Asymptotic probability of irreducible labeled objects in terms of virtual species

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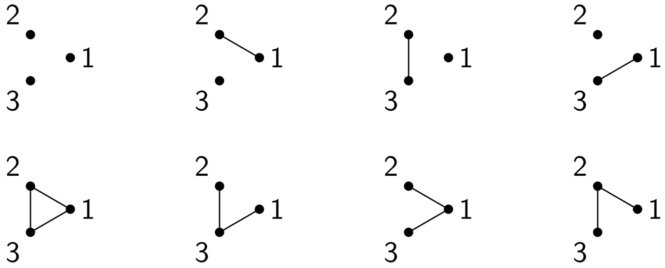
Simple labeled graphs

- $\mathfrak{g}_n$: the number of labeled graphs with n vertices,
- $\mathfrak{cg}_n$: the number of connected labeled graphs with n vertices.



$$\mathfrak{g}_n = 2^{\binom{n}{2}}$$

$$(\mathfrak{cg}_n) = 1, 1, 4, 38, 728, 26704, 1866256, \ldots$$

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{\mathfrak{cg}_n}{\mathfrak{g}_n}$ of a random graph with n vertices to be connected, as $n \to \infty$?

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- Wright, 1970:

$$p_n = 1 - \binom{n}{1}\frac{2}{2^n} - 2\binom{n}{3}\frac{2^6}{2^{3n}} - 24\binom{n}{4}\frac{2^{10}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

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- Can we have all terms at once? What is the interpretation?

Asymptotics for connected graphs

Theorem (Monteil, N., 2019)

For any positive integer r, the probability p_n that a random labeled graph of size n is connected, satisfies

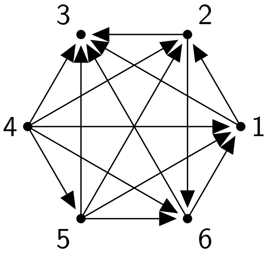
$$p_n = 1 - \sum_{k=1}^{r-1} \mathrm{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where it_k is the number of irreducible labeled tournaments of size k.

$$(\mathrm{it}_k) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$$

Tournaments

A tournament is a complete directed graph.



The number of labeled tournaments with n vertices is

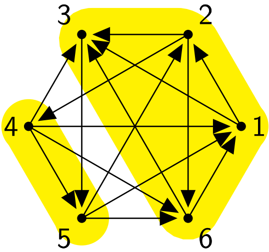
$$\mathfrak{t}_n = 2^{\binom{n}{2}}$$

Irreducible tournaments

A tournament is irreducible, if
for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B,
2. there exist an edge from B to A.

$V = \{1, 2, 3, 4, 5, 6\}$



$A = \{1, 2, 3, 6\}$
$B = \{4, 5\}$

Irreducible tournaments

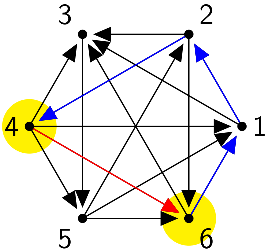
A tournament is irreducible, if
for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B,
2. there exist an edge from B to A.

Equivalently, a tournament is strongly connected: for each two vertices u and v

1. there is a path from u to v,
2. there is a path from v to u.

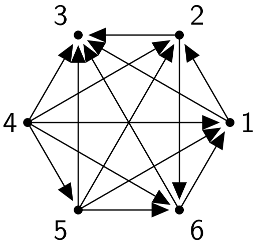
$V = \{1, 2, 3, 4, 5, 6\}$



$u = 4$
$v = 6$

Tournament as a sequence

Lemma. Every labeled tournament can be uniquely decomposed into a sequence of irreducible labeled tournaments.



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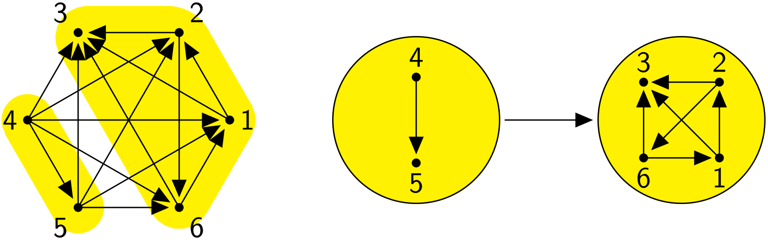

