Correlation functions in the Abelian Sandpile Model

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Graphs G and G^*

- Let $G = (V, E)$ be an undirected connected graph which may have multiple edges but loops are not allowed.
- Let $N = |V|$, that is G contains N vertices $v_1, \dots, v_N$.
- Define an extended graph $G^* = (V^*, E^*)$ such that $V^* = V \cup \{v^*\}$ and $E \subset E^*$. The vertex v^* is called the *root* or the *sink*.

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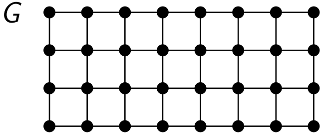
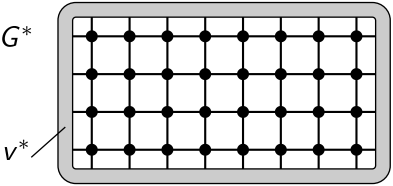
Our typical example will be a square lattice $m \times n$, $N = m \cdot n$.





Toppling matrix

- For every pair of vertices v_i and v_j we will denote by x_{ij} the number of edges that connect these vertices.
- Define a *toppling matrix* Δ by the formula below:

$$\Delta_{ij} = \begin{cases} -x_{ij}, & \text{if } i \neq j, \\ \deg v_i, & \text{if } i = j, \end{cases} \quad (1)$$

Here the size of matrix Δ is $N \times N$ though by $\deg v_i$ we mean the degree of vertex v_i in graph G^*. The value $\deg v_i$ will be called a *capacity* of v_i.

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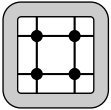
Example:

$m = n = 2$

G^* 

$$\Delta = \begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}$$

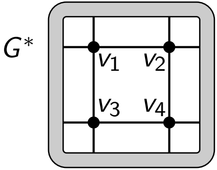
Height configurations and topplings

- A *height configuration* is a map $\eta\colon V \to \mathbb{N}$, and the set of all height configurations will be denoted by $\mathcal{H} = \mathcal{H}(G)$.
- A height configuration $\eta \in \mathcal{H}$ is *stable* if $\eta(x) \leqslant \Delta_{xx}$ for every $x \in V$.
- A site (vertex) $x \in V$ is called an *unstable site* if $\eta(x) > \Delta_{xx}$.
- The *toppling* of a site $x \in V$ is defined by

$$T_x(\eta)(z) = \eta(z) - \Delta_{xz} \tag{2}$$

  The toppling is called *legal* if the site x is unstable, otherwise it is called illegal. It is easy to see that result of the legal toppling is a height configuration again.
- The «elementary abelian property»: $T_x T_y = T_y T_x$.

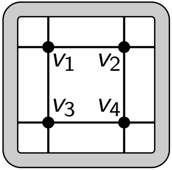
An example of topplings



5	6
2	4

An example of topplings

G^*



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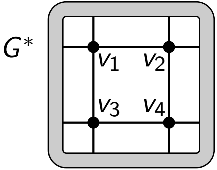

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$\xrightarrow{T_{v_1}}$

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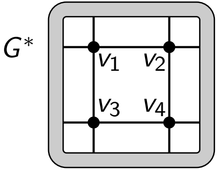

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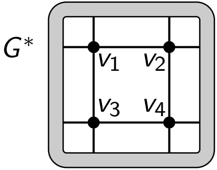

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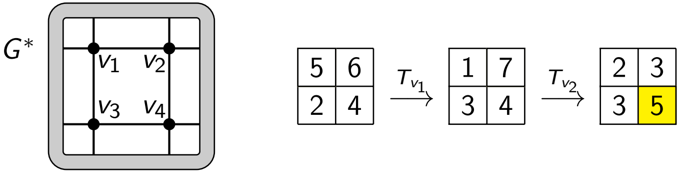
An example of topplings



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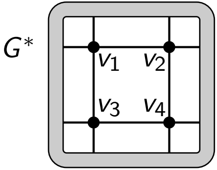

