right)^{n-2} \sim en.$$

On the other hand, the total amount of forests of size $(n+1)$ satisfies

$$\mathfrak{f}_{n+1} \geqslant \mathfrak{tr}_{n+1} + (n+1)\cdot \mathfrak{tr}_n.$$

Indeed, among the all forests of size $(n+1)$, there are trees (of size $(n+1)$) and forests with two components of size n and 1. Therefore,

$$\mathbb{P}(\text{forest is a tree}) = \frac{\mathfrak{tr}_{n+1}}{\mathfrak{f}_{n+1}} \leqslant \frac{\mathfrak{tr}_{n+1}}{\mathfrak{tr}_{n+1} + (n+1)\mathfrak{tr}_n} \sim \frac{e}{e+1} < 1,$$

meaning that, in accordance with Remark 7.2.3, $\mathcal{F}$ is not gargantuan. Note that this observation is in accordance with Example 3.1.74 showing that the class $\mathcal{TR}$ of trees is not gargantuan either, as well as with results of Bell, Bender, Cameron and Richmond [6, 7].

7.3 Objects with several SET-irreducible components

Theorem 7.3.1 (SET_m asymptotics). *Let $\mathcal{U}$ be a gargantuan labeled combinatorial class with positive coefficients, such that $\mathcal{U} = \mathrm{SET}(\mathcal{V})$ and $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ for some labeled combinatorial classes $\mathcal{V}$ and $\mathcal{W}$. Suppose that $u \in \mathcal{U}$ is a random object of size n. Then for any $m \in \mathbb{N}$,*

$$\mathbb{P}(u \text{ has } m \text{ SET-irreducible components}) \approx \sum_{k\geqslant 0} \alpha_k^{\{m\}} \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}, \tag{7.10}$$

where $\alpha_k^{\{m\}}$ are the coefficients of the exponential generating function

$$V^{\{m-1\}}(z)\big(1-W(z)\big) = \sum_{s=m-1}^{\infty} (-1)^{s+m-1}\binom{s}{m-1} V^{\{s\}}(z).$$

Proof. Apply Theorem 3.1.79 a) to the gargantuan labeled combinatorial class $\tilde{\mathcal{U}} = \mathcal{U} - \{\epsilon\}$ and the function

$$F_m(y) = \frac{\big(\ln(1+y)\big)^m}{m!}.$$

According to (7.2) and (7.4),

$$F_m\big(U(z)-1\big) = \frac{\Big(\ln\big(U(z)\big)\Big)^m}{m!} = V^{\{m\}}(z),$$

while

$$C_m(z) = \frac{\Big(\ln\big(U(z)\big)\Big)^{m-1}}{(m-1)!\,U(z)} = V^{\{m-1\}}(z)\big(1-W(z)\big).$$

Hence, the statement of Theorem 3.1.79 a) gives us

$$\mathfrak{v}_n^{\{m\}} \approx \sum_{k\geqslant 0}\binom{n}{k} c_k \mathfrak{u}_{n-k} = \sum_{k\geqslant 0}\binom{n}{k}\alpha_k^{\{m\}}\mathfrak{u}_{n-k}.$$

Dividing by $\mathfrak{u}_n$, we get desired formula (7.10).

To verify the last identity of the statement, we use relations (7.2) together with (7.4):

$$\begin{aligned} V^{\{m-1\}}(z)\big(1-W(z)\big) &= \frac{\big(V(z)\big)^{m-1}}{(m-1)!}\sum_{k=0}^{\infty}(-1)^k\frac{\big(V(z)\big)^k}{k!} \\ &= (-1)^{m-1}\sum_{k=0}^{\infty}(-1)^{k+m-1}\binom{k+m-1}{m-1}\frac{\big(V(z)\big)^{k+m-1}}{(k+m-1)!} \\ &= \sum_{s=m-1}^{\infty}(-1)^{s+m-1}\binom{s}{m-1}V^{\{s\}}(z). \end{aligned}$$

□

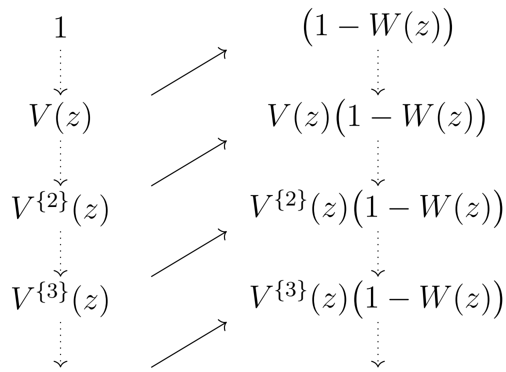
Remark 7.3.2. In the statement of Theorem 7.3.1, to obtain the asymptotic expansion (7.10), we do not need to have a labeled combinatorial class $\mathcal{W}$ such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$. In the case of absence an appropriate class, in the proof of the theorem, we use the exponential generating function $W(z)$ formally as a function defined by

$$W(z) = 1 - \frac{1}{U(z)}.$$

For a fixed $k \in \mathbb{N}$, the coefficient $\alpha_k^{\{m\}}$ does not depend on $W(z)$ and can be represented as the finite linear combination of the form

$$\alpha_k^{\{m\}} = \sum_{s=m-1}^{k}(-1)^{s+m-1}\binom{s}{m-1}\mathfrak{v}_k^{\{s\}}.$$

Remark 7.3.3. Let us define $V^{(0)}(z) = \big(V(z)\big)^0 = 1$. With such a notation, we have Theorem 7.2.1 to be the particular case of Theorem 7.3.1 for $m = 1$. The coefficients of asymptotic probabilities fit into the general scheme depicted below. In this scheme, the left column represents to the exponential generating functions of labeled combinatorial classes $\mathcal{V}^{\{m\}}$, where m runs over $\mathbb{Z}_{\geqslant 0}$. The right column corresponds to the same functions multiplied by $\big(1 - W(z)\big)$. As we have seen in Theorem 7.3.1, the coefficients involved into the asymptotic expansion of $\mathcal{V}^{\{m\}}$ constitute the exponential generating function $V^{\{m-1\}}(z)\big(1 - W(z)\big)$. To illustrate this fact we draw solid arrows pointing from the left column to the right one.



Corollary 7.3.4. *Let $\mathcal{U}$ be a gargantuan labeled combinatorial class with positive coefficients, such that $\mathcal{U} = \mathrm{SET}(\mathcal{V})$ and $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ for some labeled combinatorial classes $\mathcal{V}$ and $\mathcal{W}$. Suppose that $u \in \mathcal{U}$ is a random object of size n. Then for any $m \in \mathbb{N}$,*

$$\mathbb{P}(u \textit{ has } m \text{ SET-}\textit{irreducible components}) = \binom{n}{m-1} \cdot \frac{\mathfrak{u}_1^{m-1}\mathfrak{u}_{n-m+1}}{\mathfrak{u}_n} + O\left(n^m \cdot \frac{\mathfrak{u}_{n-m}}{\mathfrak{u}_n}\right). \tag{7.11}$$

Proof. According to Theorem 7.3.1, we need to look at the leading term of the exponential generating function $V^{\{m-1\}}(z)\big(1 - W(z)\big)$. Since $\mathfrak{v}_1 = \mathfrak{u}_1$, we have

$$V^{\{m-1\}}(z)\big(1 - W(z)\big) = \frac{\big(V(z)\big)^{m-1}\big(1 - W(z)\big)}{(m-1)!} = \mathfrak{u}_1^{m-1} \cdot \frac{z^{m-1}}{(m-1)!} + \ldots$$

Hence, the first non-zero coefficient in asymptotic probability (7.10) is

$$\mathfrak{u}_1^{m-1} \cdot \binom{n}{m-1} \cdot \frac{\mathfrak{u}_{n-m+1}}{\mathfrak{u}_n}.$$

□

Remark 7.3.5. Every labeled combinatorial object $u \in \mathcal{U} = \mathrm{SET}(\mathcal{V})$ has a fixed number m of SET-irreducible components. Therefore, for any positive integer n considered as the size of u, we have the identity

$$\mathfrak{u}_n = \sum_{m=1}^{\infty} \mathfrak{v}_n^{\{m\}}.$$

As a consequence, in the proof of Theorem 7.3.1 the sum of the series $C_m(z)$ over all $m \geqslant 1$ equals one. We can verify this property directly:

$$\sum_{m=1}^{\infty} C_m(z) = \big(1 - W(z)\big) \cdot \sum_{m=1}^{\infty} \frac{\big(V(z)\big)^{(m-1)}}{(m-1)!} = \frac{\exp\big(V(z)\big)}{U(z)} = 1.$$

Example 7.3.6. As we have seen in Example 7.2.2, the labeled combinatorial class of graphs is gargantuan and can be represented as a set of the labeled combinatorial class of connected graphs,

$$\mathcal{G} = \mathrm{SET}(\mathcal{CG}).$$

Hence, we can apply Theorem 7.3.1 to it. As a result, for any $m \in \mathbb{N}$, we obtain the asymptotic probability that a random graph belongs to the labeled combinatorial classes $\mathcal{CG}^{\{m\}}$, meaning that this graph has exactly m connected components:

$$\mathbb{P}\big(\text{graph has } m \text{ connected components}\big) \approx \sum_{k \geqslant 0} \alpha_k^{\{m\}} \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}},$$

where

$$\alpha_k^{\{m\}} = \sum_{s=m-1}^{k} (-1)^{s+m-1} \binom{s}{m-1} \mathfrak{cg}_k^{\{s\}}.$$

For $m \leqslant 5$, sequences $\big(\mathfrak{cg}_n^{\{m\}}\big)$ and $\big(\alpha_n^{\{m\}}\big)$ are listed in Table 7.1 and Table 7.2, respectively.

n	1	2	3	4	5	6	7	8	9	
$\big(\mathfrak{cg}_n\big)$	1	1	4	38	728	26 704	1 866 256	251 548 592	66 296 291 072	...
$\big(\mathfrak{cg}_n^{\{2\}}\big)$	0	1	3	19	230	5 098	207 536	15 891 372	2 343 580 752	...
$\big(\mathfrak{cg}_n^{\{3\}}\big)$	0	0	1	6	55	825	20 818	925 036	76 321 756	...
$\big(\mathfrak{cg}_n^{\{4\}}\big)$	0	0	0	1	10	125	2 275	64 673	3 102 204	...
$\big(\mathfrak{cg}_n^{\{5\}}\big)$	0	0	0	0	1	15	245	5 320	169 113	...

Table 7.1: Counting sequences $\big(\mathfrak{cg}_n^{\{m\}}\big)$ for $m \leqslant 5$.

n	1	2	3	4	5	6	7	8	9	
$\big(\alpha_n^{\{1\}}\big)$	−1	0	−2	−24	−544	−22 320	−1 677 488	−236 522 496	−64 026 088 576	...
$\big(\alpha_n^{\{2\}}\big)$	1	−1	1	14	398	18 552	1 505 644	222 306 448	61 826 469 776	...
$\big(\alpha_n^{\{3\}}\big)$	0	1	0	7	115	3 238	156 576	13 457 052	2 131 689 876	...
$\big(\alpha_n^{\{4\}}\big)$	0	0	1	2	25	455	13 783	711 788	65 405 368	...
$\big(\alpha_n^{\{5\}}\big)$	0	0	0	1	5	65	1 330	43 673	2 400 363	...

Table 7.2: Asymptotic coefficients for graphs.

7.4 Asymptotics for d-gargantuan classes

In this section, we generalize the results of Sections 7.2 and 7.3 to d-gargantuan labeled combinatorial classes, $d \in \mathbb{N}$.

Theorem 7.4.1. *Let d be a positive integer and $\mathcal{U}$ be a d-gargantuan labeled combinatorial class, such that $\mathcal{U} = \mathrm{SET}(\mathcal{V})$ and $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ for some labeled combinatorial classes $\mathcal{V}$ and $\mathcal{W}$. Suppose that $u \in \mathcal{U}$ is a random object of size dn. Then*

$$\mathbb{P}(u \textit{ is } \mathrm{SET}\textit{-irreducible}) \approx 1 - \sum_{k \geqslant 1} \mathfrak{w}_{dk} \cdot \binom{dn}{dk} \cdot \frac{\mathfrak{u}_{d(n-k)}}{\mathfrak{u}_{dn}}. \qquad (7.12)$$

Proof. Apply Theorem 3.1.79 b) to the d-gargantuan labeled combinatorial class $\tilde{\mathcal{U}} = \mathcal{U} - \{\epsilon\}$ and the function

$$F(y) = \ln(1+y),$$

and repeat the reasoning of the proof of Theorem 7.2.1. □

Theorem 7.4.2. *Let d be a positive integer and $\mathcal{U}$ be a d-gargantuan labeled combinatorial class, such that $\mathcal{U} = \mathrm{SET}(\mathcal{V})$ and $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ for some labeled combinatorial classes $\mathcal{V}$ and $\mathcal{W}$. Suppose that $u \in \mathcal{U}$ is a random object of size dn. Then for any $m \in \mathbb{N}$,*

$$\mathbb{P}(u \textit{ has } m \text{ SET-}\textit{irreducible components}) \approx \sum_{k \geqslant 0} \alpha_{dk}^{\{m\}} \cdot \binom{dn}{dk} \cdot \frac{\mathfrak{u}_{d(n-k)}}{\mathfrak{u}_{dn}}, \tag{7.13}$$

where $\alpha_{dk}^{\{m\}}$ are the coefficients of the exponential generating function

$$V^{\{m-1\}}(z)\big(1 - W(z)\big) = \sum_{s=m-1}^{\infty} (-1)^{s+m-1} \binom{s}{m-1} V^{\{s\}}(z).$$

In particular, the leading term of (7.13) *is*

$$\mathbb{P}(u \textit{ has } m \text{ SEQ-}\textit{irreducible components}) =$$

$$= \frac{1}{(m-1)!} \cdot \frac{(dn)!}{(d!)^{m-1}(d(n-m+1))!} \cdot \frac{u_d^{m-1} u_{d(n-m+1)}}{u_{dn}} + O\left(n^{dm} \cdot \frac{u_{d(n-m)}}{u_{dn}}\right). \tag{7.14}$$

Proof. To establish (7.13), apply Theorem 3.1.79 b) to the d-gargantuan labeled combinatorial class $\tilde{\mathcal{U}} = \mathcal{U} - \{\epsilon\}$ and the function

$$F_m(y) = \frac{\big(\ln(1+y)\big)^m}{m!}$$

and repeat the reasoning of the proof of Theorem 7.3.1. According to (7.13), the leading term of the desired asymptotics is determined by the exponential generating function

$$V^{\{m-1\}}(z)\big(1 - W(z)\big).$$

Since the first non-zero coefficient in the exponential generating functions $V(z)$ is equal to $\mathfrak{v}_d = \mathfrak{u}_d$, we have

$$\begin{aligned} V^{\{m-1\}}(z)\big(1 - W(z)\big) &= \frac{\big(V(z)\big)^{m-1}}{(m-1)!}\big(1 - W(z)\big) \\ &= \frac{u_d^{m-1}}{(m-1)!} \cdot \frac{z^{d(m-1)}}{(d!)^{m-1}} + \dots \\ &= \left(\frac{\big(d(m-1)\big)!}{(d!)^{m-1}} \cdot \frac{u_d^{m-1}}{(m-1)!}\right) \cdot \frac{z^{d(m-1)}}{\big(d(m-1)\big)!} + \dots . \end{aligned}$$

Thus, formula (7.13), provides us the first non-zero coefficient in the asymptotic probability that a random object has exactly m SET-irreducible components:

$$\left(\binom{dn}{d(m-1)} \cdot \frac{u_{d(n-m+1)}}{u_{dn}}\right) \cdot \left(\frac{\big(d(m-1)\big)!}{(d!)^{m-1}} \cdot \frac{u_d^{m-1}}{(m-1)!}\right).$$

Regrouping factors, we obtain exactly (7.14). □

7.5 Asymptotics for unlabeled classes

The aim of this section is to generalize the results of Section 7.2 to unlabeled combinatorial classes. In this goal, we succeed only partially. We obtain asymptotic probability concerning the constructions MSET and PSET, but we fail to do this for the restricted constructions. Another disadvantage is that we do not have interesting applications. Nevertheless, it seems important to introduce the reader to the obtained results, since they are based on Theorem 2.2.8 rather than Theorem 2.2.5. The initial idea goes back to Bender [9], while we fill it with a new interpretation, similar to what we have seen in Theorem 7.2.1.

Theorem 7.5.1 (MSET/PSET asymptotics). *Let $\mathcal{U}$ be a gargantuan unlabeled combinatorial class with positive coefficients, such that $\mathcal{U} = \mathrm{MSET}(\mathcal{V})$ (or $\mathcal{U} = \mathrm{PSET}(\mathcal{V})$) and $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ for some unlabeled combinatorial classes $\mathcal{V}$ and $\mathcal{W}$. Suppose that $u \in \mathcal{U}$ is a random object of size n. Then*

$$\mathbb{P}(u \text{ is SET-irreducible}) \approx 1 - \sum_{k\geqslant 1} \mathfrak{w}_k \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}. \tag{7.15}$$

Proof. We carry out the proof for the construction MSET only, since for PSET the reasoning is the same. The idea is to apply Theorem 2.2.8 b) to the ordinary generating function

$$A(z) = U(z) - 1$$

and the functions

$$H(x, y) = \ln(1 + y) - x$$

and

$$D(x) = \sum_{k=2}^{\infty} \frac{V(x^k)}{k}.$$

In is possible, indeed, since the sequence $(\mathfrak{u}_n)$ is gargantuan and

$$|d_n| < \sum_{k=2}^{n} a_{[n/k]} = O(a_{[\lambda n]})$$

whenever $\lambda > 1/2$. Hence,

$$H\big(D(z), A(z)\big) = \sum_{k=1}^{\infty} \frac{V(z^k)}{k} - \sum_{k=2}^{\infty} \frac{V(z^k)}{k} = V(z),$$

while

$$C(z) = \frac{1}{U(z)} = 1 - W(z),$$

and, according to the statement of Theorem 2.2.8 b),

$$\mathfrak{v}_n \approx \sum_{k\geqslant 0} c_k \mathfrak{u}_{n-k} = \mathfrak{u}_n - \sum_{k\geqslant 1} \mathfrak{w}_k \mathfrak{u}_{n-k}.$$

Dividing by $\mathfrak{u}_n$, we obtain desired formula (7.15). □

Chapter 8

Asymptotic theorems in terms of species

In the previous chapters, we have established the asymptotic expansions for the probabilities concerning structure decompositions with respect to different combinatorial constructions. We have seen that the involved coefficients could be interpreted in terms of initial and related combinatorial classes. However, in most cases, the interpretation has been possible rather in terms of linear combinations of counting sequences of some combinatorial classes, than in terms of counting sequences themselves. Moreover, for the construction SET, we have used the additional supposition that the initial combinatorial class could be constructed with the help of the construction SEQ as well, not just SET. All these observations show the limited applicability of the symbolic method.

The aim of this chapter is to cope with the above mentioned disadvantages with the help of the theory of species. Within this theory, one replaces combinatorial constructions of the symbolic method by substitutions. More precisely, we have the following list of correspondences between decompositions of labeled combinatorial classes and species substitutions, including restricted constructions (compare to Example 3.1.60).

$$\begin{array}{ccc} \text{combinatorial classes} & & \text{species of structures} \\ \begin{array}{rcl} \mathcal{A} &=& \mathrm{SEQ}(\mathcal{B}) \\ \mathcal{A} &=& \mathrm{CYC}(\mathcal{B}) \\ \mathcal{A} &=& \mathrm{SET}(\mathcal{B}) \\ \mathcal{A} &=& \mathrm{SEQ}_m(\mathcal{B}) \\ \mathcal{A} &=& \mathrm{CYC}_m(\mathcal{B}) \\ \mathcal{A} &=& \mathrm{SET}_m(\mathcal{B}) \end{array} & \Leftrightarrow & \begin{array}{rcl} \mathcal{A} &=& \mathcal{L}\circ\mathcal{B} \\ \mathcal{A} &=& \mathcal{CP}\circ\mathcal{B} \\ \mathcal{A} &=& \mathcal{E}\circ\mathcal{B} \\ \mathcal{A} &=& \mathcal{L}_m\circ\mathcal{B} \\ \mathcal{A} &=& \mathcal{CP}_m\circ\mathcal{B} \\ \mathcal{A} &=& \mathcal{E}_m\circ\mathcal{B} \end{array} \end{array}$$

In Section 8.1, we discuss asymptotics related to substitutions with $\mathcal{L}$ and $\mathcal{L}_m$. Section 8.2 is devoted to substitutions with $\mathcal{CP}$ and $\mathcal{CP}_m$. Finally, in Section 8.3, we establish results concerning substitutions with $\mathcal{E}$ and $\mathcal{E}_m$.

The reader may notice that, discussing combinatorial constructions in terms of substitution, Flajolet and Sedgewick [38] indicated the combinatorial isomorphism for the construction SEQ in the form

$$\mathrm{SEQ}(\mathcal{B}) \cong \mathcal{P}\circ\mathcal{B}.$$

It is reasonable, however, to use species $\mathcal{L}$ of linear orders instead of species $\mathcal{P}$ of permutations equipotent to $\mathcal{L}$. By doing so, we extend the applicability limits of this substitution to unlabeled structures as well.

In order to simplify the presentation of the chapter and to make it consistent with the previous ones, we use the following notations for constructions restricted to cardinality $m \in \mathbb{N}$:

$$\mathcal{B}^{(m)} = \mathcal{L}_m\circ\mathcal{B}, \qquad \mathcal{B}^{[m]} = \mathcal{CP}_m\circ\mathcal{B}, \qquad \mathcal{B}^{\{m\}} = \mathcal{E}_m\circ\mathcal{B}.$$

8.1 Asymptotic probability for SEQ-irreducibles

Theorem 8.1.1. *Let $\mathcal{A}$ be a gargantuan (weighted) species of structures, such that $\mathcal{A} = \mathcal{L} \circ \mathcal{B}$ for some (weighted) species $\mathcal{B}$. Suppose that $s \in \mathcal{A}$ is a random $\mathcal{A}$-structure on $[n]$, where $n \in \mathbb{N}$. Then*

$$\mathbb{P}(s \in \mathcal{B}) \approx 1 - \sum_{k \geqslant 1} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}, \tag{8.1}$$

where $\mathfrak{a}_n$ and $\mathfrak{c}_n$ are the total weight on $[n]$ of the species $\mathcal{A}$ and $\mathcal{C} = (\mathbf{1} - \mathcal{B})^2$, respectively.

Proof. First of all, express the species $\mathcal{B}$ in terms of $\mathcal{A}$. Since $\mathcal{A} = \mathcal{A}_+ + \mathbf{1}$, one has

$$\mathcal{A}_+ = (\mathcal{L}_+ + \mathbf{1}) \circ \mathcal{B} - \mathbf{1} = \mathcal{L}_+ \circ \mathcal{B},$$

and hence,

$$\mathcal{B} = \mathcal{L}_+^{(-1)} \circ \mathcal{A}_+.$$

Second, apply Theorem 4.2.21 for the species $\mathcal{A}$ and $F = \mathcal{L}_+^{(-1)}$. We obtain

$$\mathfrak{b}_n \approx \sum_{k \geqslant 0} \binom{n}{k} \mathfrak{c}_k \mathfrak{a}_{n-k}, \tag{8.2}$$

where $\mathfrak{c}_k$ is the total weight of $\mathcal{C} = F' \circ \mathcal{A}_+$ on the set $[k]$. Since $(\mathcal{L}_+)' = \mathcal{L}^2$ (see Example 4.1.61), the following relations hold:

$$\mathcal{C} = \left(\mathcal{L}_+^{(-1)}\right)' \circ \mathcal{A}_+ = \left((\mathcal{L}_+)'\right)^{-1} \circ \mathcal{L}_+^{(-1)} \circ \mathcal{A}_+ = \mathcal{L}^{-2} \circ \mathcal{B}.$$

Taking into account that $\mathcal{L}^{-1} = \mathbf{1} - \mathcal{Z}$ (see Example 4.2.12), we get

$$\mathcal{C} = (\mathbf{1} - \mathcal{B})^2.$$

Now, using the relation $\mathbb{P}(s \in \mathcal{B}) = \mathfrak{b}_n / \mathfrak{a}_n$, divide (8.2) by $\mathfrak{a}_n$ to finish the proof. ☐

Theorem 8.1.2. *Let $\mathcal{A}$ be a gargantuan (weighted) species of structures, such that $\mathcal{A} = \mathcal{L} \circ \mathcal{B}$ for some (weighted) species $\mathcal{B}$. Suppose that $s \in \mathcal{A}$ is a random $\mathcal{A}$-structure on $[n]$, where $n \in \mathbb{N}$. Then for any $m \in \mathbb{N}$,*

$$\mathbb{P}\left(s \in \mathcal{B}^{(m)}\right) \approx \sum_{k \geqslant 0} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}, \tag{8.3}$$

where $\mathfrak{a}_n$ and $\mathfrak{c}_n$ are the total weight on $[n]$ of the species $\mathcal{A}$ and $\mathcal{C} = m\mathcal{B}^{m-1}(\mathbf{1} - \mathcal{B})^2$, respectively.

Proof. Let us notice the following relations. First,

$$\mathcal{B}^{(m)} = \mathcal{B}^m = \mathcal{Z}^m \circ \mathcal{B} = \mathcal{Z}^m \circ \mathcal{L}_+^{(-1)} \circ \mathcal{A}_+.$$

Second,

$$\left(\mathcal{Z}^m \circ \mathcal{L}_+^{(-1)}\right)' = \left(m\mathcal{Z}^{m-1} \circ \mathcal{L}_+^{(-1)}\right) \cdot \left(\left((\mathcal{L}_+)'\right)^{-1} \circ \mathcal{L}_+^{(-1)}\right).$$

Third, as we have seen in the proof of Theorem 8.1.1,

$$\left((\mathcal{L}_+)'\right)^{-1} \circ \mathcal{L}_+^{(-1)} \circ \mathcal{A}_+ = \mathcal{L}^{-2} \circ \mathcal{B} = (\mathbf{1} - \mathcal{B})^2.$$

As a consequence,

$$\left(\mathcal{Z}^m \circ \mathcal{L}_+^{(-1)}\right)' \circ \mathcal{A}_+ = m\mathcal{B}^{m-1} \cdot (\mathbf{1} - \mathcal{B})^2.$$

Therefore, applying Theorem 4.2.21 for the species $\mathcal{A}$ and $F = \mathcal{L}_m \circ \mathcal{L}_+^{(-1)}$, we obtain

$$\mathfrak{b}_n^{(m)} \approx \sum_{k \geqslant 0} \binom{n}{k} \mathfrak{c}_k \mathfrak{a}_{n-k},$$

where $\mathfrak{c}_k$ is the total weight of

$$\mathcal{C} = F' \circ \mathcal{A}_+ = m\mathcal{B}^{m-1} \cdot (\mathbf{1} - \mathcal{B})^2$$

on $[k]$. Now, dividing this asymptotic relation by $\mathfrak{a}_n$, we get the desired asymptotic probability. □

8.2 Asymptotic probability for CYC-irreducibles

Theorem 8.2.1. *Let $\mathcal{A}$ be a gargantuan (weighted) species of structures, such that $\mathcal{A} = \mathcal{CP} \circ \mathcal{B}$ for some (weighted) species $\mathcal{B}$. Suppose that $s \in \mathcal{A}$ is a random $\mathcal{A}$-structure on $[n]$, where $n \in \mathbb{N}$. Then*

$$\mathbb{P}(s \in \mathcal{B}) \approx 1 - \sum_{k \geqslant 1} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}, \tag{8.4}$$

where $\mathfrak{a}_n$ and $\mathfrak{c}_n$ are the total weight on $[n]$ of the species $\mathcal{A}$ and $\mathcal{C} = (\mathbf{1} - \mathcal{B})$, respectively.

Proof. Since $\mathcal{A}_0 = \mathbf{0}$, we can express the species $\mathcal{B}$ as

$$\mathcal{B} = \mathcal{CP}^{(-1)} \circ \mathcal{A}.$$

Apply Theorem 4.2.21 for the species $\mathcal{A}$ and $F = \mathcal{CP}^{(-1)}$. We obtain

$$\mathfrak{b}_n \approx \sum_{k \geqslant 0} \binom{n}{k} \mathfrak{c}_k \mathfrak{a}_{n-k}, \tag{8.5}$$

where $\mathfrak{c}_k$ is the total weight of $\mathcal{C} = F' \circ \mathcal{A}$ on the set $[k]$. Since $(\mathcal{CP})' = \mathcal{L}$ (see Example 4.1.60), the species $\mathcal{C}$ has the following form:

$$\mathcal{C} = \left(\mathcal{CP}^{(-1)}\right)' \circ \mathcal{A} = \left(\mathcal{CP}'\right)^{-1} \circ \mathcal{CP}^{(-1)} \circ \mathcal{A} = \mathcal{L}^{-1} \circ \mathcal{B}.$$

Taking into account that $\mathcal{L}^{-1} = \mathbf{1} - \mathcal{Z}$ (see Example 4.2.12), we get

$$\mathcal{C} = \mathbf{1} - \mathcal{B}.$$

Now, using the relation $\mathbb{P}(s \in \mathcal{B}) = \mathfrak{b}_n/\mathfrak{a}_n$, divide (8.5) by $\mathfrak{a}_n$ to finish the proof. □

Theorem 8.2.2. *Let $\mathcal{A}$ be a gargantuan (weighted) species of structures, such that $\mathcal{A} = \mathcal{CP} \circ \mathcal{B}$ for some (weighted) species $\mathcal{B}$. Suppose that $s \in \mathcal{A}$ is a random $\mathcal{A}$-structure on $[n]$, where $n \in \mathbb{N}$. Then for any $m \in \mathbb{N}$,*

$$\mathbb{P}\left(s \in \mathcal{B}^{[m]}\right) \approx \sum_{k \geqslant 0} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}, \tag{8.6}$$

where $\mathfrak{a}_n$ and $\mathfrak{c}_n$ are the total weight on $[n]$ of the species $\mathcal{A}$ and $\mathcal{C} = \mathcal{B}^{m-1}(\mathbf{1} - \mathcal{B})$, respectively.