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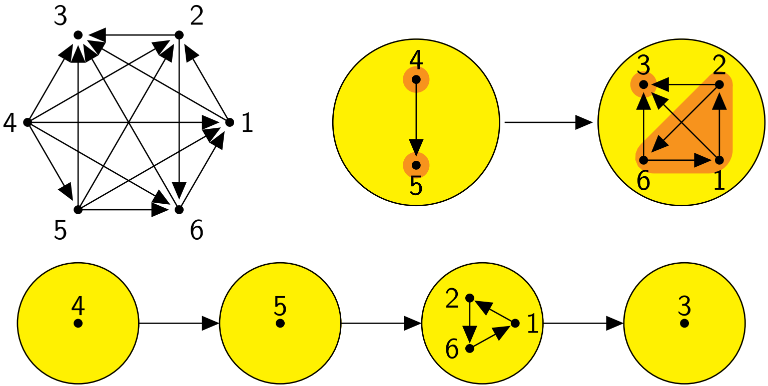

SET and SEQ decompositions

graphs $\xrightarrow[\text{as SET of}]{\text{decomposed}}$ connected graphs — counted by

$$CG(z) = \sum_{n=0}^{\infty} \mathfrak{cg}_n \frac{z^n}{n!} = \log(G(z))$$

graphs — counts

$$G(z) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{z^n}{n!} = T(z)$$

counts — tournaments

tournaments $\xrightarrow[\text{as SEQ of}]{\text{decomposed as}}$ irreducible tournaments — counted by

$$IT(z) = \sum_{n=0}^{\infty} \mathfrak{it}_n \frac{z^n}{n!} = 1 - \frac{1}{T(z)}$$

Key tool of the proof: Bender's theorem

Theorem (Bender, 1975)

- $A(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0,$
- $F(x, y)$ *is analytic in* $U(0;0)$,
- $B(z) = \sum_{n=1}^{\infty} b_n z^n = F(z, A(z)),$
- $C(z) = \sum_{n=1}^{\infty} c_n z^n = \frac{\partial F}{\partial y}(z, A(z)),$
- $\frac{a_{n-1}}{a_n} \to 0, \quad \text{as } n \to \infty,$
- $\exists r \geqslant 1 : \sum_{k=r}^{n-r} |a_k a_{n-k}| = O(a_{n-r}).$

Then $\quad b_n = \sum_{k=1}^{r-1} c_k a_{n-k} + O(a_{n-r}).$

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Then $b_n = \sum_{k=1}^{r-1} c_k a_{n-k} + O(a_{n-r}).$

Application to graphs:

- $A(z) = G(z) - 1 = T(z) - 1,$
- $F(x, y) = \log(1 + y),$
- $B(z) = \log\left(G(z)\right) = CG(z),$
- $C(z) = \frac{1}{T(z)} = 1 - IT(z),$

The last two condition hold, hence,

$$\frac{\mathfrak{cg}_n}{n!} = \frac{\mathfrak{g}_n}{n!} - \sum_{k=0}^{r-1} \frac{\mathfrak{it}_k \mathfrak{g}_{n-k}}{k!(n-k)!} + O(a_{n-r}).$$

Combinatorial constructions

1. $\mathcal{U} = \mathrm{SET}(\mathcal{V})$, $\qquad U(z) = \exp\big(V(z)\big)$.

2. $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$, $\qquad U(z) = \dfrac{1}{1 - W(z)}$.

3. $\mathcal{V} = \mathrm{CYC}(\mathcal{W})$, $\qquad V(z) = \log \dfrac{1}{1 - W(z)}$.

Gargantuan sequences and combinatorial classes

A sequence (a_n) is gargantuan, if

1. $\dfrac{a_{n-1}}{a_n} \to 0, \quad \text{as } n \to \infty,$
2. $\displaystyle\sum_{k=r}^{n-r} |a_k a_{n-k}| = O(a_{n-r}) \quad \text{for any } r \geqslant 1.$

A labeled combinatorial class $\mathcal{A}$ is gargantuan,
if the sequence $(\mathfrak{a}_n/n!)$ is gargantuan, i.e.

1. $n \cdot \dfrac{\mathfrak{a}_{n-1}}{\mathfrak{a}_n} \to 0, \quad \text{as } n \to \infty,$
2. $\displaystyle\sum_{k=r}^{n-r} \binom{n}{k} |\mathfrak{a}_k \mathfrak{a}_{n-k}| = O(n^r \mathfrak{a}_{n-r}) \quad \text{for any } r \geqslant 1.$

Asymptotics notation

For a sequence (a_n) and an integer m, we write

$$a_n \approx \sum_{k \geqslant m} f_k(n),$$

if. for any integer $r \geqslant m+1$, one has

$$a_n = \sum_{k=m}^{r-1} f_k(n) + O\big(f_r(n)\big),$$

where the sequence (f_k) satisfies $f_{k+1}(n) = o\big(f_k(n)\big)$ for any $k \in \mathbb{N}$.

