Asymptotic probability of connected surfaces

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LIPN, Paris 13

Seminar Structures on Surfaces

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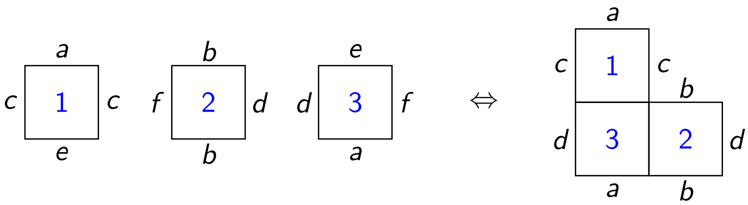
Table of contents

Square-tiled surfaces

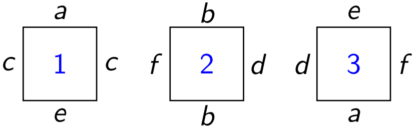
- Take n labeled squares.
- Identify their sides by translation (right side $\leftrightarrow$ left side, bottom side $\leftrightarrow$ top side).
- If the obtained surface is connected, then it is called a labeled square-tiled surface (SQS) or origami.



Square-tiled surfaces

SQS is determined by the pair of permutations $(h, v) \in S_n^2$ acting transitively on $\{1, \dots, n\}$:

- h: horizontal (right) permutation,
- v: vertical (top) permutation,
- transitive action $\leftrightarrow$ connectedness of SQS.



$\Leftrightarrow$

$h = (1)(23)$

$v = (13)(2)$

Probability of a surface to be connected

Question. What is the probability p_n of a random surface determined by $(\sigma, \tau) \in S_n^2$ to be connected as $n \to \infty$?

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- Dixon, 2005: $$p_n = 1 - \sum_{k=1}^{r-1} \frac{\mu_k}{(n)_k} + O\left(\frac{1}{n^r}\right),$$

  where $(n)_k = n(n-1)\dots(n-k+1)$ are the falling factorials.

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- Cori, 2009: the sequence

  $$(\mu_k) = 1, 1, 3, 13, 71, 461, 3447, 29093, \dots$$

  counts indecomposable permutations.

Indecomposable permutations

A permutation $\sigma \in S_n$ is

- decomposable, if there is an index $p < n$ such that $\sigma\big(\{1,\dots,p\}\big) = \{1,\dots,p\}$.
- indecomposable otherwise.

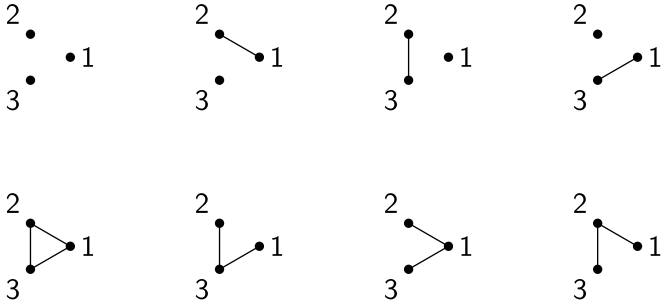
$$\begin{pmatrix} 1 & 2 & 3 & \vert & 4 & 5 \\ 3 & 1 & 2 & \vert & 5 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}$$

decomposable ($p = 3$) indecomposable

Observation. Every permutation can be uniquely decomposed into a sequence (SEQ) of indecomposable permutations.

Graphs

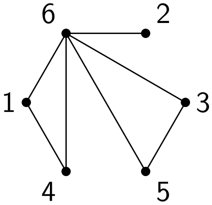
Let f_n be the number of labeled graphs with n vertices.



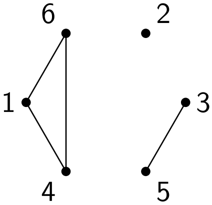
$$f_n = 2^{\binom{n}{2}}$$

Connected graphs

Let g_n be the number of connected labeled graphs with n vertices.



connected graph



disconnected graph

$$(g_n) = 1, 1, 4, 38, 728, 26704, 1866256, \ldots$$

Observation. Every graph is a disjoint union (SET) of connected graphs.