Proof. Let us notice the following relations. First,

$$\mathcal{B}^{[m]} = \mathcal{CP}_m \circ \mathcal{B} = \mathcal{CP}_m \circ \mathcal{CP}^{(-1)} \circ \mathcal{A}.$$

Second,

$$\left(\mathcal{CP}_m \circ \mathcal{CP}^{(-1)}\right)' = \left(\mathcal{CP}'_m \circ \mathcal{CP}^{(-1)}\right) \cdot \left((\mathcal{CP}')^{-1} \circ \mathcal{CP}^{(-1)}\right).$$

Third, $\mathcal{CP}'_m = \mathcal{L}_{m-1}$. Finally, as we have seen in the proof of Theorem 8.2.1,

$$(\mathcal{CP}')^{-1} \circ \mathcal{CP}^{(-1)} \circ \mathcal{A} = \mathcal{L}^{-1} \circ \mathcal{B} = \mathbf{1} - \mathcal{B}.$$

As a consequence,

$$\left(\mathcal{CP}_m \circ \mathcal{CP}^{(-1)}\right)' \circ \mathcal{A} = \mathcal{B}^{m-1} \cdot (\mathbf{1} - \mathcal{B}).$$

Therefore, applying Theorem 4.2.21 for the species $\mathcal{A}$ and $F = \mathcal{CP}_m \circ \mathcal{CP}^{(-1)}$, we obtain

$$\mathfrak{b}_n^{[m]} \approx \sum_{k \geqslant 0} \binom{n}{k} \mathfrak{c}_k \mathfrak{a}_{n-k},$$

where $\mathfrak{c}_k$ is the total weight of

$$\mathcal{C} = F' \circ \mathcal{A} = \mathcal{B}^{m-1} \cdot (1 - \mathcal{B})$$

on $[k]$. Dividing this asymptotic relation by $\mathfrak{a}_n$, we get the desired asymptotic probability. □

8.3 Asymptotic probability for SET-irreducibles

Lemma 8.3.1. *The virtual species $\mathcal{E}^{-1}$ is equipotent to the alternating sum*

$$\mathcal{E}^{-1} \equiv \mathbf{1} - \mathcal{E}_1 + \mathcal{E}_2 - \mathcal{E}_3 + \ldots$$

Proof. This follows from the fact that the corresponding exponential generating series are equal to each other:

$$\frac{1}{e^z} = e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \ldots$$

□

Remark 8.3.2. There is no isomorphism between the species mentioned in Lemma 8.3.1, since the corresponding type generating series are different:

$$\left(\frac{1}{1-z}\right)^{-1} = 1 - z \neq \frac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots$$

Lemma 8.3.3. *The virtual species $\mathcal{E}^{-1} \circ \mathcal{E}_+^{(-1)}$ and $\left(\mathbf{1} - \mathcal{L}_+^{(-1)}\right)$ are equipotent.*

Proof. To verify the target equipotency, it is sufficient to check that the corresponding exponential generating series are the same. Indeed,

$$\mathcal{E}^{-1}(z) = e^{-z}, \qquad \mathcal{E}_+^{(-1)}(z) = \log(1+z), \qquad \mathcal{L}_+^{(-1)}(z) = \frac{z}{1+z}.$$

Hence,

$$\mathcal{E}^{-1}\left(\mathcal{E}_+^{(-1)}(z)\right) + \mathcal{L}_+^{(-1)}(z) = \frac{1}{1+z} + \frac{z}{1+z} = 1.$$

□

Theorem 8.3.4. *Let $\mathcal{A}$ be a gargantuan (weighted) species of structures, such that $\mathcal{A} = \mathcal{E} \circ \mathcal{B}$ for some (weighted) species $\mathcal{B}$. Suppose that $s \in \mathcal{A}$ is a random $\mathcal{A}$-structure on $[n]$, where $n \in \mathbb{N}$. Then*

$$\mathbb{P}(s \in \mathcal{B}) \approx 1 - \sum_{k \geqslant 1} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}, \tag{8.7}$$

where $\mathfrak{a}_n$ and $\mathfrak{c}_n$ are the total weights on $[n]$ of the species $\mathcal{A}$ and $\mathcal{C} = \mathcal{E}^{-1} \circ \mathcal{B} \equiv \left(\mathbf{1} - \mathcal{L}_+^{(-1)}\right) \circ \mathcal{A}_+$, respectively.

Proof. The same way as in the proof of Theorem 8.1.1, we can express the species $\mathcal{B}$ in terms of $\mathcal{A}$:

$$\mathcal{B} = \mathcal{E}_+^{(-1)} \circ \mathcal{A}_+.$$

Apply Theorem 4.2.21 for the species $\mathcal{A}$ and $F = \mathcal{E}_+^{(-1)}$. We obtain

$$\mathfrak{b}_n \approx \sum_{k \geqslant 0} \binom{n}{k} \mathfrak{c}_k \mathfrak{a}_{n-k}, \tag{8.8}$$

where $\mathfrak{c}_k$ is the total weight of $\mathcal{C} = F' \circ \mathcal{A}_+$ on $[k]$. Since $(\mathcal{E}_+)' = \mathcal{E}$ (see Example 4.1.59), the following relations hold:

$$\mathcal{C} = \left(\mathcal{E}_+^{(-1)}\right)' \circ \mathcal{A}_+ = \left((\mathcal{E}_+)'\right)^{-1} \circ \mathcal{E}_+^{(-1)} \circ \mathcal{A}_+ = \mathcal{E}^{-1} \circ \mathcal{B}.$$

Using the relation $\mathbb{P}(s \in \mathcal{B}) = \mathfrak{b}_n / \mathfrak{a}_n$, divide (8.2) by $\mathfrak{a}_n$ to obtain the desired asymptotics. To finish the proof, notice, that

$$\mathcal{C} = \mathcal{E}^{-1} \circ \mathcal{B} = \mathcal{E}^{-1} \circ \mathcal{E}_+^{(-1)} \circ \mathcal{A}_+ \equiv \left(\mathbf{1} - \mathcal{L}_+^{(-1)}\right) \circ \mathcal{A}_+$$

by Lemma 8.3.3. ☐

Remark 8.3.5. Due to Lemma 8.3.1, we can consider $\mathcal{E}^{-1} \circ \mathcal{B}$ as the alternate sum

$$\mathbf{1} - \mathcal{E}_1 \circ \mathcal{B} + \mathcal{E}_2 \circ \mathcal{B} - \mathcal{E}_3 \circ \mathcal{B} + \ldots = \mathbf{1} - \mathcal{B}^{\{1\}} + \mathcal{B}^{\{2\}} - \mathcal{B}^{\{3\}} + \ldots$$

Interpreting the species $\mathcal{B}$ as a subclass of the species $\mathcal{A}$ consisting of connected species, we obtain the following meaning: the species $\mathcal{E}^{-1} \circ \mathcal{B}$ consists of the same structures as $\mathcal{A}$, but a structure is considered negative if the number of its connected components is odd.

Theorem 8.3.6. *Let $\mathcal{A}$ be a gargantuan (weighted) species of structures, such that $\mathcal{A} = \mathcal{E} \circ \mathcal{B}$ for some (weighted) species $\mathcal{B}$. Suppose that $s \in \mathcal{A}$ is a random $\mathcal{A}$-structure on $[n]$, where $n \in \mathbb{N}$. Then for any $m \in \mathbb{N}$,*

$$\mathbb{P}\left(s \in \mathcal{B}^{\{m\}}\right) \approx \sum_{k \geqslant 0} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}, \tag{8.9}$$

where $\mathfrak{a}_n$ and $\mathfrak{c}_n$ are the total weight on $[n]$ of the species $\mathcal{A}$ and $\mathcal{C} = \mathcal{B}^{\{m-1\}}(\mathcal{E}^{-1} \circ \mathcal{B})$, respectively.

Proof. Let us notice the following relations. First,

$$\mathcal{B}^{\{m\}} = \mathcal{E}_m \circ \mathcal{B} = \mathcal{E}_m \circ \mathcal{E}_+^{(-1)} \circ \mathcal{A}_+.$$

Second,

$$\left(\mathcal{E}_m \circ \mathcal{E}_+^{(-1)}\right)' = \left((\mathcal{E}_m)' \circ \mathcal{E}_+^{(-1)}\right) \cdot \left(\left((\mathcal{E}_+)'\right)^{-1} \circ \mathcal{E}_+^{(-1)}\right).$$

Third, as we have seen in the proof of Theorem 8.1.1,

$$\big((\mathcal{E}_+)'\big)^{-1} \circ \mathcal{E}_+^{(-1)} \circ \mathcal{A}_+ = \mathcal{E}^{-1} \circ \mathcal{B}.$$

As a consequence, taking into account that $(\mathcal{E}_m)' = \mathcal{E}_{m-1}$, we have

$$\big(\mathcal{E}_m \circ \mathcal{E}_+^{(-1)}\big)' \circ \mathcal{A}_+ = \mathcal{B}^{\{m-1\}} \cdot (\mathcal{E}^{-1} \circ \mathcal{B}).$$

Therefore, applying Theorem 4.2.21 to the species $\mathcal{A}$ and $F = \mathcal{E}_m \circ \mathcal{E}_+^{(-1)}$, we obtain

$$\mathfrak{b}_n^{\{m\}} \approx \sum_{k \geqslant 0} \binom{n}{k} \mathfrak{c}_k \mathfrak{a}_{n-k},$$

where $\mathfrak{c}_k$ is the total weight of

$$\mathcal{C} = F' \circ \mathcal{A}_+ = \mathcal{B}^{\{m-1\}} \cdot (\mathcal{E}^{-1} \circ \mathcal{B})$$

on $[k]$. Now, dividing this asymptotic relation by $\mathfrak{a}_n$, we get the desired asymptotic probability. □

Remark 8.3.7. The same way as in Remark 8.3.5, we can consider $\mathcal{B}^{\{m-1\}} \cdot (\mathcal{E}^{-1} \circ \mathcal{B})$ as the alternate sum

$$\mathcal{B}^{\{m-1\}} - \mathcal{B}^{\{m-1\}} \cdot \mathcal{B}^{\{1\}} + \mathcal{B}^{\{m-1\}} \cdot \mathcal{B}^{\{2\}} - \mathcal{B}^{\{m-1\}} \cdot \mathcal{B}^{\{3\}} + \ldots$$

Hence, the $\mathcal{B}^{\{m-1\}} \cdot (\mathcal{E}^{-1} \circ \mathcal{B})$-structures are the $\mathcal{A}$-structures whose connected components are divided into two groups such that the first group contains $(m-1)$ connected components. A structure is considered negative if the number of connected components in the second group is odd.

Remark 8.3.8. As well as for the symbolic method, we could additionally define d-gargantuan species and obtain results similar to what we have had in Sections 5.4, 6.4 and 7.4.

Part III

Applications

Chapter 9

Permutations

In this chapter, we apply the theory developed in Part II to various classes of permutations. We have seen in Example 3.2.13 that the combinatorial class of permutations formally considered as unlabeled admits a decomposition with respect to the construction SEQ. Following [22, p. 262], we call the SEQ-irreducible permutations under this decompositions *indecomposable*. In the literature, the reader can also find such designations as *irreducible permutations* [50, 62] or *connected permutations* [31, 53].

It looks like that the concept of indecomposable permutations arose, for the first time, within the monoid theory in the work of Lentin [57]. In 1972, Comtet [21] showed that their counting sequence appeared as coefficients of the multiplicative inverse of the formal power series comprised by factorials:

$$\sum_{n=1}^{\infty} \mathfrak{ip}_n z^n = 1 - \left(\sum_{n=0}^{\infty} n! z^n\right)^{-1}.$$

Moreover, he established the asymptotic expansion for the probability that a random permutation of size $n \in \mathbb{N}$ is indecomposable (see also [22]):

$$\mathbb{P}\big(\text{permutation is indecomposable}\big) =$$
$$= 1 - \frac{2}{n} - \frac{1}{(n)_2} - \frac{4}{(n)_3} - \frac{19}{(n)_4} - \frac{110}{(n)_5} - \frac{745}{(n)_6} - \frac{5752}{(n)_7} - \frac{49775}{(n)_8} - \frac{476994}{(n)_9} - \frac{5016069}{(n)_{10}} + O\left(\frac{1}{n^{11}}\right).$$

Note that there are bijections between indecomposable permutations and a number of other combinatorial objects, such as labeled Dyck paths [23] and rooted hypermaps [78]. They are also used for enumeration of certain kind of Feynman diagrams [24] and appear as indices of the basis of the Malvenuto–Reutenauer Hopf algebra [1, 31].

We start the exposition of our results with Section 9.1 showing that the above asymptotic expansion of Comtet can be established within the general framework (Theorem 9.1.11). First, we discuss the result of Comtet regarding permutations as unlabeled objects. On this way, we also obtain the asymptotic expansions for the probabilities, that a random permutation consists of exactly $m \in \mathbb{N}$ indecomposable parts, as well as the combinatorial interpretation of coefficients involved in the asymptotics. Although this point of view is practical for obtaining a short proof, it is somewhat artificial to consider permutations to be unlabeled, since the atoms of a permutation of size n are already labeled by integers from 1 to n by definition. On the other hand, the notion of indecomposability is heavily dependent on these natural labels and is not stable under relabeling. In order to cope with this obstacle, we pass to another combinatorial class, namely, pairs of linear orders $\mathcal{L} \odot \mathcal{L}$. We establish the SEQ-decomposition for this class, as well as the correspondence between SEQ-irreducible objects from the class $\mathcal{L} \odot \mathcal{L}$ and indecomposable permutations. Note that, as a bonus of using our general results related to the labeled case, we obtain important connections with other combinatorial objects (see Chapter 11).

Section 9.2 is devoted to perfect matchings. The reader might be familiar with the concept of *perfect matching in a graph* as a union of edges in this graph passing through each vertex exactly once (see, for example, [56, p. 374]). In our study, we focus on the case corresponding to complete labeled graphs. This allows us to consider perfect matchings as a subclass of the combinatorial class of permutations that consists of involutions without fixed points. In particular, the concept of indecomposability is applicable to perfect matchings as well. All the methods described in Section 9.1 work well in this case, so that we obtain asymptotic expansions for the probabilities that a perfect matching is indecomposable or comprises a fixed number of indecomposable parts, together with the combinatorial interpretation of the involved coefficients. In addition, we will see in Chapter 11 that indecomposable perfect matchings have connections with the combinatorial map model of surfaces. Note that the number of perfect matchings of size n is a double factorial $(2n-1)!!$ that appear in a variety of contexts.

In Section 9.3, we generalize the developed study to multipermutations. It is worth mentioning that, in our investigation, we introduce a multipermutation as a tuple of permutations of the same size. Thus, this notion is different from the one of Comtet [22, p. 235] and from the concept of permutations of multisets that are often called multipermutations as well. The concept of indecomposability can be naturally generalized for multipermutations, while Theorems 5.5.1 and 5.5.2 allow to obtain asymptotic expansions for the corresponding probabilities together with the combinatorial interpretation. We will see in Chapter 11 that multipermutations arise as the coefficients of the asymptotic probability that a random tuple of permutations forms a constellation.

Finally, in Section 9.4, we shortly discuss further generalizations that can be obtained within the framework of the approach considered in this chapter.

In order to make the presentation of the material more clear and self-contained, we recall all necessary definitions and properties of permutations and related objects.

9.1 Permutations

9.1.1 Permutations: definitions and properties

Definition 9.1.1. A *permutation* of size n is a bijection from the set $[n]$ onto itself.

Notation 9.1.2. We designate by $\mathcal{P}$ the labeled combinatorial class of permutations. Sometimes we formally consider permutations as unlabeled objects. In the latter case, we denote it using the corresponding index: ${}_u\mathcal{P}$. Given $n \in \mathbb{N}$, the restriction $\mathcal{P}_n$ of the class $\mathcal{P}$ to cardinality n is the standard set of permutations S_n of size n.

Remark 9.1.3. The number of permutations of size n is

$$\mathfrak{p}_n = n!,$$

meaning that the exponential generating function of the class $\mathcal{P}$ is

$$P(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

Regarding $\mathcal{P}$ as the unlabeled combinatorial class, we have its ordinary generating function to be

$${}_uP(z) = \sum_{n=0}^{\infty} n! z^n.$$

Lemma 9.1.4. *The unlabeled combinatorial class* ${}_u\mathcal{P}$ *of permutations is gargantuan.*

Proof. The statement of Lemma 9.1.4 follows from the fact that the sequence $\mathfrak{p}_n = n!$ is gargantuan (see Example 2.1.2). □

Remark 9.1.5. The sequence

$$n \cdot \frac{\mathfrak{p}_{n-1}}{\mathfrak{p}_n} = 1$$

does not tend to 0 as n tends to infinity. Hence, the labeled combinatorial class $\mathcal{P}$ is not gargantuan.

9.1.2 Indecomposable permutations

Definition 9.1.6. A permutation $\sigma \in S_n$ is *decomposable*, if there exists a positive integer $k < n$ such that $[k]$ is invariant under action of σ, i.e. $\sigma\big([k]\big) = [k]$. Otherwise, σ is said to be *indecomposable.*

Notation 9.1.7. We denote the labeled combinatorial class of indecomposable permutations by $\mathcal{IP}$. The corresponding counting sequence is $(\mathfrak{ip}_n)$.

Lemma 9.1.8. *The unlabeled combinatorial classes* ${}_u\mathcal{P}$ *of permutations and* ${}_u\mathcal{IP}$ *of indecomposable permutations satisfy the relation*

$${}_u\mathcal{P} = \mathrm{SEQ}({}_u\mathcal{IP}).$$

Proof. Let $n \in \mathbb{N}$ and $\sigma \in S_n$. Consider the partition of $[n]$ into a maximal number of consecutive intervals

$$[n] = I_1 + \ldots + I_m,$$

such that $\sigma(I_k) = I_k$ for each $k \in [m]$. The restriction of σ to I_k can be considered as an indecomposable permutation of size $|I_k|$. Hence, σ corresponds to a sequence of m indecomposable permutations and vice versa (Fig. 9.1). □

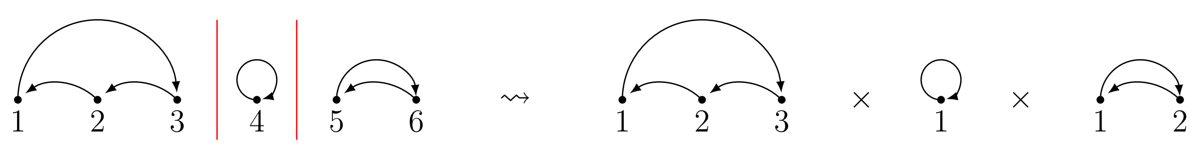


Figure 9.1: Decomposition of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 4 & 6 & 5 \end{pmatrix}$ into a sequence.

Remark 9.1.9. It follows from Lemma 9.1.8 that indecomposable parts of a permutation coincide with its SEQ-irreducible parts. In particular, a permutation is SEQ-irreducible, if and only if it is indecomposable.

Notation 9.1.10. For any $m \in \mathbb{N}$, we denote by $\mathcal{IP}^{(m)}$ the labeled combinatorial class consisting of permutations that have exactly m indecomposable parts. For unlabeled classes, we have the relation

$${}_u\mathcal{IP}^{(m)} = \mathrm{SEQ}_m({}_u\mathcal{IP})$$

We designate as $\mathfrak{ip}_n^{(m)}$ the number of permutations of size n with m indecomposable parts.

9.1.3 Asymptotic probabilities for indecomposable permutations

Theorem 9.1.11. *Let $n \in \mathbb{N}$. The asymptotic probability that a random permutation $\sigma \in S_n$ is indecomposable satisfies*

$$\mathbb{P}\big(\sigma \text{ is indecomposable}\big) \approx 1 - \sum_{k \geqslant 1} \frac{2\mathfrak{ip}_k - \mathfrak{ip}_k^{(2)}}{(n)_k}. \tag{9.1}$$

Proof. Let us formally consider $\mathcal{P}$ as an unlabeled combinatorial class. According to Lemma 9.1.8, ${}_u\mathcal{P} = \text{SEQ}({}_u\mathcal{IP})$, meaning that a permutation is SEQ-irreducible if and only if it is indecomposable. On the other hand, by Lemma 9.1.4, the class ${}_u\mathcal{P}$ is gargantuan. Hence, we can apply Theorem 5.5.1. Taking into account that

$$\frac{\mathfrak{p}_{n-k}}{\mathfrak{p}_n} = \frac{1}{n(n-1)\ldots(n-k+1)} = \frac{1}{(n)_k},$$

the statement of Theorem 5.5.1 implies exactly (9.1). □

Theorem 9.1.12. *Let $m, n \in \mathbb{N}$. The asymptotic probability that a random permutation $\sigma \in S_n$ consists of m indecomposable parts satisfies*

$$\mathbb{P}\big(\sigma \text{ has } m \text{ indecomposable parts}\big) \approx \sum_{k \geqslant 0} \frac{\beta_k^{(m)}}{(n)_k}, \tag{9.2}$$

where $\beta_k^{(m)} = m\Big(\mathfrak{ip}_k^{(m-1)} - 2\mathfrak{ip}_k^{(m)} + \mathfrak{ip}_k^{(m+1)}\Big)$. In particular,

$$\mathbb{P}\big(\sigma \text{ has } m \text{ indecomposable parts}\big) = \frac{m}{(n)_{m-1}} + O\left(\frac{1}{n^m}\right).$$

Proof. Again, consider $\mathcal{P}$ as an unlabeled combinatorial class. As we have just mentioned in the proof of Theorem 9.1.11, ${}_u\mathcal{P} = \text{SEQ}({}_u\mathcal{IP})$ and the class ${}_u\mathcal{P}$ is gargantuan. Hence, we can apply Theorem 5.5.2 that implies exactly relation (9.2). As for the behavior of the leading term, it follows from (5.19) and the fact that $\mathfrak{ip}_1 = 1$. □

Remark 9.1.13. For $m = 1$, asymptotic probability (9.2) takes the form of (9.1) with

$$\beta_k^{(1)} = \begin{cases} 1, & \text{if } k = 0, \\ -2\mathfrak{ip}_k + \mathfrak{ip}_k^{(2)}, & \text{if } k \neq 0. \end{cases}$$

9.1.4 Pairs of linear orders

As we have seen in Example 3.2.13, the labeled class $\mathcal{IP}$ is not stable under relabeling. Nevertheless, it is possible to obtain asymptotic results (9.1) and (9.2) in the framework of labeled universe. For this aim, we use the labeled class of pairs of linear orders $\mathcal{L} \odot \mathcal{L}$.

The motivation comes from the fact that

$$\mathcal{L} \odot \mathcal{L} = \mathcal{L} \odot \mathcal{P}$$

(see Example 4.1.72). From this point of view, a permutation

$$\tau = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$$

can be identified with the pair $(\prec, \tau) \in \mathcal{L} \odot \mathcal{P}$, where $\prec = (1 \prec 2 \prec \dots \prec n)$. In the spirit of the theory of species, the structure of an object does not depend on labels: we could replace the set of labels $[n]$ by any other set U via some transport function $[u] \to U$. In particular, any relabeling corresponds to $U = [n]$. For example, if

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix},$$

then the relabeling $(1, 2, 3, 4) \mapsto (3, 4, 1, 2)$ produces the transport

$$\tau \mapsto \begin{pmatrix} 3 & 4 & 1 & 2 \\ 4 & 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

and pairs

$$\left((1 \prec 2 \prec 3 \prec 4), \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}\right) \qquad \text{and} \qquad \left((3 \prec 4 \prec 1 \prec 2), \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}\right)$$

represent the same permutation τ. Understanding permutations in this way, we represent $\mathcal{IP}$ as a subclass of $\mathcal{L} \odot \mathcal{L}$, which is stable under relabelings and can be treated with Theorems 5.2.1 and 5.3.1.

Definition 9.1.14. A pair of linear orders $(\prec_1, \prec_2)$ of size n is *reducible* if there exists a partition $[n] = A \sqcup B$ such that every pair of elements $a \in A$ and $b \in B$ satisfy

$$a \prec_1 b \qquad \text{and} \qquad a \prec_2 b.$$

Otherwise, we call $(\prec_1, \prec_2)$ *irreducible.*

Notation 9.1.15. We denote by $\mathcal{ML}(2)$ the combinatorial class of pairs of linear orders. Also, we denote by $\mathcal{IML}(2)$ its subclass of irreducible pairs of linear orders. The corresponding counting sequences are designated as $(\mathfrak{ml}_n(2))$ and $(\mathfrak{iml}_n(2))$, respectively.

Remark 9.1.16. The reason for the notation $\mathcal{ML}(2)$ is that pairs of linear orders are the particular case of what we call multiple linear orders, see Definition 9.3.16.

Lemma 9.1.17. *The labeled combinatorial class $\mathcal{ML}(2)$ of pairs of linear orders is gargantuan.*

Proof. This follows from the fact that the sequence

$$\frac{\mathfrak{ml}_n(2)}{n!} = n!$$

is gargantuan (see Example 2.1.2). □

Lemma 9.1.18. *The labeled combinatorial classes $\mathcal{ML}(2)$ and $\mathcal{IML}(2)$ satisfy*

$$\mathcal{ML}(2) = \mathrm{SEQ}\big(\mathcal{IML}(2)\big),$$

meaning that any pair of linear orders can be uniquely decomposed as a sequence of irreducible ones.

Proof. Let $(\prec_1, \prec_2) \in \mathcal{ML}_n(2)$ be a pair of linear orders. In this case, the set $[n]$ can be uniquely decomposed into the partition of minimal possible number of components,

$$[n] = A_1 \sqcup \ldots \sqcup A_m,$$

such that for each $k < l$ and for any $a \in A_k$ and $a' \in A_l$:

$$a \prec_1 a' \qquad \text{and} \qquad a \prec_2 a'.$$

After relabeling, the restriction of $(\prec_1, \prec_2)$ to every A_k becomes an object from $\mathcal{IML}_{|A_k|}(2)$. $\square$

Notation 9.1.19. For any $m \in \mathbb{N}$, we denote by $\mathcal{IML}^{(m)}(2)$ the labeled combinatorial class consisting of pairs of linear orders that have exactly m irreducible parts:

$$\mathcal{IML}^{(m)}(2) = \mathrm{SEQ}_m\big(\mathcal{IML}(2)\big).$$

The corresponding counting sequence is $\big(\mathfrak{iml}_n^{(m)}(2)\big)$.

Lemma 9.1.20. *For any $m \in \mathbb{N}$, the counting sequence of the class $\mathcal{IML}^{(m)}(2)$ satisfy*

$$\mathfrak{iml}_n^{(m)}(2) = n! \cdot \mathfrak{ip}_n^{(m)}.$$

In particular,

$$\mathfrak{iml}_n(2) = n! \cdot \mathfrak{ip}_n.$$

Proof. Let us consider the map $\mathcal{ML}(2) \to \mathcal{P}$ defined as

$$\Big((u_1 \prec_1 u_2 \prec_1 \ldots \prec_1 u_n), (u_{i_1} \prec_2 u_{i_2} \prec_2 \ldots \prec_2 u_{i_n})\Big) \mapsto \begin{pmatrix} 1 & 2 & \ldots & n \\ i_1 & i_2 & \ldots & i_n \end{pmatrix}.$$

This map translates irreducible (respectively, consisting of m irreducible parts) pairs of linear orders into indecomposable (respectively, consisting of m indecomposable parts) permutations. Moreover, any indecomposable permutation of size n has exactly $n!$ preimages, which differ from each other by relabeling. This implies the statement of the lemma. $\square$

Proof of Theorems 9.1.11 and 9.1.12. We have seen that the class $\mathcal{ML}(2)$ is gargantuan and can be represented as $\mathrm{SEQ}\big(\mathcal{IML}(2)\big)$ (see Lemmas 9.1.17 and 9.1.18). Hence, Theorem 5.2.1 gives us

$$\frac{\mathfrak{iml}_n(2)}{\mathfrak{ml}_n(2)} \approx 1 - \sum_{k \geqslant 1} \big(2\mathfrak{ml}_k(2) - \mathfrak{ml}_k^{(2)}(2)\big) \binom{n}{k} \frac{\mathfrak{ml}_{n-k}(2)}{\mathfrak{ml}_n(2)}.$$

Here,

$$\binom{n}{k} \frac{\mathfrak{ml}_{n-k}(2)}{\mathfrak{ml}_n(2)} = \frac{1}{k!(n)_k}.$$

Besides, by Lemma 9.1.20,

$$\frac{\mathfrak{iml}_n(2)}{\mathfrak{ml}_n(2)} = \frac{\mathfrak{ip}_n}{\mathfrak{p}_n} \qquad \text{and} \qquad \frac{2\mathfrak{ml}_k(2) - \mathfrak{ml}_k^{(2)}(2)}{k!(n)_k} = \frac{2\mathfrak{ip}_k - \mathfrak{ip}_k^{(2)}}{(n)_k}.$$

Thus, we obtain desired asymptotics (9.1). Theorem 9.1.12 is proved in a similar way. $\square$

9.2 Perfect matchings

9.2.1 The concept of a perfect matching

Definition 9.2.1. A *perfect matching* of size n is an involution of the set $[n]$ without fixed points. In other words, it is a permutation from S_n whose cycles are transpositions.

Notation 9.2.2. We denote the labeled combinatorial class of perfect matchings by $\mathcal{M}$.

Remark 9.2.3. The number of perfect matchings of size n is

$$\mathfrak{m}_n = \begin{cases} (n-1)!!, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 9.2.4. *The unlabeled combinatorial class ${}_u\mathcal{M}$ of perfect matchings is* 2*-gargantuan.*

Proof. The statement of Lemma 9.2.4 follows from the fact that the sequence $\mathfrak{m}_n$ is 2-gargantuan (see Example 2.1.15). □

Remark 9.2.5. The sequence

$$n \cdot \frac{\mathfrak{m}_{2n}}{\mathfrak{m}_{2n}} = \frac{n}{2n-1}$$

does not tend to 0 as n tends to infinity. As a consequence, the labeled combinatorial class $\mathcal{M}$ is not 2-gargantuan.

9.2.2 Indecomposable perfect matchings

Definition 9.2.6. A perfect matching $\sigma \in S_n$ is *indecomposable*, if it is indecomposable as a permutation.

Notation 9.2.7. We denote the labeled combinatorial class of indecomposable perfect matchings and its counting sequence by $\mathcal{IM}$ and $(\mathfrak{im}_n)$, respectively.

Lemma 9.2.8. *The unlabeled combinatorial classes ${}_u\mathcal{M}$ of perfect matchings and ${}_u\mathcal{IM}$ of indecomposable perfect matchings satisfy the relation*

$${}_u\mathcal{M} = \mathrm{SEQ}({}_u\mathcal{IM}).$$

Proof. For the proof, repeat the reasoning applied in the proof of Lemma 9.1.8 or take a restriction of the identity ${}_u\mathcal{P} = \mathrm{SEQ}({}_u\mathcal{IP})$ to the subclass of perfect matchings. □

Remark 9.2.9. Indecomposable parts of a perfect matching coincide with its SEQ-irreducible parts. In particular, a perfect matching is SEQ-irreducible, if and only if it is indecomposable.

Notation 9.2.10. For any $m \in \mathbb{N}$, we denote by

$${}_u\mathcal{IM}^{(m)} = \mathrm{SEQ}_m({}_u\mathcal{IM})$$

the unlabeled combinatorial class consisting of perfect matchings that have exactly m indecomposable parts. The corresponding counting sequence is designated as $\left(\mathfrak{im}_n^{(m)}\right)$.

9.2.3 Asymptotic probabilities for indecomposable perfect matchings

Theorem 9.2.11. *The asymptotic probability that a random perfect matching $\sigma \in S_{2n}$ is indecomposable satisfies*

$$\mathbb{P}\big(\sigma \text{ is indecomposable}\big) \approx 1 - \sum_{k \geqslant 1} \Big(2\mathfrak{im}_{2k} - \mathfrak{im}_{2k}^{(2)}\Big) \frac{\big(2(n-k)-1\big)!!}{(2n-1)!!}. \tag{9.3}$$

Proof. According to Lemma 9.2.4, the unlabeled combinatorial class ${}_u\mathcal{M}$ is 2-gargantuan, while, by Lemma 9.2.8, ${}_u\mathcal{M} = \mathrm{SEQ}({}_u\mathcal{IM})$. Hence, applying Theorem 5.5.1, we get the desired result. ☐

Theorem 9.2.12. *Let $m, n \in \mathbb{N}$. The asymptotic probability that a random perfect matching $\sigma \in S_{2n}$ consists of m indecomposable parts satisfies*

$$\mathbb{P}\big(\sigma \text{ has } m \text{ indecomposable parts}\big) \approx \sum_{k \geqslant 0} \beta_{2k}^{(m)} \frac{\big(2(n-k)-1\big)!!}{(2n-1)!!}, \tag{9.4}$$

where $\beta_{2k}^{(m)} = m\Big(\mathfrak{im}_{2k}^{(m-1)} - 2\mathfrak{im}_{2k}^{(m)} + \mathfrak{im}_{2k}^{(m+1)}\Big)$. In particular,

$$\mathbb{P}\big(\sigma \text{ has } m \text{ indecomposable parts}\big) = m \cdot \frac{\big(2(n-m)+1\big)!!}{(2n-1)!!} + O\left(\frac{1}{n^m}\right).$$

Proof. As we have just mentioned in the proof of Theorem 9.2.11, ${}_u\mathcal{M} = \mathrm{SEQ}({}_u\mathcal{IM})$ and ${}_u\mathcal{M}$ is 2-gargantuan. Hence, we can apply Theorem 5.5.2 that implies exactly relation (9.4). The behavior of the leading term follows from (5.19) and the fact that $\mathfrak{im}_2 = 1$. ☐

Remark 9.2.13. The same way as it has been done for permutations, it is possible to prove Theorems 9.2.11 and 9.2.12 with the help of the labeled combinatorial class $\mathcal{ML}(2)$ of pairs of linear orders. Within $\mathcal{ML}(2)$, we distinguish the 2-gargantuan subclass $\mathcal{MLM}(2)$ satisfying the following property. For any $(\prec_1, \prec_2) \in \mathcal{MLM}(2)$, the transform

$$(\prec_1, \prec_2) \mapsto (\prec_2, \prec_1)$$

coincides with some relabeling $(1, \dots, n) \mapsto (1, \dots, n)$ that is an involution without fixed point. Defining one more subclass to be

$$\mathcal{IMLM}(2) = \mathcal{MLM}(2) \cap \mathcal{IML}(2),$$

we have

$$\mathcal{MLM}(2) = \mathrm{SEQ}\big(\mathcal{IMLM}(2)\big) \tag{9.5}$$

and the corresponding counting sequences satisfy

$$\mathfrak{mlm}_n(2) = n! \cdot \mathfrak{m}_n \qquad \text{and} \qquad \mathfrak{imlm}_n(2) = n! \cdot \mathfrak{im}_n.$$

In other words, the subclasses $\mathcal{MLM}(2)$ and $\mathcal{IMLM}(2)$ correspond to perfect matchings and indecomposable perfect matchings, respectively, in the same way as the classes $\mathcal{ML}(2)$ and $\mathcal{IML}(2)$ correspond to permutations and indecomposable permutations, respectively. All these observations allows us to apply Theorems 5.4.1 and 5.4.2 to $\mathcal{MLM}(2)$, which gives us asymptotics (9.3) and (9.4).