SET asymptotics

Theorem (Monteil, N., 2019+)

If $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are such combinatorial classes that

- *$\mathcal{U}$ is gargantuan with positive counting sequence,*
- *$\mathcal{U} = \mathrm{SET}(\mathcal{V})$ and $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$,*

then

$$p_n := \frac{\mathfrak{v}_n}{\mathfrak{u}_n} \approx 1 - \sum_{k=1}^{r-1} \mathfrak{w}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

Combinatorial meaning: p_n is the probability that a random object of size n from $\mathcal{U}$ is irreducible in terms of SET-decomposition.

SEQ asymptotics

Theorem (Monteil, N., 2019+)

If $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{W}^{(2)}$ are such combinatorial classes that

- *$\mathcal{U}$ is gargantuan with positive counting sequence,*
- $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ *and* $\mathcal{W}^{(2)} = \mathrm{SEQ}_2(\mathcal{W})$,

then

$$q_n := \frac{\mathfrak{w}_n}{\mathfrak{u}_n} \approx 1 - \sum_{k=1}^{r-1} \left(2\mathfrak{w}_k - \mathfrak{w}_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

Combinatorial meaning: q_n is the probability that a random object of size n from $\mathcal{U}$ is irreducible in terms of SEQ-decomposition.

CYC asymptotics

Theorem (Monteil, N., 2020+)

If $\mathcal{V}$ and $\mathcal{W}$ are such combinatorial classes that

- *$\mathcal{V}$ is gargantuan with positive counting sequence,*
- *$\mathcal{V} = \mathrm{CYC}(\mathcal{W})$,*

then

$$r_n := \frac{\mathfrak{w}_n}{\mathfrak{v}_n} \approx 1 - \sum_{k=1}^{r-1} \mathfrak{w}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{v}_{n-k}}{\mathfrak{v}_n}.$$

Combinatorial meaning: r_n is the probability that a random object of size n from $\mathcal{V}$ is irreducible in terms of CYC-decomposition.

Example: asymptotics for irreducible tournaments

Theorem (Monteil, N., 2019)

The probability q_n that a random labeled tournament of size n is irreducible, satisfies

$$q_n = 1 - \sum_{k=1}^{r-1} \left(2\mathrm{it}_k - \mathrm{it}_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $\mathrm{it}_k^{(2)}$ is the number of labeled tournaments of size k with two irreducible components.

$$\begin{array}{rcrrrrrrl}
(\mathrm{it}_k) & = & 1, & 0, & 2, & 24, & 544, & 22320, & \ldots \\
(\mathrm{it}_k^{(2)}) & = & 0, & 2, & 0, & 16, & 240, & 6608, & \ldots \\
(2\mathrm{it}_k - \mathrm{it}_k^{(2)}) & = & 2, & -2, & 4, & 32, & 848, & 38032, & \ldots
\end{array}$$

Application to random surfaces

	square-tiled surfaces	combinatorial map model
$\mathfrak{u}_n$	translation surfaces obtained by gluing squares, $\{(\sigma,\tau) \mid \sigma,\tau \in S_n^2\}$	surfaces obtained by gluing polygons, $\{(\sigma,\tau) \mid \tau$ is perfect matching$\}$
$\mathfrak{v}_n$	connected surfaces	connected surfaces
h_n	$\{(\sigma,\tau) \mid \tau$ is indecomposable permutation$\}$	$\{(\sigma,\tau) \mid \tau$ is indecomposable perfect matching$\}$
p_n	$\mathbb{P}\{$surface is connected$\}$	$\mathbb{P}\{$surface is connected$\}$
q_n	$\mathbb{P}\{$permutation is indecomposable$\}$	$\mathbb{P}\{$perfect matching is indecomposable$\}$
$\mathfrak{u}_n$	$n!$	$n!(n-1)!!$, n is even
$\mathfrak{v}_n$	$1, 3, 26, 426, 11064 \ldots$	$0, 2, 0, 60, 0, 8880 \ldots$
$\mathfrak{w}_n$ m_n	$\mathfrak{w}_n = n! \cdot m_n$ $(m_n) = 1, 1, 3, 13, 71, 461 \ldots$	$\mathfrak{w}_{2n} = (2n)! \cdot m_{2n}$ $(m_{2n}) = 1, 2, 10, 74, 706, \ldots$

