Tiling translation surfaces with Wang tiles

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ALEA Young — 2019

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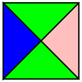
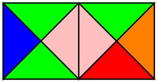
Wang tiles

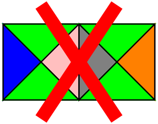
Wang tiles are squares with colored edges.

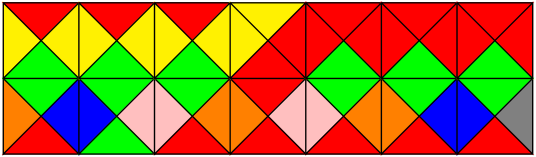


Local rules:

- Wang tiles must be placed edge-to-edge;
- colors on contiguous edges must match;
- rotations and reflections are forbidden.







Domino Problem

A protoset is a finite set of Wang tiles $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$.

Domino Problem. Can we determine if a given protoset $\mathcal{T}$ admits a valid tiling? Is there an algorithm?

A decision problem is called decidable if there is an algorithm that provides the correct yes/no answer to every input instance of the problem (otherwise, it is undecidable).

Thus, is Domino Problem decidable?

Aperiodic protosets

If a protoset $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$ admits a tiling of the plane, then there are three cases:

- every tiling is periodic;
- there are both periodic and non-periodic tilings;
- every tiling is non-periodic.

A protoset $\mathcal{T}$ is called aperiodic if it admits non-periodic tilings only. Every tiling with aperiodic protoset is called aperiodic as well.

Wang's theorem and Berger theorem

Wang's theorem. If for every $n \in \mathbb{N}$ it is possible to assemble $n \times n$ blocks of tiles with a given protoset $\mathcal{T}$, then $\mathcal{T}$ admits a tiling of the plane.

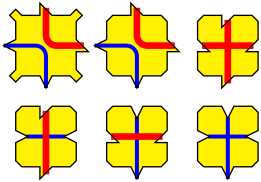
Wang's theorem and Berger theorem

Wang's theorem. If for every $n \in \mathbb{N}$ it is possible to assemble $n \times n$ blocks of tiles with a given protoset $\mathcal{T}$, then $\mathcal{T}$ admits a tiling of the plane.

Theorem (Berger, 1966). There exist aperiodic tilings.

Consequence. Domino Problem is undecidable.

Robinson tiling – 1



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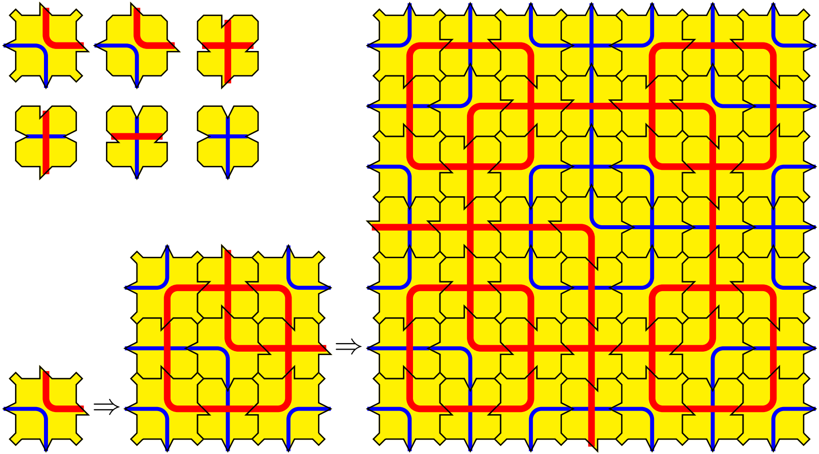
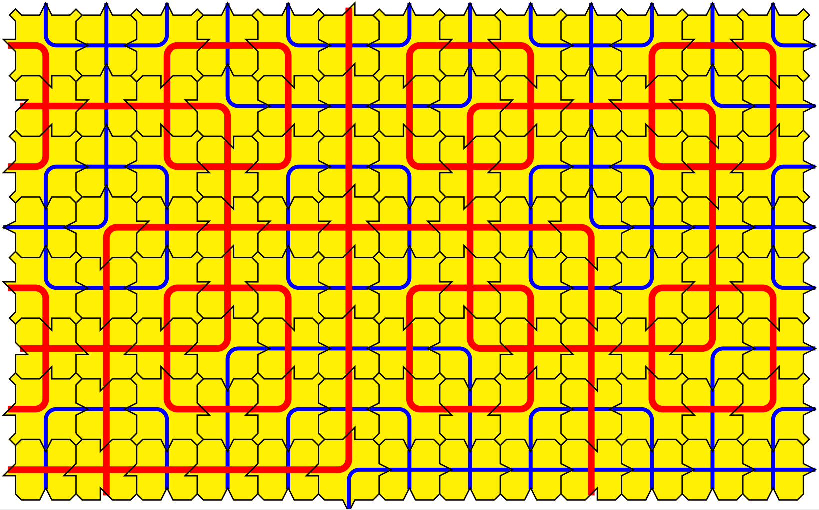
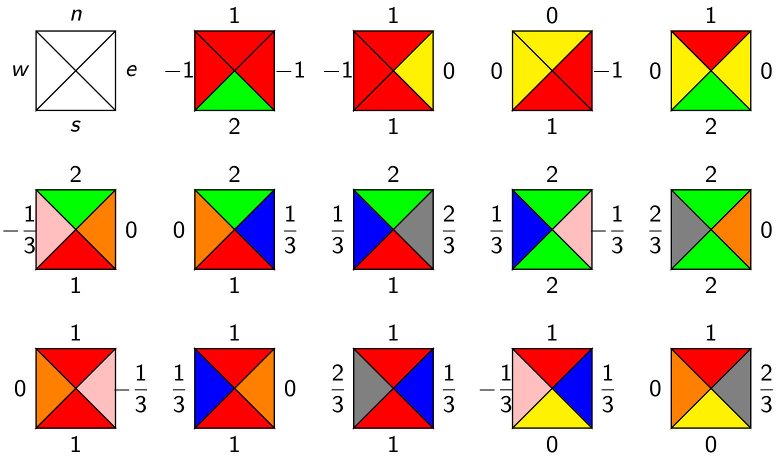


