Semi-regular polygons on regular tilings

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Content

Tilings with regular polygons

- A plane tiling is a countable family of polygons which cover the plane without gaps or overlaps. These polygons are tiles, and its vertices are vertices of the tiling.
- We will consider only edge-to-edge tilings with regular polygons, that is every tile of tiling is a regular polygon and a non-empty intersection of two tiles is either a vertex or a side of these tiles.
- A species of the tiling vertex is the set of polygons contained this vertex or, to be more precise, the set of numbers which represent the amount of sides of these polygons. And the type of the vertex is a way in which these polygons are arranged.

Regular tilings and Archimedean tilings

- A tiling is regular, if its symmetry group acts transitively on the set of vertices.
- A tiling is Archimedean, if the types of all vertices are the same. This type is the type of the tiling.

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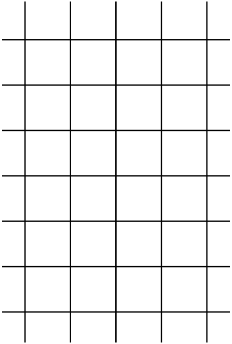
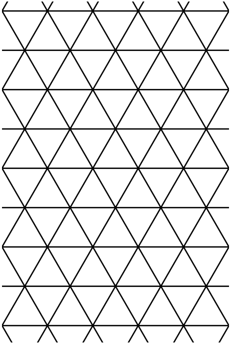
Fact. There are 11 Archimedean tilings, and every Archimedean tiling is regular.

The types of Archimedean tilings

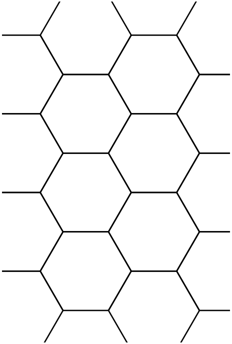
- Archimedean tilings with tiles of single shape.



(4)4



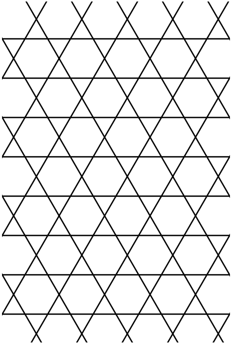
(6)3



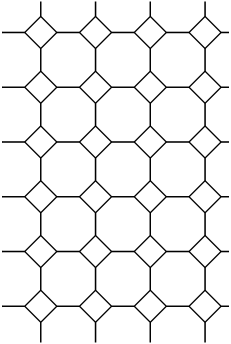
(3)6

The types of Archimedean tilings

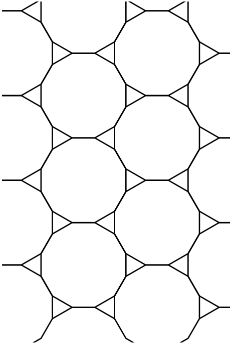
- Archimedean tilings with tiles of two different shapes.



3, 6, 3, 6



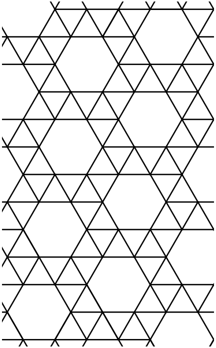
4, 8, 8



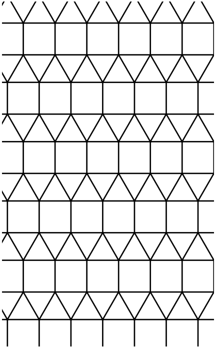
3, 12, 12

The types of Archimedean tilings

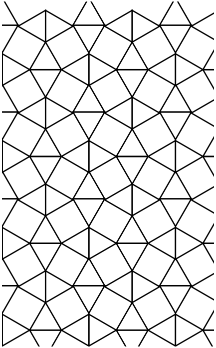
- Archimedean tilings with tiles of two different shapes.



