Asymptotics for connected graphs and irreducible tournaments

Khaydar Nurligareev (joint with Thierry Monteil)

LIPN, University of Paris 13

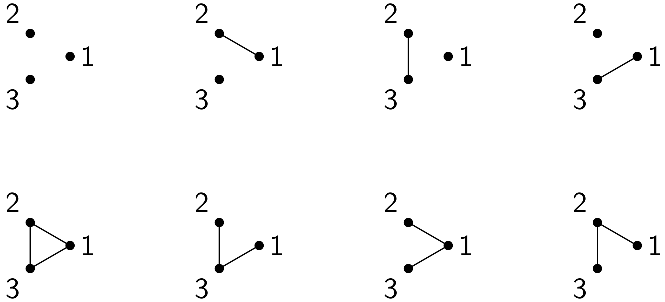
EUROCOMB 2021

September 9, 2021

Table of contents

Graphs

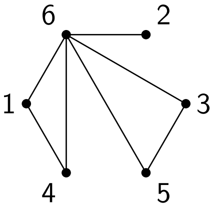
Let g_n be the number of labeled graphs with n vertices.



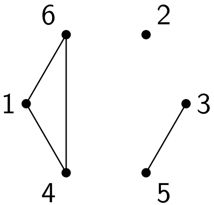
$$g_n = 2^{\binom{n}{2}}$$

Connected graphs

Let c_n be the number of connected labeled graphs with n vertices.



connected graph



disconnected graph

$$(c_n) = 1, 1, 4, 38, 728, 26704, 1866256, \ldots$$

Every graph is a disjoint union (SET) of connected graphs.

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{c_n}{g_n}$ of a random graph with n vertices to be connected as $n \to \infty$?

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{c_n}{g_n}$ of a random graph with n vertices to be connected as $n \to \infty$?

- folklore: $p_n = 1 + o(1)$

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{c_n}{g_n}$ of a random graph with n vertices to be connected as $n \to \infty$?

- folklore: $p_n = 1 + o(1)$
- Gilbert, 1959: $p_n = 1 - \frac{2n}{2^n} + O\left(\frac{n^2}{2^{3n/2}}\right)$

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{c_n}{g_n}$ of a random graph with n vertices to be connected as $n \to \infty$?

- folklore: $p_n = 1 + o(1)$
- Gilbert, 1959: $p_n = 1 - \frac{2n}{2^n} + O\left(\frac{n^2}{2^{3n/2}}\right)$
- Wright, 1970:

$$p_n = 1 - \binom{n}{1}\frac{2}{2^n} - 2\binom{n}{3}\frac{2^6}{2^{3n}} - 24\binom{n}{4}\frac{2^{10}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

Probability of a graph to be connected

Question. What is the probability $p_n = \dfrac{c_n}{g_n}$ of a random graph with n vertices to be connected as $n \to \infty$?

- folklore: $p_n = 1 + o(1)$
- Gilbert, 1959: $p_n = 1 - \dfrac{2n}{2^n} + O\left(\dfrac{n^2}{2^{3n/2}}\right)$
- Wright, 1970:

$$p_n = 1 - \binom{n}{1}\frac{2}{2^n} - 2\binom{n}{3}\frac{2^6}{2^{3n}} - 24\binom{n}{4}\frac{2^{10}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

- Can we have all terms at once? What is the interpretation?

Asymptotics for connected graphs

Theorem (1)

For any positive integer r, the probability p_n that a random labeled graph of size n is connected, satisfies

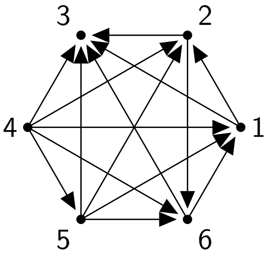
$$p_n = 1 - \sum_{k=1}^{r-1} i_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where i_k is the number of irreducible labeled tournaments of size k.

$$(i_k) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$$

Tournaments

A tournament is a complete directed graph.



The number of labeled tournaments with n vertices is equal to

$$t_n = 2^{\binom{n}{2}}$$

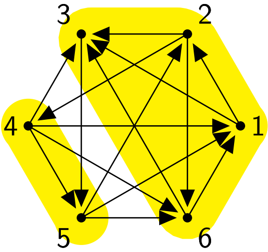
Irreducible tournaments

A tournament is called irreducible (or strongly connected tournament),

if for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B and
2. there exist an edge from B to A.