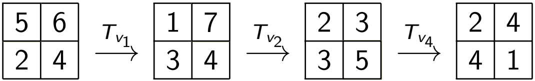

Toppling numbers

Defining the *toppling numbers* of a sequence $T_{x_1}, \ldots, T_{x_k}$ of legal topplings to be $n_x = \sum_{i=1}^{k} \mathbb{I}_{x_i = x}$ we can write the result of topplings in a following way

$$T_{x_1} \ldots T_{x_k}(\eta) = \eta - \Delta n, \tag{3}$$

where n is the column indexed by $x \in V$ with elements n_x.

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<u>Example:</u>

$$\begin{array}{|c|c|}\hline 5 & 6 \\ \hline 2 & 4 \\ \hline \end{array} \xrightarrow{T_{v_1}} \begin{array}{|c|c|}\hline 1 & 7 \\ \hline 3 & 4 \\ \hline \end{array} \xrightarrow{T_{v_2}} \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & 5 \\ \hline \end{array} \xrightarrow{T_{v_4}} \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 4 & 1 \\ \hline \end{array}$$

$$T_{v_4} T_{v_2} T_{v_1}(\eta) = \begin{pmatrix} 5 \\ 6 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 4 \\ 1 \end{pmatrix}.$$

Height configurations and topplings

- For a general height configuration we define its *stabilization* $\mathcal{S}(\eta) = T_{x_1} \dots T_{x_k}(\eta)$ by the requirement that every toppling is legal and that $\mathcal{S}(\eta)$ is stable.
- One can prove that the stabilization is well-defined, that is:
  - for every height configuration η there exists a sequence of legal topplings leading to a stable configuration,
  - the resulting stable configuration doesn't depend on the order of topplings.

Markov chain

- Let Ω be the set of all stable configurations. Then for every $x \in V$ we can define an *additional operator* $a_x : \Omega \to \Omega$ by
$$a_x(\eta) = \mathcal{S}(\eta + \delta_x). \tag{4}$$
- Let $p = p(x)$ be a probability distribution on V. Starting from $\eta_0 \in \Omega$, the state at time k is given by the random variable
$$\eta_k = \prod_{i=1}^{k} a_{X_i} \eta_0. \tag{5}$$
where $X_1, \ldots, X_k$ are i.i.d.r.v. with distribution p.
- The Markov transition operator defined on functions $f : \Omega \to \mathbb{R}$ is given by
$$Pf(\eta) = \sum_{x \in V} p(x) f(a_x \eta). \tag{6}$$

Group structure

- Let $\mathcal{R}$ be the set of all recurrent configurations of the Markov chain.
- Let $\mathcal{A}$ be the semi-group of all additional-operator products. In other words, $\mathcal{A} = \left\{ \prod_{i=1}^{k} a_{x_i} \middle| x_i \in V \right\}$.
- Define the equivalence relation on $\mathcal{A}$ by

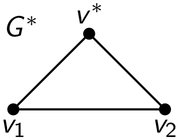
$$g_1 \sim g_2 \qquad \text{iff} \quad g_1(\eta) = g_2(\eta) \ \ \forall\, \eta \in \mathcal{R}. \tag{7}$$

Then $\mathcal{G} = \mathcal{A}/\sim$ turns out to be a group acting on $\mathcal{R}$.

Properties of the group $\mathcal{G}$ and the set $\mathcal{R}$

- The group $\mathcal{G}$ acts on $\mathcal{R}$ transitively, that is, for all $\eta \in \mathcal{G}$ the orbit $U_\eta = \{g\eta \mid g \in \mathcal{G}\} = \mathcal{R}$.
- The group $\mathcal{G}$ acts on $\mathcal{R}$ freely, that is, if $g\eta = g'\eta$ for some $g, g' \in \mathcal{G}$ and $\eta \in \mathcal{R}$ then $g = g'$.
- The stationary measure μ of Markov chain is uniform on $\mathcal{R}$. In other words,
$$\mu = \frac{1}{|\mathcal{R}|} \sum_{\eta \in \mathcal{R}} \delta_\eta.$$
- For every graph G^* we have $|\mathcal{G}| = |\mathcal{R}| = \det \Delta$.