Robinson tiling – 2

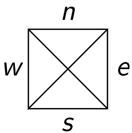


Kari-Culic tiling – 1

$nq + w = s + e$ $\qquad$ $q = 2/3,\ 2$

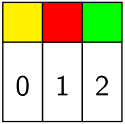


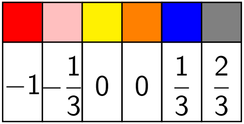
Kari-Culic tiling – 2

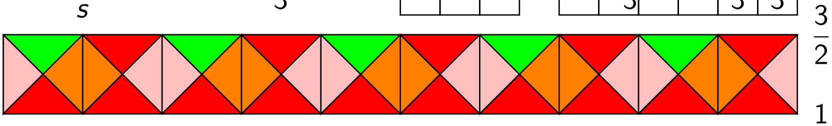


$$nq + w = s + e$$

$$q = \frac{2}{3}, 2$$







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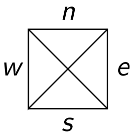
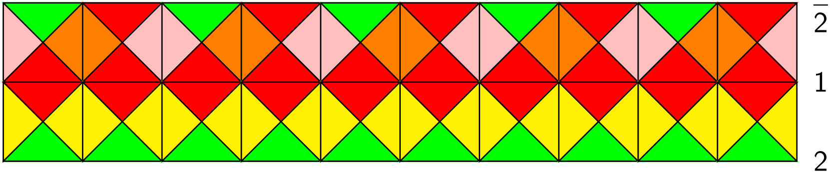


$$nq + w = s + e$$

$$q = \frac{2}{3},\ 2$$

0	1	2

-1	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{2}{3}$



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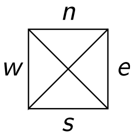

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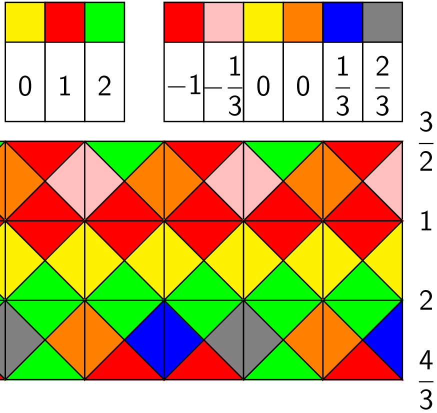