$V = \{1, 2, 3, 4, 5, 6\}$



$A = \{1, 2, 3, 6\}$

$B = \{4, 5\}$

Irreducible tournaments

A tournament is called irreducible
(or strongly connected tournament),

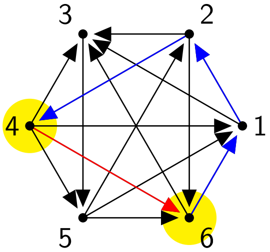
if for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B and
2. there exist an edge from B to A.

Equivalently, for each two vertices u and v

1. there is a path from u to v and
2. there is a path from v to u.

$V = \{1, 2, 3, 4, 5, 6\}$



$u = 4$

$v = 6$

Probability of a tournament to be irreducible

Question. What is the probability $q_n = \dfrac{i_n}{t_n}$ of a random tournament with n vertices to be irreducible as $n \to \infty$?

Probability of a tournament to be irreducible

Question. What is the probability $q_n = \dfrac{i_n}{t_n}$ of a random tournament with n vertices to be irreducible as $n \to \infty$?

- Moon and Moser, 1962: $q_n = 1 + o(1)$

Probability of a tournament to be irreducible

Question. What is the probability $q_n = \dfrac{i_n}{t_n}$ of a random tournament with n vertices to be irreducible as $n \to \infty$?

- Moon and Moser, 1962: $q_n = 1 + o(1)$
- Moon, 1968: $q_n = 1 - \dfrac{4n}{2^n} + O\left(\dfrac{n^2}{2^{2n}}\right)$

Probability of a tournament to be irreducible

Question. What is the probability $q_n = \frac{i_n}{t_n}$ of a random tournament with n vertices to be irreducible as $n \to \infty$?

- Moon and Moser, 1962: $q_n = 1 + o(1)$
- Moon, 1968: $q_n = 1 - \frac{4n}{2^n} + O\left(\frac{n^2}{2^{2n}}\right)$
- Wright, 1970:

$$q_n = 1 - \binom{n}{1}\frac{2^2}{2^n} + \binom{n}{2}\frac{2^4}{2^{2n}} - \binom{n}{3}\frac{2^8}{2^{3n}} - \binom{n}{4}\frac{2^{15}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

Probability of a tournament to be irreducible

Question. What is the probability $q_n = \frac{i_n}{t_n}$ of a random tournament with n vertices to be irreducible as $n \to \infty$?

- Moon and Moser, 1962: $q_n = 1 + o(1)$
- Moon, 1968: $q_n = 1 - \frac{4n}{2^n} + O\left(\frac{n^2}{2^{2n}}\right)$
- Wright, 1970:

$$q_n = 1 - \binom{n}{1}\frac{2^2}{2^n} + \binom{n}{2}\frac{2^4}{2^{2n}} - \binom{n}{3}\frac{2^8}{2^{3n}} - \binom{n}{4}\frac{2^{15}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

- Can we have all terms at once? What is the interpretation?

Asymptotics for irreducible tournaments

Theorem (2)

For any positive integer r, the probability q_n that a random labeled tournament of size n is irreducible, satisfies

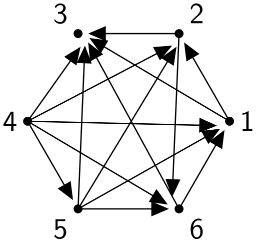
$$q_n = 1 - \sum_{k=1}^{r-1} (2i_k - i_k^{(2)}) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $i_k^{(2)}$ is the number of irreducible labeled tournaments of size k with two irreducible components.

$$\begin{array}{rcrrrrrrl} (i_k) & = & 1, & 0, & 2, & 24, & 544, & 22320, & \ldots \\ (i_k^{(2)}) & = & 0, & 2, & 0, & 16, & 240, & 6608, & \ldots \\ (2i_k - i_k^{(2)}) & = & 2, & -2, & 4, & 32, & 848, & 38032, & \ldots \end{array}$$