9.3 Multipermutations

9.3.1 The concept of a multipermutation

Definition 9.3.1. We call a *multipermutation* of rank N and size n an N-tuple of permutations

$$(\sigma_1, \ldots, \sigma_N) \in S_n^N.$$

In other words, a multipermutation is an element of the Hadamard product $\mathcal{P} \odot \ldots \odot \mathcal{P}$.

Remark 9.3.2. The reader can think of a multipermutation as a directed graph whose edges are colored. In this representation, each permutation σ_k of a tuple $(\sigma_1, \ldots, \sigma_N)$ corresponds to edges of color c_k, so that the indegree and outdegree of each color for every vertex equal to 1. For example, the graphical representation of the multipermutation (σ_1, σ_2) with

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 3 & 2 & 7 & 5 & 6 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 4 & 6 & 5 & 7 \end{pmatrix}$$

is shown on Fig. 9.2.

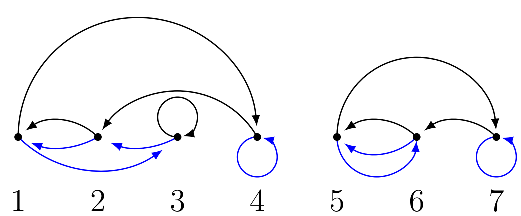


Figure 9.2: Multipermutation of rank 2 and size 7.

Notation 9.3.3. We denote by $\mathcal{MP}(N)$ and $\big(\mathfrak{mp}_n(N)\big)$ the labeled combinatorial class of multipermutations of rank N and its counting sequence, respectively.

Remark 9.3.4. The counting sequence $\big(\mathfrak{mp}_n(N)\big)$ of the class $\mathcal{MP}(N)$ satisfies

$$\mathfrak{mp}_n(N) = (n!)^N.$$

Lemma 9.3.5. *For any $N \in \mathbb{N}$, the unlabeled combinatorial class ${}_u\mathcal{MP}(N)$ of multipermutations of rank N is gargantuan.*

Proof. Recall that the sequences

$$a_n = n!$$

is gargantuan (see Example 2.1.2). Applying Lemma 2.1.8, we have the sequence

$$\mathfrak{mp}_n(N) = (n!)^N = a_n^N$$

to be gargantuan as well. Hence, the class $\mathcal{MP}(N)$ is gargantuan. □

9.3.2 Indecomposable multipermutations

Definition 9.3.6. We call a multipermutation $(\sigma_1, \ldots, \sigma_N) \in S_n^N$ of rank N *decomposable*, if there exists a positive integer $k < n$ such that $\sigma_j\big([k]\big) = [k]$ for all $j \in [N]$. Otherwise, $(\sigma_1, \ldots, \sigma_N)$ is said to be *indecomposable*.

Remark 9.3.7. Let us consider multipermutations as directed graphs with colored edges (see Remark 9.3.2), whose vertices lie on the number line and whose edges are arcs. Then a multipermutation is decomposable, if there exists a vertical line $x = x_0$ separating one component of the graph from others. For example, the multipermutation discussed in Remark 9.3.2 is decomposable with the separating line $x = 4.5$ (Fig. 9.3).

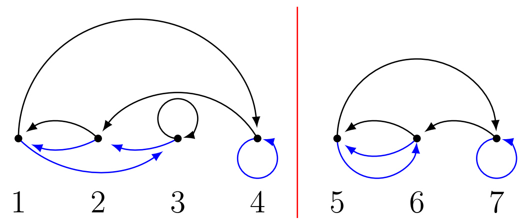


Figure 9.3: Decomposition of multipermutation of rank 2 and size 7.

Notation 9.3.8. We denote by $\mathcal{IMP}(N)$ and $\big(\mathfrak{imp}_n(N)\big)$ the labeled combinatorial class of indecomposable multipermutations of rank N and its counting sequence, respectively.

Remark 9.3.9. The counting sequence $\big(\mathfrak{imp}_n(N)\big)$ of the class $\mathcal{IMP}(N)$ has the following form:

$$\big(\mathfrak{imp}_n(N)\big) = 1,\ (2^N - 1),\ (6^N - 2^{N+1} + 1),\ (24^N - 2 \cdot 6^N - 4^N + 3 \cdot 2^N - 1),\ \ldots$$

Lemma 9.3.10. *The unlabeled combinatorial classes ${}_u\mathcal{MP}(N)$ of multipermutations of rank (N) and ${}_u\mathcal{IMP}(N)$ of indecomposable multipermutations of rank (N) satisfy the relation*

$$_u\mathcal{MP} = \mathrm{SEQ}(_u\mathcal{IMP}).$$

Proof. The proof is the same as the proof of Lemma 9.1.8. ☐

Remark 9.3.11. As well as for the unlabeled combinatorial class of permutations, indecomposable parts of a multipermutation of rank N coincide with its SEQ-irreducible parts. In particular, a multipermutation is SEQ-irreducible, if and only if it is indecomposable.

Notation 9.3.12. For any $m \in \mathbb{N}$, we denote by

$$_u\mathcal{IMP}^{(m)}(N) = \mathrm{SEQ}_m\big(_u\mathcal{IMP}(N)\big)$$

the unlabeled combinatorial class consisting of multipermutations of rank N that have exactly m indecomposable parts. The corresponding counting sequence is designated as $\big(\mathfrak{imp}_n^{(m)}(N)\big)$.

9.3.3 Asymptotic probabilities for indecomposable multipermutations

Theorem 9.3.13. *The asymptotic probability that a random multipermutation $\sigma = (\sigma_1, \ldots, \sigma_N)$ of rank N and size n is indecomposable satisfies*

$$\mathbb{P}\big(\sigma \text{ is indecomposable}\big) \approx 1 - \sum_{k \geqslant 1} \frac{2\mathfrak{imp}_k(N) - \mathfrak{imp}_k^{(2)}(N)}{\big((n)_k\big)^N}. \tag{9.6}$$

Proof. The structure of the proof is similar to the proof of Theorem 9.1.11. We notice that ${}_u\mathcal{MP}(N)$ is gargantuan (by Lemma 9.3.10) and that ${}_u\mathcal{MP}(N) = \mathrm{SEQ}\big({}_u\mathcal{IMP}(N)\big)$ (by Lemma 9.3.5). These two facts allows us to apply Theorem 5.5.1 that provides a result. □

Remark 9.3.14. Asymptotic coefficients $2\mathfrak{imp}_k(N) - \mathfrak{imp}_k^{(2)}(N)$ for the first few values of n are

$$2,\ (2^{N+1} - 3),\ (2 \cdot 6^N - 3 \cdot 2^N + 4),\ (2 \cdot 24^N - 6^{N+1} - 3 \cdot 4^N + 3 \cdot 2^{N+2} - 5),\ \ldots$$

Theorem 9.3.15. *Let $m, n, N \in \mathbb{N}$. The asymptotic probability that a random multipermutation $\sigma = (\sigma_1, \ldots, \sigma_N)$ consists of m indecomposable parts satisfies*

$$\mathbb{P}\big(\sigma \text{ has } m \text{ indecomposable parts}\big) \approx \sum_{k \geqslant 0} \frac{\beta_k^{(m)}(N)}{\big((n)_k\big)^N}, \tag{9.7}$$

where $\beta_k^{(m)}(N) = m\Big(\mathfrak{imp}_k^{(m-1)}(N) - 2\mathfrak{imp}_k^{(m)}(N) + \mathfrak{imp}_k^{(m+1)}(N)\Big)$. *In particular,*

$$\mathbb{P}\big(\sigma \text{ has } m \text{ indecomposable parts}\big) = \frac{m}{\big((n)_{m-1}\big)^N} + O\left(\frac{1}{n^{mN}}\right).$$

Proof. The same way as before, we notice that ${}_u\mathcal{MP}(N)$ is gargantuan (by Lemma 9.3.10) and that ${}_u\mathcal{MP}(N) = \mathrm{SEQ}\big({}_u\mathcal{IMP}(N)\big)$ (by Lemma 9.3.5). Hence, Theorem 5.5.2 gives us a result. The behavior of the leading term follows from (5.19) and the fact that $\mathfrak{imp}_1(N) = 1$. □

Another method to prove Theorems 9.3.13 and 9.3.15 uses tuples of linear orders. This can be done in a similar way to the second proof of Theorems 9.1.11 and 9.1.12 in Section 9.1. Here, we outline the key definitions and lemmas, but omit some details.

Definition 9.3.16. We call a *multiple linear order* of rank N and size n an N-tuple of linear orders

$$(\prec_1, \ldots, \prec_N) \in \mathcal{L} \odot \ldots \odot \mathcal{L}.$$

Definition 9.3.17. A tuple of linear orders $L = (\prec_1, \ldots, \prec_N)$ of size n is *reducible* if there exists a partition $[n] = A \sqcup B$ such that each $k \in [n]$ and every pair of elements $a \in A$ and $b \in B$ satisfy

$$a \prec_k b.$$

Otherwise, we call L *irreducible.*

Notation 9.3.18. We denote the labeled combinatorial class of multiple linear orders of rank N and its subclass of irreducible multiple linear orders by $\mathcal{ML}(N)$ and $\mathcal{IML}(N)$, respectively. The corresponding counting sequences are $\big(\mathfrak{ml}_n(N)\big)$ and $\big(\mathfrak{iml}_n(N)\big)$, respectively.

Lemma 9.3.19. *If $N \geqslant 2$, then the class $\mathcal{ML}(N)$ of multiple linear orders is gargantuan.*

Proof. This follows from Lemma 2.1.8 and Example 2.1.2 (see also Lemma 9.3.5). □

Lemma 9.3.20. *The labeled combinatorial classes $\mathcal{ML}(N)$ and $\mathcal{IML}(N)$ satisfy*

$$\mathcal{ML}(N) = \mathrm{SEQ}\big(\mathcal{IML}(N)\big).$$

Proof. This can be done the same way as the proof of Lemma 9.1.18. □

Notation 9.3.21. For any $m, N \in \mathbb{N}$, we denote by $\mathcal{IML}^{(m)}(N)$ the labeled combinatorial class consisting of multiple linear orders that have exactly m irreducible parts:

$$\mathcal{IML}^{(m)}(N) = \mathrm{SEQ}_m\big(\mathcal{IML}(N)\big).$$

The corresponding counting sequence is $\big(\mathfrak{iml}_n^{(m)}(N)\big)$

Lemma 9.3.22. *For any $m, N \in \mathbb{N}$, we have*

$$\mathfrak{iml}_n^{(m)}(N+1) = n! \cdot \mathfrak{imp}_n^{(m)}(N).$$

In particular,

$$\mathfrak{iml}_n(N+1) = n! \cdot \mathfrak{imp}_n(N).$$

Proof. This can be done the same way as the proof of Lemma 9.1.20. □

Proof of Theorems 9.3.13 and 9.3.15. Since the class $\mathcal{ML}(N+1)$ is gargantuan and can be represented as $\mathrm{SEQ}\big(\mathcal{IML}(N+1)\big)$ (see Lemmas 9.3.19 and 9.3.20), we can use Theorems 5.2.1 and 5.3.1. Applying Lemma 9.3.22 to the obtained asymptotics, we get desired (9.6) and (9.7). □

9.4 Other generalizations

Asymptotic results stated in Theorems 9.1.11 and 9.1.12 can be extended to many subclasses of the class $\mathcal{P}$ of permutations. In Section 9.2 we have studied the subclass $\mathcal{M}$ of perfect matchings, that is, involutions without fixed points. A perfect matching of size $2n$ is a product of n transpositions, i.e. cycles of length 2. From this point of view, a natural generalization of the concept of the perfect matching would be a *product of n cycles of length d*, $d \in \mathbb{N}$. Let us denote by $\mathcal{PC}(d)$ the labeled combinatorial class of products of cycles of length d. Then its counting sequence is

$$\mathfrak{pc}_n(d) = \begin{cases} \dfrac{(dk)!}{d^k k!}, & n = dk, \\ 0, & n \neq dk, \end{cases}$$

and ${}_u\mathcal{PC}(d)$ is d-gargantuan (the proof is similar to one of Lemma 9.2.4). On the other hand, by analogy of Lemma 9.2.8, one has

$${}_u\mathcal{PC}(d) = \mathrm{SEQ}\big({}_u\mathcal{IPC}(d)\big),$$

where $\mathcal{IPC}(d)$ is the labeled combinatorial class of the products of cycles of length d that are indecomposable. Let us denote, the same way as before,

$${}_u\mathcal{IPC}^{(m)}(d) = \mathrm{SEQ}_m\big({}_u\mathcal{IPC}(d)\big)$$

and let $\big(\mathfrak{ipc}_n(d)\big)$ and $\big(\mathfrak{ipc}_n^{(m)}(d)\big)$ be the counting sequences of the combinatorial classes ${}_u\mathcal{IPC}(d)$ and ${}_u\mathcal{IPC}^{(m)}(d)$, respectively. This allows us to state the following theorem.

Theorem 9.4.1. *The probability that a random object $\sigma \in \mathcal{PC}(d)$, i.e. a random product of n cycles of length d, is indecomposable satisfies*

$$\mathbb{P}\big(\sigma \text{ is indecomposable}\big) \approx 1 - \sum_{k\geqslant 1} \left(2\mathfrak{ipc}_{dk}(d) - \mathfrak{ipc}^{(2)}_{dk}(d)\right) \cdot \frac{d^k (n)_k}{(dn)_{dk}}, \tag{9.8}$$

while the probability that σ consists of m indecomposable parts, where $m \in \mathbb{N}$, is

$$\mathbb{P}\big(\sigma \text{ has } m \text{ indecomposable parts}\big) \approx \sum_{k\geqslant 0} \beta^{(m)}_{dk}(d) \cdot \frac{d^k (n)_k}{(dn)_{dk}} \tag{9.9}$$

where $\beta^{(m)}_{dk}(d) = m\Big(\mathfrak{ipc}^{(m-1)}_{dk}(d) - 2\mathfrak{ipc}^{(m)}_{dk}(d) + \mathfrak{ipc}^{(m+1)}_{dk}(d)\Big)$. In particular,

$$\mathbb{P}\big(\sigma \text{ has } m \text{ indecomposable parts}\big) = m(d!)^{m-1} \cdot \frac{(n)_{m-1}}{(dn)_{d(m-1)}} + O\left(\frac{1}{n^{(d-1)m}}\right).$$

In Section 9.3, we have seen the generalization of the concept of permutations to multipermutations. The same way, we can consider multiple perfect matchings. Namely, we call a *multiple perfect matchings* of rank N and size n an N-tuple

$$(\sigma_1, \ldots, \sigma_N) \in S_n^N,$$

where σ_k is a perfect matching for any $k \in [n]$. The labeled combinatorial class $\mathcal{MM}(N)$ of multiple perfect matchings of rank N, formally considered as unlabeled class, is 2-gargantuan and satisfies the relation

$${}_u\mathcal{MM}(N) = \mathrm{SEQ}\big({}_u\mathcal{IMM}(N)\big),$$

where $\mathcal{IMM}(N)$ is the subclass of indecomposable multiple perfect matchings of rank N. For any positive integer m, let us denote

$${}_u\mathcal{IMM}^{(m)}(N) = \mathrm{SEQ}_m\big({}_u\mathcal{IMM}(N)\big)$$

and let $\big(\mathfrak{mm}_n(N)\big)$, $\big(\mathfrak{imm}_n(N)\big)$ and $\big(\mathfrak{imm}^{(m)}_n(N)\big)$ be the counting sequences of the combinatorial classes ${}_u\mathcal{MM}(N)$, ${}_u\mathcal{IMM}(N)$ and ${}_u\mathcal{IMM}^{(m)}(N)$, respectively. With these notations, Theorems 5.5.1 and 5.5.2 imply the following result.

Theorem 9.4.2. *The probability that a random object $\sigma = (\sigma_1, \ldots, \sigma_N) \in \mathcal{MM}(N)$, i.e. a random multiple perfect matching of rank N of size $2n$, is indecomposable satisfies*

$$\mathbb{P}\big(\sigma \text{ is indecomposable}\big) \approx 1 - \sum_{k\geqslant 1} \left(2\mathfrak{imm}_{2k}(N) - \mathfrak{imm}^{(2)}_{2k}(N)\right) \left(\frac{\big(2(n-k)-1\big)!!}{(2n-1)!!}\right)^N, \tag{9.10}$$

while the probability that σ consists of m indecomposable parts, where $m \in \mathbb{N}$, satisfies

$$\mathbb{P}\big(\sigma \text{ has } m \text{ indecomposable parts}\big) \approx \sum_{k\geqslant 0} \beta^{(m)}_{2k}(N) \left(\frac{\big(2(n-k)-1\big)!!}{(2n-1)!!}\right)^N, \tag{9.11}$$

where $\beta^{(m)}_{2k}(N) = m\Big(\mathfrak{imm}^{(m-1)}_{2k}(N) - 2\mathfrak{imm}^{(m)}_{2k}(N) + \mathfrak{imm}^{(m+1)}_{2k}(N)\Big)$. In particular,

$$\mathbb{P}\big(\sigma \text{ has } m \text{ indecomposable parts}\big) = m\left(\frac{\big(2(n-m)+1\big)!!}{(2n-1)!!}\right)^N + O\left(\frac{1}{n^{mN}}\right).$$

Compiling the results of Theorems 9.4.1 and 9.4.2, one can proceed to the labeled combinatorial class $\mathcal{MPC}(d,N)$ consisting of N-tuples

$$(\sigma_1,\ldots,\sigma_N) \in S_{dn}^N$$

such that σ_k is a product of n cycles of length d for any $k \in [n]$. The similar way, one can show that ${}_u\mathcal{MPC}(d,N)$ is d-gargantuan and that

$${}_u\mathcal{MPC}(d,N) = \mathrm{SEQ}\big({}_u\mathcal{IMPC}(d,N)\big),$$

where $\mathcal{IMPC}(d,N)$ is the subclass of indecomposable objects. With the help of the natural notations $\big(\mathfrak{mpc}_n(d,N)\big)$, $\big(\mathfrak{impc}_n(d,N)\big)$ and $\big(\mathfrak{impc}_n^{(m)}(d,N)\big)$ for the counting sequences of the combinatorial classes ${}_u\mathcal{MPC}(d,N)$, ${}_u\mathcal{IMPC}(d,N)$ and ${}_u\mathcal{IMPC}^{(m)}(d,N) = \mathrm{SEQ}_m\big({}_u\mathcal{IMPC}(d,N)\big)$, respectively, we obtain the following result.

Theorem 9.4.3. *Let $d,n,N \in \mathbb{N}$. The probability that a random object $\sigma \in \mathcal{MPC}(d,N)$ is indecomposable satisfies*

$$\mathbb{P}\big(\sigma \text{ is indecomposable}\big) \approx 1 - \sum_{k\geqslant 1}\Big(2\mathfrak{impc}_{dk}(d,N) - \mathfrak{impc}_{dk}^{(2)}(d,N)\Big)\cdot\left(\frac{d^k\cdot(n)_k}{(dn)_{dk}}\right)^N, \tag{9.12}$$

while the probability that σ consists of m indecomposable parts, where $m \in \mathbb{N}$, is

$$\mathbb{P}\big(\sigma \text{ has } m \text{ indecomposable parts}\big) \approx \sum_{k\geqslant 0}\beta_{dk}^{(m)}(d,N)\cdot\left(\frac{d^k\cdot(n)_k}{(dn)_{dk}}\right)^N \tag{9.13}$$

where

$$\beta_{dk}^{(m)}(d,N) = m\Big(\mathfrak{impc}_{dk}^{(m-1)}(d,N) - 2\mathfrak{impc}_{dk}^{(m)}(d,N) + \mathfrak{impc}_{dk}^{(m+1)}(d,N)\Big).$$

In particular,

$$\mathbb{P}\big(\sigma \text{ has } m \text{ indecomposable parts}\big) = m\left((d!)^{m-1}\cdot\frac{(n)_{m-1}}{(dn)_{d(m-1)}}\right)^N + O\left(\frac{1}{n^{(d-1)mN}}\right).$$

Chapter 10

Graphs

This chapter is devoted to the applications of the theory developed in Part II to various models of labeled graphs, such as undirected graphs, directed graphs, directed acyclic graphs, tournaments, etc. We recall key definitions and properties, but often leave the details omitted as a standard well-known material (in the case of need, we refer the reader, say, to [46] and [62]).

The asymptotic results of this chapter can be divided into four groups that differ from each other in origin. The first of them comes from the well-known exponential formula for various types of connected graphs (maybe, first explicitly mentioned in the dissertation of Riddell [89]). In the language of symbolic method, the exponential formula is interpreted with the help of the construction SET. Say, for the labeled combinatorial class $\mathcal{G}$ of simple unlabeled graphs, we have

$$\mathcal{G} = \mathrm{SET}(\mathcal{CG}) \qquad \Leftrightarrow \qquad G(z) = \exp\big(CG(z)\big),$$

where $\mathcal{CG}$ is the labeled combinatorial class of connected (simple undirected) graphs. In Section 10.1, we dwell on this case in detail, since simple unlabeled graphs are the easiest and, at the same time, the most striking example from the first group. We establish the asymptotic expansions of the probabilities that a random graph is connected (Theorem 10.1.10) or consists of a given number of connected components (Theorem 10.1.11). The method used for studying connected (simple unlabeled) graphs is transferred almost unchanged to the other models, such as multigraphs (Statements 10.1.18 and Statement 10.1.19) and weakly connected graphs of various types (Statements 10.3.23 and 10.3.28).

In Section 10.2, we present another group of results which relies on the construction SEQ. Here, we illustrate the general rule by the example of tournaments,

$$\mathcal{T} = \mathrm{SEQ}(\mathcal{IT}) \qquad \Leftrightarrow \qquad T(z) = \frac{1}{1 - IT(z)},$$

where $\mathcal{T}$ and $\mathcal{IT}$ are the labeled combinatorial classes of tournaments and irreducible tournaments, respectively. The main result concerns the asymptotic expansion of the probability that a random tournament can be decomposed as a sequence of a given number of irreducible subtournaments (Theorem 10.2.11). Also, we generalize this result to the case of multitournaments (Statement 10.2.18).

The main result of Section 10.3 consists in establishing the asymptotic expansion of the probability that a random directed graph is strongly connected (Theorem 10.3.16). Unlike the others, this result does not rely on Part II, but comes directly from the symbolic method developed for strongly connected digraphs and Bender's theorem for labeled classes. Another difference is that there is no direct generalization to the probability that a random directed graph consists of a given number of strongly connected components. The reason for this obstacle is that the structure of a directed graph is determined not only by its strongly connected components, but also by the relations between them.

Finally, in Section 10.4, we discuss the Erdős–Rényi model $G(n,p)$ in the sense of Gilbert [41] (rather than in the sense of Erdős and Rényi [33]): there are n vertices in a graph, and each edge is taken with probability p independently. Graphs and connected graphs within this model are linked by the exponential formula, but the use of the combinatorial classes and constructions in the way presented in Section 10.1 is problematic. In order to cope with this obstacle, we translate the exponential formula into the language of species, so that

$$\mathcal{G}_w = \mathcal{E} \circ \mathcal{CG}_w,$$

where $\mathcal{G}_w$ and $\mathcal{CG}_w$ are weighted species of graphs and connected graphs, respectively. Thus, we obtain the asymptotic expressions of the probabilities that a random graph in $G(n,p)$ is connected (Theorem 10.4.6) or consists of a given number of connected components (Theorem 10.4.7) with the interpretation of the involved coefficients in terms of virtual species.

10.1 Undirected graphs

Let us consider the labeled combinatorial class of graphs $\mathcal{G}$. Pick a random graph G of size n from it. In this section, we are interested in the asymptotic probability that G is connected.

The traces of the beginning of this story is hidden in the depth of time. It has been known for a long time that this probability goes to 1 as n goes to ∞. In 1959, Gilbert [41] provided the following estimation up to the second term for this probability:

$$\mathbb{P}(G \text{ is connected}) = 1 - \frac{2n}{2^n} + O\left(\frac{n^2}{2^{3n/2}}\right).$$

Later, in 1970, Wright [102] computed the first four terms of the asymptotic expansion of the probability that a random graph is connected:

$$\mathbb{P}(G \text{ is connected}) = 1 - \binom{n}{1}\frac{1}{2^{n-1}} - 2\binom{n}{3}\frac{1}{2^{3n-6}} - 24\binom{n}{4}\frac{1}{2^{4n-10}} + O\left(\frac{n^5}{2^{5n}}\right).$$

Wright found a recursive way to get any finite number of asymptotic terms (a more general recursion in terms of inversion polynomial was obtained later in the paper of Konvalinka and Pak [54]). Nevertheless, there was no method to obtain them all at once. Besides, there was no combinatorial interpretation of coefficients which turned out to be integers. The first gap could be filled by methods of Bender [9], though, personally, he did not mention such an opportunity. The second gap has been filled recently in our work [65] which constitutes the main result of this section.

The structure of the section presentation is the following. We begin with recalling all necessary concepts and properties. Most of them have been already used in the previous chapters as examples. Thus, here we only need to put them together and fill the gaps in order to make the presentation consistent. Having finished this task, we proceed to the main goal, namely, to the asymptotic expansions for the probabilities that a random graph is connected or consists of a fixed number of connected components, and to the combinatorial interpretation of the involved coefficients. Finally, we discuss a natural generalization of these results for multigraphs.

10.1.1 Definitions and properties

Definition 10.1.1. A *labeled (simple undirected) graph* of size n consists of the set of *vertices* $[n] = \{1, \ldots, n\}$ together with a set E of *edges* that are unordered pairs of distinct vertices. A graph is *connected*, if any two its vertices are joined by a path.

Remark 10.1.2. Rigorously, one usually define *adjacent* vertices (as vertices joined by an edge) first, and then a *path* (as a sequence of vertices such that consecutive vertices are adjacent). Next step is to define an equivalence relation $\sim$ on the set of vertices, such that $i \sim j$ if and only if the vertices i and j are joined by a path. Then *connected components* of a graph are equivalence classes under $\sim$. We do not go into the details, since it is well-known; the reader can consult the material, for example, in [46].

Notation 10.1.3. We denote the labeled combinatorial class of graphs and of connected graphs by $\mathcal{G}$ and $\mathcal{CG}$, respectively. Their counting sequences are designated as $(\mathfrak{g}_n)$ and $(\mathfrak{cg}_n)$, respectively.

Remark 10.1.4. The number of labeled graphs of size n is determined by the cardinality of the set of unordered pairs of $\{1, \ldots, n\}$ which is $\binom{n}{2}$. Since any pair can be either present or absent in a graph, the counting sequence $(\mathfrak{g}_n)$ of $\mathcal{G}$ satisfies

$$\mathfrak{g}_n = 2^{\binom{n}{2}}.$$

Lemma 10.1.5. *The labeled combinatorial class $\mathcal{G}$ of graphs is gargantuan.*

Proof. This follows directly from the fact, proved in Example 2.1.7, that the sequence

$$\frac{\mathfrak{g}_n}{n!} = \frac{2^{\binom{n}{2}}}{n!}$$

is gargantuan (see also Example 3.1.73). □

Lemma 10.1.6. *Any graph can be uniquely decomposed into an unordered collection of connected components, meaning that*

$$\mathcal{G} = \mathrm{SET}(\mathcal{CG}).$$

Proof. Let us count all graphs of size $n \in \mathbb{N}$. To this end, for each graph, consider an n-sequence $(c_1, \ldots, c_n)$ such that c_k is the number of graph connected components of size k (in particular, $c_1 + 2c_2 + \ldots + nc_n = n$). Given an n-sequence $(c_1, \ldots, c_n)$, how many graphs correspond to it? To choose an appropriate graph, we need to choose

1. labels for each element:
$$\frac{n!}{(1!)^{c_1} \cdot \ldots \cdot (n!)^{c_n}} \quad \text{choices;}$$

2. elements themselves:
$$\mathfrak{cg}_1^{c_1} \cdot \ldots \cdot \mathfrak{cg}_n^{c_n} \quad \text{choices.}$$

Since the order of components is not important, we need to divide the product of the two above factors by $c_1! \cdot \ldots \cdot c_n!$. Summing up over all possible n-sequences $(c_1, \ldots, c_n)$, we get

$$\mathfrak{g}_n = \sum_{C_n} \frac{n!}{(1!)^{c_1} \cdot \ldots \cdot (n!)^{c_n}} \cdot \frac{1}{c_1! \cdot \ldots \cdot c_n!} \cdot \left(\mathfrak{cg}_1^{c_1} \cdot \ldots \cdot \mathfrak{cg}_n^{c_n}\right).$$

In terms of exponential generating functions, this relation turns into

$$G(z) = e^{CG(z)},$$

meaning that $\mathcal{G} = \mathrm{SET}(\mathcal{CG})$. □

Remark 10.1.7. It follows from Lemma 10.1.6 that connected components of a graph coincide with its SET-irreducible components. In particular, a graph is SET-irreducible, if and only if it is connected.

Notation 10.1.8. For any $m \in \mathbb{N}$, we denote by

$$\mathcal{CG}^{\{m\}} = \mathrm{SET}_m(\mathcal{CG})$$

the labeled combinatorial class consisting of labeled graphs that have exactly m connected components. The corresponding counting sequence is $\left(\mathfrak{cg}_n^{\{m\}}\right)$.

Example 10.1.9. There are eight labeled graphs of size 3, see Fig. 10.1. Four of them depicted at the top row of Fig. 10.1 are connected. On the contrary, the four graphs at the bottom row are disconnected. First of them is totally disconnected, meaning that its number of connected components coincides with the size. Three other disconnected graphs of size 3 have two connected components each.

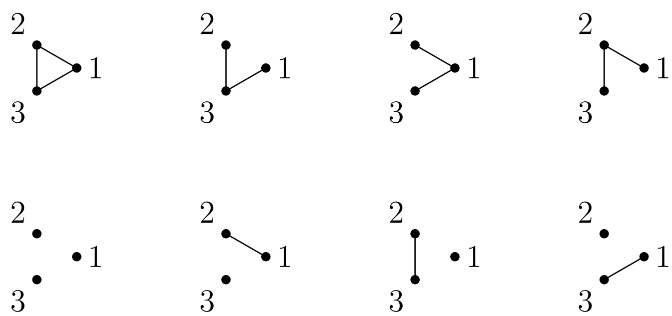


Figure 10.1: Labeled graphs of size 3.