(4)3, 6



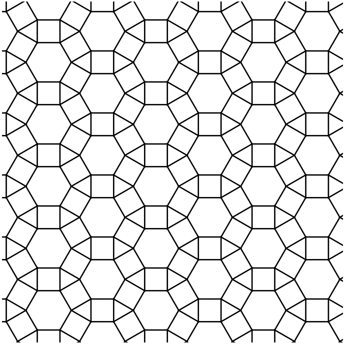
(3)3, (2)4



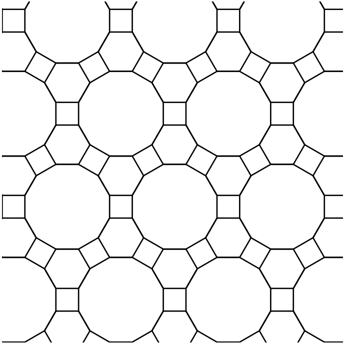
(2)3, 4, 3, 4

The types of Archimedean tilings

- Archimedean tilings with tiles of three different shapes.



$3, 4, 6, 4$



$4, 6, 12$

Semi-regular polygons

- A polygon is equilateral, if all its sides have the same length.
- A polygon is equiangular, if all its angles are equal.
- We say that a polygon F is located on the tiling T, if every vertex of F is a vertex of T as well.

Semi-regular polygons, located on the (4)4-tiling

Theorem (Folklor)

The only regular polygons, that can be located on the (4)4-tiling, are squares.

Theorem (Ball, 1973)

Equilateral polygon can be located on the (4)4-tiling iff the number of its sides is even.

Theorem (Ball, 1973)

The only equiangular polygons, that can be located on the (4)4-tiling, are squares and equiangular octagons.

Regular polygons, located on Archimedean tilings

Theorem (I. Sedoshkin, E. Mychka, 2001)

For every Archimedean tiling T it is possible to locate some regular polygon on T iff "it is visible to the naked eye" (that is, this polygon is a tile or the union of tiles).

Stating the problem

- Given Archimedean tiling T, what kind of equilateral polygons can be located on T?
- Given Archimedean tiling T, what kind of equiangular polygons can be located on T?

Equilateral polygons located on Archimedean tilings

Theorem (Nurligareev, 2011)

- *Let* $\alpha \in \{(4)4;\ 4,8,8\}$, $n > 2$. *Then there is an equilateral n-gon located on the* α*-tiling iff n is even.*
- *For any other Archimedean tiling* T *and for every* $n > 2$ *there is an equilateral n-gon located on* T.

Corollary

For every $n > 2$ there is an equilateral n-gon located on the lattice $\mathbb{Z}^3$.

Equiangular polygons located on Archimedean tilings

Theorem (Nurligareev, 2011)

There are equiangular n-gons located on the α-tiling iff

Tiling type (α)	(4)4	4, 8, 8	(3)3, (2)4
Number of sides (n)	4, 8	4, 8, 16	3, 4, 6, 8

Tiling type (α)	(6)3 (3)6 3, 6, 3, 6 (4)3, 6	3, 12, 12 (2)3, 4, 3, 4 3, 4, 6, 4 4, 6, 12
Number of sides (n)	3, 4, 6, 12	3, 4, 6, 8, 12, 24

Equilateral polygons

- Given $\alpha \in \{(4)4;\ 4,8,8\}$ and even $n > 2$, provide an example of an equilateral n-gon located on α-tiling.
- Given $\alpha \in \{(4)4;\ 4,8,8\}$ and odd $n > 2$, prove that an equilateral n-gon cannot located on α-tiling (apply number theory methods).