SEQ asymptotics, revisited

Theorem (Monteil, N., 2020+)

If $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{W}^{(m)}$, $m \in \mathbb{N}$, are such combinatorial classes that

- *$\mathcal{U}$ is gargantuan with positive counting sequence,*
- $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ *and* $\mathcal{W}^{(m)} = \mathrm{SEQ}_m(\mathcal{W})$,

then
$$q_n^{(m)} := \frac{\mathfrak{w}_n^{(m)}}{\mathfrak{u}_n} \approx \sum_{k=0}^{r-1} \beta_k^{(m)} \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

where
$$\beta_k^{(m)} = m\left(\mathfrak{w}_k^{(m-1)} - 2\mathfrak{w}_k^{(m)} + \mathfrak{w}_k^{(m+1)}\right).$$

Combinatorial meaning: $q_n^{(m)}$ is the probability that a random object of size n from $\mathcal{U}$ has exactly m irreducible components in terms of SEQ-decomposition.

Combinatorial classes: limited applicability

- Coefficients can be negative (hence, there is no direct combinatorial meaning in terms of combinatorial classes).
- In certain cases, there is a decomposition

$$\mathcal{U} = \mathrm{SET}(\mathcal{V}),$$

but we have no class $\mathcal{W}$ such that

$$\mathcal{U} = \mathrm{SEQ}(\mathcal{W}),$$

and our theorem is not applicable. We would like to have an "anti-SEQ" operator to create this class.

Species of structures

A species of structures is a rule F such that

1. for each finite set U, F produces a finite set $F[U]$;
2. for each bijection $\sigma\colon U \to V$, F produces a transport function

$$F[\sigma]\colon F[U] \to F[V] :$$

such that

1. for all bijections $\sigma\colon U \to V$ and $\tau\colon V \to W$,

$$F[\tau \circ \sigma] = F[\tau] \circ F[\sigma];$$

2. for the identity map:

$$F[\mathrm{Id}_U] = \mathrm{Id}_{F[U]};$$

3. if F is weighted, the weights are preserved:

$$w_U = w_V \circ F[\sigma].$$

Species operations

Species admit natural operations:

- sum,
- product,
- substitusion,
- derivative...

These operations satisfy natural properties, say,

- $(F+G)' = F' + G'$,
- $(F \cdot G)' = F' \cdot G + F \cdot G'$,
- $(F \circ G)' = (F' \circ G) \cdot G' \ldots$

Examples of species

- $\mathcal{Z}$ — characteristic of singletons,

$$\mathcal{Z}[U] = \begin{cases} \{U\} & \text{if } |U| = 1 \\ \emptyset & \text{if } |U| \neq 1, \end{cases} \qquad \mathcal{Z}(z) = z.$$

- $\mathcal{E}$ — species of sets, $\quad \mathcal{E}[U] = \{U\}, \qquad \mathcal{E}(z) = e^z.$
- $\mathcal{L}$ — species of linear orders,

$$\mathcal{L}[U] = \{\text{linear orders on } U\}, \qquad \mathcal{L}(z) = \frac{1}{1-z}.$$

- $\mathcal{P}$ — species of permutations,

$$\mathcal{P}[U] = \{\text{bijections } U \to U\}, \qquad \mathcal{P}(z) = \frac{1}{1-z}.$$

- $\mathcal{CP}$ — species of cyclic permutations

$$\mathcal{CP}[U] = \{\text{cyclic bijections } U \to U\}, \qquad \mathcal{CP}(z) = \log\frac{1}{1-z}.$$

Decomposition and gargantuan species

- Each species F admits a decomposition

$$F = F_0 + F_1 + F_2 + F_3 + \dots,$$

where F_n is the restriction of F to cardinality n, i.e.

$$F_n[U] = \begin{cases} F[U] & \text{if } |U| = n \\ \emptyset & \text{if } |U| \neq n. \end{cases}$$

- A species F is gargantuan, if the sequence $\big(\big|F[n]\big|/n!\big)$ is gargantuan, where $[n] = \{1, 2, \dots, n\}$.