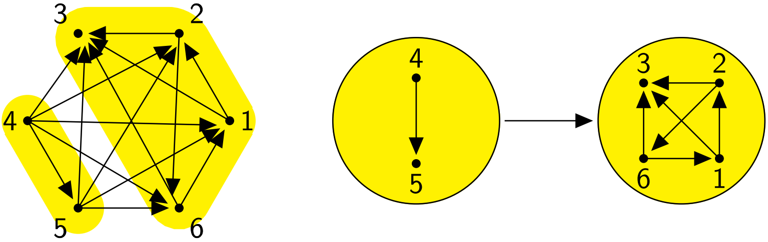
Tournament as a sequence

Lemma. Every labeled tournament can be uniquely decomposed into a sequence (SEQ) of irreducible labeled tournaments.



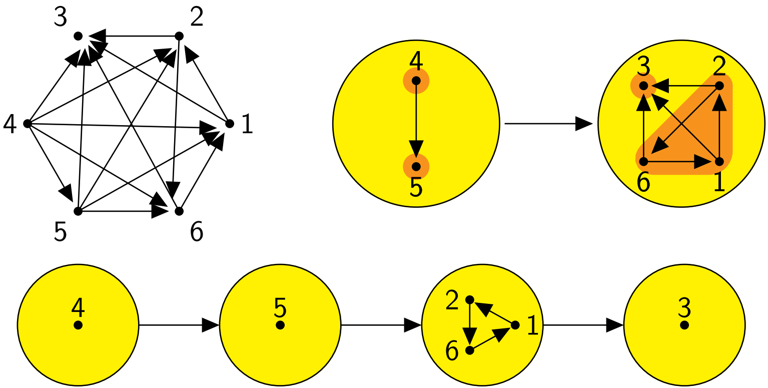
Tournament as a sequence

Lemma. Every labeled tournament can be uniquely decomposed into a sequence (SEQ) of irreducible labeled tournaments.



Tournament as a sequence

Lemma. Every labeled tournament can be uniquely decomposed into a sequence (SEQ) of irreducible labeled tournaments.



SET and SEQ decompositions

graphs — decomposed as SET of → connected graphs — counted by →

$$C(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!} = \log(G(z))$$

counts ↑

$$G(z) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{z^n}{n!} = T(z)$$

counts ↓

tournaments — decomposed as as SEQ of → irreducible tournaments — counted by →

$$I(z) = \sum_{n=0}^{\infty} i_n \frac{z^n}{n!} = 1 - \frac{1}{T(z)}$$

Main tool: Bender's theorem

Theorem (Bender, 1975)

- $A(z) = \sum_{n=1}^{\infty} a_n z^n$ *is a formal power series,* $\forall n \in \mathbb{N} : a_n \neq 0$;
- $F(x, y)$ *is a function analytic in a neighborhood of* $(0; 0)$;
- $B(z) = \sum_{n=1}^{\infty} b_n z^n = F(z, A(z))$;
- $D(z) = \sum_{n=1}^{\infty} d_n z^n = \frac{\partial F}{\partial y}(z, A(z))$.

If (i) $\frac{a_{n-1}}{a_n} \to 0$ *and* (ii) $\exists r \geqslant 1 : \sum_{k=r}^{n-r} |a_k a_{n-k}| = O(a_{n-r})$,

then, as $n \to \infty$,
$$b_n = \sum_{k=0}^{r-1} d_k a_{n-k} + O(a_{n-r}).$$

Applying Bender's theorem for graphs

Take

- $A(z) = G(z) - 1 = T(z) - 1$,
- $F(x, y) = \ln(1 + y)$.

Then

- $B(z) = F(z, A(z)) = \ln\big(G(z)\big) = C(z)$,
- $D(z) = \dfrac{\partial F}{\partial y}(z, A(z)) = \dfrac{1}{T(z)} = 1 - I(z)$.
- The statement of Bender's theorem transforms into

$$b_n = \frac{c_n}{n!} = \frac{g_n}{n!} - \frac{1}{n!} \sum_{k=1}^{r-1} \binom{n}{k} i_k g_{n-k} + O\left(\frac{g_{n-r}}{(n-r)!} \right)$$