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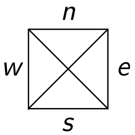

$$nq + w = s + e$$

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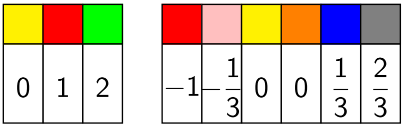
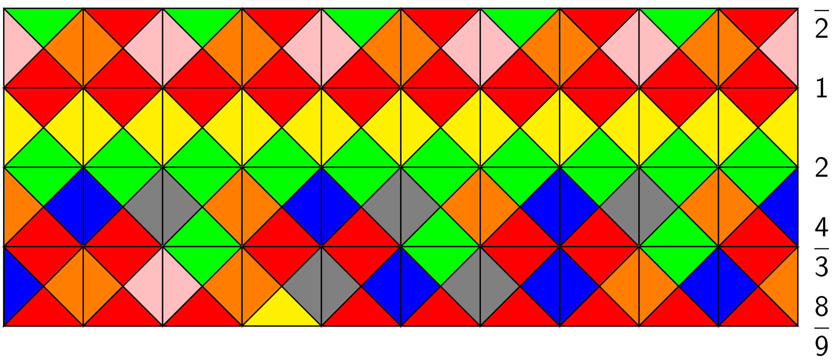
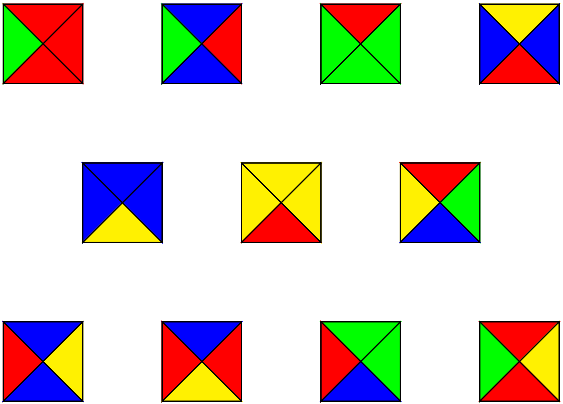




Jeandel-Rao tiling



Main question

Some famous aperiodic tilings of the plane:

- the first aperiodic tiling by Robert Berger (1966);
- classical tiling by Raphael Robinson (1971);
- amazing tiling by Jarkko Kari (1995);
- tilings with the smallest protoset by Emmanuel Jeandel and Michael Rao (2015).

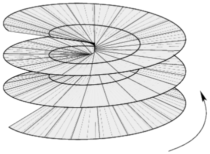
Main question. What is the nature of aperiodic tilings? How to distinguish them?

Idea. Let us tile translation surfaces.

Tilings of translation surfaces

A translation surface is a pair (S, ω), where S is a compact Riemann surface and ω is a holomorfic 1-form on S.

- The form ω has finitely many zeroes (so called singularities).
- A zero of order k corresponds to a cone angle of $2(k+1)\pi$.



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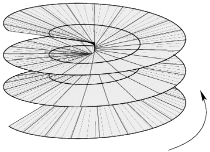
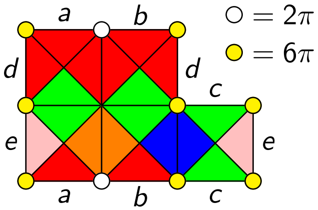


Tiling of the translation surface:

- take a set of Wang tiles,
- glue them according local rules;
- vertices $\rightarrow$ singularities.



Stating the problem

Process:

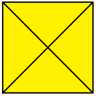
- take n Wang tiles;
- identify their edges at random (keepeng the local rules);
- obtain a translation surface.

Questions.

- Is this surface connected?
- What are typical:
  - genus;
  - number of singularities;
  - diameter?

Connectedness

Let all edges of Wang tiles have the same color.



Lemma. (Dixon, 2005) The translation surface is connected with probability 1 as $n \to \infty$. Moreover:

$$\mathbb{P}(\text{surface is connected}) = 1 - \frac{1}{n} - \sum_{k=1}^{\infty} \frac{a_k}{(k-1)! \cdot (n)_{k+1}} =$$

$$= 1 - \frac{1}{n} - \frac{1}{n^2} - \frac{4}{n^3} - \frac{23}{n^4} - \frac{171}{n^5} - \frac{1542}{n^6} - \ldots$$

- a_k is the number of connected gluings of k Wang tiles;
- $(n)_{k+1} = n(n-1)\ldots(n-k)$.