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{g_n}{f_n}$ of a random graph with n vertices to be connected as $n \to \infty$?

Probability of a graph to be connected

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- folklore: $p_n = 1 + o(1)$
- Gilbert, 1959: $p_n = 1 - \frac{2n}{2^n} + O\left(\frac{n^2}{2^{3n/2}}\right)$
- Wright, 1970:

$$p_n = 1 - \binom{n}{1}\frac{2}{2^n} - \binom{n}{3}\frac{2^7}{2^{3n}} - 3\binom{n}{4}\frac{2^{13}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{g_n}{f_n}$ of a random graph with n vertices to be connected as $n \to \infty$?

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Can we have all terms at once? What is the interpretation?

Asymptotics for p_n

- Monteil, N., 2019:

  as $n \to \infty$, for every $r \geqslant 1$

$$p_n = 1 - \sum_{k=1}^{r-1} h_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

Asymptotics for p_n

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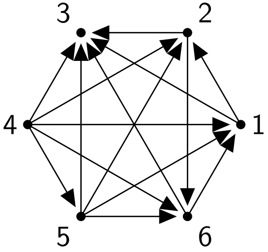
where h_k counts irreducible labeled tournaments of size k,

$$(h_k) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$$

Tournaments

A tournament is a complete directed graph.



The number of labeled tournaments with n vertices is equal to

$$f_n = 2^{\binom{n}{2}}$$

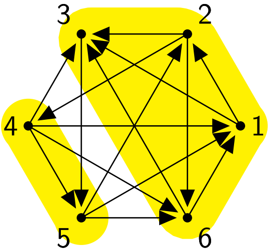
Irreducible tournaments

A tournament is called irreducible (or strongly connected tournament),

if for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B and
2. there exist an edge from B to A.

$V = \{1, 2, 3, 4, 5, 6\}$



$A = \{1, 2, 3, 6\}$
$B = \{4, 5\}$

Irreducible tournaments

A tournament is called irreducible (or strongly connected tournament),

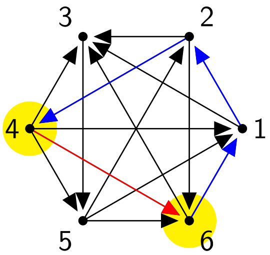
if for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B and
2. there exist an edge from B to A.

Equivalently, for each two vertices u and v

1. there is a path from u to v and
2. there is a path from v to u.

$V = \{1, 2, 3, 4, 5, 6\}$

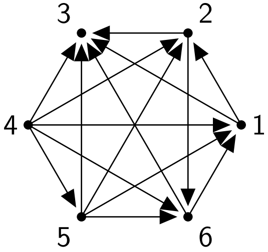


$u = 4$

$v = 6$

Tournament as a sequence

Lemma. Every labeled tournament can be uniquely decomposed into a sequence (SEQ) of irreducible labeled tournaments.



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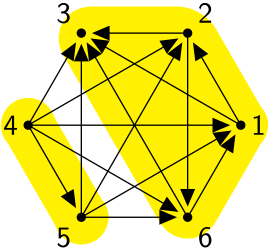