Applying Bender's theorem for tournaments

Take

- $A(z) = T(z) - 1,$
- $F(x,y) = -\dfrac{1}{1+y}.$

Then

- $B(z) = F(z, A(z)) = -\dfrac{1}{T(z)} = -1 + I(z),$
- $D(z) = \dfrac{\partial F}{\partial y}(z, A(z)) = \dfrac{1}{\big(T(z)\big)^2} = \big(1 - I(z)\big)^2.$
- The statement of Bender's theorem transforms into

$$b_n = \frac{i_n}{n!} = \frac{t_n}{n!} - \frac{1}{n!}\sum_{k=1}^{r-1}\binom{n}{k}\left(2i_k - i_k^{(2)}\right)t_{n-k} + O\left(\frac{t_{n-r}}{(n-r)!}\right)$$

Asymptotics for graphs

Theorem (forthcoming)

For any positive m and r, the probability $p_n^{(m+1)}$ that a random labeled graph of size n has exactly $(m+1)$ connected components, satisfies

$$p_n^{(m+1)} = 1 - \sum_{k=1}^{r-1} \alpha_k^{(m+1)} \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where

$$\alpha_k^{(m+1)} = \sum_{s=1}^{k} (-1)^s \binom{s}{m} c_k^{(s)}$$

and $c_k^{(s)}$ is the number of labeled graphs of size k with s connected components.

Asymptotics for tournaments

Theorem (forthcoming)

For any positive m and r, the probability $q_n^{(m+1)}$ that a random labeled tournament of size n has exactly $(m+1)$ irreducible components, satisfies

$$q_n^{(m+1)} = 1 - \sum_{k=1}^{r-1} \beta_k^{(m+1)} \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $$\beta_k^{(m+1)} = (m+1)\left(i_k^{(m)} - 2i_k^{(m+1)} + i_k^{(m+2)}\right)$$

and $i_k^{(s)}$ is the number of labeled tournaments of size k with s irreducible components.

Asymptotics for the Erdös-Rényi model $G(n, p)$

Theorem (forthcoming)

For any positive r, the probability p_n that a random labeled graph $G(n, p)$ is connected, satisfies

$$p_n = 1 - \sum_{k=1}^{r-1} P_k(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}} + O\big(n^r q^{nr}\big),$$

where $\rho = 1/(1-p)$ and a sequence of polynomials $P_k(\rho)$ has an explicit combinatorial interpretation.

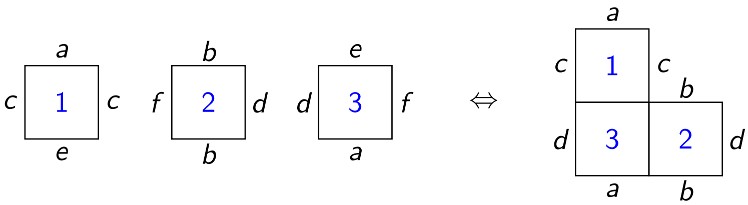
$$\big(P_k(\rho)\big) = 1,\ \rho - 2,\ \rho^3 - 6\rho + 6,\ \rho^6 - 8\rho^3 - 6\rho^2 + 36\rho - 24, \ldots$$

Many thanks to all listeners

Thank you for your attention!

Square-tiled surfaces

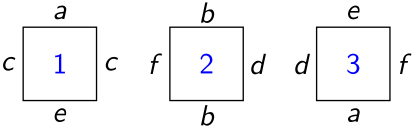
- Take n labeled squares.
- Identify their sides by translation (right side $\leftrightarrow$ left side, bottom side $\leftrightarrow$ top side).
- If obtained surface is connected, then it is called a labeled square-tiled surface (SQS) or origami.



Square-tiled surfaces

SQS is determined by the pair of permutations $(h, v) \in S_n^2$ acting transitively on $\{1, \ldots, n\}$:

- h: horizontal (right) permutation,
- v: vertical (top) permutation,
- transitive action $\leftrightarrow$ connectedness of SQS.



$\Leftrightarrow$

$h = (1)(23)$

$v = (13)(2)$

Probability to obtain a square-tiled surface

- g_n counts surfaces generated by pairs $(\sigma, \tau) \in S_n^2$,
- c_n counts connected surfaces (SQS),
- $i_n = n! \cdot \mu_n$, where μ_n counts indecomposable permutations.