10.1.2 Asymptotic probabilities for simple connected graphs

Theorem 10.1.10. *The asymptotic probability that a random labeled simple graph G with n vertices is connected satisfies*

$$\mathbb{P}\big(G \textit{ is connected}\big) \approx 1 - \sum_{k \geqslant 1} \mathfrak{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{kn}}. \tag{10.1}$$

Proof. Due to Lemma 10.1.5, the labeled combinatorial class $\mathcal{G}$ is gargantuan. On the one hand, $\mathcal{G} = \mathrm{SET}(\mathcal{CG})$ by Lemma 10.1.6. On the other hand, as we have seen in Examples 3.1.14 and 3.1.33, the labeled combinatorial class of graphs is combinatorially equivalent to the labeled combinatorial class of tournaments, $\mathcal{G} \cong \mathcal{T}$, and any tournament can be represented as a sequence of irreducible subtournaments, $\mathcal{T} = \mathrm{SEQ}(\mathcal{IT})$ (see also Lemma 10.2.7). Hence, we can apply Theorem 7.2.1 which gives us

$$\mathbb{P}\big(G \text{ is SET-irreducible}\big) \approx 1 - \sum_{k \geqslant 1} \mathfrak{it}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{g}_{n-k}}{\mathfrak{g}_n}.$$

To finish the proof, we use the fact that a labeled graph is SET-irreducible if and only if it is connected and the relation

$$\frac{\mathfrak{g}_{n-k}}{\mathfrak{g}_n} = \frac{2^{k(k+1)/2}}{2^{kn}}.$$

□

Theorem 10.1.11. *The asymptotic probability that a random labeled simple graph G with n vertices consists of $m \in \mathbb{N}$ connected components satisfies*

$$\mathbb{P}\big(G \text{ has } m \text{ connected components}\big) \approx \sum_{k\geqslant 1} \alpha_k^{\{m\}} \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{kn}}, \tag{10.2}$$

where

$$\alpha_k^{\{m\}} = \sum_{s=m-1}^{\infty} (-1)^{s+m-1} \binom{s}{m-1} \mathfrak{cg}_k^{\{s\}}(z).$$

In particular,

$$\mathbb{P}\big(G \text{ has } m \text{ connected components}\big) = \binom{n}{m-1} \frac{2^{m(m-1)/2}}{2^{(m-1)n}} + O\left(\frac{n^m}{2^{mn}}\right).$$

Proof. The same way, as in the proof of Theorem 10.1.10, we notice that the labeled combinatorial classes $\mathcal{G} \cong \mathcal{T}$ are gargantuan, while $\mathcal{G} = \mathrm{SET}(\mathcal{CG})$ and $\mathcal{T} = \mathrm{SEQ}(\mathcal{IT})$. As a consequence, the asymptotic probability (10.2) can be obtained with the help of Theorem 7.3.1. To establish the leading term of this asymptotics, we apply Corollary 7.3.4. □

Remark 10.1.12. For $m = 1$, asymptotic probability (10.2) takes the form of (10.1) with

$$\alpha_k^{(1)} = \begin{cases} 1, & \text{if } k = 0, \\ -\mathfrak{it}_k, & \text{if } k \neq 0. \end{cases}$$

10.1.3 Multigraphs and their asymptotics

Definition 10.1.13. We call a *(labeled) multigraph of rank N* an element of the Hadamard product $\mathcal{G} \odot \ldots \odot \mathcal{G}$ of N copies of the class $\mathcal{G}$. In other words, a multigraph of rank N and size n is an N-tuple of graphs

$$(G_1, \ldots, G_N) \in \mathcal{G}_n^N$$

such that the vertices with the same labels are identified with each other.

Remark 10.1.14. In the general case, one usually defines a labeled multigraph of size n as a function from the set of unordered pairs $\big\{\{i,j\} \mid i,j \in [n]\big\}$ to the set of non-negative integers (see [80]). With this definition, two vertices in a labeled multigraph may be joined by any finite number of edges; however, loops are forbidden. Even more generally, one can consider graphs endowed with edge-weight functions, assigning real weights to their edges.

The reader may think about a multigraph of rank N as a multigraph whose edges bear colors from the set $\{1, \ldots, N\}$. The concept of connectivity is transferred to multigraphs of rank N in a natural way. Comparing to the general case, multigraphs of rank N possess the additional restriction: fixing a color $c \in [N]$ and removing all the edges of other colors, we get the remaining structure to be a simple graph.

Notation 10.1.15. We denote the labeled combinatorial class of multigraphs of rank N and its subclass of connected multigraphs by $\mathcal{MG}(N)$ and $\mathcal{CMG}(N)$, respectively. We designate the corresponding counting sequences as $\big(\mathfrak{mg}_n(N)\big)$ and $\big(\mathfrak{cmg}_n(N)\big)$, respectively.

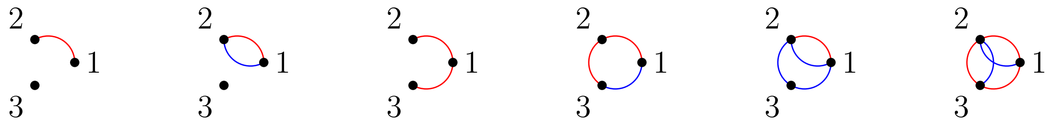


Figure 10.2: Some labeled multigraphs of size 3.

Example 10.1.16. Some labeled multigraphs of rank 2 and size 3 are presented on Fig. 10.2. The first two of them consist of two connected components each, while the other four multigraphs are connected.

All the above discussed properties of simple graphs are generalized to the case of multigraphs in a natural way. Thus, the counting sequence $\big(\mathfrak{mg}_n(N)\big)$ of the class $\mathcal{MG}(N)$ satisfies

$$\mathfrak{mg}_n(N) = (\mathfrak{g}_n)^N = 2^{N\binom{n}{2}}.$$

Also, any multigraph of rank N can be uniquely decomposed into an unordered collection of connected multigraphs of rank N, meaning that

$$\mathcal{MG}(N) = \mathrm{SET}\big(\mathcal{CMG}(N)\big). \tag{10.3}$$

As for simple graphs, the connected components of a multigraph of rank N coincide with its SET-irreducible parts. In particular, a multigraph is SET-irreducible if and only if it is connected. Similarly, we denote by

$$\mathcal{CMG}^{\{m\}}(N) = \mathrm{SET}_m\big(\mathcal{CMG}(N)\big)$$

the labeled combinatorial class consisting of labeled multigraphs of rank N that have exactly m connected components, where m is an arbitrary positive integer, and $\big(\mathfrak{cmg}_n^{\{m\}}(N)\big)$ its counting sequence, respectively. Finally, the class $\mathcal{MG}(N)$ is gargantuan. We highlight the latter property as a separate lemma, for its proof is not a direct generalization of the proof of Lemma 10.1.5.

Lemma 10.1.17. *For any $N \in \mathbb{N}$, the labeled combinatorial class $\mathcal{MG}(N)$ of multigraphs of rank N is gargantuan.*

Proof. Recall, that sequences

$$a_n = n! \qquad \text{and} \qquad b_n = \frac{2^{\binom{n}{2}}}{n!}$$

are gargantuan (see Examples 2.1.2 and 2.1.7). Taking into account Lemma 2.1.8, this implies that the sequence

$$\frac{\mathfrak{mg}_n(N)}{n!} = a_n^{N-1} b_n^N = \frac{2^{N\binom{n}{2}}}{n!}$$

is gargantuan as well. Hence, in accordance with Definition 3.1.72, the class $\mathcal{MG}(N)$ is gargantuan. □

All the mentioned properties allow us to state the results concerning the asymptotic behavior of multigraphs. These results naturally generalize the statements of Theorem 10.1.10 and 10.1.11. This is the reason why we omit their proofs that proceed exactly the same way. The only thing we need to mention is that, in this case, the involved coefficients $\big(\mathfrak{imt}_k(N)\big)$ count irreducible multitournaments, i.e. the structure that generalizes the concept of irreducible tournaments (see Section 10.2 and, particularly, Definition 10.2.14).

Statement 10.1.18. *The asymptotic probability that a random labeled multigraph G of rank N with n vertices is connected satisfies*

$$\mathbb{P}\big(G \text{ is connected}\big) \approx 1 - \sum_{k\geqslant 1} \mathfrak{imt}_k(N) \cdot \binom{n}{k} \cdot \frac{2^{Nk(k+1)/2}}{2^{Nkn}}. \tag{10.4}$$

Statement 10.1.19. *The asymptotic probability that a random labeled multigraph G of rank N with n vertices consists of $m \in \mathbb{N}$ connected components satisfies*

$$\mathbb{P}\big(G \text{ has } m \text{ connected components}\big) \approx \sum_{k\geqslant 0} \alpha_k^{\{m\}} \cdot \binom{n}{k} \cdot \frac{2^{Nk(k+1)/2}}{2^{Nkn}}, \tag{10.5}$$

where

$$\alpha_k^{\{m\}} = \sum_{s=m-1}^{\infty} (-1)^{s+m-1} \binom{s}{m-1} \mathfrak{cmg}_k^{\{s\}}(N).$$

In particular,

$$\mathbb{P}\big(G \text{ has } m \text{ connected components}\big) = \binom{n}{m-1} \frac{2^{Nm(m-1)/2}}{2^{N(m-1)n}} + O\left(\frac{n^m}{2^{Nmn}}\right).$$

10.2 Tournaments

In this section, we consider the labeled combinatorial class of tournaments $\mathcal{T}$. Pick a random tournament T of size n from it. What is the asymptotic probability that T is irreducible?

In 1962, Moon and Moser [68] gave a first estimation of this probability:

$$\left|\mathbb{P}(T \text{ is reducible}) - \frac{2n}{2^{n-1}}\right| < \frac{1}{2^{n-1}} \qquad \text{for } n \geqslant 14.$$

Few years later, it Moon [67] improved the above result and obtained

$$\left|\mathbb{P}(T \text{ is reducible}) - \frac{2n}{2^{n-1}}\right| < \frac{1}{2}\left(\frac{n}{2^{n-1}}\right)^2.$$

In 1970, Wright [103] computed the first four terms of the asymptotic expansion of the probability that a labeled tournament is irreducible:

$$\mathbb{P}(T \text{ is irreducible}) = 1 - \binom{n}{1}2^{2-n} + \binom{n}{2}2^{4-2n} - \binom{n}{3}2^{8-3n} - \binom{n}{4}2^{15-4n} + O\left(n^5 2^{-5n}\right).$$

Although Wright found a way to get as many terms of asymptotics as one would like, his method relied on a recursion. In other words, it was impossible, using this method, to have all the terms at once. Besides, one could see that all the coefficients were integers, but they were not given a combinatorial interpretation. The both gaps have been filled recently in our work [65], which constitutes the main content of this section.

The structure of this section is similar to the one of Section 10.1. First, we recall all concept and properties. Second, we establish the main result, i.e. Theorem 10.2.11. Third, we discuss a natural generalization for multitournaments.

10.2.1 Definitions and properties

Definition 10.2.1. A *(labeled) tournament* of size n is a directed graph with the set of vertices $[n]$, such that each pair of vertices $i \neq j$ is joined by exactly one of two directed edges $\overrightarrow{ij}$ or $\overrightarrow{ji}$.

Definition 10.2.2. A tournament is *reducible*, if there exists a partition of its vertices into two nonempty parts A and B such that any pair of vertices $(a, b) \in A \times B$ are joined by the edge that goes from A to B, i.e $\overrightarrow{ab}$. A tournament is *irreducible*, if it is not reducible.

Remark 10.2.3. A tournament T is irreducible if and only if it is strongly connected as a directed graph (see Definition 10.3.1). Indeed, if T is reducible, then there is a partition of its vertices $A \sqcup B$, such that A is not reachable from B, and hence, A and B are in different strongly connected components. Conversely, if T is not strongly connected then there is a pair of vertices a and b, such that a is not reachable from b. Let B be the set of vertices that can be reached from B, and A be the rest. Then, on the one hand, $a \in A$ and $b \in B$, and hence, A and B are nonempty. On the other hand, all edges go from A to B, meaning that T is reducible.

Notation 10.2.4. We denote the labeled combinatorial classes of tournaments and irreducible tournaments by $\mathcal{T}$ and $\mathcal{IT}$, respectively. The corresponding counting sequences are $(\mathfrak{t}_n)$ and $(\mathfrak{it}_n)$, respectively.

Remark 10.2.5. Since for any pair of vertices there are two ways of choosing the direction of the edge that joins these vertices, the counting sequence $(\mathfrak{t}_n)$ satisfies

$$\mathfrak{t}_n = 2^{\binom{n}{2}}.$$

Lemma 10.2.6. *The labeled combinatorial class $\mathcal{T}$ of tournaments is gargantuan.*

Proof. This follows from Lemma 10.1.5, since the counting sequences of the classes $\mathcal{T}$ and $\mathcal{G}$ are the same (see also Example 3.1.73). □

Lemma 10.2.7. *Any tournament can be uniquely decomposed into a sequence of irreducible tournaments, meaning that*

$$\mathcal{T} = \mathrm{SEQ}(\mathcal{IT}).$$

Proof. For the proof, see Example 3.1.33 (see also [65]). □

Remark 10.2.8. The notion of SEQ-irreducibility for tournaments coincides with the one of irreducibility.

Notation 10.2.9. For any $m \in \mathbb{N}$, we denote by

$$\mathcal{IT}^{(m)} = \mathrm{SEQ}_m(\mathcal{IT})$$

the labeled combinatorial class consisting of labeled tournaments that have exactly m irreducible parts. We designate its counting sequence as $\left(\mathfrak{it}_n^{(m)}\right)$.

Example 10.2.10. All possible eight labeled tournaments of size 3 are depicted on Fig. 10.3. Among them, the only irreducible tournaments are cycles: the second tournament at the top row and the third one at the bottom row. The other six tournaments can be interpreted as sequences of three irreducible tournaments of size 1 each. For instance, the top left tournament corresponds to the sequence $\{2\} \to \{1\} \to \{3\}$.

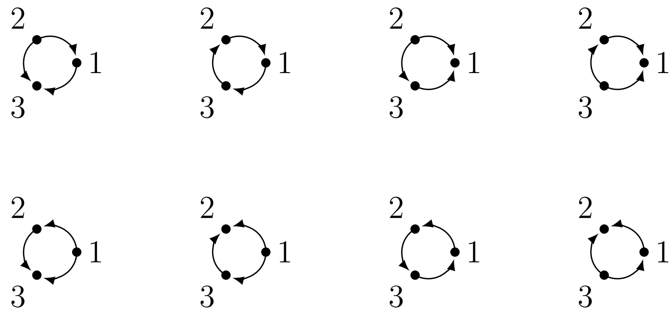


Figure 10.3: Labeled tournaments of size 3.

10.2.2 Asymptotic probabilities for irreducible tournaments

Theorem 10.2.11. *The asymptotic probability that a random labeled tournament T with n vertices consists of $m \in \mathbb{N}$ irreducible parts satisfies*

$$\mathbb{P}\big(T \text{ has } m \text{ irreducible parts}\big) \approx \sum_{k \geqslant 0} \beta_k^{(m)} \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{kn}}, \tag{10.6}$$

where $\beta_k^{(m)} = m\left(\mathfrak{it}_k^{(m-1)} - 2\mathfrak{it}_k^{(m)} + \mathfrak{it}_k^{(m+1)}\right)$. *In particular,*

$$\mathbb{P}\big(T \text{ has } m \text{ irreducible parts}\big) = m \cdot (n)_{m-1} \cdot \frac{2^{m(m-1)/2}}{2^{(m-1)n}} + O\left(\frac{n^m}{2^{mn}}\right).$$

Proof. Due to Lemma 10.2.6, the labeled combinatorial class $\mathcal{T}$ is gargantuan. At the same time, $\mathcal{T} = \mathrm{SEQ}(\mathcal{IT})$ by Lemma 10.2.7. Hence, in accordance with Theorem 5.3.1, we have

$$\mathbb{P}\big(T \text{ has } m \text{ SEQ-irreducible parts}\big) \approx \sum_{k \geqslant 0} \beta_k^{(m)} \cdot \binom{n}{k} \cdot \frac{\mathfrak{t}_{n-k}}{\mathfrak{t}_n}.$$

To finish the proof, note that SEQ-irreducibility of a labeled tournament coincides with its irreducibility, and

$$\frac{\mathfrak{t}_{n-k}}{\mathfrak{t}_n} = \frac{2^{k(k+1)/2}}{2^{kn}}.$$

The behavior of the leading term of this asymptotics follows from Corollary 5.3.4. $\square$

Corollary 10.2.12. *The asymptotic probability that a random labeled tournament T with n vertices is irreducible satisfies*

$$\mathbb{P}\big(T \text{ is irreducible}\big) \approx 1 - \sum_{k \geqslant 1} \left(2\mathfrak{it}_k - \mathfrak{it}_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{kn}}. \tag{10.7}$$

Remark 10.2.13. For $m = 1$, asymptotic expression (10.6) in the statement of Theorem 10.2.11 takes the form of (10.7) with

$$\beta_k^{(1)} = \begin{cases} 1, & \text{if } k = 0, \\ -2\mathfrak{it}_k + \mathfrak{it}_k^{(2)}, & \text{if } k \neq 0. \end{cases}$$

10.2.3 Multitournaments and their asymptotics

Definition 10.2.14. We call a *(labeled) multitournament* of rank N an element of the Hadamard product $\mathcal{T} \odot \ldots \odot \mathcal{T}$ of N copies of the class $\mathcal{T}$. In other words, a multitournament of rank N of size n is an N-tuple of tournaments

$$(T_1, \ldots, T_N) \in \mathcal{T}_n^N$$

such that the vertices with the same labels are identified with each other.

Remark 10.2.15. The concept of multitournaments generalizes the concept of tournaments the same way as multigraphs generalize simple graphs (see Remark 10.1.14 and Remark 10.3.21). In particular, the usual notion of (ir)reducibility can be used. Namely, a multitournament is *reducible*, if there exists a partition of its vertices into two nonempty parts A and B such that any pair of vertices $(a, b) \in A \times B$ are joined by N edges that go from A to B, and *irreducible* otherwise.

Notation 10.2.16. We denote the labeled combinatorial class of multitournament of rank N and its subclass of irreducible multitournament by $\mathcal{MT}(N)$ and $\mathcal{IMT}(N)$, respectively. We designate their counting sequences as $\big(\mathfrak{mt}_n(N)\big)$ and $\big(\mathfrak{imt}_n(N)\big)$, respectively.

The properties of tournaments are transferred to multitournaments in a natural way. Thus, any multitournament of rank N can be uniquely decomposed into a sequence of irreducible multitournaments of rank N, meaning that

$$\mathcal{MT}(N) = \mathrm{SEQ}\big(\mathcal{IMT}(N)\big). \tag{10.8}$$

As a consequence, as well as for simple tournaments, irreducible parts of a multitournament of rank N coincide with its SEQ-irreducible parts. In particular, a multitournament is SEQ-irreducible, if and only if it is irreducible. We denote the labeled combinatorial class consisting of labeled multigraphs of rank N that have exactly m irreducible parts, where m is a positive integer, by

$$\mathcal{IMT}^{(m)}(N) = \mathrm{SEQ}_m\big(\mathcal{IMT}(N)\big).$$

The corresponding counting sequence is designated as $\big(\mathfrak{imt}_n^{(m)}(N)\big)$.

Example 10.2.17. Some of the 64 labeled multigraphs of rank 2 of size 3 are depicted on Fig. 10.4. The first of them is identified with the sequence of three irreducible multitournaments of size 1 each. The second and the third are decomposed into a sequence of two irreducible parts, while the last three multitournaments are irreducible. Note that the fourth multitournament is irreducible, but both its restrictions on the first and second element of the tuple are reducible. In other words, both the tournament with red edges and the tournament with blue edges are reducible.

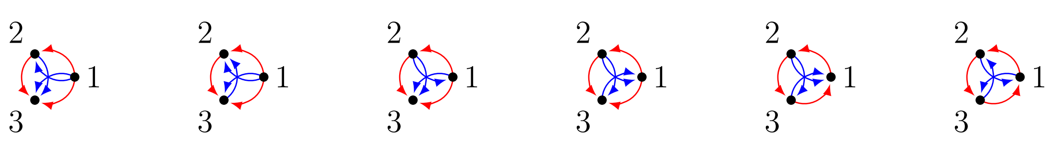


Figure 10.4: Some labeled multitournaments of size 3.

The counting sequence $\big(\mathfrak{mt}_n(N)\big)$ of the class $\mathcal{MT}(N)$ satisfies

$$\mathfrak{mt}_n(N) = \mathfrak{mg}_n(N) = (\mathfrak{g}_n)^N = 2^{N\binom{n}{2}}.$$

In particular, there is a combinatorial equivalence

$$\mathcal{MT}(N) \cong \mathcal{MG}(N),$$

implying by Lemma 10.1.17, that the class $\mathcal{MT}(N)$ is gargantuan. Taking into account decomposition (10.8), this fact, due to Theorem 5.3.1, leads to the following result which generalizes Theorem 10.2.11.

Statement 10.2.18. *The asymptotic probability that a random labeled multitournament T of rank N with n vertices consists of $m \in \mathbb{N}$ irreducible parts satisfies*

$$\mathbb{P}\big(T \text{ has } m \text{ irreducible parts}\big) \approx \sum_{k\geqslant 0} \beta_k^{(m)} \cdot \binom{n}{k} \cdot \frac{2^{Nk(k+1)/2}}{2^{Nkn}}, \tag{10.9}$$

where $\beta_k^{(m)} = m\left(\mathfrak{imt}_k^{(m-1)}(N) - 2\mathfrak{imt}_k^{(m)}(N) + \mathfrak{imt}_k^{(m+1)}(N)\right)$. *In particular,*

$$\mathbb{P}\big(T \text{ has } m \text{ irreducible parts}\big) = m\cdot(n)_{m-1}\cdot\frac{2^{Nm(m-1)/2}}{2^{N(m-1)n}} + O\left(\frac{n^m}{2^{Nmn}}\right).$$

Corollary 10.2.19. *The asymptotic probability that a random labeled multitournament T of rank N with n vertices is irreducible satisfies*

$$\mathbb{P}\big(T \text{ is irreducible}\big) \approx 1 - \sum_{k\geqslant 1}\left(2\mathfrak{imt}_k(N) - \mathfrak{imt}_k^{(2)}(N)\right)\cdot\binom{n}{k}\cdot\frac{2^{Nk(k+1)/2}}{2^{Nkn}}. \tag{10.10}$$

10.3 Directed graphs

Tournaments discussed in the previous section are the particular case of directed graphs. In this section, we discuss other subclasses of the labeled combinatorial class $\mathcal{D}$ of directed graphs, including strongly connected, weakly connected and directed acyclic graphs.

Among the topics of this section, the most interesting question, probably, relates to the enumeration of strongly connected digraphs. This question arose in 1960 in the paper of Harary [44], see also [45]. The first approach was made in 1966, when Moon and Moser [69] showed that the number $\mathfrak{sd}_n$ of strongly connected digraphs of size n satisfies

$$\mathfrak{sd}_n \sim 2^{n(n-1)},$$

as $n \to \infty$. Three years later, Liskovets [58] provided three recurrences allowing to calculate the sequence $(\mathfrak{sd}_n)$ iteratively, see also [59]. Basing on the results of Liskovets, in 1971, Wright [104] established the asymptotic behavior of the number of strongly connected digraphs in the form

$$\mathfrak{sd}_n = 2^{n(n-1)}\left(\sum_{k=0}^{r-1} 2^{-kn}\omega_k(n)\frac{n!}{(n+[k/2]-k)!} + O\big(n^r 2^{-rn}\big)\right), \tag{10.11}$$

where

$$\omega_k(n) = \sum_{\nu=0}^{[k/2]} \gamma_\nu \xi_{k-2\nu} 2^{k(k+1)/2-\nu(k-\nu)}(n+[k/2]-k)\ldots(n+\nu+1-k),$$

the numbers γ_ν are defined recursively by

$$\gamma_0 = 1, \qquad \gamma_\nu = \sum_{s=0}^{\nu-1} \frac{\gamma_s \eta_{n-s}}{(\nu - s)!}, \tag{10.12}$$

the numbers ξ_ν are the coefficients of the formal power series

$$\sum_{\nu=0}^{\infty} \xi_\nu z^\nu = \left(1 - \sum_{n=0}^{\infty} \frac{\eta_n}{2^{n(n-1)/2}} \frac{z^n}{n!}\right)^2 \tag{10.13}$$

and the numbers η_n are defined recursively by

$$\eta_1 = 1, \qquad \eta_n = 2^{n(n-1)} - \sum_{t=1}^{n-1} \binom{n}{t} 2^{(n-1)(n-t)} \eta_t. \tag{10.14}$$

According to Wright, the polynomials $\omega_k(n)$ up to $k = 5$ are

$$\omega_0(n) = 1, \qquad \omega_1(n) = -4, \qquad \omega_2(n) = 4(2n-1),$$
$$\omega_3(n) = -\frac{32(4n-5)}{3}, \qquad \omega_4(n) = -\frac{64(64n^2 - 326n + 393)}{3},$$
$$\omega_5(n) = -\frac{1024(3392n^2 - 2372n + 40659)}{15}.$$

Also, he mentioned that he had not found any combinatorial interpretation of involved coefficients though the recurrences looked as if they should possess one.

Further simplifications were done by Robinson [90] who showed that η_n are the coefficients of the exponential generating function

$$H(z) = \Delta^{-1}\left(1 - \frac{1}{\Delta D(z)}\right),$$

and Bender [9] who provided a simple proof of Wright's result, using exponential generating functions instead of recurrences. The direct interpretation of η_n in terms of tournaments was given in 1975 by Liskovets [60] who indicated that $2^{-\binom{n}{2}}\eta_n$ is the number of irreducible tournaments of size n; see also recent paper of Archer, Gessel, Graves and Liang [3] who showed that, after dividing by $2^{\binom{n}{2}}$, (10.14) becomes a well-known formula

$$\mathfrak{it}_1 = 1, \qquad \mathfrak{it}_n = 2^{\binom{n}{2}} - \sum_{t=1}^{n-1} \binom{n}{t} 2^{\binom{n-t}{2}} \mathfrak{it}_t$$

for irreducible tournaments [68, (1)]. Thus, $\eta_n = \mathfrak{t}_n \mathfrak{it}_n$.

In this section, relying on the enumerative methods of Robinson [90] and Dovgal and de Panafieu [29], we make further simplifications and rewrite (10.11) in a shorter form. We give combinatorial interpretation to all the terms involved into our asymptotic expansion as well as for Wright's coefficients γ_n, ξ_n and η_n. In addition, we need to indicate that Wright's expression for $\omega_5(n)$ contains a typo. The correct version should be

$$\omega_5(n) = -\frac{1024(3392n^2 - 23724n + 40659)}{15},$$

while in Wright's version the last digit in the number 23724 is omitted.

Another problem that is worth mentioning is the enumeration of weakly connected directed acyclic graphs. Using the recursive approach of Wright [102], Robinson [90] showed that the number of weakly connected directed acyclic graphs satisfies

$$\mathfrak{wdag}_n \approx \mathfrak{dag}_n + \sum_{k \geqslant 1} g_k \binom{n}{k} \mathfrak{dag}_{n-k},$$

where g_k are the coefficients of the exponential generating function $g(z)$ such that

$$DAG(z)\big(1 + g(z)\big) = 1.$$

We put this result of Robinson in the general context, give interpretation to the coefficients g_n and generalize it by establishing the probability that a random directed acyclic graph consists of a given number of weakly connected components.

The structure of the section is the following. First, we recall necessary definitions and properties of directed graphs. In the second part of the section, we discuss the symbolic method for strongly connected graphs and the asymptotic expansions for the probability that a random directed graph is strongly connected. Next, we establish the asymptotic probability that a random directed graph consists of a fixed number of weakly connected components and generalizes this result for multidigraphs. Finally, we finish this section providing asymptotics and interpretations for the probability that a random directed acyclic graph has a given number of weakly connected components.

10.3.1 Definitions and properties

Definition 10.3.1. A *(labeled) directed graph* (or simply *digraph*) of size n consists of the set of *vertices* $[n] = \{1, \ldots, n\}$ together with a set E of *(directed) edges* that are ordered pairs of distinct vertices. Among different type of digraphs, we consider the following:

- *weakly connected digraphs*, that is, digraphs whose underlying undirected graphs are connected,
- *strongly connected digraphs*, which are determined by the property that every vertex can be reached from any other vertex by a directed path,
- *semi-strong digraphs* whose weakly connected components are strongly connected,
- *acyclic digraphs* that do not contain directed cycles.

Notation 10.3.2. For different labeled combinatorial classes and their counting sequences, we employ the following notations.

- For directed graphs, we use $\mathcal{D}$ and $(\mathfrak{d}_n)$, respectively.
- For weakly connected directed graphs, we use $\mathcal{WD}$ and $(\mathfrak{wd}_n)$, respectively.
- For strongly connected directed graphs, we use $\mathcal{SD}$ and $(\mathfrak{sd}_n)$, respectively.
- For semi-strong directed graphs, we use $\mathcal{SSD}$ and $(\mathfrak{ssd}_n)$, respectively.
- For directed acyclic graphs, we use $\mathcal{DAG}$ and $(\mathfrak{dag}_n)$, respectively.
- For weakly connected directed acyclic graphs, we use $\mathcal{WDAG}$ and $(\mathfrak{wdag}_n)$, respectively.

Remark 10.3.3. The number of labeled digraphs of size n is determined by the cardinality of the set of ordered pairs of $[n] = \{1, \ldots, n\}$ which is $n(n-1)$. Since any pair can be either present or absent in a graph, the counting sequence $(\mathfrak{d}_n)$ of the class $\mathcal{D}$ satisfies

$$\mathfrak{d}_n = 2^{n(n-1)}.$$

As a consequence, there is a combinatorial equivalence

$$\mathcal{D} \cong \mathcal{MG}(2)$$

and the graphic generating function of $\mathcal{D}$ coincides with the exponential generating function of $\mathcal{G}$:

$$\Delta(z) = G(z) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{z^n}{n!}.$$

Lemma 10.3.4. *The labeled combinatorial class $\mathcal{D}$ of directed graphs is gargantuan.*

Proof. Since $\mathfrak{d}_n = \mathfrak{mg}_n(2)$, the statement of this lemma follows from Lemma 10.1.17. □

Remark 10.3.5. The concept of weakly connectedness of digraphs is similar to the concept of connectedness of undirected graphs. In particular, the following decompositions take place:

$$\mathcal{D} = \mathrm{SET}(\mathcal{WD}), \tag{10.15}$$

$$\mathcal{SSD} = \mathrm{SET}(\mathcal{SD}), \tag{10.16}$$

and

$$\mathcal{DAG} = \mathrm{SET}(\mathcal{WDAG}). \tag{10.17}$$

Hence, as for undirected graphs, weakly connected components of a directed graph coincide with its SET-irreducible parts. For any $m \in \mathbb{N}$, we denote by

$$\mathcal{WD}^{\{m\}} = \mathrm{SET}_m(\mathcal{WD}) \qquad \text{and} \qquad \mathcal{WDAG}^{\{m\}} = \mathrm{SET}_m(\mathcal{WDAG})$$

the labeled combinatorial classes consisting of digraphs and acyclic digraphs, respectively, that have exactly m weakly connected components. Their counting sequences are $\big(\mathfrak{wd}_n^{\{m\}}\big)$ and $\big(\mathfrak{wdag}_n^{\{m\}}\big)$, respectively.

The concept of strongly connectedness is less ordinary. On the one hand, any digraph can be uniquely decomposed into a finite collection of strongly connected components, i.e. the strongly connected subgraphs that are maximal in the sense of inclusion. On the other hand, two different components may be joined by edges, but, in this case, all the edges go in the same direction. Enumerating strongly connected graphs is a non-trivial task; we shortly discuss it in the next subsection (see Lemma 10.3.14).

Remark 10.3.6. Due to relations (10.3) and (10.15),

$$\mathrm{SET}(\mathcal{WD}) = \mathcal{D} \cong \mathcal{MG}(2) = \mathrm{SET}\big(\mathcal{CMG}(2)\big).$$

Hence, for every $m \in \mathbb{N}$, we have the combinatorial equivalences

$$\mathcal{WD} \cong \mathcal{CMG}(2) \qquad \text{and} \qquad \mathcal{WD}^{\{m\}} \cong \mathcal{CMG}^{\{m\}}(2).$$

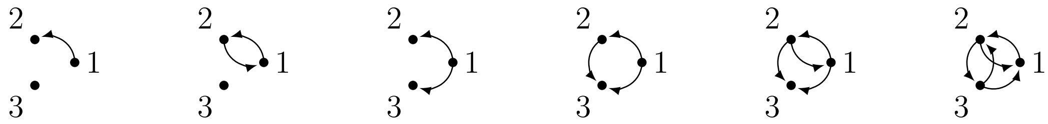


Figure 10.5: Some labeled digraphs of size 3.