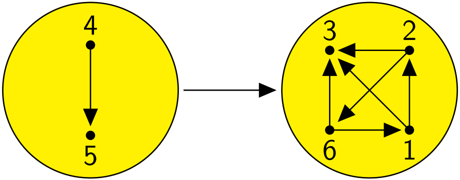

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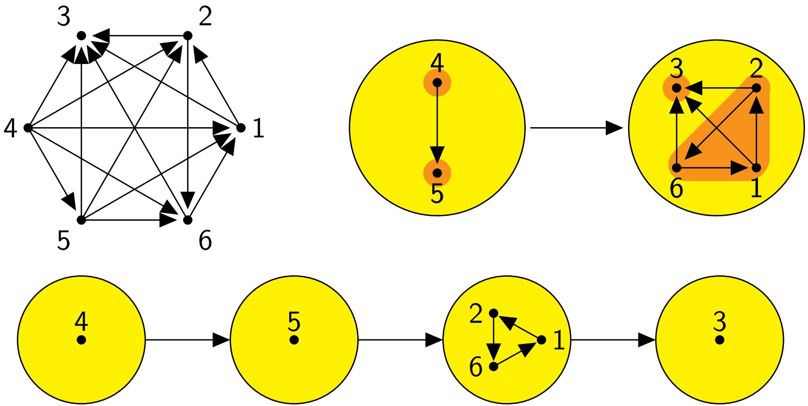


SET vs SEQ

graphs — decomposed as SET of → connected graphs — counted by →

$$G(x) = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!} = \log(F(x))$$

graphs ↑ counts

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!}$$

counts ↓ tournaments

$$H(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!} = 1 - \frac{1}{F(x)}$$

tournaments — decomposed as as SEQ of → irreducible tournaments — counted by →

SET asymptotics

Theorem (Monteil, N., 2019+)

Let $\mathcal{F} = \mathrm{SET}(\mathcal{G})$ *and* $\mathcal{F} = \mathrm{SEQ}(\mathcal{H})$.
If $f_n \neq 0$ *for all* $n \in \mathbb{N}$ *and there exists* $r \geqslant 1$ *such that*

$$\text{(i)}\ \ n \cdot \frac{f_{n-1}}{f_n} \to 0 \qquad \text{and} \qquad \text{(ii)}\ \ \sum_{k=r}^{n-r} \binom{n}{k} f_k f_{n-k} = O\big(n^r f_{n-r}\big),$$

Then, as $n \to \infty$,

$$p_n = \frac{g_n}{f_n} = 1 - \sum_{k=1}^{r-1} h_k \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right).$$

Combinatorial meaning: p_n is the probability of a random object of size n to be connected (in terms of SET-decomposition).

Example 1: square-tiled surfaces

- f_n counts surfaces generated by pairs $(\sigma, \tau) \in S_n^2$,
- g_n counts connected surfaces (SQS),
- $h_n = n! \cdot \mu_n$, where μ_n counts indecomposable permutations.

$\mathbb{P}\{\text{surface is connected}\} =$

$$= \frac{g_n}{f_n} = 1 - \sum_{k=1}^{r-1} \frac{\mu_k}{(n)_k} + O\left(\frac{1}{n^r}\right),$$

where $(n)_k = n(n-1)\dots(n-k+1)$ are the falling factorials

and $(\mu_k) = 1, 1, 3, 13, 71, 461, 3447, 29093, \dots$

Example 2: connected graphs

- f_n counts labeled graphs / tournaments,
- g_n counts connected labeled graphs,
- h_n counts irreducible labeled tournaments.

$\mathbb{P}\{\text{graph is connected}\} =$

$$= \frac{g_n}{f_n} = 1 - \sum_{k=1}^{r-1} h_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $(h_k) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$

SET_m asymptotics

Theorem (Monteil, N., 2020+)

Let $\mathcal{F} = \mathrm{SET}(\mathcal{G})$ *and* $\mathcal{G}_m = \mathrm{SET}_m(\mathcal{G})$.
If $f_n \neq 0$ *for all* $n \in \mathbb{N}$ *and there exists* $r \geqslant 1$ *such that*

$$\text{(i)} \;\; n \cdot \frac{f_{n-1}}{f_n} \to 0 \qquad \text{and} \qquad \text{(ii)} \;\; \sum_{k=r}^{n-r} \binom{n}{k} f_k f_{n-k} = O\big(n^r f_{n-r}\big),$$

Then for all $m \geqslant 1$, *as* $n \to \infty$,

$$\frac{g_n^{(m+1)}}{f_n} = \sum_{k=1}^{r-1} c_k^{(m+1)} \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right),$$

where

$$c_k^{(m+1)} = \sum_{s=1}^{k} (-1)^{s-m} \binom{s}{m} g_k^{(s)}.$$

Example 1: square-tiled surfaces

- f_n counts surfaces generated by pairs $(\sigma, \tau) \in S_n^2$,
- $g_n^{(2)}$ counts surfaces with exactly 2 connected components.