$\mathbb{P}\{\text{surface is connected}\} =$

$$= \frac{c_n}{g_n} = 1 - \sum_{k=1}^{r-1} \frac{\mu_k}{(n)_k} + O\left(\frac{1}{n^r}\right),$$

where $(n)_k = n(n-1)\dots(n-k+1)$ are the falling factorials

and $(\mu_k) = 1, 1, 3, 13, 71, 461, 3447, 29093, \dots$

Indecomposable permutations

A permutation $\sigma \in S_n$ is

- decomposable, if there is an index $p < n$ such that $\sigma\big(\{1,\ldots,p\}\big) = \{1,\ldots,p\}$.
- indecomposable otherwise.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}$$

decomposable ($p = 3$) indecomposable

Observation. Every permutation can be uniquely decomposed into a sequence (SEQ) of indecomposable permutations.

Probability of a permutation to be indecomposable

- $t_n = (n!)^2$ counts pairs of permutations,
- $i_n = n! \cdot \mu_n$, where μ_n counts indecomposable permutations.
- $i_n^{(2)} = n! \cdot \mu_n^{(2)}$, where $\mu_n^{(2)}$ counts permutations that have exactly 2 indecomposable parts.

$\mathbb{P}\{\text{permutation is indecomposable}\} =$

$$= \frac{i_n}{t_n} = \frac{\mu_n}{n!} = 1 - \sum_{k=1}^{r-1} \frac{2\mu_k - \mu_k^{(2)}}{(n)_k} + O\left(\frac{1}{n^r}\right),$$

$$\begin{array}{rcl} (\mu_k) & = & 1,\ 1,\ 3,\ 13,\ 71,\ 461,\ 3447,\ \ldots \\ (\mu_k^{(2)}) & = & 0,\ 1,\ 2,\ 7,\ 32,\ 177,\ 1142,\ \ldots \\ (2\mu_k - \mu_k^{(2)}) & = & 2,\ 1,\ 4,\ 19,\ 110,\ 745,\ 5752,\ \ldots \end{array}$$

Other applications

	combinatorial map model	$(D+1)$-colored graphs
f_n	surfaces obtained by gluing polygons $\{(\sigma,\tau) \mid \tau$ is perfect matching$\}$	bipartite regular graphs with colored edges $(\sigma_1,\ldots,\sigma_{D+1}) \in S_n^{D+1}$
g_n	connected surfaces	connected graphs
h_n	$\{(\sigma,\tau) \mid \tau$ is indecomposable perfect matching$\}$	$(\tau_1,\ldots,\tau_{D-1})$ is indecomposable tuple of permutations
p_n	$\mathbb{P}\{$surface is connected$\}$	$\mathbb{P}\{$graph is connected$\}$
$p_n^{(1)}$	$\mathbb{P}\{$perfect matching is indecomposable$\}$	$\mathbb{P}\{$tuple of permutations is indecomposable$\}$
f_{2n}	$(2n)!(2n-1)!!$	$(2n)! \cdot (n!)^{D-1}$
g_{2n}	$2, 60, 8880, 3558240\ldots$	$2, 12(2^D-1), \ldots$
μ_{2n} h_{2n}	$h_{2n} = (2n)! \cdot \mu_{2n}$ $(\mu_{2n}) = 1, 2, 10, 74, 706\ldots$	$h_{2n} = (2n)! \cdot \mu_{2n}$ $1, 2^{D-1}-1, 6^{D-1}-2^D+1, \ldots$

Literature I

Moon J.W.
Topics on tournament.
New-York: Holt, Rinehart and Winston VIII, 1968.

Bender E.A.
An asymptotic expansion for some coefficients of some formal power series
J. Lond. Math. Soc., Ser. 9 (1975), pp. 451-458.

Gilbert E.N.
Random graphs
Ann. Math. Stat., Vol. 30, N. 4 (1959), pp. 1141-1144.

Literature II

Moon J.W., Moser L.
Almost all tournaments are irreducible
Can. Math. Bull., 5 (1962), 61-65.

Wright E.M.
Asymptotic relations between enumerative functions in graph theory
Proc. Lond. Math. Soc., Vol. s3-20, Issue 3 (1970), pp. 558-572.

Wright E.M.
The Number of Irreducible Tournaments
Glasg. Math. J., Vol. 11, Issue 2 (1970), pp. 97-101.