Irreducibility of combinatorial objects: asymptotic probability and interpretation

Khaydar Nurligareev (joint with Thierry Monteil)

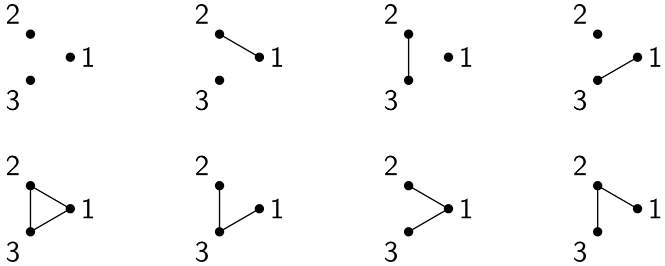
LIPN, University Paris 13

Séminaire LIB

February 23, 2023

Simple labeled graphs

- $\mathfrak{g}_n$: the number of labeled graphs with n vertices,
- $\mathfrak{cg}_n$: the number of connected labeled graphs with n vertices.



$$\mathfrak{g}_n = 2^{\binom{n}{2}}$$

$$(\mathfrak{cg}_n) = 1, 1, 4, 38, 728, 26704, 1866256, \ldots$$

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{\mathfrak{cg}_n}{\mathfrak{g}_n}$ that a random graph with n vertices is connected, as $n \to \infty$?

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{\mathfrak{cg}_n}{\mathfrak{g}_n}$ that a random graph with n vertices is connected, as $n \to \infty$?

1. folklore: $p_n = 1 + o(1)$

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{\mathfrak{cg}_n}{\mathfrak{g}_n}$ that a random graph with n vertices is connected, as $n \to \infty$?

1. folklore: $p_n = 1 + o(1)$
2. Gilbert, 1959: $p_n = 1 - \frac{2n}{2^n} + O\left(\frac{n^2}{2^{3n/2}}\right)$

Probability of a graph to be connected

Question. What is the probability $p_n = \dfrac{\mathfrak{cg}_n}{\mathfrak{g}_n}$ that a random graph with n vertices is connected, as $n \to \infty$?

1. folklore: $p_n = 1 + o(1)$
2. Gilbert, 1959: $p_n = 1 - \dfrac{2n}{2^n} + O\left(\dfrac{n^2}{2^{3n/2}}\right)$
3. Wright, 1970:

$$p_n = 1 - \binom{n}{1}\frac{2}{2^n} - 2\binom{n}{3}\frac{2^6}{2^{3n}} - 24\binom{n}{4}\frac{2^{10}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{\mathfrak{cg}_n}{\mathfrak{g}_n}$ that a random graph with n vertices is connected, as $n \to \infty$?

1. folklore: $p_n = 1 + o(1)$
2. Gilbert, 1959: $p_n = 1 - \frac{2n}{2^n} + O\left(\frac{n^2}{2^{3n/2}}\right)$
3. Wright, 1970:

$$p_n = 1 - \binom{n}{1}\frac{2}{2^n} - 2\binom{n}{3}\frac{2^6}{2^{3n}} - 24\binom{n}{4}\frac{2^{10}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

4. Can we see the structure? What is the interpretation?

Asymptotics for p_n

Theorem

For every $r \geqslant 1$, the probability p_n that a random labeled graph of size n is connected satisfies

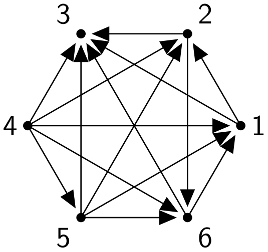
$$p_n = 1 - \sum_{k=1}^{r-1} \mathrm{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where it_k is the number of irreducible labeled tournaments of size k.

$$(\mathrm{it}_k) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$$

Tournaments

A **tournament** is a complete directed graph.



The number of labeled tournaments with n vertices is

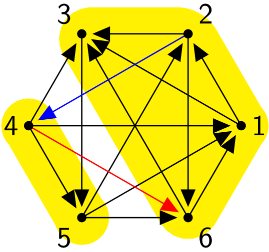
$$\mathfrak{t}_n = 2^{\binom{n}{2}}$$

Irreducible tournaments

A tournament is **irreducible**, if
for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B,
2. there exist an edge from B to A.