An example of group action



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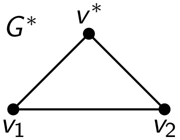

$$\Delta = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

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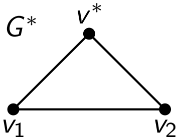

$$\Delta = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\Omega = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$$

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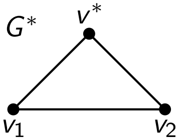

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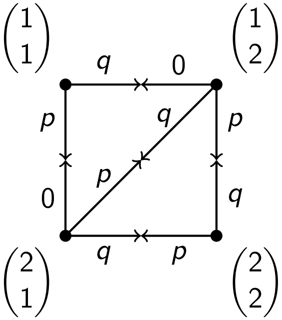

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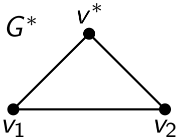

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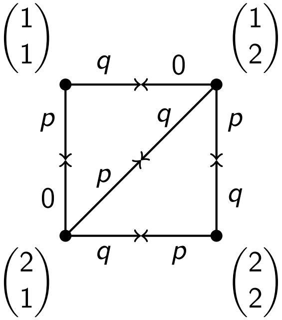
$$\mathcal{R} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}, \quad \mathcal{G} \cong \mathbb{Z}_3$$



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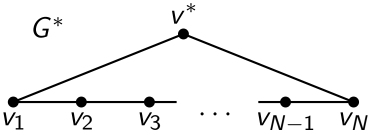

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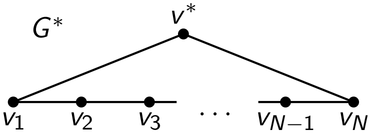
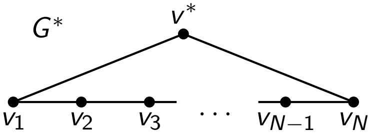


$$\Delta = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & & 0 \\ 0 & -1 & 2 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}$$

Another example of group action



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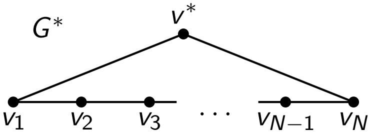
$$\det \Delta = N + 1$$

Another example of group action



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$$\det \Delta = N + 1 \qquad \mathcal{G} \cong \mathbb{Z}_{N+1}$$

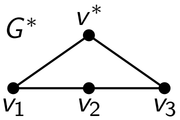
$$\mathcal{R} = \left\{ \begin{pmatrix} 2 \\ 2 \\ 2 \\ \dots \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \\ \dots \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \\ \dots \\ 1 \\ 2 \end{pmatrix}, \dots, \begin{pmatrix} 2 \\ 2 \\ 1 \\ \dots \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ \dots \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \\ \dots \\ 2 \\ 2 \end{pmatrix} \right\}$$

Matrix Tree Theorem and burning algorithm

- Matrix Tree Theorem: The number of spanning trees of graph G^* is $\det \Delta$.

- There is explicit bijection between recurrent configurations and spanning trees of G^*. This bijection is called *the burning algorithm* and proceed as follows. Given a height configuration $\eta \in \mathcal{R}$, in the first step remove («burn») from V all sites x from V which have a height $\eta(x)$ strictly bigger than the number of neighbors of x in V. After the first burning we are left with the set V_1, and we then repeat the same procedure with V replaced by V_1, and so on until no more sites can be burnt. We say that the site has *burning time* k if it is removed on the step k.