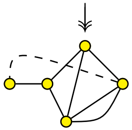
Connectedness (additional information)

$$\frac{a_{n-1}}{(n-2)!} = \frac{1}{n!} \cdot \sum_{C} (-1)^{(c_1+\ldots+c_n)-1} \cdot \frac{n!}{(1!)^{c_1} \cdot \ldots \cdot (n!)^{c_n}} \cdot \frac{a_1^{c_1} \cdot \ldots \cdot a_n^{c_n}}{c_1! \cdot \ldots \cdot c_n!}$$

where $C = \{(c_1, \ldots, c_n) \mid c_1 + 2c_2 + \ldots + nc_n = n\}$

Rooted maps

- A map is a connected graph, such that
  - half-edges incident to each vertex are in order,
  - multiple edges and loops are allowed.
- Each pair of adjacent vertex form a corner.
- A rooted map is a map with a marked corner.



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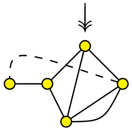
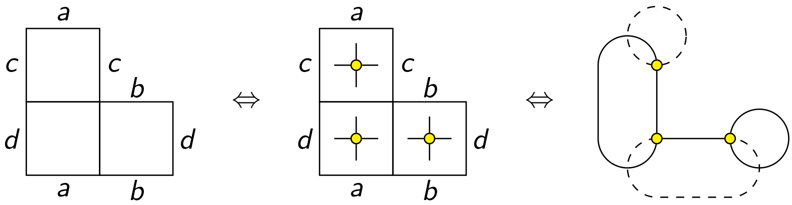


Correspondance between square tiled surfaces and maps:



Related investigations

Budzinski, Curien, Petri [BCP-2019]:

- take several polygons with the total perimeter $2n$;
- glue their edges at random;
- obtain (oriented) surface.

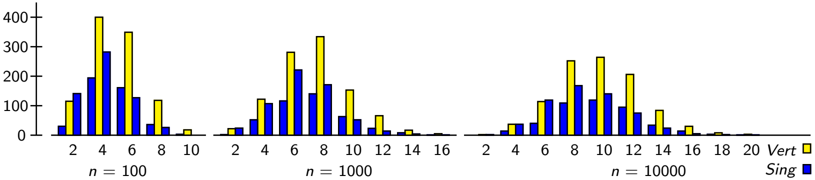
Bodini, Courtiel, Dovgal and Hwang [BCDH-2018]:

- take rooted random map with n edges.

Average number of vertices and singularities

Simulations:

- $n = 100$, $n = 1000$ and $n = 10000$;
- sample size: 1000 for each case.

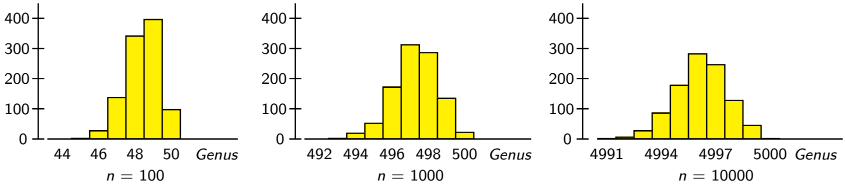


[BCDH-2018] and [BCP-2019]: $$\frac{\text{Vertices}(\mathbb{M}_n) - \log n}{\sqrt{\log n}} \xrightarrow[n\to\infty]{(d)} \mathcal{N}_1$$

Average genus

Simulations:

- $n = 100$, $n = 1000$ and $n = 10000$;
- sample size: 1000 for each case.



[BCP-2019]: $$\frac{\mathrm{Genus}(\mathbb{M}_n) - \frac{n}{2} + \log n}{\sqrt{\log n}} \xrightarrow[n\to\infty]{(d)} -\frac{\mathcal{N}_1 + \mathcal{N}_2}{2}$$

Average diameter

Simulations:

- $n = 100$, $n = 1000$ and $n = 10000$;
- sample size: 1000 for each case.

	diam $= 1$	diam $= 2$
$n = 100$	318	682
$n = 1000$	113	887
$n = 10000$	40	960

[BCP-2019]: there exist a constant $\xi \in (0, 1)$, such that

$$\lim_{n\to\infty} \left(\text{Diameter}(\mathbb{M}_n) = 3\right) = 1 - \lim_{n\to\infty} \left(\text{Diameter}(\mathbb{M}_n) = 2\right) = \xi$$

Thank you for attention!

Literature I

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Asymptotic Distribution of Parameters in Random Maps
https://arxiv.org/abs/1802.07112

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The Electronic Journal of Combinatorics, 12 (2005), #R56.

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Random Structures & Algorithms, 28 (2006), N3, 247-288.