Regular plane multi-tilings

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Tilings

- Tiling is a countable family of (regular) polygons which cover the plane without gaps or overlaps.
- Tiles are (regular) polygons constituting a tiling.
- Tiling is edge-to-edge if a non-empty intersection of two tiles is either an edge or a vertex for both tiles.
- Vertices and edges of a tiling are the vertices and the edges of its tiles respectively.

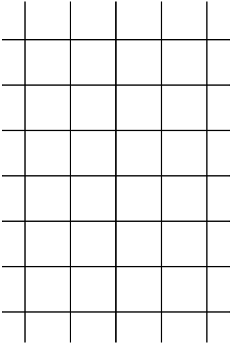
Regular tilings

- Vertex type is the order of polygons around the vertex (up to cyclic shift and direction).
- Regular tiling has the symmetry group that acts transitively on the set of vertices.
- All the vertices of a regular tiling have the same type, that is, the type of the regular tiling.

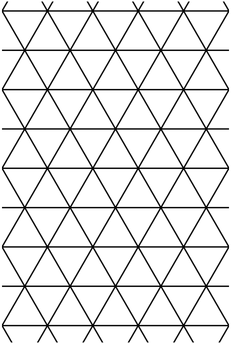
Fact. There are exactly 11 regular tilings (Archimedean tilings).

Regular tiling types

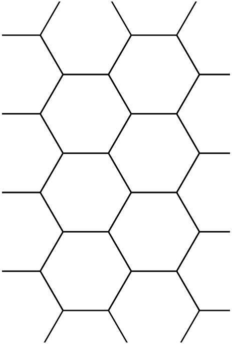
- Tilings constituted of congruent tiles.



(4)4



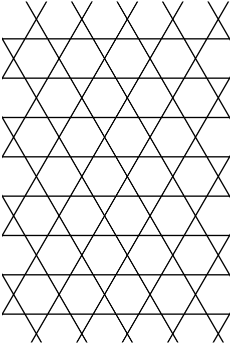
(6)3



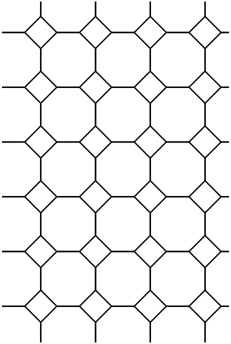
(3)6

Regular tiling types

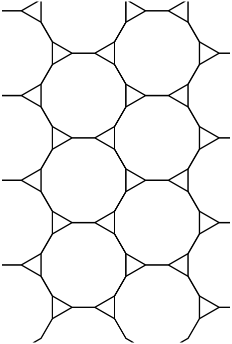
- Tilings constituted of tiles of two different kinds.



3, 6, 3, 6



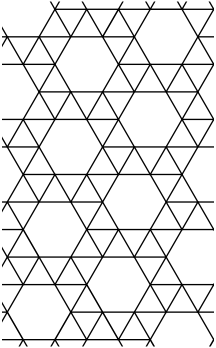
4, 8, 8

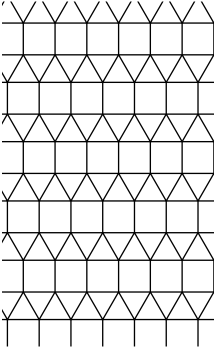


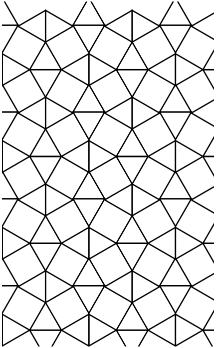
3, 12, 12

Regular tiling types

- Tilings constituted of tiles of two different kinds.

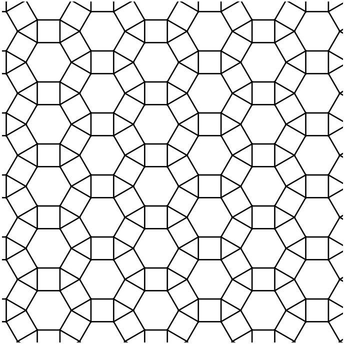

(4)3, 6


(3)3, (2)4

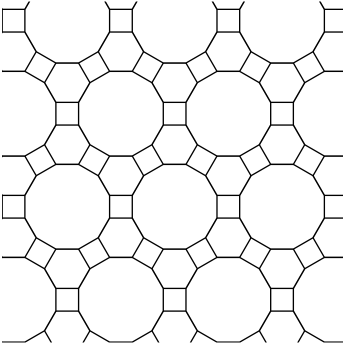

(2)3, 4, 3, 4

Regular tiling types

- Tilings constituted of tiles of three different kinds.