Virtual species

- Species constitute a semi-ring Spe.
- Virtual species are elements of

$$\mathrm{Virt} = (\mathrm{Spe} \times \mathrm{Spe})/\sim,$$

where $\quad (F, G) \sim (H, K) \quad \Leftrightarrow \quad F + K = G + H.$

- Virtual species is a commutative ring with
  - zero — the empty species 0,
  - one — the characteristic of the empty set 1.
- Some additional operations in Virt:
  - subtraction,
  - multiplicative inverse F^{-1} (if $F_0 = 1$),
  - the inverse under substitution $F^{(-1)}$ (if $F_0 = 0$ and $F_1 = \mathcal{Z}$).

Correspondance between combinatorial classes and species

combinatorial classes		species of structures
$\mathcal{A} = \mathrm{SET}(\mathcal{B})$		$\mathcal{A} = \mathcal{E} \circ \mathcal{B}$
$\mathcal{A} = \mathrm{SEQ}(\mathcal{B})$		$\mathcal{A} = \mathcal{L} \circ \mathcal{B}$
$\mathcal{A} = \mathrm{CYC}(\mathcal{B})$	$\Leftrightarrow$	$\mathcal{A} = \mathcal{CP} \circ \mathcal{B}$
$\mathcal{A} = \mathrm{SET}_m(\mathcal{B})$		$\mathcal{A} = \mathcal{E}_m \circ \mathcal{B}$
$\mathcal{A} = \mathrm{SEQ}_m(\mathcal{B})$		$\mathcal{A} = \mathcal{L}_m \circ \mathcal{B}$
$\mathcal{A} = \mathrm{CYC}_m(\mathcal{B})$		$\mathcal{A} = \mathcal{CP}_m \circ \mathcal{B}$

SET asymptotics in terms of species

Theorem (Monteil, N., 2021+)

If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{B}^{\{m\}}$, $m \in \mathbb{N}$, are such (weighted) species that

- *$\mathcal{A}$ is gargantuan with positive total weights on $[n]$, $n \in \mathbb{N}$,*
- *$\mathcal{A} = \mathcal{E} \circ \mathcal{B}$ and $\mathcal{B}^{\{m\}} = \mathcal{E}_m \circ \mathcal{B}$,*

then

$$p_n^{\{m\}} := \frac{\mathfrak{b}_n^{\{m\}}}{\mathfrak{a}_n} \approx \sum_{k=0}^{r-1} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}.$$

where $$\mathcal{C} = \mathcal{B}^{\{m-1\}}(\mathcal{E}^{-1} \circ \mathcal{B}) \equiv \mathcal{B}^{\{m-1\}}\Big(\big(1 - \mathcal{L}_+^{(-1)}\big) \circ \mathcal{A}_+\Big).$$

Combinatorial meaning: $p_n^{\{m\}}$ is the probability that a random object of size n from $\mathcal{U}$ has exactly m irreducible components in terms of SET-decomposition.

SEQ asymptotics in terms of species

Theorem (Monteil, N., 2021+)

If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{B}^{(m)}$, $m \in \mathbb{N}$, are such (weighted) species that

- *$\mathcal{A}$ is gargantuan with positive total weights on $[n]$, $n \in \mathbb{N}$,*
- *$\mathcal{A} = \mathcal{L} \circ \mathcal{B}$ and $\mathcal{B}^{(m)} = \mathcal{L}_m \circ \mathcal{B}$,*

then

$$q_n^{(m)} := \frac{\mathfrak{b}_n^{(m)}}{\mathfrak{a}_n} \approx \sum_{k=0}^{r-1} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}.$$

where $\mathcal{C} = m\mathcal{B}^{m-1}(1-\mathcal{B})^2$.

Combinatorial meaning: $q_n^{(m)}$ is the probability that a random object of size n from $\mathcal{U}$ has exactly m irreducible components in terms of SEQ-decomposition.

CYC asymptotics in terms of species

Theorem (Monteil, N., 2021+)

If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{B}^{[m]}$, $m \in \mathbb{N}$, are such (weighted) species that

- *$\mathcal{A}$ is gargantuan with positive total weights on $[n]$, $n \in \mathbb{N}$,*
- *$\mathcal{A} = \mathcal{CP} \circ \mathcal{B}$ and $\mathcal{B}^{[m]} = \mathcal{CP}_m \circ \mathcal{B}$,*

then

$$r_n^{[m]} := \frac{\mathfrak{b}_n^{[m]}}{\mathfrak{a}_n} \approx \sum_{k=0}^{r-1} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}.$$

where $\mathcal{C} = \mathcal{B}^{m-1}(1 - \mathcal{B})$.