Equilateral polygons

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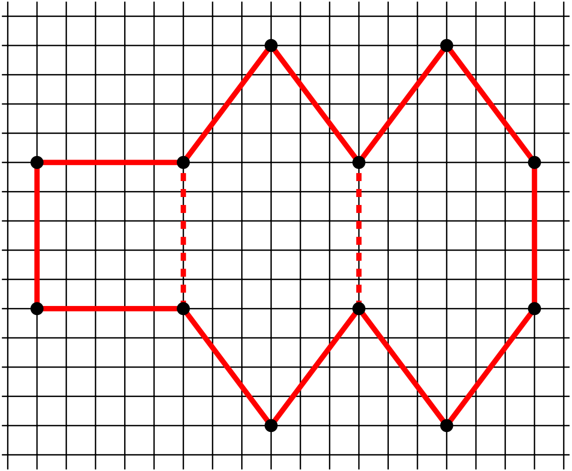
- Given $\alpha \in \{(4)4;\ 4,8,8\}$ and odd $n > 2$, prove that an equilateral n-gon cannot located on α-tiling (apply number theory methods).
- Given $\alpha \in \{(2)3,4,3,4;\ (3)3,(2)4;\ (6)3\}$ and $n > 2$, provide an example of an equilateral n-gon located on α-tiling.
- For all other α, indicate the subset of vertices of α-tiling that coincides with the set of vertices for $(6)3$-tiling.

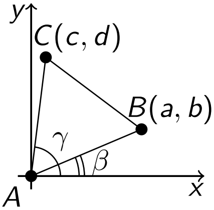
Example for (4)4-tiling



Equiangular polygons

- For points $A(0,0)$, $B(a,b)$ and $C(c,d)$ we have

$$\tan\angle BAC = \tan(\gamma-\beta) = \frac{ad-bc}{ac+bd}.$$



- Each interior angle of equiangular n-gon equals $\dfrac{(n-2)\pi}{n}$.
- Each exterior angle of equiangular n-gon equals $\dfrac{2\pi}{n}$.

Values of $\tan \frac{2\pi}{n}$

- Ratios of coordinates of tiling vertices are in the subset M:

Set M	Type of tiling α
$\{p \mid p \in \mathbb{Q}\}$	$(4)4$
$\{q\sqrt{3} \mid q \in \mathbb{Q}\}$	$(6)3;\ (3)6;\ 3,6,3,6;\ (4)3,6$
$\{p + q\sqrt{2} \mid p, q \in \mathbb{Q}\}$	$4,8,8$
$\{p + q\sqrt{3} \mid p, q \in \mathbb{Q}\}$	$(3)3,(2)4;\ 3,12,12;\ (2)3,4,3,4;\ 3,4,6,4;\ 4,6,12$

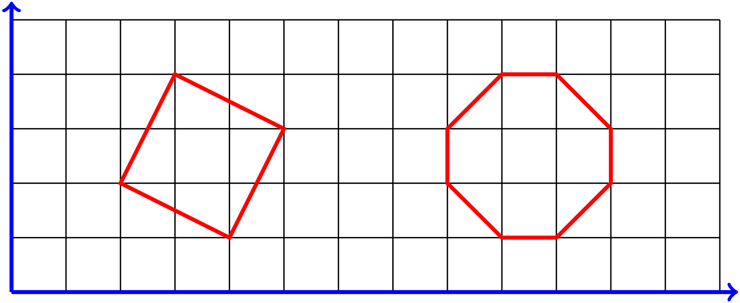
- Possible set of values $n \in \mathbb{N}$ with the condition $\tan \frac{2\pi}{n} \in M$:

M	p	$q\sqrt{3}$	$p + q\sqrt{2}$	$p + q\sqrt{3}$
n	1,2,4,8	1,2,3,4,6,12	1,2,4,8,16	1,2,3,4,6,8,12,24