3, 4, 6, 4



4, 6, 12

Classification of regular tilings

- A regular tiling is determined by the type of its vertices.
- Hence, for classifying regular tilings, we need to classify all possible vertices types.
- A vertex type corresponds to some solution of the Diophantine equation.

Example

- Let a k-gon, an l-gon, an m-gon and an n-gon meet in a vertex. Hence,

$$\frac{(k-2)\pi}{k} + \frac{(l-2)\pi}{l} + \frac{(m-2)\pi}{m} + \frac{(n-2)\pi}{n} = 2\pi.$$

- In other words, $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1.$

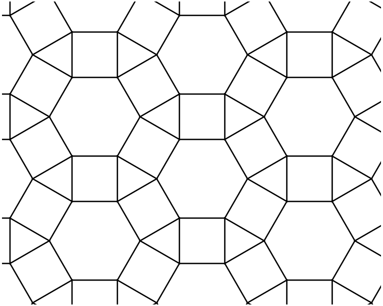
- This equation has a solution $(3,4,4,6)$.

Example, continued

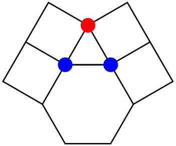
(a) Vertex type $3, 4, 6, 4$ corresponds to a regular tilings.

(b) Vertex type $3, 4, 4, 6$ cannot lead to a regular tiling.

(a)



(b)



Main problem

Consider multiple tilings determined by all possible types of vertices.

1. How many regular multiple tilings have a finite order of multiplicity?
2. What is the multiplicity, if it is finite?

Regular multiple tiling classification

Theorem (Nurligareev, 2012)

There are 21 regular multiple tiling. Among them,

- *there are 11 tilings with multiplicity* 1*:*
  (6)3; (4)4; (3)6; 3,12,12; 4,8,8; 3,6,3,6; (4)3,6; 3,4,6,4; (3)3,(2)4; (2)3,4,3,4; 4,6,12;
- *there is 1 tiling with multiplicity* 8*: 3,3,6,6;*
- *there are 9 tilings with infinite multiplicity: 3,7,42; 3,8,24; 3,9,18; 3,10,15; 4,5,20; 5,5,10; 3,3,4,12; 3,4,3,12; 3,4,4,6.*

Order of rotations

- Denote E the set of all rigid motions of the plane.
- Let $V \subset \mathbb{R}^2$ be a discrete subset with the property: $\forall\, a, b \in V \;\exists\, g \in E$ such that $g(a) = b$ and $g(V) = V$.
- Let $G = \{g \in E \mid g(V) = V\}$.
- **Lemma 1 (Coxeter).** If $g \in G$ is a rotation, then its order is in the set $\{2, 3, 4, 6\}$.

Side number restriction

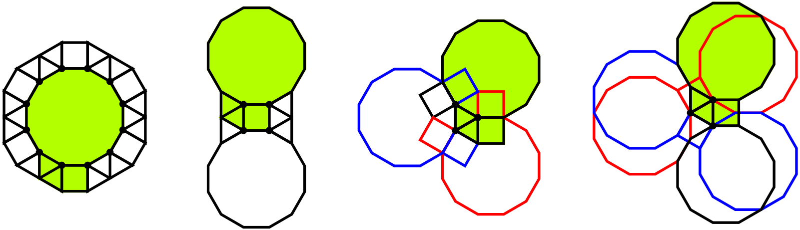
- **Lemma 2.** If a regular n-gon is a tile of some regular multiple tiling, and the type of the tiling include n exactly once, then $n \in \{3, 4, 6, 8, 12\}$.
- **Corollary.** Regular multiple tilings 3,7,42; 3,8,24; 3,9,18; 3,10,15; 4,5,20; 5,5,10 have infinite multiplicity.

Tilings 3,3,4,12; 3,4,3,12; 3,4,4,6

- To show that tilings 3,3,4,12; 3,4,3,12; 3,4,4,6 have infinite multiplicity, it is enough to establish that their sets of vertices are not discrete.
- The idea is to find translations by incommensurable vectors in the symmetry group of each tiling.

Tiling 3,3,4,12

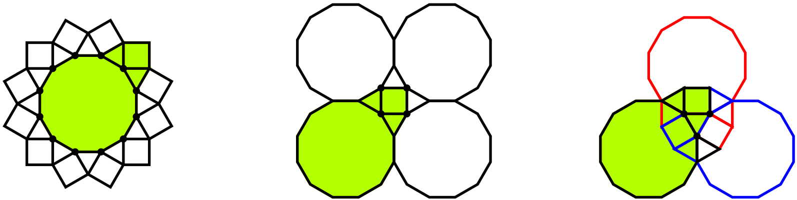
Neighbourhoods of tiles in $3,3,4,12$.