Combinatorial meaning: $r_n^{[m]}$ is the probability that a random object of size n from $\mathcal{U}$ has exactly m irreducible components in terms of CYC-decomposition.

Applications: Erdös-Rényi model $G(n,p)$

Fix $p \in (0,1)$, $q = 1 - p$.

Consider a random labeled graph G:

- p is the probability of edge presence;
- $q = 1 - p$ is the probability of edge absence;
- $\rho = pq^{-1} = q^{-1} - 1$.

$$\mathbb{P}(G) = p^{|E(G)|} q^{\binom{n}{2} - |E(G)|} = \frac{\rho^{|E(G)|}}{(\rho+1)^{\binom{n}{2}}}$$

Applications: Erdös-Rényi model $G(n,p)$

Fix $p \in (0,1)$, $q = 1 - p$.

Consider a random labeled graph G:

- p is the probability of edge presence;
- $q = 1 - p$ is the probability of edge absence;
- $\rho = pq^{-1} = q^{-1} - 1$.

Define:

- weight of the graph: $w(G) = \rho^{|E(G)|}$.
- $\mathfrak{g}_n = \sum_{|V(G)|=n} w(G) = q^{-\binom{n}{2}}$ — total weight.
- $\mathfrak{cg}_n = \sum_{G \text{ is connected}} w(G)$ — weight of connected graphs.

Applications: Erdös-Rényi model, continued

Theorem (Monteil, N., 2020+)

a) The probability p_n that a random graph with n vertices is connected satisfies

$$p_n \approx 1 - \sum_{k \geqslant 1} P_k(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}},$$

where

$$P_k(\rho) = \sum_{|V(G)|=k} (-1)^{\pi_0(G)-1} w(G).$$

Applications: Erdös-Rényi model, continued

Theorem (Monteil, N., 2020+)

b) The probability $p_n^{\{m\}}$ that a random graph with n vertices has exactly m connected components satisfies

$$p_n^{\{m\}} \approx \sum_{k \geqslant 0} P_k^{\{m\}}(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}},$$

where

$$P_k^{\{m\}}(\rho) = \sum_{|V(G)|=k} (-1)^{\pi_0(G)-m} \binom{\pi_0(G)}{m-1} w(G).$$

Applications: Erdös-Rényi model, continued

$+\rho^0$

$$P_1(\rho) = 1$$

Applications: Erdös-Rényi model, continued

$+\rho^0$

$+\rho^1$

$-\rho^0$

$P_1(\rho) = 1$

$P_2(\rho) = \rho - 1$

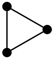
Applications: Erdös-Rényi model, continued

$+\rho^0$

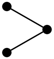
$+\rho^1$

$-\rho^0$

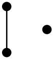
$$P_1(\rho) = 1 \qquad P_2(\rho) = \rho - 1$$



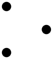
$+\rho^3$



$+3\rho^2$



$-3\rho^1$



$+\rho^0$

$$P_3(\rho) = \rho^3 + 3\rho^2 - 3\rho + 1$$

Unlabeled objects

Question 1. What are the asymptotic expansions for unlabeled combinatorial constructions?

There are results for constructions SEQ, MSET, PSET and SEQ_m. For CYC, MSET_m, PSET_m and CYC_m the answer is unknown.

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The theory of species treats both labeled and unlabeled cases.

Question 2. Can we combine the results for labeled and unlabeled cases within the framework of the theory of species?

Another type of irreducibles

The notion of irreducibility can be understood broader (see Beissinger). For instance, ordinary generating functions of "noncrossing compositions" satisfy

$$A(z) = 1 + I\big(zA(z)\big).$$

Question 3. Can we have any combinatorial interpretation for the coefficients arising in the asymptotic expansions of the probability that a random object is irreducible in the sense of the above relation?

The Erdös-Rényi model

Question 4. Can we interpret the coefficients $P_k(\rho)$ as a generalization of irreducible tournaments?

The natural generalization fails. Archer, Gessel, Graves and Liang showed that, instead of exponential generating functions, we need to use Eulerian ones for tournaments counted by descents (i.e. directed edges $\overrightarrow{st}$ such that $s > t$).

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Question 5. Can we build a fruitful theory of phase transition for asymptotic expansions?

Many thanks to listeners

Thank you for your attention!

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