$\mathbb{P}\{\text{surface has exactly 2 connected components}\} =$

$$= \frac{g_n^{(2)}}{f_n} = \sum_{k=1}^{r-1} \frac{c_k^{(2)}}{k! \cdot (n)_k} + O\left(\frac{1}{n^r}\right),$$

where $(n)_k = n(n-1)\ldots(n-k+1)$ are the falling factorials

and $(c_k^{(2)}) = 1, 1, 11, 214, 6314, 259956, 14174292, \ldots$

Example 2: graphs with two connected connected

- f_n counts labeled graphs,
- $g_n^{(2)}$ counts labeled graphs with exactly 2 connected components.

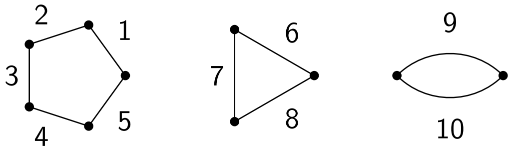
$\mathbb{P}\{\text{surface has exactly 2 connected components}\} =$

$$= \frac{g_n^{(2)}}{f_n} = \sum_{k=1}^{r-1} c_k^{(2)} \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $(c_k^{(2)}) = 1, -1, 1, 14, 398, 18552, 1505644, \ldots$

Combinatorial map model

- Take several labeled polygons of total perimeter $N = 2n$ (1-gons and 2-gons are allowed).
- Identify their sides randomly to obtain a surface.
- Each surface is defined by a pair $(\phi, \alpha) \in S_N^2$, where α is a perfect matching.



$$\phi = (1\,2\,3\,4\,5)(6\,7\,8)(9\,10), \quad \alpha = (1\,3)(2\,6)(4\,10)(5\,9)(7\,8)$$

Combinatorial map model asymptotics

Question. What is the probability of a random surface determined by $(\phi, \alpha) \in S_N^2$ to be connected as $N \to \infty$?

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- Budzinski, Curien and Petri, 2019:

$$\mathbb{P}(\text{surface is connected}) = 1 - \frac{1}{2n} + O\left(\frac{1}{n^2}\right).$$

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$$\mathbb{P}(\text{surface is connected}) = 1 - \frac{1}{2n} + O\left(\frac{1}{n^2}\right).$$

- Monteil, N., 2020:

$$\mathbb{P}(\text{surface is connected}) = 1 - \sum_{k=1}^{r-1} \mu_{2k} \cdot \frac{(2(n-k)-1)!!}{(2n-1)!!} + O\left(\frac{1}{n^r}\right)$$

where (μ_{2k}) counts indecomposable perfect matchings.

Combinatorial map model asymptotics, continued

- f_n counts surfaces generated by pairs $(\phi, \alpha) \in S_N^2$, where α is a perfect matching.
- $g_n^{(2)}$ counts surfaces with exactly 2 connected components.

$\mathbb{P}\{\text{surface has exactly 2 connected components}\} =$

$$= \frac{g_n^{(2)}}{f_n} = \sum_{k=1}^{r-1} c_k^{(2)} \cdot \frac{(2(n-k)-1)!!}{(2k)! \cdot (2n-1)!!} + O\left(\frac{1}{n^r}\right),$$

where $(c_k^{(2)}) = 2, 36, 5640, 2456160, 2192823310, \ldots$

Many thanks to all listeners

Thank you for your attention!

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Proceedings of the London Mathematical Society, Volume s3-20, Issue 3 (April 1970), pp. 558-572.