An example of burning algorithm

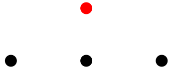


$\eta = (122)$

1	2	2

burning time

spanning tree



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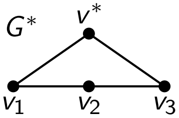

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		1

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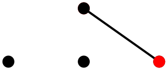

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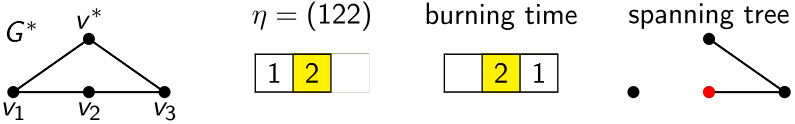

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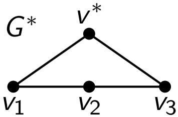


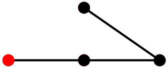
$\eta = (122)$

1		

burning time

3	2	1

spanning tree



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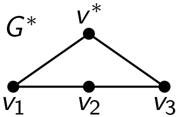

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burning time

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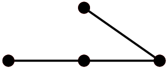
spanning tree



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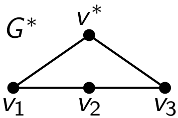

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burning time

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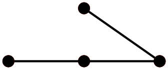
spanning tree



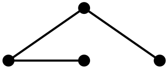
A site with burning time $k+1$ has as an ancestor a site with burning time k. If there are several neighbours of burning time k we choose the ancestor according to preference-rule defined by the height $\eta(x)$. Say the left site has lower priority.

$\eta = (212)$

burning time

1	2	1

spanning tree

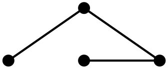


$\eta = (222)$

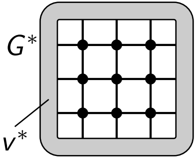
burning time

1	2	1

spanning tree



Another example of burning algorithm



preference-rule: N<W<E<S

η

4	2	4
2	1	3
3	2	2

η

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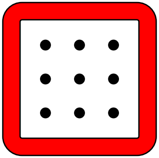
burning time

spanning tree



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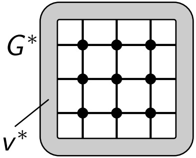

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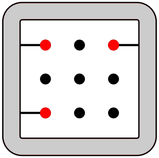
burning time

1		1
1		

spanning tree



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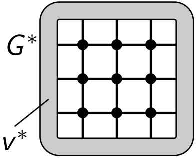

η

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2	1	3
3	2	2

preference-rule: N<W<E<S

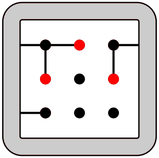
η

	2	
2	1	3
	2	2

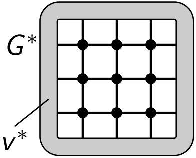
burning time

1	2	1
2		2
1		

spanning tree



Another example of burning algorithm



preference-rule: N<W<E<S

η

4	2	4
2	1	3
3	2	2

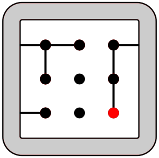
η

	1	
	2	2

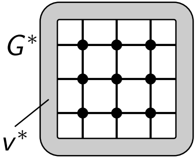
burning time

1	2	1
2		2
1		3

spanning tree



Another example of burning algorithm



preference-rule: N<W<E<S

η

4	2	4
2	1	3
3	2	2

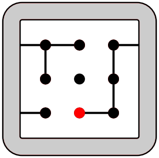
η

	1	
	2	

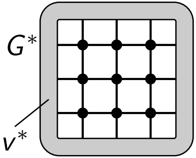
burning time

1	2	1
2		2
1	4	3

spanning tree



Another example of burning algorithm



preference-rule: N<W<E<S

η

4	2	4
2	1	3
3	2	2

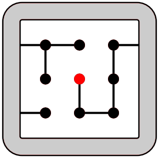
η

	1	

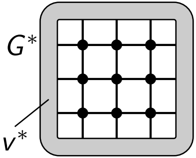
burning time

1	2	1
2	5	2
1	4	3

spanning tree



Another example of burning algorithm



preference-rule: N<W<E<S

η

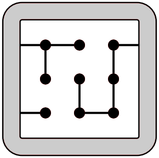
4	2	4
2	1	3
3	2	2

η

burning time

1	2	1
2	5	2
1	4	3

spanning tree



Forbidden subconfigurations

- Note that we can apply the burning procedure to any stable configuration (and obtain some tree). But for every $\eta \in \Omega \setminus \mathcal{R}$ there exist some sites that remain unburnt.
- The unburnt sites form so called *forbidden subconfigurations*, that is, pairs (W, η_W) (where $W \in V$ and $\eta_W = \eta|_W$) satisfied following requirement