Example 10.3.7. Some of the 64 labeled directed graphs of size 3 are depicted on Fig. 10.5. The last of them is strongly connected. The second and fifth digraphs contain two strongly connected components, namely, $\{1, 2\}$ and $\{3\}$. For the other three digraphs, the connected components coincide with their vertices.

Example 10.3.8. The undirected graphs which are underlying the labeled directed graphs from Fig. 10.5 are depicted on Fig. 10.6. The first two of them are disconnected, while the others are connected. Hence, on Fig. 10.5, the last four digraphs are weakly connected and the first two contain two connected components each. At the same time, the second and last digraphs are semi-strong. The first digraph is not semi-strong, since it has the weakly connected component $\{1, 2\}$ which is not strongly connected. The same way, the other three digraphs are weakly connected, but not strongly connected, and hence, they are not semi-strong.

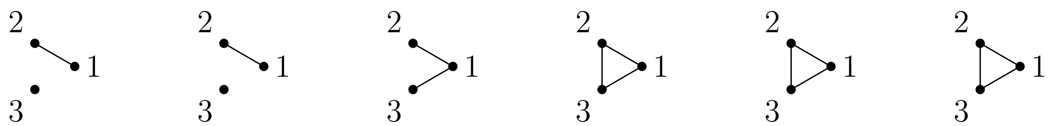


Figure 10.6: Undirected graphs underlying the labeled digraphs from Fig. 10.5.

Example 10.3.9. On Fig. 10.7, we depict some acyclic digraphs of size 3. The last four of these digraphs are weakly connected, while the first and second contain three and two weakly connected components, respectively. Note that Fig. 10.7 illustrates all possible structures, meaning that any acyclic digraph of size 3 can be obtained from one of the depicted digraphs by relabeling.

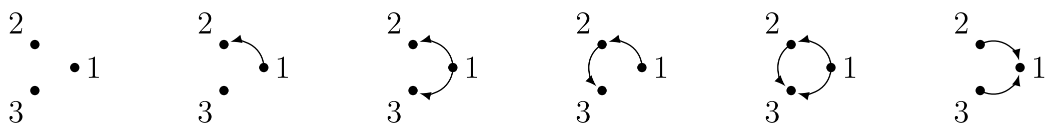


Figure 10.7: Some labeled acyclic digraphs of size 3.

10.3.2 Asymptotic probabilities for strongly connected digraphs

Lemma 10.3.10. *Let $\mathcal{A} \subset \mathcal{SD}$ be a labeled combinatorial subclass of strongly connected digraphs and let $\mathcal{D}_{\mathcal{A}}$ be the labeled combinatorial class of digraphs whose connected components belong to $\mathcal{A}$. In this case, we have*

$$\Delta D_{\mathcal{A}}(z) = \left(\Delta e^{-A(z)}\right)^{-1}.$$

Proof. For the proof we invite the reader to consult [29, Theorem 3.4] or [90, Theorem 1]. □

Notation 10.3.11. Denote by $\mathcal{H}$ the subclass of the labeled combinatorial class $\mathcal{MT}(2)$ of multi-tournaments of rank 2 whose second element is irreducible. In other words,

$$\mathcal{H} = \mathcal{T} \odot \mathcal{IT}.$$

The corresponding counting sequence is designated as $(\mathfrak{h}_n)$.

Remark 10.3.12. The counting sequence $(\mathfrak{h}_n)$ of the class $\mathcal{H}$ satisfies

$$\mathfrak{h}_n = \mathfrak{t}_n \mathfrak{i}\mathfrak{t}_n,$$

and its exponential generating function has the form

$$H(z) = T(z) \odot IT(z).$$

Lemma 10.3.13. *The labeled combinatorial class $\mathcal{H}$ is gargantuan.*

Proof. Recall, that the sequences

$$a_n = n!$$

and

$$b_n = \frac{\mathfrak{t}_n}{n!}$$

are gargantuan (see Example 2.1.2 and Lemma 10.2.6, respectively). Since

$$\mathfrak{i}\mathfrak{t}_n = \mathfrak{t}_n + o(\mathfrak{t}_n)$$

by Corollary 10.2.12, Lemma 2.1.12 implies that the sequence

$$c_n = \frac{\mathfrak{i}\mathfrak{t}_n}{n!}$$

is gargantuan as well. Hence, by Lemma 2.1.8, their product

$$\frac{\mathfrak{h}_n}{n!} = a_n b_n c_n$$

is gargantuan, meaning that the class $\mathcal{H}$ is gargantuan indeed. □

Lemma 10.3.14. *The exponential generating functions of the labeled combinatorial classes $\mathcal{SD}$ of strongly connected directed graphs and $\mathcal{SSD}$ of semi-strong directed graphs satisfy*

$$SD(z) = -\log\Big(1 - H(z)\Big)$$

and

$$SSD(z) = \frac{1}{1 - H(z)},$$

respectively.

Proof. Apply Lemma 10.3.10 to the class $\mathcal{A} = \mathcal{SD}$. Then

$$\Delta D_{\mathcal{A}}(z) = \left(\Delta e^{-A(z)}\right)^{-1} = \left(\Delta e^{-SD(z)}\right)^{-1}.$$

On the other hand,

$$\Delta D_{\mathcal{A}}(z) = \Delta D(z) = G(z).$$

Hence,

$$SD(z) = -\log\left(\Delta^{-1}\frac{1}{\Delta D_{\mathcal{A}}(z)}\right) = -\log\left(G(z) \odot \frac{1}{G(z)}\right).$$

Meanwhile, by Lemma 10.2.7,

$$\frac{1}{G(z)} = \frac{1}{T(z)} = 1 - IT(z).$$

Therefore,

$$G(z) \odot \frac{1}{G(z)} = T(z) \odot \big(1 - IT(z)\big) = 1 - T(z) \odot IT(z) = 1 - H(z),$$

that implies the first part of the lemma. To finish the proof, note that the expression for $SSD(z)$ follows from the expression for $SD(z)$ and relation (10.16). ☐

Remark 10.3.15. Lemma 10.3.14 means that there are combinatorial isomorphisms of the following combinatorial classes:

$$\mathcal{SD} \cong \mathrm{CYC}(\mathcal{H}) \qquad \text{and} \qquad \mathcal{SSD} \cong \mathrm{SEQ}(\mathcal{H}).$$

In 2020, in their work [3], Archer, Gessel, Graves and Liang noticed that the equality

$$SD(z) = -\log\Big(1 - T(z) \odot IT(z)\Big).$$

could be explained combinatorially by a bijection from strongly connected directed graphs to cycles of strong tournaments with some additional structure. Unfortunately, they were not able to find such a bijection, neither were we, so far.

Theorem 10.3.16. *The asymptotic probability that a random labeled directed graph D with n vertices is strongly connected satisfies*

$$\mathbb{P}\big(D \textit{ is strong}\big) \approx \sum_{k\geqslant 0} \mathfrak{ssd}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{n(n-1)/2}} \cdot \frac{\mathfrak{it}_{n-k}}{2^{nk}}. \tag{10.18}$$

Proof. Let us apply Theorem 3.1.79 to the gargantuan labeled combinatorial class $\mathcal{H}$ and the function

$$F(x, y) = \log\frac{1}{1-y}.$$

According to Lemma 10.3.14, we have

$$B(z) = \log\frac{1}{1 - H(z)} = SD(z)$$

and

$$C(z) = \frac{1}{1 - H(z)} = SSD(z).$$

Therefore, Theorem 3.1.79 gives the following asymptotics:

$$\mathfrak{sd}_n \approx \sum_{k\geqslant 0} \mathfrak{ssd}_k \binom{n}{k} \mathfrak{h}_{n-k}.$$

Now, to obtain the target probability, it is sufficient divide both sides of the above formula by the number of digraphs $\mathfrak{d}_n$ of size n and to notice that

$$\frac{\mathfrak{h}_{n-k}}{\mathfrak{d}_n} = \frac{2^{k(k+1)/2}}{2^{n(n-1)/2}} \cdot \frac{\mathfrak{it}_{n-k}}{2^{nk}}.$$

□

Remark 10.3.17. Since $\mathfrak{d}_n = \mathfrak{mt}_n(2)$ and

$$\frac{\mathfrak{h}_{n-k}}{\mathfrak{d}_n} = \left(\frac{\mathfrak{t}_{n-k}}{\mathfrak{t}_n}\right)^2 \frac{\mathfrak{it}_{n-k}}{\mathfrak{t}_{n-k}} = \frac{2^{k(k+1)}}{2^{2nk}} \cdot \frac{\mathfrak{it}_{n-k}}{\mathfrak{t}_{n-k}},$$

we can rewrite the statement of Theorem 10.3.16 as

$$\mathbb{P}\big(D \text{ is strong}\big) \approx \sum_{k\geqslant 0} \mathfrak{ssd}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{mt}_{n-k}(2)}{\mathfrak{mt}_n(2)} \cdot \frac{\mathfrak{it}_{n-k}}{\mathfrak{t}_{n-k}}$$

or as

$$\mathbb{P}\big(D \text{ is strong}\big) \approx \sum_{k\geqslant 0} \mathfrak{ssd}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)}}{2^{2nk}} \cdot \frac{\mathfrak{it}_{n-k}}{\mathfrak{t}_{n-k}}.$$

Corollary 10.3.18. *The asymptotic probability that a random labeled directed graph D with n vertices is strongly connected satisfies*

$$\mathbb{P}\big(D \textit{ is strong}\big) = \sum_{k=0}^{2r-1} \frac{2^{k(k+1)/2}}{2^{kn}} \sum_{\nu=0}^{[k/2]} \binom{n}{\nu, k-2\nu} \frac{\mathfrak{ssd}_\nu \beta_{k-2\nu}}{2^{\nu(k-\nu)}} + O\left(\frac{n^{2r}}{2^{2nr}}\right), \tag{10.19}$$

where

$$\beta_k = \begin{cases} 1, & \textit{if } k=0,\\ -2\mathfrak{it}_k + \mathfrak{it}_k^{(2)}, & \textit{if } k\neq 0. \end{cases}$$

Proof. As we have seen in Remark 10.3.17,

$$\mathbb{P}\big(D \text{ is strong}\big) \approx \sum_{\nu\geqslant 0} \mathfrak{ssd}_\nu \cdot \binom{n}{\nu} \cdot \frac{2^{\nu(\nu+1)}}{2^{2n\nu}} \cdot \frac{\mathfrak{it}_{n-\nu}}{\mathfrak{t}_{n-\nu}}.$$

On the other hand, Corollary 10.2.12 provides us the asymptotic behavior for

$$\frac{\mathfrak{it}_{n-\nu}}{\mathfrak{t}_{n-\nu}} \approx 1 - \sum_{s\geqslant 1} \left(2\mathfrak{it}_s - \mathfrak{it}_s^{(2)}\right) \cdot \binom{n-\nu}{s} \cdot \frac{2^{s(s+1)/2}}{2^{s(n-\nu)}},$$

since it is the probability that a random tournament of size $(n-\nu)$ is irreducible. Using the sequence (β_s), we can rewrite it in the form

$$\frac{\mathfrak{it}_{n-\nu}}{\mathfrak{t}_{n-\nu}} \approx \sum_{s\geqslant 0} \beta_s \cdot \binom{n-\nu}{s} \cdot \frac{2^{s(s+1)/2}}{2^{s(n-\nu)}}.$$

Substitute this ratio into the initial formula for the probability that D is strongly connected. Denoting $k = s + 2\nu$ and changing the order of summation, we obtain exactly (10.19). □

Remark 10.3.19. Asymptotic probability (10.19) is a simpler form of (10.11) found by Wright [104]. The main advantage of (10.19) is that all involved terms have precise combinatorial sense. Let us determine the meaning of Wright's coefficients γ_n, ξ_n, η_n. As we have already mentioned, it follows from [3] and [90] that

$$\eta_n = \mathfrak{h}_n = \mathfrak{t}_n \mathfrak{it}_n.$$

Also, we can see it directly from (10.14). Indeed, relation (10.14) can be interpreted as a decomposition of any object of $\mathcal{MT}(2)$ with respect to its second element. More precisely, for any pair of tournaments $(T', T) \in \mathcal{MT}(2)$, decompose T into a sequence (T_1, T_2) where T_1 is irreducible. Then $(T_1', T_1) \in \mathcal{H}$, while $(T_2', T_2) \in \mathcal{MT}(2)$, and hence,

$$\mathfrak{mt}_n(2) = \sum_{t=1}^{n} \binom{n}{t} \mathfrak{h}_t \mathfrak{mt}_{n-t}(2),$$

which is exactly (10.14).

Next, with the help of identity $\eta_n = \mathfrak{h}_n$, recurrent relation (10.12) can be rewritten in terms of exponential generating functions as

$$\sum_{n=0}^{\infty} \gamma_n z^n = \frac{1}{1 - H(z)}.$$

According to Lemma 10.3.14, this means that

$$\gamma_n = \frac{\mathfrak{ssd}_n}{n!}.$$

Finally, using the same identity $\eta_n = \mathfrak{h}_n$, we can interpret relation (10.13) as

$$\sum_{n=0}^{\infty} \xi_n z^n = \big(1 - IT(z)\big)^2 = 1 - 2IT(z) + IT^{(2)}(z),$$

which immediately gives us

$$\xi_n = \frac{\beta_n}{n!} = \begin{cases} 1, & \text{if } n = 0, \\ -\dfrac{2\mathfrak{it}_n + \mathfrak{it}_n^{(2)}}{n!}, & \text{if } n \neq 0. \end{cases}$$

10.3.3 Asymptotic probabilities for weakly connected digraphs

Studying the asymptotic behavior of weakly connected directed graphs is similar to the case of connected undirected graphs. That is why we do it in the utmost generality, i.e. for weakly connected multidigraphs, and omit some details.

Definition 10.3.20. We call a *(labeled) multidigraph* of rank N an element of the Hadamard product $\mathcal{D} \odot \ldots \odot \mathcal{D}$ of N copies of the class $\mathcal{D}$. In other words, a multidigraph of rank N and size n is an N-tuple of directed graphs

$$(D_1, \ldots, D_N) \in \mathcal{D}_n^N$$

such that the vertices with the same labels are identified with each other.

Remark 10.3.21. In the general case, one can define a labeled multidigraph of size n as a function from the set of ordered pairs $\{(i,j) \mid i,j \in [n]\}$ to the set of non-negative integers (compare to Remark 10.1.14). The same way as for multigraphs, it is convenient to think about a multidigraph of rank N as a multidigraph whose edges bear colors $\{1,\ldots,N\}$, such that the restriction to the set of edges of a fixed color is a digraph. The concepts of weakly connected and strongly connected multidigraphs of rank N are defined naturally.

Notation 10.3.22. We denote the labeled combinatorial class of multidigraphs of rank N and its subclasses of weakly connected and strongly connected multidigraphs by $\mathcal{MD}(N)$, $\mathcal{WMD}(N)$ and $\mathcal{SMD}(N)$, respectively. Their counting sequences are $\big(\mathfrak{md}_n(N)\big)$, $\big(\mathfrak{wmd}_n(N)\big)$ and $\big(\mathfrak{smd}_n(N)\big)$, respectively.

The counting sequence $\big(\mathfrak{md}_n(N)\big)$ of the class $\mathcal{MD}(N)$ satisfies

$$\mathfrak{md}_n(N) = (\mathfrak{d}_n)^N = 2^{Nn(n-1)}.$$

In particular,

$$\mathcal{MD}(N) \cong \mathcal{MG}(2N)$$

and $\mathcal{MD}(N)$ is gargantuan for any $N \in \mathbb{N}$. Any multidigraph of rank N can be uniquely decomposed into an unordered collection of weakly connected components, meaning that

$$\mathcal{MD}(N) = \operatorname{SET}\big(\mathcal{WMD}(N)\big). \tag{10.20}$$

Similarly to other combinatorial classes, for any $m \in \mathbb{N}$, we denote by

$$\mathcal{WMD}^{\{m\}}(N) = \operatorname{SET}_m\big(\mathcal{WMD}(N)\big)$$

and $\big(\mathfrak{wmd}_n^{\{m\}}(N)\big)$ the labeled combinatorial class consisting of labeled multidigraphs of rank N that have exactly m weakly connected components and its counting sequence, respectively. For any $N \in \mathbb{N}$,

$$\operatorname{SET}\big(\mathcal{WMD}(N)\big) = \mathcal{MD}(N) \cong \mathcal{MG}(2N) = \operatorname{SET}\big(\mathcal{CMG}(2N)\big),$$

hence, we have combinatorial equivalences

$$\mathcal{WMD}(N) \cong \mathcal{CMG}(2N)$$

and, for every $m \in \mathbb{N}$,

$$\mathcal{WMD}^{\{m\}}(N) \cong \mathcal{CMG}^{\{m\}}(2N).$$

As an example, some labeled multidigraphs of rank 2 and size 3 are depicted on Fig. 10.8. All of them, except for the first one, are weakly connected. The first and second multidigraphs have three strongly connected components of size 1 each. The third and fourth multidigraphs have two strongly connected components, namely, $\{1,2\}$ and $\{3\}$. Finally, the last two multidigraphs are strongly connected.

Since $\mathcal{MD}(N) \cong \mathcal{MG}(2N)$ and $\mathcal{WMD}(N) \cong \mathcal{CMG}(2N)$, Statements 10.1.18 and 10.1.19 give us the following result.

Statement 10.3.23. *The asymptotic probability that a random labeled multidigraph D of rank N and size n is weakly connected satisfies*

$$\mathbb{P}\big(D \text{ is weakly connected}\big) \approx 1 - \sum_{k \geqslant 1} \mathfrak{imt}_k(2N) \cdot \binom{n}{k} \cdot \frac{2^{Nk(k+1)}}{2^{2Nkn}}. \tag{10.21}$$

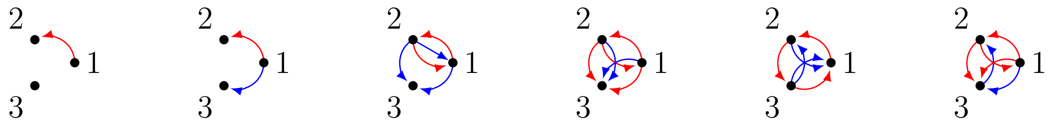


Figure 10.8: Some multidigraphs of rank 2 of size 3.

Moreover, for any $m \in \mathbb{N}$, the asymptotic probability that D consists of m weakly connected components satisfies

$$\mathbb{P}\big(D \text{ has } m \text{ weakly connected components}\big) \approx \sum_{k \geqslant 0} \alpha_k^{\{m\}} \cdot \binom{n}{k} \cdot \frac{2^{Nk(k+1)}}{2^{2Nkn}}, \tag{10.22}$$

where

$$\alpha_k^{\{m\}} = \sum_{s=m-1}^{\infty} (-1)^{s+m-1} \binom{s}{m-1} \mathfrak{wmd}_k^{\{s\}}(N).$$

In particular,

$$\mathbb{P}\big(D \text{ has } m \text{ weakly connected components}\big) = \binom{n}{m-1} \frac{2^{Nm(m-1)}}{2^{2N(m-1)n}} + O\left(\frac{n^m}{2^{2Nmn}}\right).$$

Corollary 10.3.24. *The asymptotic probability that a random labeled digraph D with n vertices is weakly connected satisfies*

$$\mathbb{P}\big(D \text{ is weakly connected}\big) \approx 1 - \sum_{k \geqslant 1} \mathfrak{imt}_k(2) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)}}{2^{2kn}}. \tag{10.23}$$

Moreover, for any $m \in \mathbb{N}$, the asymptotic probability that D consists of m weakly connected components satisfies

$$\mathbb{P}\big(D \text{ has } m \text{ weakly connected components}\big) \approx \sum_{k \geqslant 0} \alpha_k^{\{m\}} \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)}}{2^{2kn}}, \tag{10.24}$$

where

$$\alpha_k^{\{m\}} = \sum_{s=m-1}^{\infty} (-1)^{s+m-1} \binom{s}{m-1} \mathfrak{wd}_k^{\{s\}}.$$

In particular,

$$\mathbb{P}\big(D \text{ has } m \text{ weakly connected components}\big) = \binom{n}{m-1} \frac{2^{m(m-1)}}{2^{2(m-1)n}} + O\left(\frac{n^m}{2^{2mn}}\right).$$

10.3.4 Asymptotic probabilities for weakly connected acyclic digraphs

As we have seen before, directed acyclic graphs satisfy relation (10.17). This implies that the probability that a random directed acyclic graph is weakly connected can be established with the help of our standard schema by Theorem 7.2.1. However, unlike the cases discussed above, there is no explicit expression for the counting sequence $(\mathfrak{dag}_n)$. This fact makes the analysis of the asymptotic behavior of the target probability more complicated. Here, we trace out the way leading to the result.

Lemma 10.3.25. *The graphical generating function of $\mathcal{DAG}$ satisfies*

$$\Delta DAG(z) = \left(\Delta e^{-z}\right)^{-1}.$$

Proof. Apply Lemma 10.3.10, taking into account that an directed acyclic graph is strongly connected if and only if its size is equal to 1. □

Lemma 10.3.26. *The counting sequence of the labeled combinatorial class $\mathcal{DAG}$ satisfies*

$$\mathfrak{dag}_n = n! 2^{\binom{n}{2}} \sum_{k=0}^{\infty} \frac{c_k}{z_k^n},$$

where $0 < z_0 < z_1 < z_2 < \ldots$ are zeros of

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k! 2^{\binom{k}{2}}} = 0$$

and constants c_k are bounded by

$$0 < |c_k| < \frac{1}{2^{\binom{k+1}{2}}} \prod_{j=0}^{\infty} \left(1 - \frac{1}{2^j}\right)^{-2}.$$

Proof. For the proof, see [90]. □

Lemma 10.3.27. *The labeled combinatorial class $\mathcal{DAG}$ of directed acyclic graphs is gargantuan.*

Proof. Since the sequences

$$a_n = n! \qquad \text{and} \qquad b_n = \frac{2^{\binom{n}{2}}}{n!}$$

are gargantuan (see Examples 2.1.2 and 2.1.7), so is the sequence

$$\frac{a_n b_n c_0}{z_0^n} = \frac{c_0 2^{\binom{n}{2}}}{z_0^n}$$

by Lemmas 2.1.8 and 2.1.10. To finish the proof, notice that

$$\frac{\mathfrak{dag}_n}{n!} = \frac{a_n b_n c_0}{z_0^n} + o\left(\frac{a_n^2 b_n}{z_0^n},\right)$$

and hence, we can apply Lemma 2.1.12. □

Statement 10.3.28. *The asymptotic probability that a random labeled directed acyclic graph A with n vertices consists of $m \in \mathbb{N}$ weakly connected components satisfies*

$$\mathbb{P}\big(A \text{ has } m \text{ weakly connected components}\big) \approx \sum_{k \geqslant 0} \alpha_k^{\{m\}} \cdot \binom{n}{k} \cdot \frac{\mathfrak{dag}_{n-k}}{\mathfrak{dag}_n}, \tag{10.25}$$

where

$$\alpha_k^{\{m\}} = \sum_{s=m-1}^{\infty} (-1)^{s+m-1} \binom{s}{m-1} \mathfrak{wdag}_k^{\{s\}}.$$

In particular,

$$\mathbb{P}\big(A \text{ has } m \text{ weakly connected components}\big) = \frac{z_0^{m-1}}{(m-1)!} \cdot \frac{2^{m(m-1)/2}}{2^{(m-1)n}} + O\left(\frac{1}{2^{mn}}\right),$$

where $z_0 = 1.4880785\ldots$ *is the lowest root of the equation*

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k! 2^{\binom{k}{2}}} = 0.$$

Proof. By Lemma 10.3.27, the labeled combinatorial class $\mathcal{DAG}$ is gargantuan. At the same time, according to (10.17), $\mathcal{DAG} = \mathrm{SET}(\mathcal{WDAG})$. Hence, we can apply Theorem 7.3.1 providing us relation (10.25).

Let us determine the leading term of (10.25). It follows from Corollary 7.3.4 that

$$\mathbb{P}\big(A \text{ has } m \text{ weakly connected components}\big) = \binom{n}{m-1} \frac{\mathfrak{dag}_{n-m+1}}{\mathfrak{dag}_n} + O\left(n^m \cdot \frac{\mathfrak{dag}_{n-m}}{\mathfrak{dag}_n}\right).$$

According to Lemma 10.3.26,

$$\mathfrak{dag}_n = n! 2^{\binom{n}{2}} \sum_{k=0}^{\infty} \frac{c_k}{z_k^n},$$

where $0 < z_0 < z_1 < z_2 < \ldots$ are zeroes of

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k! 2^{\binom{k}{2}}} = 0$$

and the set of constants $\{c_k \mid k \in \mathbb{Z}_{\geqslant 0}\}$ is bounded. Hence, calculations give us

$$\binom{n}{m-1} \frac{\mathfrak{dag}_{n-m+1}}{\mathfrak{dag}_n} = \frac{z_0^{m-1}}{(m-1)!} \cdot \frac{2^{m(m-1)/2}}{2^{(m-1)n}} + O\left(\frac{1}{2^{(m-1)n}} \left(\frac{z_0}{z_1}\right)^n\right).$$

At the same time,

$$n^m \cdot \frac{\mathfrak{dag}_{n-m}}{\mathfrak{dag}_n} = O\left(\frac{1}{2^{mn}}\right).$$

Also, Robinson showed in [90] that $z_1 > 2z_0$. Therefore,

$$\frac{1}{2^{mn}} > \frac{1}{2^{(m-1)n}} \left(\frac{z_0}{z_1}\right)^n,$$

and as a consequence,

$$\binom{n}{m-1} \frac{\mathfrak{dag}_{n-m+1}}{\mathfrak{dag}_n} + O\left(n^m \cdot \frac{\mathfrak{dag}_{n-m}}{\mathfrak{dag}_n}\right) = \frac{z_0^{m-1}}{(m-1)!} \cdot \frac{2^{m(m-1)/2}}{2^{(m-1)n}} + O\left(\frac{1}{2^{mn}}\right).$$

□

10.4 The Erdős–Rényi model

Let $p \in (0,1)$ be a fixed real number and let $n \in \mathbb{N}$. The *Erdős–Rényi model* $G(n,p)$ of a random labeled graph with n vertices is constructed as follows. Each pair of vertices is joined by an edge independently with probability p. In particular, a labeled graph with n vertices and k edges is picked with probability

$$p^k q^{\binom{n}{2}-k},$$

where $q = 1-p$.

In this section, we are interested the following question. Let G be a random graph in the Erdős–Rényi model $G(n,p)$. Fix the parameter p and tend n to infinity. What is the asymptotic probability that G is connected and, more generally, for a fixed $m \in \mathbb{N}$, what is the asymptotic probability that G consists of exactly m connected components?

In 1959, Gilbert [41] provided the following estimation for this probability:

$$\mathbb{P}(G \text{ is connected}) = 1 - nq^{n-1} + O\big(n^2 q^{3n/2}\big).$$

We provide the whole asymptotic expansion and explain the meaning of the involved coefficients. In order to do so, first, we assign weights to the graphs within the Erdős–Rényi model. Second, we show that this assigning allows us to get gargantuan weighted species of graphs. Finally, we proceed with obtaining the asymptotic probabilities that a random graph G consists of a given number of connected components and providing the interpretation of the coefficients.

10.4.1 Definition and properties of the Erdős–Rényi model

Let G be a graph in $G(n,p)$ model. If we denote by $E(G)$ the set of its edges, then the probability to pick this particular graph G can be written in the following form

$$\mathbb{P}(G) = p^{|E(G)|} q^{\binom{n}{2}-|E(G)|} = \frac{p^{|E(G)|} q^{\binom{n}{2}-|E(G)|}}{(p+q)^{\binom{n}{2}}} = \frac{(p/q)^{|E(G)|}}{\big((p/q)+1\big)^{\binom{n}{2}}}.$$

If we also use the notation

$$\rho = \frac{p}{q} = q^{-1} - 1,$$

then, in terms of the parameter ρ, we have

$$\mathbb{P}(G) = \frac{\rho^{|E(G)|}}{(\rho+1)^{\binom{n}{2}}}.$$

This leads us to the following definition (see also Example 4.1.7).

Definition 10.4.1. The *weighted species $\mathcal{G}_w$ of graphs* is the species $\mathcal{G}$ of graphs with the assigned weight

$$w(G) = \rho^{|E(G)|}.$$

The weighted species $\mathcal{CG}_w$ of *connected graphs* is the subspecies $\mathcal{CG} \subset \mathcal{G}$ with the above defined weight function.

Remark 10.4.2. For every $n \in \mathbb{N}$, the total weight of $\mathcal{G}_w[n]$ is

$$|\mathcal{G}_w[n]| = (\rho+1)^{\binom{n}{2}} = q^{-\binom{n}{2}}.$$

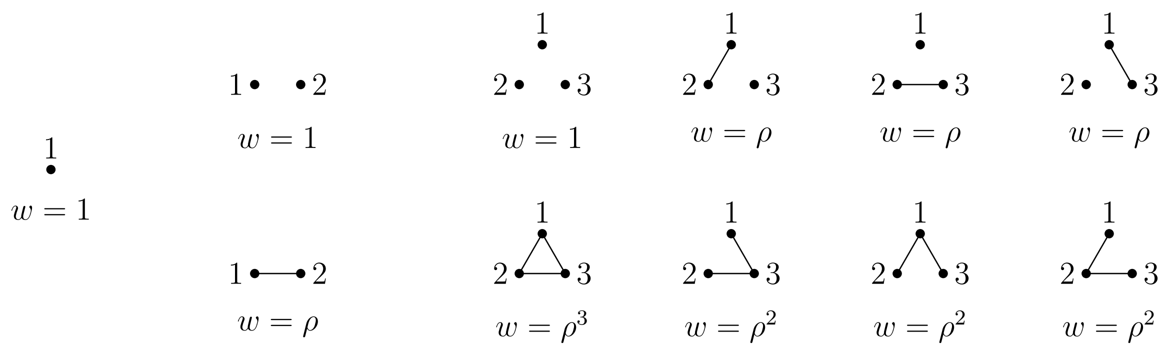


Figure 10.9: Weights of the labeled graphs whose size does not exceed 3.

Example 10.4.3. On Fig. 10.9, we indicate the graphs of size 3 and less, together with their weights.

Lemma 10.4.4. *The weighted species $\mathcal{G}_w$ of graphs and $\mathcal{CG}_w$ of connected graphs satisfy the relation*

$$\mathcal{G}_w = \mathcal{E} \circ \mathcal{CG}_w.$$

Proof. This follows from the construction of the Erdős–Rényi model and from the definition of the weight. □

Lemma 10.4.5. *The weighted species $\mathcal{G}_w$ of graphs is gargantuan.*

Proof. According to Definition 4.2.20, we need to show that the sequence

$$a_n = \frac{|\mathcal{G}_w[n]|}{n!} = \frac{(\rho+1)^{\binom{n}{2}}}{n!}$$

is gargantuan. For the case $\rho = 1$, which corresponds to $p = 1/2$, we have seen the proof in Example 2.1.7 (see also Lemma 10.1.5). In the general case, the proof is pretty similar, so we just remind the key steps. The main idea is to apply Lemma 2.1.4. In order to do so, we need to check two conditions. The first of them reads

$$\frac{n a_{n-1}}{a_n} = \frac{n}{(1+\rho)^n} \to 0,$$

as $n \to \infty$. For the second, verify that the sequence

$$x_k = a_k a_{n-k}$$

is decreasing for $k < n/2$. Indeed, for large n, we have

$$\frac{x_{k+1}}{x_k} \leqslant 1 \quad \Leftrightarrow \quad \frac{(1+\rho)^{k+1}}{(k+1)} \leqslant \frac{(1+\rho)^{n-k}}{(n-k)} \quad \Leftrightarrow \quad k+1 \leqslant n-k,$$

since the function

$$f(x) = \frac{(1+\rho)^x}{x}$$

is increasing for large x (see Remark 2.1.6). Hence, Lemma 2.1.4 is applicable and (a_n) is gargantuan. □

10.4.2 Asymptotic probabilities in the Erdős–Rényi model

Theorem 10.4.6. *Let $p \in (0,1)$. The asymptotic probability that a random graph G in the Erdős–Rényi model $G(n,p)$ is connected satisfies*

$$\mathbb{P}(G \text{ is connected}) \approx 1 - \sum_{k\geqslant 1} P_k(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}}, \tag{10.26}$$

where $q = 1-p$, $\rho = p/q$,

$$P_k(\rho) = \sum_{H \in (\mathcal{G}_w)_k} (-1)^{\pi_0(H)-1} w(H)$$

is the sum of weights of graphs $H \in (\mathcal{G}_w)_k$ taken with the sign $(-1)^{\pi_0(H)-1}$, where $\pi_0(H)$ is the number of connected components of the graph H.