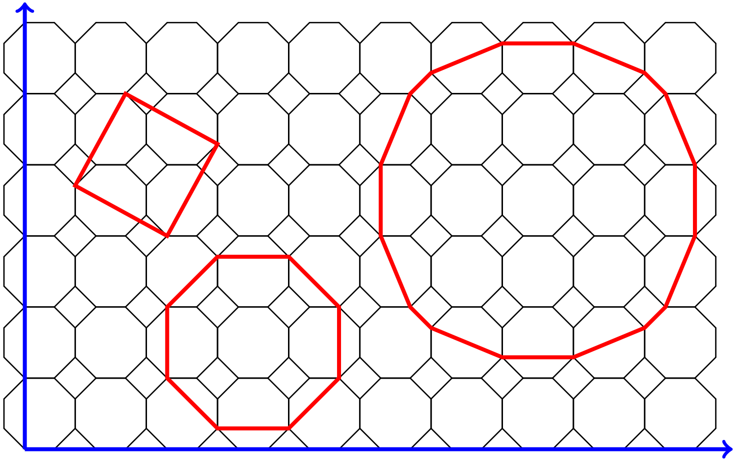
Exceptional tiling (2)3,4,3,4

- For almost every Archimedean tiling it is sufficient to give a number of examples.
- The only exception is (2)3,4,3,4: it is impossible to locate an equiangular n-gon on the tiling (2)3,4,3,4 for $n \in \{12, 24\}$.
- The proof is based on the classification of all possible angles of values $\frac{2\pi}{12}$ and $\frac{2\pi}{24}$ located on (2)3,4,3,4.

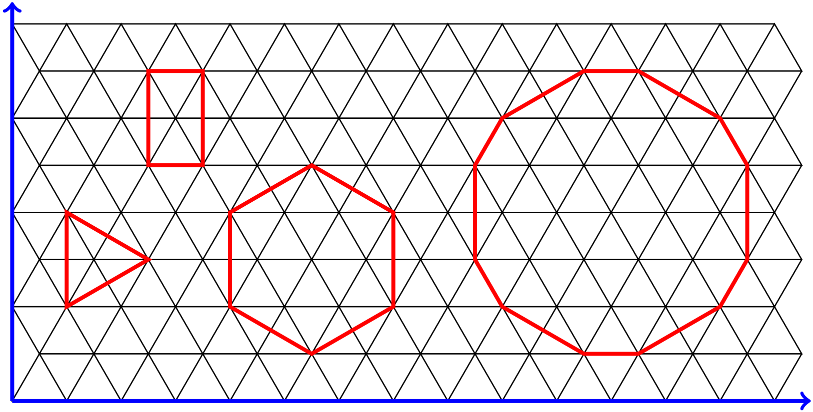
Equiangular polygons located on the tiling (4)4



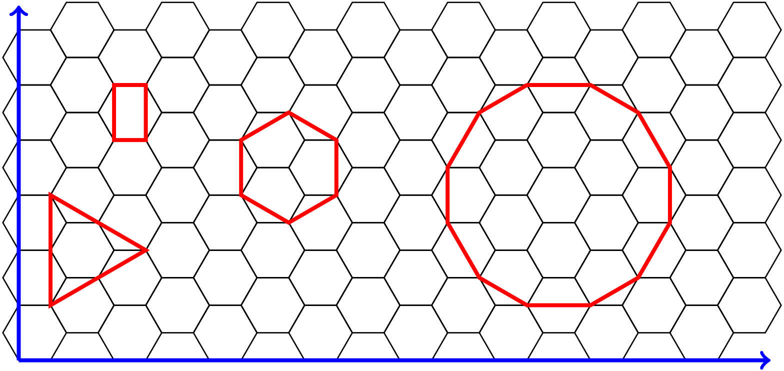
Equiangular polygons located on the tiling 4,8,8



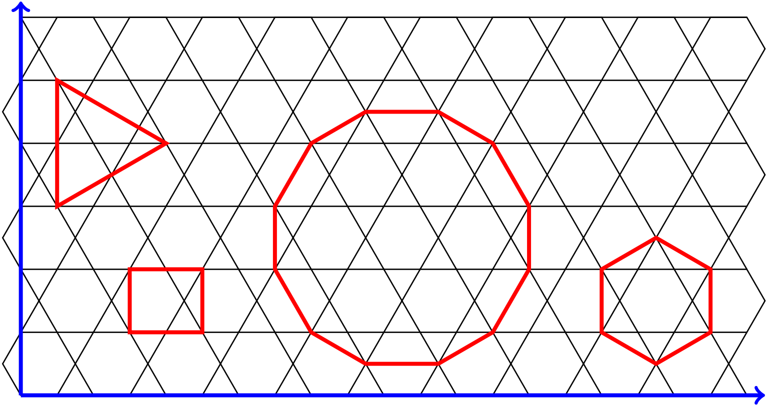
Equiangular polygons located on the tiling (6)3