$$\eta(x) \leqslant \sum_{y \in W \setminus \{x\}} (-\Delta_{xy}). \qquad (8)$$

for all sites $x \in W$.

Examples of FSC:

1	1

1	
2	1

2	2
1	2

2	2
2	2

The Bombey trick – 1

- The set $\{\eta \in \mathcal{R} \mid \eta(x) = 1\}$ is in one-to-one correspondence with the set S_1 of spanning trees that satisfy $\deg x = 1$, that is, x is a leaf of any spanning tree in S_1.
- The set S_1 can be considered as the set of all spanning trees for graph G' that is obtained from G^* by removing $(\deg x - 1)$ edges leading to site x.
- Denoting the toppling matrix of G' by Δ' with the Matrix Tree Theorem we have

$$P_1 = \mathbb{P}(\eta(x) = 1) = \frac{\det \Delta'}{\det \Delta}. \qquad (9)$$

- Defining by B the difference $\Delta' - \Delta$ we can rewrite formula (9) in the form

$$P_1 = \det(E + \Delta^{-1} B). \qquad (10)$$

The Bombey trick – 2

The same idea is used to evaluate the probability $P_{11} = \mathbb{P}(\eta(x) = \eta(y) = 1)$.

- The set $\{\eta \in \mathcal{R} \mid \eta(x) = \eta(y) = 1\}$ is in one-to-one correspondence with the set S_{11} of spanning trees that satisfy $\deg x = \deg y = 1$.
- The set S_1 can be considered as the set of all spanning trees for graph $\tilde{G}$ that is obtained from G^* by removing $(\deg x - 1)$ edges leading to site x and $(\deg y - 1)$ edges leading to site y.
- Defining the matrices $\tilde{\Delta}$ and $\tilde{B}$ as above we obtain

$$P_{11} = \frac{\det \tilde{\Delta}}{\det \Delta} = \det(E + \Delta^{-1}\tilde{B}). \qquad (11)$$

Correlation functions in the thermodynamic limit

- For the square lattice $m \times n$ with $m, n \to \infty$ $(m/n \to 1)$ the matrix Δ^{-1} turns out to be the Green function.
- Computations deliver following results.

$$\lim_{V \to \mathbb{Z}^2} \mathbb{P}(\eta(x) = 1) = \frac{2(\pi - 2)}{\pi^3} \tag{12}$$

$$\lim_{V \to \mathbb{Z}^2} (\mathbb{P}(\eta(x) = \eta(y) = 1) - (\mathbb{P}(\eta(x) = 1))^2) \simeq |x - y|^{-4}$$

- One can establish similar formulae for d-dimensional lattice.

$$\lim_{V \to \mathbb{Z}^d} (\mathbb{P}(\eta(x) = \eta(y) = 1) - (\mathbb{P}(\eta(x) = 1))^2) \simeq |x - y|^{-2d}$$

Bibliography I

Redig F.
Mathematical Aspects of the Abelian Sandpile Model..
Les Houches lecture notes. — 2005.

Priezzhev V.B.
Structure of Two-Dimensional Sandpile. I. Height Probabilities.
J. of Stat. Phys., 1994. Vol. 74. Nos. 5/6. P. 955-979.

Приезжев В.Б.
Задача о димерах и теорема Кирхгофа.
Успехи физических наук., 1985. Т. 147. Вып. 4. с. 747-765.