SEQ asymptotics

Theorem (Monteil, N., 2019+)

Let $\mathcal{F} = \mathrm{SEQ}(\mathcal{H})$ *and* $\mathcal{H}^{(2)} = \mathrm{SEQ}_2(\mathcal{H})$.
If $f_n \neq 0$ *for all* $n \in \mathbb{N}$ *and there exists* $r \geqslant 1$ *such that*

$$(i)\ n \cdot \frac{f_{n-1}}{f_n} \to 0 \qquad and \qquad (ii)\ \sum_{k=r}^{n-r} \binom{n}{k} f_k f_{n-k} = O\big(n^r f_{n-r}\big),$$

Then, as $n \to \infty$,

$$\frac{h_n}{f_n} = 1 - \sum_{k=1}^{r-1} \left(2h_k - h_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right).$$

Combinatorial meaning: it is the probability of a random object of size n to be irreducible in the sense of SEQ-decomposition.

Example 1: indecomposable permutations

- $f_n = (n!)^2$ counts pairs of permutations,
- $h_n = n! \cdot \mu_n$, where μ_n counts indecomposable permutations.
- $h_n^{(m)} = n! \cdot \mu_n^{(m)}$, where $\mu_n^{(m)}$ counts permutations that have exactly m indecomposable parts.

$\mathbb{P}\{\text{permutation is indecomposable}\} =$

$$= \frac{h_n}{f_n} = \frac{\mu_n}{n!} = 1 - \sum_{k=1}^{r-1} \frac{2\mu_k - \mu_k^{(2)}}{(n)_k} + O\left(\frac{1}{n^r}\right),$$

where

$$\begin{array}{rcllllllll} (\mu_k) & = & 1, & 1, & 3, & 13, & 71, & 461, & 3447, & \dots \\ (\mu_k^{(2)}) & = & 0, & 1, & 2, & 7, & 32, & 177, & 1142, & \dots \\ (c_k^{(1)}) & = & 2, & 1, & 4, & 19, & 110, & 745, & 5752, & \dots \end{array}$$

Example 2: irreducible tournaments

- f_n counts labeled tournaments,
- h_n counts irreducible labeled tournaments.
- $h_n^{(m)}$ counts labeled tournaments that have exactly m irreducible components.

$\mathbb{P}\{\text{tournament is irreducible}\} =$

$$= \frac{h_n}{f_n} = 1 - \sum_{k=1}^{r-1} \left(2h_k - h_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where

$$\begin{array}{rcrrrrrrl} (h_k) & = & 1, & 0, & 2, & 24, & 544, & 22320, & \dots \\ \left(h_k^{(2)}\right) & = & 0, & 2, & 0, & 16, & 240, & 6608, & \dots \\ \left(c_k^{(1)}\right) & = & 2, & -2, & 4, & 32, & 848, & 38032, & \dots \end{array}$$

SEQ_m asymptotics

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Let $\mathcal{F} = \mathrm{SEQ}(\mathcal{H})$ *and* $\mathcal{H}_m = \mathrm{SEQ}_m(\mathcal{H})$.
If $f_n \neq 0$ *for all* $n \in \mathbb{N}$ *and there exists* $r \geqslant 1$ *such that*

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Then for all $m \geqslant 1$*, as* $n \to \infty$*,*

$$(c)\ \ p_n^{(m+1)} = \frac{h_n^{(m+1)}}{f_n} = \sum_{k=1}^{r-1} c_k^{(m+1)} \cdot \binom{n}{k} \cdot \frac{f_{n-k}}{f_n} + O\left(n^r \cdot \frac{f_{n-r}}{f_n}\right),$$

where $$c_k^{(m+1)} = (m+1)\Big(h_k^{(m)} - 2h_k^{(m+1)} + h_k^{(m+2)}\Big).$$

Example 1: permutations with m indecomposable parts

- $f_n = (n!)^2$ counts pairs of permutations,
- $\mu_n^{(m)}$ counts permutations that have exactly m indecomposable parts.