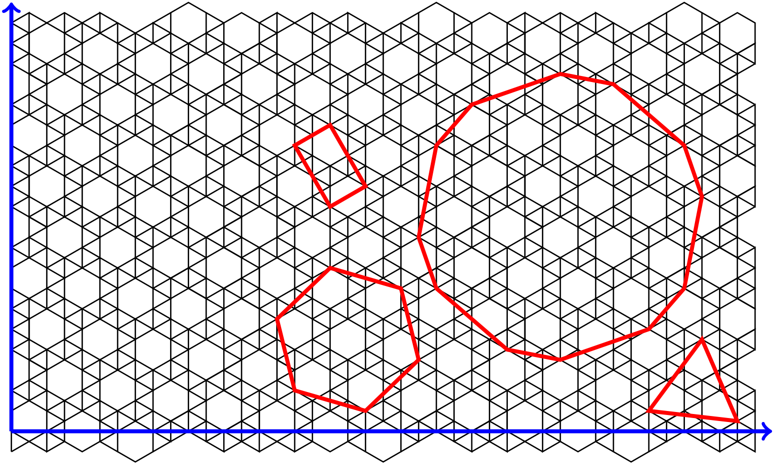
Equiangular polygons located on the tiling (3)6



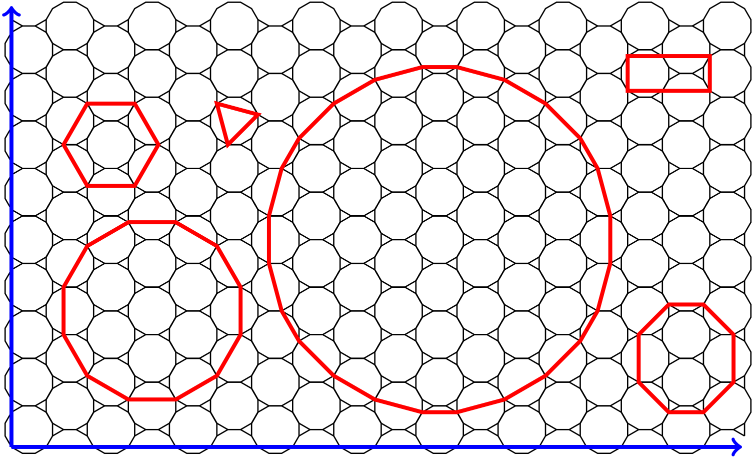
Equiangular polygons located on the tiling 3,6,3,6



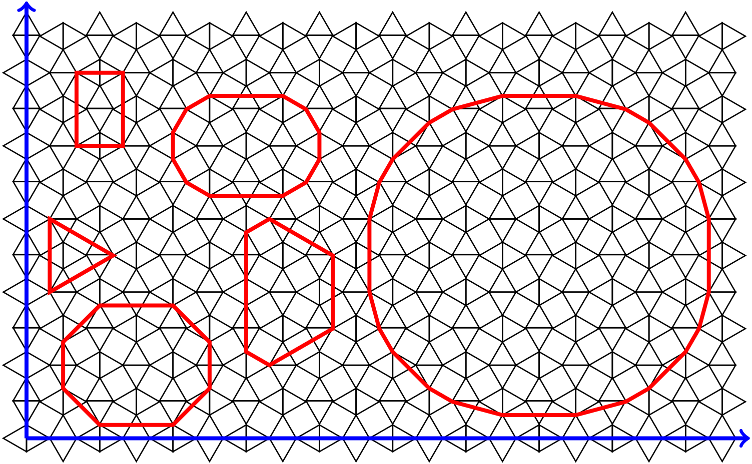
Equiangular polygons located on the tiling (4)3,6