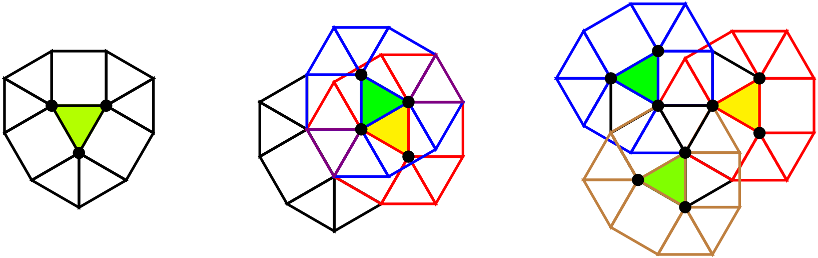
- Let the length of each edge equal 1.
- There are shifts by $\sqrt{3}$ and by $(1+\sqrt{3})$ in the same direction.

Tiling 3,4,3,12

Neighbourhoods of tiles in $3, 4, 3, 12$.



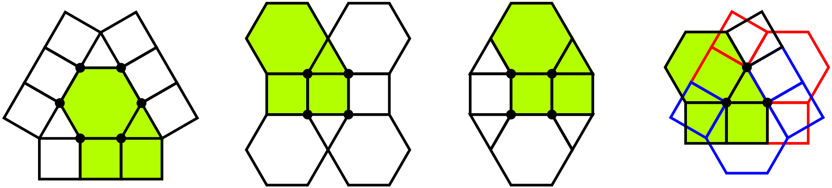
Tiling 3,4,3,12



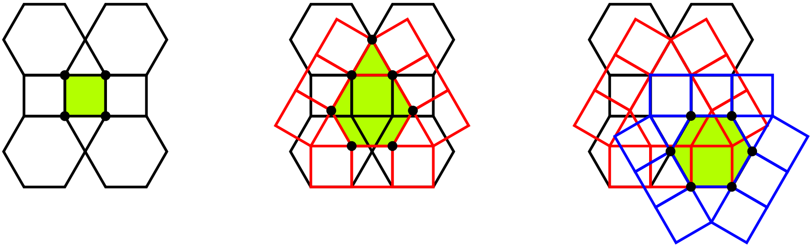
- Let the length of each edge equal 1.
- There are shifts by $\sqrt{3}$ and by $(2+\sqrt{3})$ in the same direction.

Tiling 3,4,4,6

Neighbourhoods of tiles in $3, 4, 4, 6$.



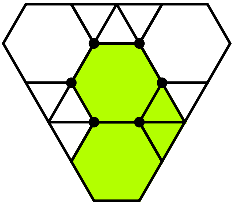
Tiling 3,4,4,6

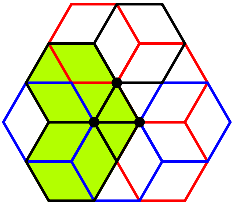


- Let the length of each edge equal 1.
- There are shifts by $\sqrt{3}$ and by $\dfrac{3+\sqrt{3}}{2}$ in the same direction.

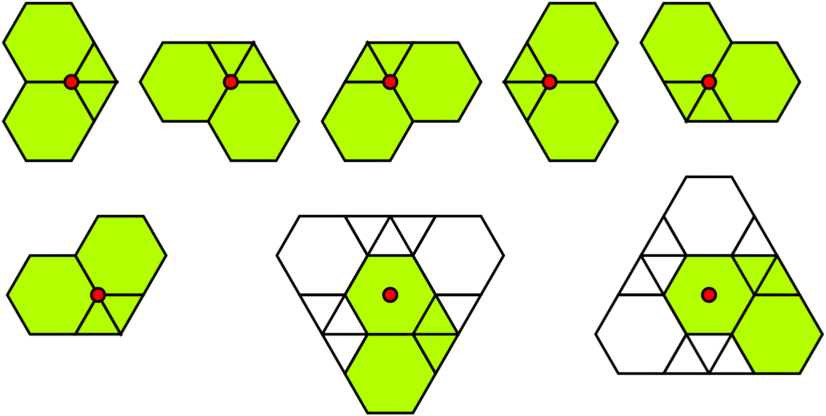
Tiling 3,3,6,6

- The set of vertices of tiling $3,3,6,6$ is discrete (it coincides with the set of vertices of tiling $(6)3$).
- Neighbourhoods of tiles in $3,3,6,6$:





Layers of tiling 3,3,6,6



Further problems

- What are regular multiple tilings of the sphere?
- What are regular multiple tilings of the hyperbolic plane?
- Is it possible to generalize results for the 3-dimensional space?

Literature I

Coxeter H.S.M.
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Grünbaum Branko, Shephard G.C.
Tilings and Patterns

Nurligareev Kh.
Regular multiple tilings (rus)
Mathematical Education, 2012, 1(61), P. 23–29.