$\mathbb{P}\{\text{permutation has exactly 2 indecomposable parts}\} =$

$$= \frac{h_n^{(2)}}{f_n} = \frac{\mu_n^{(2)}}{n!} = \sum_{k=1}^{r-1} \frac{2\left(\mu_k^{(1)} - 2\mu_k^{(2)} + \mu_k^{(3)}\right)}{(n)_k} + O\left(\frac{1}{n^r}\right),$$

where

$$\begin{array}{cclllllll} \left(\mu_k^{(1)}\right) & = & 1, & 1, & 3, & 13, & 71, & 461, & 3447, & \ldots \\ \left(\mu_k^{(2)}\right) & = & 0, & 1, & 2, & 7, & 32, & 177, & 1142, & \ldots \\ \left(\mu_k^{(3)}\right) & = & 0, & 0, & 1, & 3, & 12, & 58, & 327, & \ldots \\ \left(c_k^{(2)}\right) & = & 2, & -2, & 0, & 4, & 38, & 330, & 2980, & \ldots \end{array}$$

Example 1: permutations with m indecomposable parts

- $f_n = (n!)^2$ counts pairs of permutations,
- $\mu_n^{(m)}$ counts permutations that have exactly m indecomposable parts.

$\mathbb{P}\{\text{permutation has exactly 3 indecomposable parts}\} =$

$$= \frac{h_n^{(3)}}{f_n} = \frac{\mu_n^{(3)}}{n!} = \sum_{k=1}^{r-1} \frac{3\left(\mu_k^{(2)} - 2\mu_k^{(3)} + \mu_k^{(4)}\right)}{(n)_k} + O\left(\frac{1}{n^r}\right),$$

where

$$\begin{array}{rcrrrrrrrl}
\left(\mu_k^{(2)}\right) & = & 0, & 1, & 2, & 7, & 32, & 177, & 1142, & \dots \\
\left(\mu_k^{(3)}\right) & = & 0, & 0, & 1, & 3, & 12, & 58, & 327, & \dots \\
\left(\mu_k^{(4)}\right) & = & 0, & 0, & 0, & 1, & 4, & 18, & 92, & \dots \\
\left(c_k^{(3)}\right) & = & 0, & 3, & 0, & 6, & 36, & 237, & 1740, & \dots
\end{array}$$

Example 1: permutations with m indecomposable parts

- $f_n = (n!)^2$ counts pairs of permutations,
- $\mu_n^{(m)}$ counts permutations that have exactly m indecomposable parts.

$\mathbb{P}\{\text{permutation has exactly 4 indecomposable parts}\} =$

$$= \frac{h_n^{(4)}}{f_n} = \frac{\mu_n^{(4)}}{n!} = \sum_{k=1}^{r-1} \frac{4\left(\mu_k^{(3)} - 2\mu_k^{(4)} + \mu_k^{(5)}\right)}{(n)_k} + O\left(\frac{1}{n^r}\right),$$

where

$$\begin{array}{lcllllllll}
\left(\mu_k^{(3)}\right) & = & 0, & 0, & 1, & 3, & 12, & 58, & 327, & \ldots \\
\left(\mu_k^{(4)}\right) & = & 0, & 0, & 0, & 1, & 4, & 18, & 92, & \ldots \\
\left(\mu_k^{(5)}\right) & = & 0, & 0, & 0, & 0, & 1, & 5, & 25, & \ldots \\
\left(c_k^{(4)}\right) & = & 0, & 0, & 4, & 4, & 20, & 108, & 672, & \ldots
\end{array}$$

Example 1: permutations with m indecomposable parts

- $f_n = (n!)^2$ counts pairs of permutations,
- $\mu_n^{(m)}$ counts permutations that have exactly m indecomposable parts.