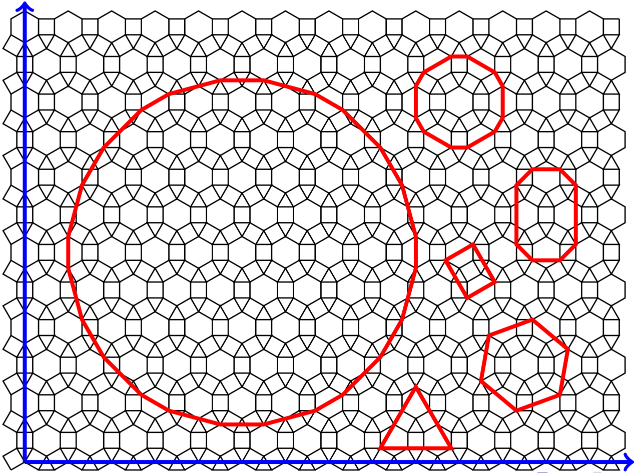
Equiangular polygons located on the tiling 3,12,12



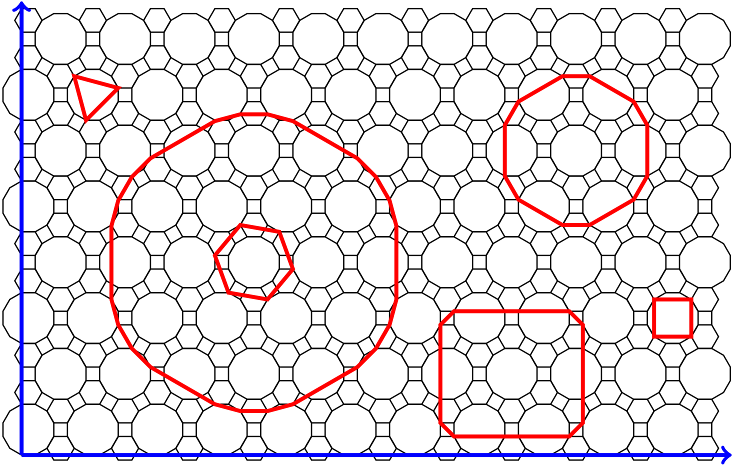
Equiangular polygons located on the tiling (2)3,4,3,4



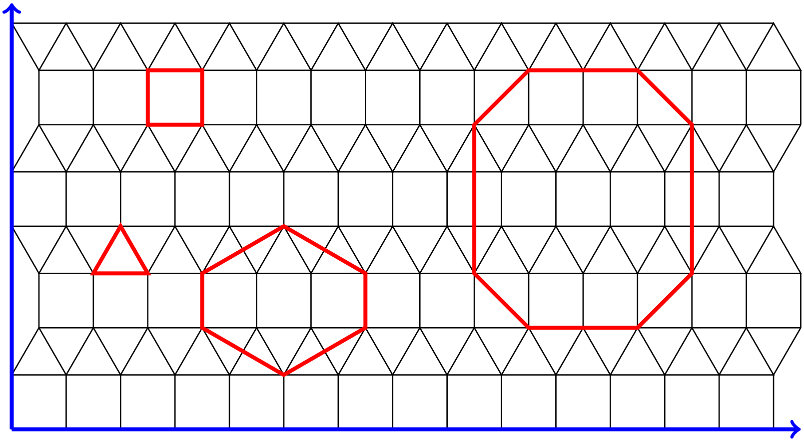
Equiangular polygons located on the tiling 3,4,6,4



Equiangular polygons located on the tiling 4,6,12



Equiangular polygons located on the tiling (3)3,(2)4



Further problems

There are several questions in connection with the problem above.

- For which $n \in \mathbb{N}$ there exists a closed equilateral broken line with the vertices in $\mathbb{Z}^3$?
- What kind of equilateral (equiangular) polygons can be located on the Penrose tiling?
- What kind of equilateral (equiangular) polygons can be located on the plane such a way that all their vertices have coordinates of $\mathbb{Q}(\alpha)$, where α is a given algebraic number?

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