Proof. According to Lemma 10.4.4, the weighted species $\mathcal{G}_w$ of graphs and $\mathcal{CG}_w$ of connected graphs are linked with the relation

$$\mathcal{G}_w = \mathcal{E} \circ \mathcal{CG}_w.$$

Due to Lemma 10.4.5, $\mathcal{G}_w$ is gargantuan. Hence, we can apply Theorem 8.3.4, which gives us

$$\mathbb{P}(G \in \mathcal{CG}_w) \approx 1 - \sum_{k\geqslant 1} P_k \cdot \binom{n}{k} \cdot \frac{|\mathcal{G}_w[n-k]|}{|\mathcal{G}_w[n]|},$$

where $P_k = P_k(\rho)$ is the total weight of $(\mathcal{E}^{-1} \circ \mathcal{CG}_w)[k]$. Since, $|\mathcal{G}_w[n]| = q^{-\binom{n}{2}}$, we have

$$\frac{|\mathcal{G}_w[n-k]|}{|\mathcal{G}_w[n]|} = q^{\binom{n}{2}-\binom{n-k}{2}} = \frac{q^{nk}}{q^{k(k+1)/2}}.$$

On the other hand, due to Lemma 8.3.1,

$$\mathcal{E}^{-1} \equiv \mathbf{1} - \mathcal{E}_1 + \mathcal{E}_2 - \mathcal{E}_3 + \dots$$

As long as the composition $\mathcal{E}_m \circ \mathcal{CG}_w = \mathcal{CG}_w^{\{m\}}$ consists of weighted graphs with m connected components, the total weight of $(\mathcal{E}_m \circ \mathcal{CG}_w)[k]$ is

$$\left|(\mathcal{E}_m \circ \mathcal{CG}_w)[k]\right| = \sum_{\substack{H \in (\mathcal{G}_w)_k \\ \pi_0(H)=m}} (-1)^{m-1} w(H).$$

Moving on to the sum over all $m \in \mathbb{N}$, we get exactly

$$P_k = \left|(\mathcal{E}^{-1} \circ \mathcal{CG}_w)[k]\right| = \sum_{H \in (\mathcal{G}_w)_k} (-1)^{\pi_0(H)-1} w(H).$$

□

Theorem 10.4.7. *Let $p \in (0,1)$ and $m \in \mathbb{N}$. The asymptotic probability that a random graph G in the Erdős–Rényi model $G(n,p)$ has m connected components satisfies*

$$\mathbb{P}(G \text{ has } m \text{ connected components}) \approx \sum_{k\geqslant 0}^{r-1} P_k^{\{m\}}(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}}, \tag{10.27}$$

where $q = 1 - p$, $\rho = p/q$,

$$P_k^{\{m\}}(\rho) = \sum_{H \in (\mathcal{G}_w)_k} (-1)^{\pi_0(H)-m} \binom{\pi_0(H)}{m-1} w(H)$$

and $\pi_0(H)$ *is the number of connected components of the graph* H*. In particular,*

$$\mathbb{P}(G \text{ has } m \text{ connected components}) = \binom{n}{m-1} \cdot \frac{q^{n(m-1)}}{q^{m(m-1)/2}} + O(n^m q^{mn}).$$

Proof. The proof is proceeding the same way as for Theorem 10.4.7, but we use Theorem 8.3.6 instead of Theorem 8.3.4. To obtain the leading term of (10.27), note that it comes from the smallest graph H such that $\pi_0(H) = m - 1$, that is, from the totally disconnected graph with $m - 1$ vertices. □

Remark 10.4.8. The polynomials $P_k^{\{m\}}(\rho)$ for small values of m and k are listed in Table 10.1. In particular,

$$\mathbb{P}(G \text{ is connected}) = 1 - \binom{n}{1} q^{n-1} - (\rho - 1)\binom{n}{2} q^{2n-3} - (\rho^3 + 3\rho^2 - 3\rho + 1)\binom{n}{3} q^{3n-6} + O(n^4 q^{4n}).$$

$P_k^{\{m\}}(\rho)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$m=1$	1	-1	$-\rho+1$	$-\rho^3 - 3\rho^2 + 3\rho - 1$	$-\rho^6 - 6\rho^5 - 15\rho^4 - 12\rho^3 + 15\rho^2 - 6\rho + 1$
$m=2$	0	1	$\rho - 2$	$\rho^3 + 3\rho^2 - 6\rho + 3$	$\rho^6 + 6\rho^5 + 15\rho^4 + 8\rho^3 - 30\rho^2 + 18\rho - 4$
$m=3$	0	0	1	$3\rho - 3$	$4\rho^3 + 15\rho^2 - 18\rho + 6$
$m=4$	0	0	0	1	$6\rho - 4$
$m=5$	0	0	0	0	1

Table 10.1: Polynomials $P_k^{\{m\}}(\rho)$ for $k \leqslant 4$ and $m \leqslant 5$.

Chapter 11

Surfaces

In this chapter, we discuss the applications of our main theorems to various surface models. In particular, Sections 11.1 and 11.2 are devoted to square-tiled surfaces and combinatorial map model, respectively. The common idea for these two sections is to obtain a surface by gluing together a collection of polygons in accordance with certain rules. In general, the resulting surface can be disconnected, but the probability of this event tends to 0, as the size of surface tends to infinity. We establish the asymptotic expansion for the probabilities that a random surface is connected or consists of a fixed number of connected components and indicate combinatorial meaning of the involved coefficients.

Both connected square-tiled surfaces and combinatorial maps can be thought as particular cases of more general objects, namely, constellations. In Section 11.3, we study the probability to obtain a constellation by picking a random tuple of permutations. Also, we discuss the model of $(D+1)$-colored graphs whose asymptotic probability to be connected shows a similar behavior.

11.1 Square-tiled surfaces

This section is devoted to *square-tiled surfaces*, also known as *origamis*. Roughly speaking, each object, within the considered model, is obtained by gluing several unit squares such that the result is a so-called translation surface (possibly, not connected), i.e. a surface with the globally defined notions of north, west, south and east. According to Zmiaikou [105], the concept of connected square-tiled surfaces appeared in 1970-1980s in works of Thurston [96] and Veech [97] on the moduli spaces of curves. It looks like that the name "square-tiled surface" was proposed by Eskin and used for the first time by Zorich [106] for counting the Teichmüller volumes of the moduli spaces of Abelian differentials. The name "origami" is ascribed to Lochak [61] who studied the moduli spaces of curves.

We are interested in the asymptotic probability that a surface randomly glued from n unit squares is connected. From the algebraic point of view this question is equivalent to the following. Pick a random pair $(\sigma, \tau) \in S_n^2$ of permutations. What is the probability t_n that the subgroup $\langle \sigma, \tau \rangle$ is transitive? For the first time, this question was studied by Dixon [27], who showed in 1969 that

$$t_n = 1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

In 2005, Dixon [28] improved his own result and obtained the full asymptotics expansion in the form

$$t_n \sim 1 - \frac{1}{n} - \frac{1}{n^2} - \frac{4}{n^3} - \frac{23}{n^4} - \frac{171}{n^5} - \frac{1542}{n^6} - \dots.$$

Along the way, he proved that

$$t_n \sim 1 - \frac{1}{(n)_1} - \frac{1}{(n)_2} - \frac{3}{(n)_3} - \frac{13}{(n)_4} - \frac{71}{(n)_5} - \frac{461}{(n)_6} - \ldots.$$

and it followed from the work of Comtet [21] that the coefficients in numerators counted indecomposable permutations (see also [23]).

In this section, we explain how to obtain the latter result within the general framework. Also, we establish the asymptotic expansion for the probability that a random square-tiled surface consists of a fixed number of connected components (in terms of group action, we find the probability that a random pair of permutations generates a group whose action has a fixed number of orbits). As usual, we indicate the combinatorial interpretation of coefficients as well.

11.1.1 Definitions and properties

Definition 11.1.1. A *square-tiled surface* or *origami* is a finite collection of labeled unit squares whose sides are identified in such a way that

- the right side of each square is identified to the left side of some square (possibly, the same one),
- the top side of each square is identified to the bottom side of some square.

The *size* of a square-tiled surface is the number of unit squares constituting this surface.

Notation 11.1.2. We denote by $\mathcal{O}$ the labeled combinatorial class of (possibly, not connected) square-tiled surfaces. The subclass of connected ones is designated by $\mathcal{CO}$. We designate the corresponding counting sequences as $(\mathfrak{o}_n)$ and $(\mathfrak{co}_n)$, respectively.

Remark 11.1.3. We can identify square-tiled surfaces of size n with labeled directed graphs of size n with colored edges. To do this, we join vertices i and j of the graph by a red (respectively, blue) edge $\overrightarrow{ij}$, if and only if the right (respectively, top) side of the square i is identified with the left (respectively, bottom) side of the square j. Note that this graph can have loops and multiple edges, but any vertex has the ingoing and outgoing degrees of each color equal to one.

Each square-tiled surface of size $n \in \mathbb{N}$ can be encoded by a pair of permutations $(\sigma_v, \sigma_h) \in S_n^2$. These two permutations determine the identification rule for vertical and horizontal sides of unit squares, respectively. More precisely, unit squares bear labels from 1 to n, and

- $\sigma_v(i) = j$, if the right side of the i-th square is identified to the left side of the j-th square,
- $\sigma_h(i) = j$, if the top side of the i-th square is identified to the bottom side of the j-th square.

As a consequence, the counting sequence $(\mathfrak{o}_n)$ of the class $\mathcal{O}$ satisfies

$$\mathfrak{o}_n = (n!)^2.$$

Since the class $\mathcal{ML}(2)$ has the same counting sequence, we get the following lemma.

Lemma 11.1.4. *The labeled combinatorial classes $\mathcal{O}$ of square-tiled surfaces and $\mathcal{ML}(2)$ of pairs of linear orders are combinatorially isomorphic.*

Corollary 11.1.5. *The labeled combinatorial class $\mathcal{O}$ of square-tiled surfaces is gargantuan.*

Proof. Follows directly from Lemmas 9.1.17 and 11.1.4. □

Lemma 11.1.6. *The labeled combinatorial classes $\mathcal{O}$ of square-tiled surfaces and $\mathcal{CO}$ of connected square-tiled surfaces satisfy*

$$\mathcal{O} = \mathrm{SET}(\mathcal{CO}).$$

Proof. The proof proceed in the same way as the proof of Lemma 10.1.6. □

Notation 11.1.7. For any $m \in \mathbb{N}$, we denote by

$$\mathcal{CO}^{\{m\}} = \mathrm{SET}_m(\mathcal{CO})$$

the labeled combinatorial class consisting of square-tiled surfaces that have exactly m connected components. The corresponding counting sequence is $\big(\mathfrak{co}_n^{\{m\}}\big)$.

11.1.2 Asymptotic expansions

Theorem 11.1.8. *The asymptotic probability that a square-tiled surface S obtained by gluing n unit squares randomly is connected satisfies*

$$\mathbb{P}\big(S \text{ is connected}\big) \approx 1 - \sum_{k\geqslant 1} \frac{\mathfrak{ip}_k}{(n)_k}. \tag{11.1}$$

Proof. Apply Theorem 7.2.1 to the labeled combinatorial class $\mathcal{O} \cong \mathcal{ML}(2)$. It is possible, since

- $\mathcal{O} = \mathrm{SET}(\mathcal{CO})$ by Lemma 11.1.6,
- $\mathcal{ML}(2) = \mathrm{SEQ}\big(\mathcal{IML}(2)\big)$ by Lemma 9.1.18,
- $\mathcal{O}$ is gargantuan by Corollary 11.1.5.

In the case in hand, asymptotic formula (7.9) has the form

$$\mathbb{P}(S \text{ is SET-irreducible}) \approx 1 - \sum_{k\geqslant 1} \mathfrak{iml}_k(2) \cdot \binom{n}{k} \cdot \frac{\mathfrak{o}_{n-k}}{\mathfrak{o}_n}.$$

Taking into the account that $\mathfrak{iml}_k(2) = k! \cdot \mathfrak{ip}_k$ by Lemma 9.1.20 and

$$\binom{n}{k} \cdot \frac{\mathfrak{o}_{n-k}}{\mathfrak{o}_n} = \frac{n!}{k!(n-k)!} \cdot \frac{\big((n-k)!\big)^2}{(n!)^2} = \frac{1}{k! \cdot (n)_k},$$

we get exactly (11.1). □

Theorem 11.1.9. *The asymptotic probability that a square-tiled surface S obtained by gluing n unit squares randomly has m connected components satisfies*

$$\mathbb{P}(S \text{ has } m \text{ connected components}) \approx \sum_{k\geqslant 0} \frac{\alpha_k^{\{m\}}}{k! \cdot (n)_k}, \tag{11.2}$$

where

$$\alpha_k^{\{m\}} = \sum_{s=m-1}^{\infty} (-1)^{s+m-1} \binom{s}{m-1} \mathtt{co}_k^{\{s\}}.$$

In particular,

$$\mathbb{P}(S \text{ has } m \text{ connected components}) = \frac{1}{(m-1)!} \cdot \frac{1}{(n)_{m-1}} + O\left(\frac{1}{n^m}\right).$$

Proof. We have seen that, by Corollary 11.1.5 and Lemma 11.1.6, the class $\mathcal{O}$ is gargantuan and $\mathcal{O} = \mathrm{SET}(\mathcal{CO})$. Hence, we can apply Theorem 7.3.1, which gives us exactly (11.2). To get the main term of the asymptotics, it is sufficient to apply Corollary 7.3.4. □

11.2 Combinatorial map model

In this section, we discuss the concept of connectedness within different models of random surfaces that have relations with combinatorial maps. Our aim is to provide asymptotic expansions that a random surface is connected or consists of a fixed number of connected components.

The simplest version of such a random surface model was introduced in 2004 by Brooks and Makover [14] in order to study the "typical" Riemann surfaces with high genus. Almost at the same time, the same model was independently investigated by Pippenger and Schleich [83] who were motivated by the needs of quantum gravity. Pippenger and Schleich showed that a random gluing of n equal triangles formed a connected surface with probability

$$1 - \frac{5}{18n} + O\left(\frac{1}{n^2}\right) \tag{11.3}$$

and studied their topological characteristics.

Next, came the time for generalizations. First, Gamburd [40] replaced triangles by p-gons for the fixed integer $p \geqslant 3$. Second, Chmutov and Pittel [19] extended the model to polygons with different number of sides. Namely, they glued together l polygons of total perimeter n and showed that the probability to get a connected surface was

$$1 - O\left(\frac{1}{n}\right).$$

Finally, in 2019, Budzinski, Curien and Petri [15] considered a model with 1-gons and 2-gons permitted, and found that the probability to obtain a connected surface was

$$1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right). \tag{11.4}$$

The main goal of the above authors was to study topological characteristics of the proposed models, such as Euler characteristic, genus, diameter, etc.

In all the mentioned models, glued polygons are assumed to be oriented and the identified sides are oriented opposite-wise. In the model introduced by Budzinski, Curien and Petri a collection of polygons of total perimeter n, together with the identification of sides, corresponds to the pair of permutations $(\sigma, \alpha) \in S_n^2$, such that α is a perfect matching. Connected surfaces within this model are in one-to-one correspondence with combinatorial maps. In this case, the family of probability measures are uniform measures on $\mathcal{P}_n \times \mathcal{M}_n$. For this reason, we call it *combinatorial map model.*

The model of Chmutov and Pittel is slightly different. They fix the set of all possible numbers of sides J and all the numbers l_j of polygons with $j \in J$ sides. This choice determines the cycle structure of the permutation σ completely. The probability measures in this case are uniform on the corresponding conjugacy classes. Note that the models of Gamburd and of Pippenger and Schleich are the particular cases for $J = \{k\}$ and $J = \{3\}$, respectively.

We start our exposition with drawing parallels between surfaces glued from polygons and combinatorial maps. Discussing combinatorial maps, we follow the book of Lando and Zvonkin [56] which we could recommend to the reader for an extensive account of this topic. We show that any surface is determined by the pair $(\sigma, \alpha) \in \mathcal{P}_n \odot \mathcal{M}_n$ and that connected surfaces correspond to combinatorial maps. In the second subsection, we establish asymptotics for the combinatorial map model of Budzinski, Curien and Petri and show that the involved coefficients count indecomposable perfect matchings. The last subsection is devoted to the asymptotics within the models of Pippenger and Schleich and of Gamburd.

11.2.1 Surfaces and combinatorial maps

Let us start with an informal construction of the combinatorial map model introduced by Budzinski, Curien and Petri [15]. Our initial object is a collection of polygons of total perimeter n whose sides are labeled with integers from 1 to n (1-gons and 2-gons are permitted). We represent each polygon with k sides as a vertex on the plane with k half-edges attached to this vertex. Half-edges are labeled with the same labels as the corresponding polygon sides (we draw a label to the left of the half-edge, see Fig. 11.1). The cyclic order of half-edges is important and corresponds to one of polygon sides. Taking half-edges in the counter-clockwise direction, we associate with the initial collection of polygons the permutation $\sigma \in S_n$. For example, the permutation $\sigma = (1, 3, 5, 6)(2)(4)$ is associated to the collection of two 1-gons and a square shown on Fig. 11.1.

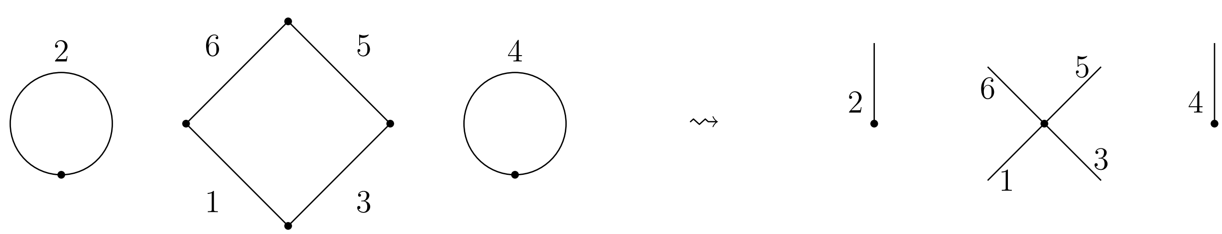


Figure 11.1: From a collection of polygons to vertices and half-edges.

Gluing sides of polygons together corresponds to identifying half-edges. This process is naturally encoded by the perfect matching α, that is, a convolution without fixed points. For example, the gluing depicted on Fig. 11.2 corresponds to $\alpha = (1, 2)(3, 4)(5, 6)$. If the resulting surface is connected, then the corresponding object is a *combinatorial map.* That is why we call the described model of surfaces the *combinatorial map model.* In terms of permutations, connectedness of a surface means that the subgroup $\langle \sigma, \alpha \rangle \subset S_n$ acts transitively on $[n]$.

Every combinatorial map is uniquely determined by two permutations σ and α, such that α is a perfect matching. On the other hand, the converse is false: it is necessary to additionally require the underlying graph to be connected. Also, note that a combinatorial map is not determined by the underlying graph with labels in a unique way. We illustrate this fact on Fig. 11.3, presenting two

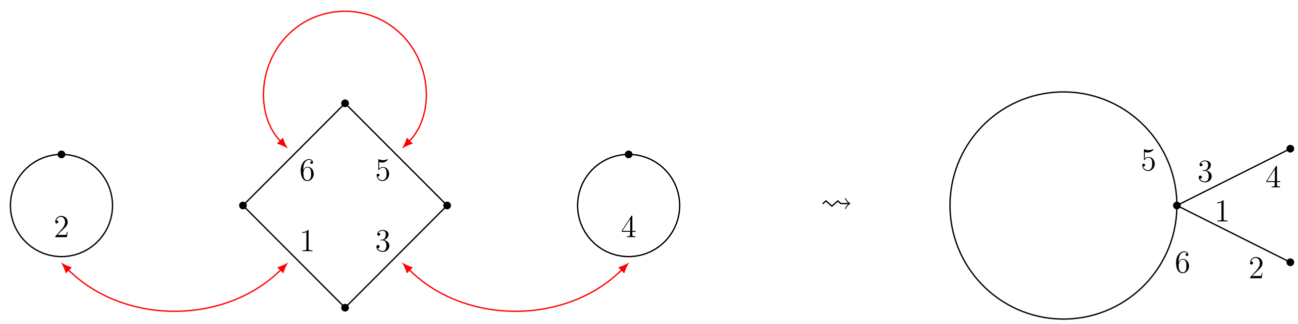


Figure 11.2: From gluing polygons to combinatorial maps.

different combinatorial maps with the same underlying graph. The combinatorial map depicted on the left-hand side corresponds to the pair

$$\sigma = (1,3,5,6)(2)(4) \qquad \text{and} \qquad \alpha = (1,2)(3,4)(5,6),$$

while the one on the right-hand side is determined by

$$\sigma = (1,5,3,6)(2)(4) \qquad \text{and} \qquad \alpha = (1,2)(3,4)(5,6).$$

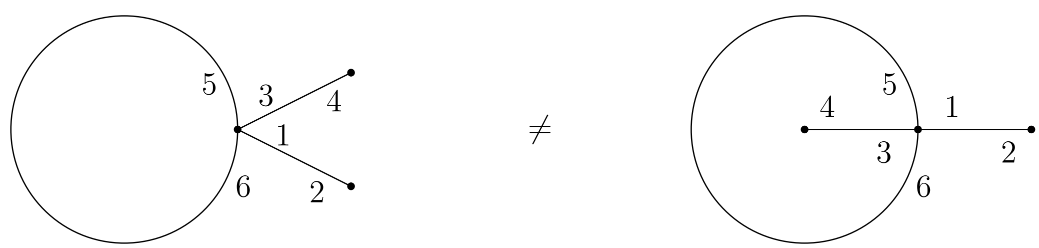


Figure 11.3: Different combinatorial maps with the same underlying graph.

We invite the reader to pay attention the way we draw the labels on the above figures. If we start at a vertex and move along a half-edge, then we draw its label on the left-hand side. Gluing two half-edges, we get an edge marked with two labels which are located on different sides of the edge (see Fig. 11.4). This allows us to associate one more permutation with a combinatorial map.

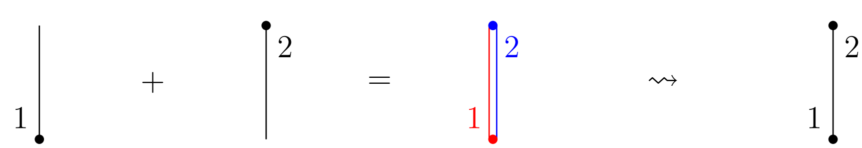


Figure 11.4: Nature of half-edges.

Indeed, each combinatorial map divide the plane into several *faces*, i.e. connected parts of the plane. The faces are bounded by half-edges, and hence, can be represented as cycles of half-edges taken in the counter-clockwise direction. We denote by φ the product of these cycles. For example, for combinatorial maps depicted on Fig. 11.3, φ are equal to

$$(1,2,6,3,4)(5) \qquad \text{and} \qquad (1,2,6)(3,4,5).$$

Given a combinatorial map, the permutation φ is determined by σ and α by

$$\varphi = \alpha^{-1}\sigma^{-1}.$$

This observation lead us to the following definition.

Definition 11.2.1. A *combinatorial map* is a triple $(\sigma, \alpha, \varphi) \in S_n^3$, where $n \in \mathbb{N}$, such that

- α is a perfect matching;
- the group $\langle \sigma, \alpha, \varphi \rangle$ acts transitively on the set $[n]$;
- $\sigma\alpha\varphi = \mathrm{id}$ is the identity permutation.

Notation 11.2.2. We denote by $\mathcal{CM}$ and $(\mathfrak{cm}_n)$ the labeled combinatorial class of combinatorial maps and its counting sequence, respectively.

Lemma 11.2.3. *The set of the class $\mathcal{CM}$ of combinatorial maps is combinatorially isomorphic to the subclass $\mathcal{MLM}(2) \subset \mathcal{ML}(2)$:*

$$\mathrm{SET}(\mathcal{CM}) \cong \mathcal{MLM}(2).$$

Proof. The counting sequence of the class $\mathcal{MLM}(2)$ is $\mathfrak{mlm}_n(2) = n! \cdot \mathfrak{m}_n$ (see Remark 9.2.13). In particular, $\mathcal{MLM}(2)$ is combinatorially isomorphic to the class $\mathcal{P} \odot \mathcal{M}$ of pairs of permutations (σ, α) of the same size, such that α is a perfect matching. Any pair of such type represents a gluing of collection of polygons which can be interpreted as a graph with additional structure. If the graph is connected, then the pair determines a combinatorial map. Hence, to establish the desired result, we can use the reasoning of the proof of Lemma 10.1.6. □

11.2.2 Asymptotics for the combinatorial map model

In this section, we study random surfaces within the combinatorial map model of Budzinski, Curien and Petri. We establish the asymptotic expansion for the probability that a random surface is connected or consists of a fixed number of connected components. Given $n \in \mathbb{N}$, we suppose that a surface is generated by a pair of permutations $(\sigma, \alpha) \in \mathcal{P}_n \odot \mathcal{M}_n$ taken at random uniformly. Then the probability that this surface is connected corresponds to the fact that the group $\langle \sigma, \alpha \rangle$ acts transitively on $[n]$ or, in other words, that $(\sigma, \alpha, \alpha^{-1}\sigma^{-1})$ is a combinatorial map. The asymptotics of this probability as the following form.

Theorem 11.2.4. *The asymptotic probability that a random surface S obtained by gluing polygons of total perimeter $2n$ within the combinatorial map model is connected satisfies*

$$\mathbb{P}\big(S \textit{ is connected}\big) \approx 1 - \sum_{k \geqslant 1} \mathfrak{im}_{2k} \cdot \frac{\big(2(n-k)-1\big)!!}{(2n-1)!!}. \tag{11.5}$$

Proof. Let us apply Theorem 7.4.1 to the labeled combinatorial class $\mathcal{MLM}(2) \cong \mathcal{P} \odot \mathcal{M}$. It is possible, since

- $\mathcal{MLM}(2) \cong \mathrm{SET}(\mathcal{CM})$ by Lemma 11.2.3,
- $\mathcal{MLM}(2) = \mathrm{SEQ}\big(\mathcal{IMLM}(2)\big)$ by Remark 9.2.13,

- $\mathcal{MLM}(2)$ is 2-gargantuan by Remark 9.2.13.

In the case in hand, asymptotic formula (7.12) turns into

$$\mathbb{P}(S \text{ is SET-irreducible}) \approx 1 - \sum_{k\geqslant 1} \mathfrak{imlm}_{2k} \cdot \binom{2n}{2k} \cdot \frac{\mathfrak{mlm}_{2(n-k)}}{\mathfrak{mlm}_{2n}}.$$

Taking into the account that $\mathfrak{mlm}_{2n} = (2n)!(2n-1)!!$, $\mathfrak{imlm}_{2k} = (2k)! \cdot \mathfrak{im}_{2k}$ and

$$\binom{2n}{2k} \cdot \frac{\mathfrak{mlm}_{2(n-k)}}{\mathfrak{mlm}_{2n}} = \frac{1}{(2k)!} \cdot \frac{\big(2(n-k)-1\big)!!}{(2n-1)!!},$$

we get exactly (11.5). □

Corollary 11.2.5. *Let $(\sigma, \alpha) \in \mathcal{P}_{2n} \odot \mathcal{M}_{2n}$ be a random pair of permutations such that α is a perfect matching. Denote $\varphi = \alpha^{-1}\sigma^{-1}$. In this case, the asymptotic probability that $(\sigma, \alpha, \varphi)$ is a combinatorial map satisfies*

$$\mathbb{P}\big((\sigma, \alpha, \varphi) \textit{ is a combinatorial map}\big) \approx 1 - \sum_{k\geqslant 1} \mathfrak{im}_{2k} \cdot \frac{\big(2(n-k)-1\big)!!}{(2n-1)!!}. \tag{11.6}$$

Remark 11.2.6. The counting sequence $(\mathfrak{im}_n)$ of indecomposable perfect matchings is given by

$$(\mathfrak{im}_n) = 1,\ 2,\ 10,\ 74,\ 706,\ 8\,162,\ 110\,410,\ 1\,708\,394,\ 29\,752\,066,\ 576\,037\,442,\ \ldots$$

(A000698 in the OEIS [76]). This leads to the following asymptotic probability to obtain a connected surface by gluing random polygons of total perimeter n:

$$1 - \frac{1}{(n-1)} - \frac{2}{(n-1)(n-3)} - \frac{10}{(n-1)(n-3)(n-5)} - \frac{74}{(n-1)(n-3)(n-5)(n-7)} - \ldots$$

or, in the more standard form,

$$1 - \frac{1}{n} - \frac{3}{n^2} - \frac{19}{n^3} - \frac{191}{n^4} - \frac{2551}{n^5} - \ldots$$

In particular, as we could expect, the first two terms coincide with asymptotics (11.4) found by Budzinski, Curien and Petri.

Theorem 11.2.7. *The asymptotic probability that a random surface S obtained by gluing polygons of total perimeter $2n$ within the combinatorial map model has m connected components satisfies*

$$\mathbb{P}(S \textit{ has } m \textit{ connected components}) \approx \sum_{k\geqslant 0} \frac{\alpha_{2k}^{\{m\}}}{(2k)!} \cdot \frac{\big(2(n-k)-1\big)!!}{(2n-1)!!}, \tag{11.7}$$

where

$$\alpha_{2k}^{\{m\}} = \sum_{s=m-1}^{\infty} (-1)^{s+m-1} \binom{s}{m-1} \mathfrak{cm}_{2k}^{\{s\}}$$

and $\big(\mathfrak{cm}_n^{\{s\}}\big)$ is the counting sequence of the labeled combinatorial class $\mathcal{CM}^{\{s\}} = \mathrm{SET}_s(\mathcal{CM})$ that corresponds to surfaces consisting of s connected components. In particular,

$$\mathbb{P}(S \textit{ has } m \textit{ connected components}) = \frac{1}{(m-1)!} \cdot \frac{\big(2(n-m)+1\big)!!}{(2n-1)!!} + O\left(\frac{1}{n^m}\right).$$

Proof. We have seen that $\mathrm{SET}(\mathcal{CM}) \cong \mathcal{MLM}(2)$ and the class $\mathcal{MLM}(2)$ is 2-gargantuan (see Lemma 11.2.3 and Remark 9.2.13). Hence, we can apply Theorem 7.4.2. Taking into account the calculations made in the proof of Theorem 11.2.4, we obtain (11.7). The leading terms comes from (7.14). □

11.2.3 Asymptotics for other models

In this subsection, we shortly discuss the asymptotic expansions for the probability that a random surface generated within the model of Gamburd consists of a fixed number $m \in \mathbb{N}$ of components. In particular, for $m = 1$, we have the probability that a random surface is connected.

First of all, recall the model we are interested in. Fix an integer $p \geqslant 3$. Denote by $\mathcal{S}(p)$ and $\big(\mathfrak{s}_n(p)\big)$ the labeled combinatorial class of orientable surfaces generated by identifying sides of labeled p-gons together and its counting sequence, respectively. More precisely, to obtain an orientable surface of size n, we take n oriented p-gons and, respecting the orientation, identify their pn sides in pairs. In particular, for $p = 3$, we get the model studied by Pippenger and Schleich.