$\mathbb{P}\{\text{permutation has exactly } (m+1) \text{ indecomposable parts}\} =$

$$= \frac{h_n^{(m+1)}}{f_n} = \frac{\mu_n^{(m+1)}}{n!} = \frac{(m+1)}{(n)_m} + O\left(\frac{1}{n^{m+1}}\right),$$

where $$(n)_m = n(n-1)(n-2)\ldots(n-m+1).$$

Example 2: tournaments with m irreducible components

- f_n counts labeled tournaments,
- $h_n^{(m)}$ counts labeled tournaments that have exactly m irreducible components.

$\mathbb{P}\{\text{tournament has exactly 2 irreducible components}\} =$

$$= \frac{h_n^{(2)}}{f_n} = \sum_{k=1}^{r-1} 2\left(h_k^{(1)} - 2h_k^{(2)} + h_k^{(3)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where

$$\begin{aligned}
\left(h_k^{(1)}\right) &= 1, \quad 0, \quad 2, \quad 24, \quad 544, \quad 22320, \quad \ldots \\
\left(h_k^{(2)}\right) &= 0, \quad 2, \quad 0, \quad 16, \quad 240, \quad 6608, \quad \ldots \\
\left(h_k^{(3)}\right) &= 0, \quad 0, \quad 6, \quad 0, \quad 120, \quad 2160, \quad \ldots \\
\left(c_k^{(2)}\right) &= 2, \quad -8, \quad 16, \quad -16, \quad 368, \quad 22528, \quad \ldots
\end{aligned}$$

Example 2: tournaments with m irreducible components

- f_n counts labeled tournaments,
- $h_n^{(m)}$ counts labeled tournaments that have exactly m irreducible components.

$\mathbb{P}\{$tournament has exactly 3 irreducible components$\} =$

$$= \frac{h_n^{(3)}}{f_n} = \sum_{k=1}^{r-1} 3\left(h_k^{(2)} - 2h_k^{(3)} + h_k^{(4)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where

$$\begin{aligned}
\left(h_k^{(2)}\right) &= 0, \quad 2, \quad 0, \quad 16, \quad 240, \quad 6608, \quad \ldots \\
\left(h_k^{(3)}\right) &= 0, \quad 0, \quad 6, \quad 0, \quad 120, \quad 2160, \quad \ldots \\
\left(h_k^{(4)}\right) &= 0, \quad 0, \quad 0, \quad 24, \quad 0, \quad 960, \quad \ldots \\
\left(c_k^{(3)}\right) &= 0, \quad 6, \quad -36, \quad 120, \quad 0, \quad 9744, \quad \ldots
\end{aligned}$$

Example 2: tournaments with m irreducible components

- f_n counts labeled tournaments,
- $h_n^{(m)}$ counts labeled tournaments that have exactly m irreducible components.

$\mathbb{P}\{\text{tournament has exactly 4 irreducible components}\} =$

$$= \frac{h_n^{(4)}}{f_n} = \sum_{k=1}^{r-1} 4\left(h_k^{(3)} - 2h_k^{(4)} + h_k^{(5)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where

$$\begin{array}{lcrrrrrrl}
\left(h_k^{(3)}\right) & = & 0, & 0, & 6, & 0, & 120, & 2160, & \ldots \\
\left(h_k^{(4)}\right) & = & 0, & 0, & 0, & 24, & 0, & 960, & \ldots \\
\left(h_k^{(5)}\right) & = & 0, & 0, & 0, & 0, & 120, & 0, & \ldots \\
\left(c_k^{(4)}\right) & = & 0, & 0, & 24, & -192, & 960, & 960, & \ldots
\end{array}$$

Example 2: tournaments with m irreducible components

- f_n counts labeled tournaments,
- $h_n^{(m)}$ counts labeled tournaments that have exactly m irreducible components.

$$\mathbb{P}\{\text{tournament has exactly } (m+1) \text{ irreducible components}\} =$$

$$= \frac{h_n^{(m+1)}}{f_n} = (n)_m \cdot \frac{2^{m(m+1)/2}}{2^{nm}} + O\left(\frac{n^{m+1}}{2^{n(m+1)}}\right),$$

where $$(n)_m = n(n-1)(n-2)\dots(n-m+1).$$