Asymptotics for the probability of labeled objects to be connected

Khaydar Nurligareev (with Thierry Monteil)

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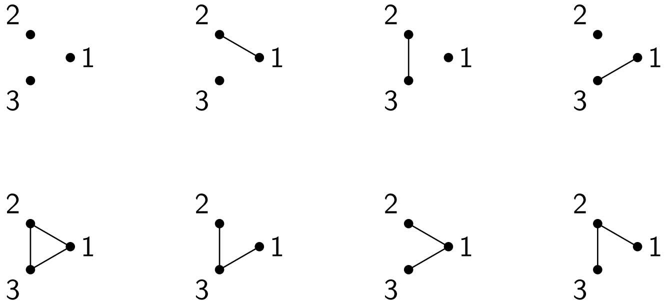
Graphs

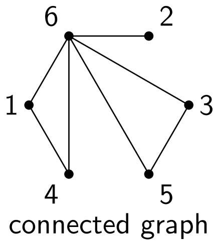
Let f_n be the number of labeled graphs with n vertices.

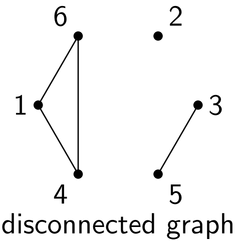


$$f_n = 2^{\binom{n}{2}}$$

Connected graphs

Let g_n be the number of connected labeled graphs with n vertices.





$$(g_n) = 1, 1, 4, 38, 728, 26704, 1866256, \ldots$$

Every graph is a disjoint union (SET) of connected graphs.

Probability of graph to be connected

Question. What is the probability $p_n = \dfrac{g_n}{f_n}$ of a random graph with n vertices to be connected as $n \to \infty$?

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$$p_n = 1 - \binom{n}{1}\frac{2}{2^n} - \binom{n}{3}\frac{2^7}{2^{3n}} - 3\binom{n}{4}\frac{2^{13}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

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- Can we have all terms at once? What is the interpretation?

Asymptotics for p_n

- Monteil, N., 2019:

  as $n \to \infty$, for every $r \geqslant 1$

$$p_n = 1 - \sum_{k=1}^{r-1} h_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

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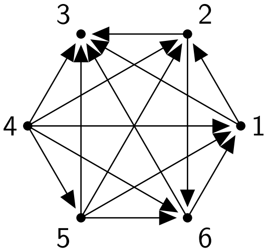
where h_k counts irreducible labeled tournaments of size k.

$$(h_k) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$$

Tournaments

A tournament is a complete directed graph.



The number of labeled tournaments with n vertices is equal to

$$f_n = 2^{\binom{n}{2}}$$

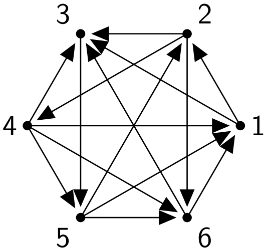
Irreducible tournaments

A tournament is called irreducible
(or strongly connected tournament),

if for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B and
2. there exist an edge from B to A.

$V = \{1, 2, 3, 4, 5, 6\}$



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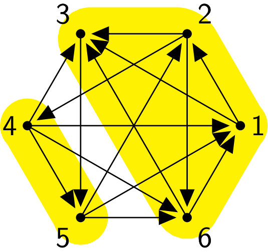

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Equivalently, for each two vertices u and v

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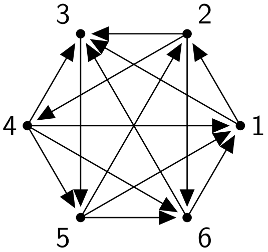

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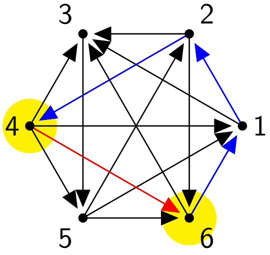
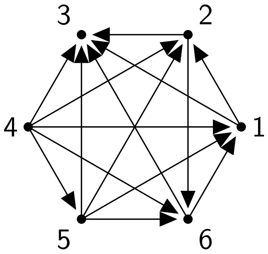


$u = 4$

$v = 6$

Tournaments as a sequence

Lemma. Every labeled tournament can be uniquely decomposed into a sequence (SEQ) of irreducible labeled tournaments.