The behavior of the counting sequence $\big(\mathfrak{s}_n(p)\big)$ depends on the parity of p. If p is even, then

$$\mathfrak{s}_n(p) = (pn-1)!!,$$

and we can show with the help of Lemma 2.1.4 that the class $\mathcal{S}(p)$ is gargantuan. In the case when p is odd, we have

$$\mathfrak{s}_n(p) = \begin{cases} (pn-1)!!, & p \text{ is even,} \\ 0, & p \text{ is odd,} \end{cases}$$

and $\mathcal{S}(p)$ is 2-gargantuan. In both cases,

$$\mathcal{S}(p) = \mathrm{SET}\big(\mathcal{CS}(p)\big),$$

where $\mathcal{CS}(p)$ is the corresponding labeled combinatorial subclass of connected surfaces. As usual, for any positive integer m, we denote by

$$\mathcal{CS}^{\{m\}}(p) = \mathrm{SET}_m\big(\mathcal{CS}(p)\big)$$

the labeled combinatorial subclass of surfaces consisting of exactly m connected components. Also, we designate as $\big(\mathfrak{cs}_n(p)\big)$ and $\big(\mathfrak{cs}_n^{\{m\}}(p)\big)$ the counting sequences corresponding to the classes $\mathcal{CS}(p)$ and $\mathcal{CS}^{\{m\}}(p)$, respectively. All these observations allow us to apply Theorems 7.3.1 and 7.4.2 for obtaining the following result.

Theorem 11.2.8. *If $p \geqslant 4$ is even, then the asymptotic probability that a surface S obtained by gluing n unit p-gons randomly has m connected components satisfies*

$$\mathbb{P}(S \textit{ has } m \textit{ connected components}) \approx \sum_{k \geqslant 0} \alpha_k^{\{m\}} \cdot \binom{n}{k} \cdot \frac{(p(n-k)-1)!!}{(pn-1)!!}. \tag{11.8}$$

If $p \geqslant 3$ is odd, then the asymptotic probability that a surface S obtained by gluing $2n$ unit p-gons randomly has m connected components satisfies

$$\mathbb{P}(S \textit{ has } m \textit{ connected components}) \approx \sum_{k \geqslant 0} \alpha_{2k}^{\{m\}} \cdot \binom{2n}{2k} \cdot \frac{(2p(n-k)-1)!!}{(2pn-1)!!}, \tag{11.9}$$

In both cases,

$$\alpha_k^{\{m\}} = \sum_{s=m-1}^{\infty} (-1)^{s+m-1} \binom{s}{m-1} \mathfrak{cs}_k^{\{s\}}$$

and $\mathcal{CS}^{\{s\}}$ is the subclass of $\mathcal{S}$ that comprise surfaces with exactly s connected components.

Example 11.2.9. For $p = 3$ and $m = 1$, the first few elements of the sequence $\left(\alpha_{2k}^{\{m\}}\right)$ are

$$\left(\alpha_{2k}^{\{m\}}\right) = 15,\ 9\,045,\ 30\,085\,425,\ 282\,543\,711\,975,\ \ldots$$

Hence, the asymptotic probability that a random surface obtained by gluing n unit triangles is

$$1 - \binom{n}{2}\frac{15}{(3n-1)(3n-3)(3n-5)} - \binom{n}{4}\frac{9045}{(3n-1)(3n-3)(3n-5)(3n-7)(3n-9)(3n-11)} - \ldots$$

This also can be rewritten in a standard form as

$$1 - \frac{5}{18n} - \frac{695}{648n^2} - \ldots$$

In particular, we can see the first two terms of (11.3) found by Pippenger and Schleich.

11.3 Higher dimensional models

This section is devoted to two higher dimensional models. First of them, constellations, can be considered as a generalization for both square-tiled surfaces and combinatorial maps. The first ideas leading to the model date back to the eighteenth century and concern some classical results, for instance, Euler's polyhedron formula for the numbers of vertices, edges and faces of a polyhedron. Nowadays, combinatorial maps and constellations are objects of interest not only for combinatorialists and topologists, but also for the specialists in various domains that are often far from each other, such as algebraic number theory, quantum physics, Galois theory. For further information and bibliography, we refer the reader to [56].

The second model, $(D+1)$-colored graphs, appeared in the work of Pezzana and his group [81, 82], regarding piecewise topological structures. Their main goal was to find a "minimal" atlas for topological manifolds, which led them to the concept of "contracted triangulation" and to the graph-theoretical tool that they called "crystallization" (see also survey [35]). An important subclass to study was bipartite $(D+1)$-colored graphs, since they encode D-dimensional orientable colored complexes (see [39]). In the last decades, $(D+1)$-colored graphs attracted attention of theoretical physicists with respect to quantum gravity and colored tensor models (see review [43]). The topological characteristics of $(D+1)$-colored graphs with a fixed number of vertices n was studied by Carrance [17]. In particular, she showed that a random $(D+1)$-colored graph is connected with probability

$$1 - \frac{1}{n^{D-1}} + O\left(\frac{1}{n^{2(D-1)}}\right). \tag{11.10}$$

In the first part of this section, we establish the asymptotic probability that a random tuple of permutations of the same size forms a constellation, as the size tends to infinity, and provide a combinatorial interpretation for its terms. In the second part, we discuss the asymptotic probabilities that a random $(D+1)$-colored graph is connected or consists of a fixed number of connected components. Also, we do the same for the subclass of bipartite $(D+1)$-colored graphs.

11.3.1 Asymptotic expansions for constellations

Definition 11.3.1. Let $N \in \mathbb{N}$. An N-tuple $(\sigma_1, \ldots, \sigma_N) \in S_n^N$ of permutations of the same size n is called N-*constellation* (or simply *constellation*), if the following two conditions hold:

- the group $\langle \sigma_1, \dots, \sigma_N \rangle$ acts transitively on the set $[n]$,
- the product of σ_k is the identity permutation, i.e. $\sigma_1 \dots \sigma_N = \mathrm{id}$.

Example 11.3.2. For $N = 1$, the only constellation is (id) and it exists if and only if $n = 1$. For $N = 2$, every constellation has the form (σ, σ^{-1}), where σ is a cyclic permutation. As a particular example of 3-constellation, we can consider any combinatorial map.

Each N-constellation is uniquely determined by the first $(N-1)$ permutations. Hence, we could consider constellations as a subclass of the labeled combinatorial class $\mathcal{MP}(N-1)$ of multipermutations of rank $(N-1)$. In order to understand when a tuple of permutations form a constellation, let us come back to square-tiled surfaces. We have seen that each square-tiled surface is determined by two permutations σ_v and σ_h of the same size, and that the surface is connected if and only if the group

$$\langle \sigma_v, \sigma_h \rangle = \langle \sigma_v, \sigma_h, \sigma_h^{-1} \sigma_v^{-1} \rangle$$

is transitive. This facts allows to establish a bijection between connected square-tiled surfaces and 3-constellations. The same way, we can consider the correspondence between N-constellations of size n and connected translation manifolds obtained by gluing n unit $(N-1)$-dimensional cubes. Each gluing is determined by $(N-1)$ permutations $(\sigma_1, \dots, \sigma_{N-1})$ of size n each, and the result is connected if and only if

$$(\sigma_1, \dots, \sigma_{N-1}, \sigma_{N-1}^{-1} \dots \sigma_1^{-1})$$

is an N-constellation.

Notation 11.3.3. We denote by $\mathcal{CN}(N)$ the labeled combinatorial class of N-constellations (that correspond to connected translation manifolds obtained by gluing $(N-1)$-dimensional cubes). We designate its counting sequence as $\big(\mathfrak{cn}_n(N)\big)$.

Lemma 11.3.4. *Let $N \geqslant 3$. The labeled combinatorial classes $\mathcal{CN}(N)$ of N-constellations and $\mathcal{ML}(N-1)$ of multiple linear orders of rank $(N-1)$ satisfy the following relation:*

$$\mathrm{SET}\big(\mathcal{CN}(N)\big) \cong \mathcal{ML}(N-1).$$

Proof. The class $\mathrm{SET}\big(\mathcal{CN}(N)\big)$ is combinatorially isomorphic to the labeled combinatorial class of all possible translation gluings of $(N-1)$-dimensional cubes. Since there is a bijection between such gluings of n unit $(N-1)$-dimensional cubes and S_n^{N-1}, the counting sequence of $\mathrm{SET}\big(\mathcal{CN}(N)\big)$ is $(n!)^{N-1}$. To finish the proof, note that the class $\mathcal{ML}(N-1)$ has the same counting sequence. □

Remark 11.3.5. Another proof of Lemma 11.3.4 can be derived from Dixon [27] and [28].

Theorem 11.3.6. *Let $N \geqslant 3$. The asymptotic probability that a random collection $\sigma = (\sigma_1, \dots, \sigma_N)$ of N permutations of size n is a constellation satisfies*

$$\mathbb{P}\big(\sigma \text{ is a constellation}\big) \approx 1 - \sum_{k \geqslant 1} \frac{\mathfrak{imp}_k(N-2)}{\big((n)_k\big)^{N-2}}. \tag{11.11}$$

Proof. Let us apply Theorem 7.2.1 to the labeled combinatorial class $\mathcal{ML}(N-1)$. It is possible, since $N \geqslant 3$, and hence,

- $\mathcal{ML}(N-1) \cong \mathrm{SET}\big(\mathcal{CN}(N)\big)$ by Lemma 11.3.4,

- $\mathcal{ML}(N-1) = \operatorname{SEQ}\big(\mathcal{IML}(N-1)\big)$, see Lemma 9.3.20,
- $\mathcal{ML}(N-1)$ is gargantuan, see Lemma 9.3.19.

In the case in hand, asymptotic formula (7.9) turns into

$$\mathbb{P}(\sigma \text{ is SET-irreducible}) \approx 1 - \sum_{k\geqslant 1} \mathfrak{iml}_k(N-1) \cdot \binom{n}{k} \cdot \frac{\mathfrak{ml}_{n-k}(N-1)}{\mathfrak{ml}_n(N-1)}.$$

Taking into the account that $\mathfrak{iml}_k(N-1) = k! \cdot \mathfrak{imp}_k(N-2)$ by Lemma 9.3.22 and

$$\binom{n}{k} \cdot \frac{\mathfrak{ml}_{n-k}(N-1)}{\mathfrak{ml}_n(N-1)} = \frac{n!}{k!(n-k)!} \cdot \frac{\big((n-k)!\big)^{(N-1)}}{(n!)^{(N-1)}} = \frac{1}{k! \cdot \big((n)_k\big)^{(N-2)}},$$

we get exactly (11.11). □

Remark 11.3.7. The question whether a collection of permutations generates a transitive group was studied by Dixon [28]. He showed the way to obtain the asymptotic expression for the probability that the generated group is transitive which was similar to (11.11). However, the meaning of the coefficients was not known.

11.3.2 Asymptotic expansions for colored graphs

Definition 11.3.8. A labeled graph is called $(D+1)$-*colored*, if each vertex has a degree $(D+1)$ and each edge is colored into one of $(D+1)$ colors such that every vertex is incident to edges of all colors. Additionally, a $(D+1)$-colored labeled graph is *bipartite*, if the set of its vertices is divided into two equal parts of "positive" and "negative" vertices such that each edge links the vertices of different type.

Notation 11.3.9. For different labeled combinatorial classes and their counting sequences, we employ the following notations.

- For $(D+1)$-colored graphs, we use $\mathcal{COG}(D+1)$ and $\big(\mathfrak{cog}_n(D+1)\big)$, respectively.
- For bipartite $(D+1)$-colored graphs, we use $\mathcal{BCG}(D+1)$ and $\big(\mathfrak{bcg}_n(D+1)\big)$, respectively.
- For connected $(D+1)$-colored graphs, we use $\mathcal{CCOG}(D+1)$ and $\big(\mathfrak{ccog}_n(D+1)\big)$, respectively.
- For connected bipartite $(D+1)$-colored graphs, we use $\mathcal{CBCG}(D+1)$ and $\big(\mathfrak{cbcg}_n(D+1)\big)$, respectively.

Remark 11.3.10. Since the set of edges of any fixed color determines a perfect matching, the counting sequence of the class $\mathcal{COG}(D+1)$ satisfies

$$\mathfrak{cog}_n(D+1) = \begin{cases} \big((2k-1)!!\big)^{D+1}, & n = 2k, \\ 0, & n = 2k+1. \end{cases}$$

On the other hand, each $(D+1)$-colored bipartite labeled graph consists of two parts of the same size $k \in \mathbb{N}$. To determine the graph, first of all, we need to fix the labels of these parts. Then the graph is described by a set of $(D+1)$ permutations, meaning that the number of graphs with $2k$ vertices is

$$\binom{2k}{k} \cdot (k!)^{D+1} = (2k)! \cdot (k!)^{D-1}.$$

Hence, the counting sequence of the class $\mathcal{BCG}(D+1)$ satisfies

$$\mathfrak{bcg}_n(D+1) = \begin{cases} (2k)! \cdot (k!)^{D-1}, & n = 2k, \\ 0, & n = 2k+1. \end{cases}$$

Lemma 11.3.11. *If $D \geqslant 2$, then the labeled combinatorial classes $\mathcal{COG}(D+1)$ and $\mathcal{BCG}(D+1)$ are 2-gargantuan.*

Proof. We have seen in the proof of Lemma 9.3.5, that for $D \geqslant 2$, the sequence $(n!)^{D-1}$ is gargantuan. This fact implies directly that the labeled combinatorial class $\mathcal{BCG}(D+1)$ is 2-gargantuan by Definition 2.1.14. To prove that the class $\mathcal{COG}(D+1)$ is 2-gargantuan, let us apply Lemma 3.1.78 b). In this case, condition (i)' reads

$$n^3 \mathfrak{cog}_{2n-2}(D+1) = O\big(\mathfrak{cog}_{2n}(D+1)\big),$$

which is true, since $D \geqslant 2$. To verify condition (ii)', we need to check that, for fixed large integer n, the sequence

$$\binom{2n}{2k} \mathfrak{cog}_{2k}(D+1)\mathfrak{cog}_{2(n-k)}(D+1)$$

is decreasing for $k < n/2$. Indeed,

$$\begin{array}{rcll} \binom{2n}{2k} \mathfrak{cog}_{2k}(D+1)\mathfrak{cog}_{2(n-k)}(D+1) & \geqslant & \binom{2n}{2(k+1)} \mathfrak{cog}_{2(k+1)}(D+1)\mathfrak{cog}_{2(n-k-1)}(D+1) & \Leftrightarrow \\ \dfrac{\big(2(n-k)-1\big)^{D+1}}{n-k} & \geqslant & \dfrac{\big(2(k+1)-1\big)^{D+1}}{k+1} & \Leftrightarrow \\ (n-k) & \geqslant & (k+1). & \end{array}$$

This is true, since the function $(2x-1)^{D+1}/x$ is increasing for $x > 0$. Hence, the class $\mathcal{COG}(D+1)$ is 2-gargantuan. □

Lemma 11.3.12. *The labeled combinatorial class $\mathcal{COG}(D+1)$ of $(D+1)$-colored graphs and its subclasses $\mathcal{CCOG}(D+1)$, $\mathcal{BCG}(D+1)$ and $\mathcal{CBCG}(D+1)$ satisfy the following relations:*

$$\mathcal{COG}(D+1) = \mathrm{SET}\big(\mathcal{CCOG}(D+1)\big) \qquad \textit{and} \qquad \mathcal{BCG}(D+1) = \mathrm{SET}\big(\mathcal{CBCG}(D+1)\big).$$

Proof. The proof proceeds the same way as the proof of Lemma 10.1.6. □

Notation 11.3.13. For any $m \in \mathbb{N}$, we denote by

$$\mathcal{CCOG}^{\{m\}}(D+1) = \mathrm{SET}_m\big(\mathcal{CCOG}(D+1)\big) \qquad \text{and} \qquad \mathcal{CBCG}^{\{m\}}(D+1) = \mathrm{SET}_m\big(\mathcal{CBCG}(D+1)\big)$$

the labeled combinatorial classes consisting of $(D+1)$-colored graphs and bipartite $(D+1)$-colored graphs, respectively, that have exactly m connected components. The corresponding counting sequences are $\big(\mathfrak{ccog}_n^{\{m\}}(D+1)\big)$ and $\big(\mathfrak{cbcg}_n^{\{m\}}(D+1)\big)$, respectively.

Theorem 11.3.14. *a) Let $D \geqslant 2$. The asymptotic probability that a random $(D+1)$-colored graph G of size $2n$ has m connected components satisfies*

$$\mathbb{P}(G \textit{ has } m \textit{ connected components}) \approx \sum_{k \geqslant 0} \alpha_{2k}^{\{m\}} \cdot \binom{2n}{2k} \cdot \left(\frac{(2(n-k)-1)!!}{(2n-1)!!} \right)^{D+1}, \qquad (11.12)$$

where

$$\alpha_{2k}^{\{m\}} = \sum_{s=m-1}^{\infty} (-1)^{s+m-1} \binom{s}{m-1} \mathfrak{ccog}_{2k}^{\{s\}}(D+1).$$

In particular,

$$\mathbb{P}(G \textit{ has } m \textit{ connected components}) = \frac{(2n)_{2(m-1)}}{2^{m-1}(m-1)!} \cdot \left(\frac{(2(n-m)+1)!!}{(2n-1)!!} \right)^{D+1} + O\left(\frac{1}{n^{(D-1)m}} \right).$$

b) Let $D \geqslant 2$. The asymptotic probability that a random bipartite $(D+1)$-colored graph G_b of size $2n$ has m connected components satisfies

$$\mathbb{P}(G_b \textit{ has } m \textit{ connected components}) \approx \sum_{k \geqslant 0} \frac{\beta_{2k}^{\{m\}}}{(2k)!! \cdot \big((n)_k\big)^{D-1}}, \qquad (11.13)$$

where

$$\beta_{2k}^{\{m\}} = \sum_{s=m-1}^{\infty} (-1)^{s+m-1} \binom{s}{m-1} \mathfrak{cbcg}_{2k}^{\{s\}}(D+1).$$

In particular,

$$\mathbb{P}(G_b \textit{ has } m \textit{ connected components}) = \frac{1}{(m-1)!} \cdot \frac{1}{\big((n)_{m-1}\big)^{D-1}} + O\left(\frac{1}{n^{(D-1)m}} \right).$$

Proof. We have seen that, by Lemma 11.3.11, the labeled combinatorial classes $\mathcal{COG}(D+1)$ and $\mathcal{BCG}(D+1)$ are 2-gargantuan whenever $D \geqslant 2$. At the same time, $\mathcal{COG}(D+1) = \mathrm{SET}\big(\mathcal{CCOG}(D+1)\big)$ and $\mathcal{BCG}(D+1) = \mathrm{SET}\big(\mathcal{CBCG}(D+1)\big)$ by Lemma 11.3.12. Hence, we can apply Theorem 7.4.2 in the case $d = 2$ to $\mathcal{COG}(D+1)$ and $\mathcal{BCG}(D+1)$, which gives us (11.12) and (11.13), respectively. The leading terms come from formula (7.14). □

In the case of bipartite $(D+1)$-colored graphs, there is a connection with indecomposable multipermutations, which finds its reflection in the following theorem.

Theorem 11.3.15. *Let $D \geqslant 2$. The asymptotic probability that a random bipartite $(D+1)$-colored graph G_b of size $2n$ is connected satisfies*

$$\mathbb{P}(G_b \textit{ has } m \textit{ connected components}) \approx 1 - \sum_{k \geqslant 1} \frac{\mathfrak{imp}_k(D-1)}{\big((n)_k\big)^{D-1}}. \qquad (11.14)$$

Proof. According to Lemma 11.3.11, the labeled combinatorial class $\mathcal{BCG}(D+1)$ is 2-gargantuan for $D \geqslant 2$. Moreover, by Lemma 11.3.12, $\mathcal{BCG}(D+1) = \mathrm{SET}\big(\mathcal{CBCG}(D+1)\big)$. At the same time, the exponential generating function of the class $\mathcal{BCG}(D+1)$ coincide with the exponential generating function of the labeled combinatorial class $\mathcal{ML}(D)$ of multiple linear orders of rank D taken at the point z^2:

$$BCG(D+1)(z) = \sum_{n=0}^{\infty}(2n)!(n!)^{D-1}\frac{z^{2z}}{(2n)!} = \sum_{n=0}^{\infty}(n!)^{D}\frac{z^{2z}}{n!} = ML(D)(z^2).$$

Taking into account that

$$\mathcal{ML}(D) = \mathrm{SEQ}\big(\mathcal{IML}(D)\big)$$

(see Lemma 9.3.20) and

$$\mathfrak{iml}_n(D) = n! \cdot \mathfrak{imp}_n(D-1)$$

(see Lemma 9.3.22), we have

$$ML(D)(z^2) = \exp\left(\sum_{n=0}^{\infty}\mathfrak{iml}_n(D)\cdot\frac{z^{2z}}{n!}\right) = \exp\left(\sum_{n=0}^{\infty}(2n)!\cdot\mathfrak{imp}_n(D-1)\cdot\frac{z^{2z}}{(2n)!}\right).$$

Hence, repeating the reasoning of Theorem 7.4.1, we get

$$\mathfrak{cbcg}_{2n}(D+1) \approx 1 - \sum_{k\geqslant 1}\binom{2n}{2k}\cdot\big((2k)!\cdot\mathfrak{imp}_k(D-1)\big)\cdot\mathfrak{bcg}_{2(n-k)}(D+1).$$

Dividing the above relation by $\mathfrak{bcg}_{2n}(D+1)$ and using the equality

$$\binom{2n}{2k}\cdot(2k)!\cdot\frac{(2n-2k)!\cdot\big((n-k)!\big)^{D-1}}{(2n)!\cdot(n!)^{D-1}} = \frac{1}{\big((n)_k\big)^{D-1}},$$

we obtain exactly (11.14). □

Remark 11.3.16. Both Theorems 11.3.14 and 11.3.15 are in agreement with asymptotics (11.10) and extend results of Carrance by establishing full asymptotic expansions and their structures.

Part IV

Conclusion

Chapter 12

Possible directions for further research

In this chapter, we discuss open problems related to the thesis and outline directions for further research. Section 12.1 is devoted to those of them that come from the general results described in Part II. The most important of them concern probabilities whose asymptotic expansions are currently unknown. First of all, we need to mention probabilities related to unlabeled constructions (Section 12.1.1) and species (Section 12.1.2). Second, we discuss the structures defined with the help of different compositions, including those that are given implicitly (Section 12.1.4). Finally, there are a quantum analogue of the construction SEQ and related problems (Section 12.1.5. Also, it is worth mentioning algorithmic aspects related to the developed theory.

In Section 12.2 we discuss questions related to some concrete applications. More precisely, Section 12.2.2 is devoted to strongly connected directed graphs, and Section 12.2.3 concerns open problems related to the Erdős–Rényi model.

12.1 Possible generalizations of the main results

12.1.1 Asymptotics for unlabeled constructions

As we have mentioned before, the universe of the symbolic method consists of structures of two different types: labeled and unlabeled. In Part II we have obtained the asymptotic probabilities and combinatorial interpretation of their coefficients for labeled combinatorial classes regarding the principal constructions SEQ, CYC and SET, as well as the restricted constructions SEQ_m, CYC_m and SET_m, where m is an arbitrary positive integer. The situation related to the unlabeled part of the universe is not so encouraging. There are results regarding the constructions SEQ, MSET, PSET and SEQ_m only. Thus, we come to the following question.

Question 12.1.1. *Let $\mathcal{V}$ be a gargantuan unlabeled combinatorial class, such that $V = \mathrm{CYC}(\mathcal{W})$ for some unlabeled combinatorial class $\mathcal{W}$. What is the asymptotic expansion of the probability that a random object $v \in \mathcal{V}$ belongs to $\mathcal{W}$ as well (that is, v is* CYC*-irreducible)? The same question for the constructions CYC_m, MSET_m and PSET_m, where $m \in \mathbb{N}$.*

The obstacle consists in the applicability of Bender's theorem. In the labeled case, the asymptotics are obtained with the help of its simplest version (i.e. Theorem 2.2.5) and the way to do it is straightforward. So is the case for the constructions SEQ and SEQ_m of the unlabeled universe, which is caused by rather simple forms of the corresponding relations for the generating functions. In contrast to this, the relations for the other constructions are complicated and cannot be treated by Theorem 2.2.5. Thus, we need to implement Theorem 2.2.8 b) which is not direct, meaning that the implementation is not determined by the relation in a unique way: the outer function in the composition can be chosen in different ways. However, we cannot choose this function in a completely arbitrary way, since there are certain restrictions imposed on it. Following Bender [9], we make such a choice for the constructions MSET and PSET, but for the others the question is still open.

12.1.2 Asymptotics for species

Another question related to the unlabeled case concerns the theory of species. We have seen that the language of species somehow encloses the labeled and unlabeled cases together. Thus, it is natural to unite asymptotic results for the labeled and unlabeled constructions within the species theory. In other words, we have the following question.

Question 12.1.2. *Can we combine the results for labeled and unlabeled cases within the framework of the theory of species? What could be conditions for that?*

For instance, for the construction SEQ the view and interpretation of the coefficients do not depend on the type of combinatorial classes (labeled or unlabeled): Theorems 5.2.1 and 5.5.1 give the same sequence. This fact allows us to conjecture that, under certain conditions, Theorem 8.1.1 could serve as a generalization for them both. The case related to the construction SET looks similar, though it is more complicated (see Theorems 7.2.1, 7.5.1 and 8.3.4). For most of the other constructions, before making guesses, we need to answer Question 12.1.1.

One more question related to species comes from Lemma 8.3.3.

Question 12.1.3. *Is it true that the virtual species* $\mathcal{E}^{-1} \circ \mathcal{E}_+^{(-1)}$ *and* $\left(1 - \mathcal{L}_+^{(-1)}\right)$ *are equal?*

Lemma 8.3.3 states the equipotency of the above species. If the answer to Question 12.1.3 is positive, then there is a simple form of the "anti-SEQ" operator $\mathcal{L}_+^{(-1)}$ which possibly describes the unlabeled case as well.

12.1.3 Positivity question

Theorem 7.2.1 states that if a gargantuan labeled combinatorial class $\mathcal{U}$ has two different decompositions

$$\mathcal{U} = \mathrm{SET}(\mathcal{V}) \qquad \text{and} \qquad \mathcal{U} = \mathrm{SEQ}(\mathcal{W}),$$

then the coefficients $\mathfrak{w}_k$ (which are positive) arise in the asymptotic expansion of the probability that a random object $u \in \mathcal{U}$ is SET-irreducible. However, it is reasonable to ask whether u is SET-irreducible even in the case when there is no labeled combinatorial class $\mathcal{W}$ such that $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$. In the latter case, the asymptotics is given by Theorem 8.3.4 and the involved coefficients relate to the virtual species $\mathcal{L}_+^{(-1)}$. In terms of the symbolic method, the composition with $\mathcal{L}_+^{(-1)}$ can be considered as a construction that is the inverse of the construction SEQ. This leads to the following question.

Question 12.1.4. *Is it true that if* $\mathcal{U}$ *is a gargantuan labeled combinatorial class then there exists a labeled combinatorial class* $\mathcal{W}$ *such that* $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$*? If not, what could be conditions for the existence of such a class? The same in terms of species: is it true that if* $\mathcal{A}$ *is a gargantuan species of structures then* $\mathcal{L}_+^{(-1)} \circ \mathcal{A}$ *is positive (i.e. coincides with its positive part)?*

The property of a sequence to be gargantuan does not depend on any fixed finite number of its elements. On the other hands, the first several terms of the asymptotics depend on the beginning of the counting sequence $(\mathfrak{u}_n)$ only. This is the reason why the answer to the first part of Question 12.1.4 is negative. For example, if the counting sequence of the class $\mathcal{V}$ starts with

$$(\mathfrak{v}_n) = 0,\, 3,\, 2,\, 2,\, \ldots,$$

then the beginning of the counting sequence of $\mathcal{U} = \mathrm{SET}(\mathcal{V})$ is

$$(\mathfrak{u}_n) = 1,\, 3,\, 11,\, 47,\, \ldots$$

and the formal power series

$$W(z) = 1 - \frac{1}{U(z)} = 3 \cdot z - 7 \cdot \frac{z^2}{2!} + 11 \cdot \frac{z^3}{3!} + \ldots$$

possesses at least one negative coefficient. Thus, it is the second part of the question which is significant and of interest.

12.1.4 Asymptotics of compositions

Let us have a look to Bender's theorem from another point of view. Roughly speaking, its essence is to establish the asymptotic behavior of the composition $F(A(z))$, where $F(z)$ and $A(z)$ are known. In the initial work [9], reflected in Theorems 2.2.5 and 2.2.8, the function $F(z)$ is analytic, while the coefficient of $A(z)$ form a gargantuan sequence. In other words, the coefficients of the outer function $F(z)$ grow slowly, while the coefficients of the inner function $A(z)$ grow very fast. However, one can imagine a different behavior. For instance, in 1982, Bender and Richmond [11] established the following result.

Theorem 12.1.5. *Let $A(z)$ be a formal power series whose coefficients satisfy $na_{n-1} = o(a_n)$. Let, furthermore, $F(z)$ be a power series with non-zero radius of convergence, such that $f_0 = 0$ and $f_1 \neq 0$. Then for every $r \in \mathbb{N}$, the asymptotic behavior of $B(z) = A(F(z))$ satisfies*

$$b_n = \sum_{k=0}^{r-1} e_k a_{n-k} + O(f_1^n n^r a_{n-r}),$$

where $e_k = [z^n]\big(F(z)\big)^{n-k}$.

In other words, comparing to Bender's theorem, we can obtain the asymptotic behavior for the inverse situation, the coefficients of the outer function $A(z)$ grow very fast, while the coefficients of the inner function $F(z)$ grow slowly. This observation, with the help of Lagrange inversion, allowed Bender and Richmond to obtain asymptotic behavior of various combinatorial structures whose generating functions were defined implicitly. In particular, as it was mentioned in [11], this method could be applied for irreducibilities coming from "noncrossing compositions" of Beissinger [5] whose ordinary generating functions satisfy

$$A(z) = 1 + I\big(zA(z)\big) \tag{12.1}$$

(here, $A(z)$ and $I(z)$ are ordinary generating functions of some combinatorial class and its irreducible subclass, respectively). The list of possible applications includes irreducible partitions and permutations without invariant subintervals whose generating functions satisfy (12.1), and, more generally, different subclasses of graphs such as biconnected and bridgeless graphs.

Question 12.1.6. *Can we have any combinatorial interpretation for the coefficients arising in the asymptotic expansions for the probability that a random object is irreducible in the sense of* (12.1) *or, more generally, in the sense of other generating functions defined implicitly (such as bridgeless graphs and biconnected graphs)?*

The composition of two generating functions shows a much more complex asymptotic behavior in the case when both functions are of the same growth rate. There is no known result in the general case. However, Borinsky [13] obtained the asymptotic behavior of the composition $F\big(G(z)\big)$ in the case when both $F(z)$ and $G(z)$ belong to class $\mathbb{R}[[z]]^{\alpha}_{\beta}$ of generating functions whose coefficients have the growth of

$$\alpha^{n+\beta}\Gamma(n+\beta)\left(d_0+\frac{d_1}{(n+\beta-1)}+\frac{d_2}{(n+\beta-1)(n+\beta-2)}+\ldots\right),$$

where $\alpha,\beta\in\mathbb{R}_{>0}$ and $d_k\in\mathbb{R}$. Let $\mathcal{A}^{\alpha}_{\beta}\colon\mathbb{R}[[z]]^{\alpha}_{\beta}\to\mathbb{R}[[z]]$ be a linear map which maps a power series to its own asymptotic expansion. Borinsky showed that $\mathcal{A}^{\alpha}_{\beta}$ is a derivation satisfying a Leibniz rule

$$\big(\mathcal{A}^{\alpha}_{\beta}(F\cdot G)\big)(z)=F(z)\cdot\big(\mathcal{A}^{\alpha}_{\beta}G\big)(z)+\big(\mathcal{A}^{\alpha}_{\beta}F\big)(z)\cdot G(z)$$

and a chain rule

$$\big(\mathcal{A}^{\alpha}_{\beta}(F\circ G)\big)(z)=F\big(G(z)\big)\cdot\big(\mathcal{A}^{\alpha}_{\beta}G\big)(z)+\left(\frac{z}{G(z)}\right)^{\beta}\cdot\exp\left(\frac{G(z)-z}{\alpha zG(z)}\right)\cdot\big(\mathcal{A}^{\alpha}_{\beta}F\big)\big(G(z)\big).$$

In particular, one has the following relation for the compositional inverse:

$$\big(\mathcal{A}^{\alpha}_{\beta}G^{-1}\big)(z)=-\big(G^{-1}\big)'(z)\cdot\left(\frac{z}{G^{-1}(z)}\right)^{\beta}\cdot\exp\left(\frac{G^{-1}(z)-z}{\alpha zG^{-1}(z)}\right)\cdot\big(\mathcal{A}^{\alpha}_{\beta}G\big)\big(G^{-1}(z)\big).$$

With the help of the above formulas, Borinsky obtained asymptotic expansions of connected chord diagrams and simple permutations (also called connected, see [50]). In particular, the probability that a random chord diagram C is connected can be written in the form

$$\mathbb{P}(C\text{ is connected})\approx\frac{1}{e}\left(1-\sum_{k\geqslant1}c_{2k}\frac{\mathfrak{ch}_{2(n-k)}}{\mathfrak{ch}_{2n}}\right),\tag{12.2}$$

where $\mathfrak{ch}_{2n}=(2n-1)!!$ is the number of chord diagrams with $2n$ points and the first several terms of the sequence (c_{2k}) are

$$\frac{5}{2},\ \frac{43}{8},\ \frac{579}{16},\ \frac{44477}{128},\ \frac{5326191}{1280},\ \ldots$$

In the same way, the probability that a random permutation σ is simple has the form

$$\mathbb{P}(\sigma\text{ is simple})\approx\frac{1}{e^2}\left(1-\sum_{k\geqslant1}d_k\frac{\mathfrak{p}_{n-k}}{\mathfrak{p}_n}\right),\tag{12.3}$$

where the first terms of the sequence (d_k) are

$$4,\ -2,\ \frac{40}{3},\ \frac{182}{3},\ \ldots$$

Except for the constants e^{-1} and e^{-2}, asymptotic expansions (12.2) and (12.3) look very similar to those expressed by Theorems 5.2.1, 6.2.1 and 7.2.1. This observation leads us to the following question.

Question 12.1.7. *Can we have any combinatorial interpretation for the coefficients* (c_{2k}) *involved in asymptotic relation* (12.2)*? More generally, can we have any combinatorial interpretation for the coefficients involved in the asymptotic expansions obtained by Borinsky's approach?*

Note that a possible combinatorial explanation should be contained in the above mentioned chain rule. Note also, that the constant e^{-1} comes from the particular form of the implicit relation that determines the ordinary generating function of connected chord diagrams. This situation differs from the behavior of structures defined by the construction SET. For the latter, Bell [6] showed that a non-trivial constant does not occur when the radius of convergence is zero.

Another, more ambitious, question concerns the asymptotic behavior of the compositions of larger classes of generating functions.

Question 12.1.8. *Can we obtain any analogue of Borinsky's results for the composition* $F\big(G(z)\big)$ *in the case when coefficients of generating functions* $F(z)$ *and* $G(z)$ *asymptotically behave as* $2^{\binom{n}{2}}/n!$*? Can we find an analogue for any other class of functions?*

An answer to Question 12.1.8 would give another opportunity to get the asymptotic expansions for different subclasses of graphs whose generating functions are defined implicitly.

12.1.5 Quantum analogues

Let us denote by

$$n!_q = 1 \cdot (1+q) \cdot (1+q+q^2) \cdot \ldots \cdot (1+q+\ldots+q^{n-1})$$

the q-factorials, with $0!_q = 1$, and

$$\binom{n}{k}_q = \frac{n!_q}{k!_q (n-k)!_q}$$

the q-binomial coefficients. These q-analogues of factorials and binomial coefficients play significant role in generalizations of various combinatorial constructions. Typically, they are used for specifications. For instance, let $\mathfrak{p}_{n,k}$ be the number of permutations with k inversions. Then the corresponding ordinary generating function has the form

$$P(z,q) = \sum_{n=0}^{\infty} \sum_{k=0}^{\binom{n}{2}} \mathfrak{p}_{n,k} q^k z^n = \sum_{n=0}^{\infty} n!_q z^n.$$

In this case, the decomposition

$$\mathcal{P} = \mathrm{SEQ}(\mathcal{IP})$$

is preserved and the relation

$$IP(z,q) = 1 - \frac{1}{P(z,q)}$$

for generating functions is valid, where $IP(z,q)$ is the ordinary generating function for indecomposable permutations (see [26] and [94, Ex 1.129]). The same way, we can consider the Erdős–Rényi model as a q-deformation of undirected graphs with

$$G(z,q) = \sum_{n=0}^{\infty} \sum_{k=0}^{\binom{n}{2}} \mathfrak{g}_{n,k} q^k \frac{z^n}{n!} = \sum_{n=0}^{\infty} (1+q)^{\binom{n}{2}} \frac{z^n}{n!},$$

where $\mathfrak{g}_{n,k}$ and $\mathfrak{cg}_{n,k}$ are the numbers of graphs and connected graphs, respectively, with n vertices and k edges, and $G(z,q)$ and $CG(z,q)$ are the corresponding exponential generating functions.[1] The decomposition

$$\mathcal{G} = \mathrm{SET}(\mathcal{CG})$$

is preserved in this case, so that

$$G(z,q) = \exp\big(CG(z,q)\big).$$

However, in certain cases, passing to the specifications, we need to change the standard construction. In 2020, Archer, Gessel, Graves and Liang [3] presented their work devoted to counting labeled directed graphs by descents, where a *descent* is a directed edge $\overrightarrow{st}$ such that $s > t$. Let us denote by $\mathfrak{t}_{n,k}$ and $\mathfrak{it}_{n,k}$ the number of tournaments and irreducible tournaments, respectively, with n vertices and k descents. Then it turns out that the standard SEQ-decomposition for tournaments is transformed into

$$\diamond T(z,q) = \frac{1}{1 - \diamond IT(z,q)},$$

where

$$\diamond T(z,q) = \sum_{n=0}^{\infty} \sum_{k=0}^{\binom{n}{2}} \mathfrak{t}_{n,k} q^k \frac{z^n}{n!_q} = \sum_{n=0}^{\infty} (1+q)^{\binom{n}{2}} \frac{z^n}{n!_q}$$

and

$$\diamond IT(z,q) = \sum_{n=0}^{\infty} \sum_{k=0}^{\binom{n}{2}} \mathfrak{it}_{n,k} q^k \frac{z^n}{n!_q}$$

are so-called *Eulerian generating functions* of tournaments and irreducible tournaments, respectively. The similar way, one can obtain generalizations in terms of Eulerian generating functions for strongly connected directed graphs and directed acyclic graphs.

Question 12.1.9. *Can we obtain an interesting generalization of Theorem 5.2.1 for q-deformed* SEQ-*decompositions?*

The use of the Eulerian generating functions is caused by the special form of convolution rule,

$$c_n = \sum_{k=0}^{n} \binom{n}{k}_q a_k b_{n-k},$$

which is needed for counting directed graphs by descents. In particular, this is the reason why the numbers of tournaments counted by descents differ from the asymptotic coefficients in the Erdős–Rényi model.

12.1.6 Algorithmic aspects

In Section 7.1 we have discussed the combinatorial approach to establishing the asymptotic expansion for the probability that a random element $u \in \mathcal{U}$ is SET-irreducible. The idea was that the main contribution is given by reducible objects possessing large irreducible components. In other words, the first approximation is determined by the objects whose components are of sizes $(n-1)$ and 1.

[1] Note that in Chapter 10 the parameter q has different meaning. Here, we denote by q the value that was denoted by ρ in Chapter 10.

To find the second approximation, one has additionally to take into account objects possessing an irreducible component of size $(n-2)$, etc. To obtain first terms of the asymptotic expansion, we can go from the other side and count objects possessing small irreducible components, which is easier. For example, in the case of undirected graphs, a random graph G of size n contains at least one isolated vertex with a probability of

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{\mathfrak{g}_{n-k}}{\mathfrak{g}_n},$$

while the probability that G has exactly m isolated vertices is

$$\sum_{k=m}^{n}(-1)^{k+m}\binom{n}{k}\binom{k}{m}\frac{\mathfrak{g}_{n-k}}{\mathfrak{g}_n}=\binom{n}{m}\sum_{k=0}^{n-m}(-1)^{k}\binom{n-m}{k}\frac{\mathfrak{g}_{n-m-k}}{\mathfrak{g}_n}.$$

Also, with the help of the symbolic method or the inclusion-exclusion principle, one can find the probability that G possesses a given subconfiguration of size k. This probability is of order $n^k\mathfrak{g}_{n-k}/\mathfrak{g}_n$.

This observation potentially may lead to an algorithm for generating random connected graphs. In principle, this problem is hard, since the notion of connectedness is a global property of a graph. However, one can think about the rejection method: to generate random connected graphs, we typically generate a random graph and reject until we find a connected one. If we pick a random graph from $\mathcal{G}_n$, we get a disconnected graph with a probability of order $n\cdot 2^{-n}$, which is not much for large n. However, if we create a procedure eliminating graphs with isolated vertices, then we get a disconnected graph with probability of smaller order, $n^2\cdot 2^{-2n}$, etc. The more subconfigurations we eliminate, the closer to 1 the probability to obtain a connected graph is.

Question 12.1.10. *Can we create a rejection algorithm for producing connected graphs randomly, so that we reject with a probability of a smaller order?*

The same question takes place for SEQ-irreducibles and CYC-irreducibles.

12.2 Applications and concrete problems

12.2.1 Applicability of theorems

In the labeled case, our general results relate to the constructions SEQ, CYC and SET, as well as the restricted constructions SEQ_m, CYC_m and SET_m for any $m\in\mathbb{Z}_{\geqslant 0}$. While Theorems 5.2.1 and 7.2.1 concerning SEQ and SET, respectively, have numerous applications studied in Part III, we do not know any natural application of Theorem 6.2.1.

Question 12.2.1. *Is there any application of Theorem 6.2.1 that would be of interest? In other words, is there any interesting gargantuan labeled combinatorial class that admits a decomposition with respect to the construction* CYC*?*

As for the unlabeled case, the situation is even more complicated. The problem is that explicit expressions for the counting sequences of unlabeled combinatorial classes are quite rare. Thus, for instance, it is known due to the results of Oberschelp [72] and Wright [101] that the unlabeled combinatorial class ${}_u\mathcal{G}$ of graphs is gargantuan and that its counting sequence $({}_u\mathfrak{g}_n)$ satisfies

$$ {}_u\mathfrak{g}_n\approx\frac{\mathfrak{g}_n}{n!}\left(1+\sum_{k\geqslant 1}\frac{\varphi_k(n)}{2^{kn}}\right), \tag{12.4}$$

where $\varphi_k(n)$ is a polynomial of degree $2k$ in n possessing the factor $(n)_{k+1} = n(n-1)\dots(n-k)$. This means that we could apply Theorem 7.5.1 to the class ${}_u\mathcal{G}$, so that we obtain the probability for a random unlabeled graph G of size n to be connected to be

$$\mathbb{P}(G \text{ is connected}) \approx 1 - \sum_{k\geqslant 1} \mathfrak{w}_k \frac{{}_u\mathfrak{g}_{n-k}}{{}_u\mathfrak{g}_n}. \tag{12.5}$$

However, the behavior of this expression is not as visible as in the labeled case; to understand it, we need to apply (12.4) first. Moreover, in the unlabeled case, the number ${}_u\mathfrak{g}_n$ of graphs of size n differs from the number of tournaments ${}_u\mathfrak{t}_n$ of the same size (see Example 3.2.19). For instance, if $n = 2$, then

$${}_u\mathfrak{g}_2 = 2 \neq 1 = {}_u\mathfrak{t}_2.$$

Hence, there is no combinatorial equivalence of the corresponding classes, ${}_u\mathcal{G} \ncong {}_u\mathcal{T}$. As a consequence, the meaning of coefficients $\mathfrak{w}_k$ in (12.5) is rather abstract, it is not the number of irreducible tournaments anymore.

Similar observations, which take place in other cases, lead us to the following question.

Question 12.2.2. *Does there exist any application of Theorem 7.5.1 that would be of interest? In particular, are there two rather interesting unlabeled combinatorial classes ${}_u\mathcal{V}$ and ${}_u\mathcal{W}$ such that*

$$\mathrm{SET}({}_u\mathcal{V}) \cong \mathrm{SEQ}({}_u\mathcal{W})?$$

12.2.2 Strongly connected directed graphs

As we have discussed in Remark 10.3.15, and as it was mentioned by Archer, Gessel, Graves and Liang [3], the result of Lemma 10.3.14, i.e.

$$SD(z) = -\log\Big(1 - G(z) \odot IT(z)\Big),$$

means that there is a combinatorial isomorphism between the labeled combinatorial classes $\mathcal{SD}$ of strongly connected directed graphs and $\mathrm{CYC}(\mathcal{T} \odot \mathcal{IT})$, where $\mathcal{T}$ and $\mathcal{IT}$ represents the classes of tournaments and irreducible tournaments, respectively. The same way, the relation

$$SSD(z) = \frac{1}{1 - G(z) \odot IT(z)}$$

means that the labeled combinatorial classes $\mathcal{SSD}$ of semi-strong directed graphs and $\mathrm{SEQ}(\mathcal{T} \odot \mathcal{IT})$ are combinatorially isomorphic as well.

Question 12.2.3. *Given a positive integer n, is there a natural bijection between*

- *$\mathcal{SD}_n$ and $\big(\mathrm{CYC}(\mathcal{T} \odot \mathcal{IT})\big)_n$, i.e. between strongly connected digraphs of size n and cycles of pairs of tournaments of size n (such that the second tournament is irreducible),*
- *$\mathcal{SSD}_n$ and $\big(\mathrm{SEQ}(\mathcal{T} \odot \mathcal{IT})\big)_n$, i.e. between semi-strong digraphs of size n and sequences of pairs of tournaments of size n (such that the second tournament is irreducible)?*

Another question comes from formula (10.19). The asymptotic probability for strongly connected digraphs looks rather complicated and involves semi-strong digraphs and irreducible tournaments. There should be a combinatorial reason for the particular form of this expression, just the same way as the asymptotic probability coefficients of connected graphs and irreducible tournaments can be explained with the help of the inclusion-exclusion principle.

Question 12.2.4. *What is an explanation of the components of relation* (10.19) *in terms of the inclusion-exclusion principle?*

The underlying idea is the same as in Sections 5.1, 6.1 and 7.1: the main contribution to asymptotics (10.19) is made by large components. The obstacle is that the structure of a directed graph is determined not only by the number of strongly connected components, but also by the way they interact with each other.

The same problem occurs when trying to answer the following question.

Question 12.2.5. *Could we find the full asymptotic expansion for the probability that a random directed graph consists of a given number of strongly connected components?*

12.2.3 The Erdős–Rényi model

Relation with tournaments

As we have seen in Theorem 10.1.10, the coefficients involved in the asymptotic expression for connected graphs count irreducible tournaments. Since simple graphs of size n are the particular case of the Erdős–Rényi model $G(n,p)$ with $p = 1/2$, it is reasonable to ask whether the coefficients $P_k(\rho)$ appearing in the asymptotics given by (10.26) can be, somehow, interpreted in terms of tournaments.

Question 12.2.6. *Can we interpret the coefficients $P_k(\rho)$ involved in asymptotic expression* (10.26) *for connected graphs within the Erdős–Rényi model as a generalization of irreducible tournaments?*

We have already mentioned in Section 12.1.5 that the most natural generalization does not take place. In other words, these coefficients are not coincide with irreducible tournaments counted by descents (see [3]). Thus, a possible generalization should be of a more complicated form.

Phase transition

Another question concerns the phase transition of the Erdős–Rényi model $G(n,p)$. In the case when $p \in (0,1)$, $q = 1-p$ and $\rho = p/q$ are constants, formula (10.26) is well-defined. The situation becomes more complicated if p decrease with $n \to \infty$. Since the work of Erdős and Rényi [34] it is known that

$$p = \frac{\ln n}{n}$$

is a sharp threshold for the connectedness of $G(n,p)$. In other words, for any $\varepsilon > 0$, if

$$p > \frac{(1+\varepsilon)\ln n}{n},$$

then a random graph in the Erdős–Rényi model is almost surely connected, while in the case

$$p < \frac{(1-\varepsilon)\ln n}{n},$$

a graph almost surely contains isolated vertices. It is seen that our asymptotics is in agreement with these results. Indeed, for

$$p = \frac{(1+\varepsilon)\ln n}{n}$$

the k-th term of (10.26) can be approximated as

$$f_k(n) = P_k(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}} \sim n^{-\varepsilon k}, \tag{12.6}$$

since

$$\begin{aligned}
\ln f_k(n) &= \ln \binom{n}{k} + \ln \left(1 - \frac{(1+\varepsilon)\ln n}{n}\right)^{nk} \bigl(1 + o(1)\bigr) + O\left(\frac{\ln n}{n}\right) \\
&= k \ln n + kn \left(-\frac{(1+\varepsilon)\ln n}{n}\right) \bigl(1 + o(1)\bigr) \\
&= -\varepsilon k \ln n \bigl(1 + o(1)\bigr).
\end{aligned}$$

In particular, $f_{k+1}(n) = o\bigl(f_k(n)\bigr)$ in this case. On the other hand, for $p = \ln n/n$, formulas (10.26) and (12.6) are no longer valid.

Question 12.2.7. *Can we build a fruitful theory of phase transition for asymptotic expansions?*

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Labeled combinatorial classes

$\mathcal{BCG}(D+1)$	bipartite $(D+1)$-colored graphs, 188
$\mathcal{CBCG}(D+1)$	connected bipartite $(D+1)$-colored graphs, 188
$\mathcal{CBCG}^{\{m\}}(D+1)$	bipartite $(D+1)$-colored graphs with m components, 189
$\mathcal{CCOG}(D+1)$	connected $(D+1)$-colored graphs, 188
$\mathcal{CCOG}^{\{m\}}(D+1)$	$(D+1)$-colored graphs with m components, 189
$\mathcal{CG}$	connected graphs, 152
$\mathcal{CG}^{\{m\}}$	graphs with m connected components, 153
$\mathcal{CM}$	combinatorial maps, 182
$\mathcal{CM}^{\{m\}}$	surfaces with m connected components from the combinatorial map model, 184
$\mathcal{CMG}(N)$	connected multigraphs of rank N, 154
$\mathcal{CMG}^{\{m\}}(N)$	multigraphs of rank N with m connected components, 155
$\mathcal{CN}(N)$	N-constellations, 187
$\mathcal{CO}$	connected square-tiled surfaces, 178
$\mathcal{CO}^{\{m\}}$	square-tiled surfaces with m connected components, 179
$\mathcal{COG}(D+1)$	$(D+1)$-colored graphs, 188
$\mathcal{CP}$	cyclic permutations, 44
$\mathcal{CS}(p)$	connected surfaces generated by p-gons, 185
$\mathcal{CS}^{\{m\}}(p)$	surfaces generated by p-gons with m connected components, 185
$\mathcal{D}$	directed graphs, 162
$\mathcal{DAG}$	directed acyclic graphs, 162
$\mathcal{E}$	totally disconnected graphs, 44
$\mathcal{F}$	forests, 53
$\mathcal{G}$	(simple undirected) graphs, 152
$\mathcal{H}$	Hadamard product $\mathcal{T} \odot \mathcal{IT}$, 165
$\mathcal{IM}$	indecomposable perfect matchings, 142
$\mathcal{IML}(2)$	irreducible pairs of linear orders, 140
$\mathcal{IML}(N)$	irreducible multiple linear orders of rank N, 146
$\mathcal{IML}^{(m)}(2)$	pairs of linear orders with m irreducible parts, 141
$\mathcal{IML}^{(m)}(N)$	multiple linear orders of rank N with m irreducible parts, 147
$\mathcal{IMLM}(2)$	irreducible pairs of linear orders represented by perfect matchings, 143
$\mathcal{IMM}(N)$	indecomposable multiple perfect matchings of rank N, 148
$\mathcal{IMP}(N)$	indecomposable multipermutations of rank N, 145
$\mathcal{IMPC}(d,N)$	indecomposable multiple products of cycles of length d of rank N, 149
$\mathcal{IMT}(N)$	irreducible multitournaments of rank N, 159
$\mathcal{IMT}^{(m)}(N)$	multitournaments of rank N with m irreducible parts, 159
$\mathcal{IP}$	indecomposable permutations, 138

$\mathfrak{iml}_n(2)$	irreducible pairs of linear orders, 140
$\mathfrak{iml}_n(N)$	irreducible multiple linear orders of rank N, 146
$\mathfrak{iml}_n^{(m)}(2)$	pairs of linear orders with m irreducible parts, 141
$\mathfrak{iml}_n^{(m)}(N)$	multiple linear orders of rank N with m irreducible parts, 147
$\mathfrak{imlm}_n(2)$	irreducible pairs of linear orders represented by perfect matchings, 143
$\mathfrak{imm}_n(N)$	indecomposable multiple perfect matchings of rank N, 148
$\mathfrak{imm}_n^{(m)}(N)$	multiple perfect matchings of rank N with m indecomposable parts, 148
$\mathfrak{imp}_n(N)$	indecomposable multipermutations of rank N, 145
$\mathfrak{imp}_n^{(m)}(N)$	multipermutations of rank N with m indecomposable parts, 145
$\mathfrak{impc}_n(d, N)$	indecomposable multiple products of cycles of length d of rank N, 149
$\mathfrak{impc}_n^{(m)}(d, N)$	multiple products of cycles of length d of rank N with m indecomposable parts, 149
$\mathfrak{imt}_n(N)$	irreducible multitournaments of rank N, 159
$\mathfrak{imt}_n^{(m)}(N)$	multitournaments of rank N with m irreducible parts, 159
$\mathfrak{ip}_n$	indecomposable permutations, 138
$\mathfrak{ip}_n^{(m)}$	permutations with m indecomposable parts, 138
$\mathfrak{ipc}_n(d)$	indecomposable products of cycles of length d, 147
$\mathfrak{ipc}_n^{(m)}(d)$	products of cycles of length d with m indecomposable parts, 147
$\mathfrak{it}_n$	irreducible tournaments, 157
$\mathfrak{it}_n^{(m)}$	tournaments with m irreducible parts, 157
$\mathfrak{l}_n$	linear orders, 45
$\mathfrak{m}_n$	perfect matchings, 142
$\mathfrak{md}_n(N)$	multidigraphs of rank N, 169
$\mathfrak{mg}_n(N)$	multigraphs of rank N, 154
$\mathfrak{ml}_n(2)$	pairs of linear orders, 140
$\mathfrak{ml}_n(N)$	multiple linear orders of rank N, 146
$\mathfrak{mlm}_n(2)$	pairs of linear orders represented by perfect matchings, 143
$\mathfrak{mm}_n(N)$	multiple perfect matchings of rank N, 148
$\mathfrak{mp}_n(N)$	multipermutations of rank N, 144
$\mathfrak{mpc}_n(d, N)$	multiple products of cycles of length d of rank N, 149
$\mathfrak{mt}_n(N)$	multitournaments of rank N, 159
$\mathfrak{o}_n$	square-tiled surfaces, 178
$\mathfrak{p}_n$	permutations, 137
$\mathfrak{p}_{n,k}$	permutations with k inversions, 196
$\mathfrak{pc}_n(d)$	products of cycles of length d, 147
$\mathfrak{po}_n(p)$	monic polynomials over $\mathbb{F}_p$, 64
$\mathfrak{s}_n(p)$	surfaces generated by p-gons, 184
$\mathfrak{sd}_n$	strongly connected directed graphs, 162
$\mathfrak{smd}_n(N)$	strongly connected multidigraphs of rank N, 169
$\mathfrak{ssd}_n$	semi-strong directed graphs, 162
$\mathfrak{t}_n$	tournaments, 157
$\mathfrak{t}_{n,k}$	irreducible tournaments with k descents, 197
$\mathfrak{t}_{n,k}$	tournaments with k descents, 197
$\mathfrak{tr}_n$	trees, 59
$\mathfrak{wd}_n$	weakly connected directed graphs, 162
$\mathfrak{wd}_n^{\{m\}}$	directed graphs with m weakly connected components, 163
$\mathfrak{wdag}_n$	weakly connected directed acyclic graphs, 162

$\mathfrak{wdag}_n^{\{m\}}$	directed acyclic graphs with m weakly connected components, 163
$\mathfrak{wmd}_n(N)$	weakly connected multidigraphs of rank N, 169
$\mathfrak{wmd}_n^{\{m\}}(N)$	multidigraphs of rank N with m weakly connected components, 169
$\mathfrak{w}_n^{(m)}$	counting sequence of combinatorial class $\mathcal{W}^{(m)}$, 96
$\mathfrak{w}_n^{[m]}$	counting sequence of combinatorial class $\mathcal{W}^{[m]}$, 110
$\mathfrak{w}_n^{\{m\}}$	counting sequence of combinatorial class $\mathcal{W}^{\{m\}}$, 118

Labeled constructions

CYC	cycle, 51
CYC_m	m-cycle, 50
CYC_Ω	cycle restricted to Ω, 53
SEQ	sequence, 49
SEQ_m	m-sequence, 49
SEQ_Ω	sequence restricted to Ω, 53
SET	set, 52
SET_m	m-set, 51
SET_Ω	set restricted to Ω, 53
Θ	pointing, 53
$+$	disjoint union, 46
$\circ$	substitution, 55
$\odot$	Hadamard product, 56
$\star$	labeled product, 47

Unlabeled constructions

CYC	cycle, 66
CYC_m	m-cycle, 66
MSET	multiset, 67
MSET_m	m-multiset, 67
PSET	powerset, 67
PSET_m	m-powerset, 67
SEQ	sequence, 65
SEQ_m	m-sequence, 65
$\times$	Cartesian product, 64

Species of structures

$\mathbf{0}$	empty species, 71
$\mathbf{1}$	characteristic of the empty set, 71
$\mathcal{CG}_w$	connected weighted graphs, 173
$\mathcal{CP}$	cyclic permutations, 72
$\mathcal{DE}$	derangements, 82
$\mathcal{E}$	sets, 72
$\mathcal{G}_w$	weighted graphs, 173
$\mathcal{G}$	graphs, 79
$\mathcal{L}$	linear orders, 72
$\mathcal{P}$	permutations, 72
$\mathcal{T}$	tournaments, 79
$\mathcal{Z}$	characteristic of singletons, 71
$\mathcal{B}^{(m)}$	species $\mathcal{L}_m \circ \mathcal{B}$, where $\mathcal{B}$ is a given species of structure and $m \in \mathbb{N}$, 129
$\mathcal{B}^{[m]}$	species $\mathcal{CP}_m \circ \mathcal{B}$, where $\mathcal{B}$ is a given species of structure and $m \in \mathbb{N}$, 129
$\mathcal{B}^{\{m\}}$	species $\mathcal{E}_m \circ \mathcal{B}$, where $\mathcal{B}$ is a given species of structure and $m \in \mathbb{N}$, 129

$\Psi^{(-1)}$	substitutional inverse of the virtual species Ψ satisfying $\Psi_0 = \mathbf{0}$ and $\Psi_1 = \mathcal{Z}$, 92
F^{-1}	multiplicative inverse of the species F satisfying $F_0 = \mathbf{1}$, 90
F_+	species $F - \mathbf{1}$, where F is a given species of structure satisfying $F_0 = \mathbf{1}$, 90

Species operations and generating series

$F(z)$	exponential generating series of a species of structures F, 74
nF	sum of n copies of the species F, 81
$Z_F(z_1, z_2, z_3, \ldots)$	cycle index series of a species of structures F, 77
$\widetilde{F}(z)$	type generating series of a species of structures F, 75
$'$	derivative, 85
$+$	sum, 80
$\cdot$	product, 81
$\circ$	substitution, 83
$\times$	Cartesian product, 87
$\bullet$	pointing, 86

Other symbols

$\operatorname{Fix}\sigma$	set of fixed points of the permutation σ, 76
Id_X	identity map $X \to X$, 71
Spe	semi-ring of species, 82
Virt	ring of virtual species, 89
$G(n,p)$	Erdős–Rényi model, 173
$(n)_k$	falling factorials, that is, $n(n-1)\ldots(n-k+1)$, 22
$[n]$	the set $\{1, \ldots, n\}$, 43
$\widehat{A}(z)$	graphic generating function of the labeled combinatorial class $\mathcal{A}$, 57
Δ	operator converting an exponential generating function to the graphic generating function, 57
ϵ	neutral object, 43
$\pi_0(G)$	number of connected components of the graph G, 175
$\approx$	asymptotic behavior, 37
$\cong$	combinatorial isomorphism, 45
$\equiv$	equipotency of species of structures, 78
$\odot$	Hadamard product (of formal power series), 55
$\simeq$	isomorphism of species of structures, 79

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Titre : Irréductibilité des objets combinatoires : probabilité asymptotique et interprétation

Mots-clés : Expansion asymptotique, probabilité, classe combinatoire, espèces de structures, fonction généra-trice, théorème de Bender, permutation indécomposable, couplage parfait, graphe connexe, tournoi irréduc-tible, graphe orienté fortement connexe, graphe orienté acyclique, modèle Erdős-Rényi, surface à petits carreaux, carte combinatoire, constellation, multigraphe, multitournoi, multipermutation.

Résumé : De nombreuses structures combinatoires admettent, au sens large, une notion d'irréductibilité : les graphes peuvent être connexes, les permutations indécomposables, les polynômes irréductibles, etc. Dans cette thèse, nous nous intéressons à la probabilité qu'un tel objet pris au hasard soit irréductible, lorsque sa taille tend vers l'infini. On obtient des développements asymptotiques complets pour ces probabilités ; l'irréductibilité est comprise à travers les constructions combinatoires SET, SEQ et CYC dans le contexte de la méthode symbolique. Nous appliquons notre approche aux graphes connexes, aux tournois irréductibles, aux permutations indécomposables et aux couplages parfaits. En outre, nous établissons des asymptotiques pour plusieurs modèles de surfaces connexes comprenant les surfaces à petits carreaux, les cartes combinatoires et certains objets de dimension supérieure tels que les constellations et les modèles de tenseurs colorés. Nous montrons que les coefficients apparaissant dans ces asymptotiques sont entiers et qu'ils peuvent être interprétés comme des suites de comptage d'autres classes combinatoires « dérivées ». Par exemple, les graphes connexes conduisent aux tournois irréductibles, les surfaces à petits carreaux aux permutations indécomposables, les cartes combinatoires aux couplages parfaits indécomposables. De plus, nous obtenons certaines probabilités asymptotiques qu'un objet combinatoire aléatoire ait un nombre donné de composantes irréductibles. Passant de la méthode symbolique à la théorie des espèces, nous traitons également le modèle $G(n, p)$ de Erdős–Rényi. Nous établissons également la probabilité qu'un graphe orienté aléatoire soit fortement connexe en utilisant une décomposition plus complexe qui implique des graphes acycliques orientés.

Title: Irreducibility of combinatorial objects: asymptotic probability and interpretation

Key words: Asymptotic expansion, probability, combinatorial class, species of structures, generating function, Bender's theorem, indecomposable permutation, perfect matching, connected graph, irreducible tournament, strongly connected directed graph, directed acyclic graph, Erdős-Rényi model, square-tiled surface, combinatorial map, constellation, multigraph, multitournament, multipermutation.

Abstract: Various combinatorial structures admit, in a broad sense, a notion of irreducibility: graphs can be connected, permutations can be indecomposable, polynomials can be irreducible, etc. In this thesis, we are interested in the probability that any such object picked randomly is irreducible, as its size tends to infinity. We obtain complete asymptotic expansions for these probabilities; irreducibility is understood via the combinatorial constructions SET, SEQ, and CYC within the symbolic method. We apply our approach to connected graphs, irreducible tournaments, indecomposable permutations, and perfect matchings. Also, we establish asymptotics for several models of connected surfaces, including square-tiled surfaces, combinatorial maps, and higher dimensional objects such as constellations, and colored tensor models. We show that the coefficients appearing in those asymptotics are integers and can be interpreted as the counting sequences of other "derivative" combinatorial classes. For instance, connected graphs lead to irreducible tournaments, square-tiled surfaces lead to indecomposable permutations, combinatorial maps lead to indecomposable perfect matchings. Moreover, we obtain asymptotic probabilities that a random combinatorial object has a given number of irreducible components. Switching from the symbolic method to the theory of species, we treat the Erdős–Rényi $G(n, p)$ model as well. We also establish the probability that a random directed graph is strongly connected using a more complex decomposition that involves directed acyclic graphs.