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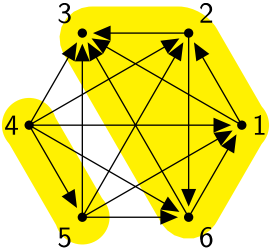

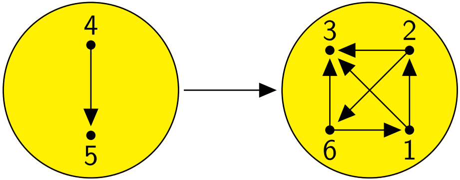

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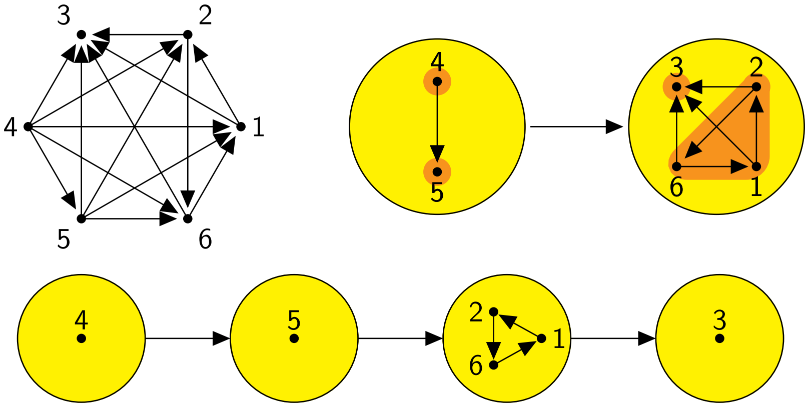

SET vs SEQ

graphs —decomposed as SET of→ connected graphs —counted by→

$$G(x) = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!} = \log(F(x))$$

graphs ←counts—

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!}$$

—counts→ tournaments

$$H(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!} = 1 - \frac{1}{F(x)}$$

tournaments —decomposed as as SEQ of→ irreducible tournaments —counted by→ $H(x)$

General result

Let $G(x) = \log(F(x)),\ \ H(x) = 1 - \dfrac{1}{F(x)},\ \ H^{(2)}(x) = 1 - \dfrac{1}{F^2(x)}.$

Theorem (Monteil, N., 2019)

If $f_n \neq 0$ for all $n \in \mathbb{N}$ and there exists $r \geqslant 1$ such that

$$\text{(i)}\ \ n \cdot \frac{f_{n-1}}{f_n} \to 0 \qquad \text{and} \qquad \text{(ii)}\ \ \sum_{k=r}^{n-r} \binom{n}{k} f_k f_{n-k} = O\left(n^r f_{n-r}\right),$$

Then

$$(a) \qquad p_n = \frac{g_n}{f_n} = 1 - \sum_{k=1}^{r-1} h_k \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right).$$

$$(b) \qquad p_n^{(1)} := \frac{h_n}{f_n} = 1 - \sum_{k=1}^{r-1} h_k^{(2)} \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right).$$

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Let $G(x) = \log(F(x)),\ H(x) = 1 - \dfrac{1}{F(x)},\quad H^{(m)}(x) = 1 - \dfrac{1}{F^m(x)}.$

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$$(b') \quad p_n^{(m)} := \frac{1}{m}\frac{h_n^{(m)}}{f_n} = 1 - \sum_{k=1}^{r-1} h_k^{(m+1)} \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right).$$

Probability for graphs and tournaments

- f_n counts labeled graphs / tournaments,
- g_n counts connected labeled graphs,
- h_n counts irreducible labeled tournaments.

$\mathbb{P}\{\text{graph is connected}\} =$

$$= \frac{g_n}{f_n} = 1 - \sum_{k=1}^{r-1} h_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right).$$

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$\mathbb{P}\{\text{tournament is irreducible}\} =$

$$= \frac{h_n}{f_n} = 1 - \sum_{k=1}^{r-1} h_k^{(2)} \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $\quad (h_k^{(2)}) = 2, -2, 4, 32, 848, 38032 \ldots$

Other applications

	square-tiled surfaces	polygons model
f_n	translation surfaces obtained by gluing squares $\{(\sigma,\tau) \mid \sigma,\tau \in S_n^2\}$	surfaces obtained by gluing polygons $\{(\sigma,\tau) \mid \tau$ is perfect matching$\}$
g_n	connected surfaces	connected surfaces
h_n	$\{(\sigma,\tau) \mid \tau$ is indecomposable permutation$\}$	$\{(\sigma,\tau) \mid \tau$ is indecomposable perfect matching$\}$
p_n	$\mathbb{P}\{$surface is connected$\}$	$\mathbb{P}\{$surface is connected$\}$
$p_n^{(1)}$	$\mathbb{P}\{$permutation is indecomposable$\}$	$\mathbb{P}\{$perfect matching is indecomposable$\}$
f_n	$n!$	$n!(n-1)!!$, n is even
g_n	$1, 3, 26, 426, 11064 \ldots$	$0, 2, 0, 60, 0, 8880 \ldots$
h_n	$h_n = n! \cdot m_n$ $(m_n) = 1, 1, 3, 13, 71, 461 \ldots$	$h_n = n! \cdot m_n$ $(m_n) = 0, 1, 0, 2, 0, 10, 0, 74 \ldots$

Happy end

Thank you for attention!