$V = \{1, 2, 3, 4, 5, 6\}$



$A = \{1, 2, 3, 6\}$
$B = \{4, 5\}$

Irreducible tournaments

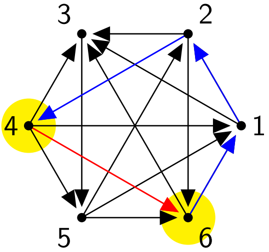
A tournament is **irreducible**, if
for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B,
2. there exist an edge from B to A.

Equivalently, a tournament is **strongly connected**: for each two vertices u and v

1. there is a path from u to v,
2. there is a path from v to u.

$V = \{1, 2, 3, 4, 5, 6\}$



$u = 4$

$v = 6$

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

$CG(z) = \log G(z)$

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

$CG(z) = \log G(z)$

Bender, 1975:

1. $A(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0$
2. $F(x,y)$ is analytic in $U(0;0)$
3. $B(z) = \sum_{n=0}^{\infty} b_n z^n = F(z, A(z))$
4. $C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial F}{\partial y}(z,y)\right]_{y=A(z)}$
5. $\frac{a_{n-1}}{a_n} \to 0, \text{ as } n \to \infty$
6. $\exists r \geqslant 1 : \sum_{k=r}^{n-r} |a_k a_{n-k}| = O(a_{n-r})$

Then $\quad b_n = \sum_{k=0}^{r-1} c_k a_{n-k} + O(a_{n-r}).$

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

$CG(z) = \log G(z)$

Bender, 1975:

1. $A(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0$
2. $F(x, y)$ is analytic in $U(0; 0)$
3. $B(z) = \sum_{n=0}^{\infty} b_n z^n = F(z, A(z))$
4. $C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial F}{\partial y}(z, y)\right]_{y=A(z)}$
5. the sequence (a_n) is gargantuan

Then $b_n = \sum_{k=0}^{r-1} c_k a_{n-k} + O(a_{n-r})$.

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

$CG(z) = \log G(z)$

Bender, 1975:

1. $A(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0$
2. $F(x, y)$ is analytic in $U(0; 0)$
3. $B(z) = \sum_{n=0}^{\infty} b_n z^n = F(z, A(z))$
4. $C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial F}{\partial y}(z, y)\right]_{y=A(z)}$
5. the sequence (a_n) is gargantuan

Then $\quad b_n \approx \sum_{k \geqslant 0} c_k a_{n-k}.$

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

$CG(z) = \log G(z)$

$F(y) = \log(y)$

$\frac{\partial F}{\partial y} = \frac{1}{y}$

Bender, 1975:

1. $A(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0$
2. $F(x, y)$ is analytic in $U(0;0)$
3. $B(z) = \sum_{n=0}^{\infty} b_n z^n = F(z, A(z))$
4. $C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial F}{\partial y}(z, y)\right]_{y=A(z)}$
5. the sequence (a_n) is gargantuan

Then $b_n \approx \sum_{k \geqslant 0} c_k a_{n-k}$.

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

$CG(z) = \log G(z)$

$F(y) = \log(y)$

$\frac{\partial F}{\partial y} = \frac{1}{y}$

$\frac{1}{1-y} = 1 + y + y^2 + \dots$

$\frac{\mathfrak{cg}_n}{\mathfrak{g}_n} \approx 1 - \sum_{k \geqslant 0} \mathfrak{it}_k \binom{n}{k} \frac{\mathfrak{g}_{n-k}}{\mathfrak{g}_n}$

Bender, 1975:

1. $A(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0$
2. $F(x, y)$ is analytic in $U(0;0)$
3. $B(z) = \sum_{n=0}^{\infty} b_n z^n = F(z, A(z))$
4. $C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial F}{\partial y}(z, y)\right]_{y=A(z)}$
5. the sequence (a_n) is gargantuan

Then $\quad b_n \approx \sum_{k \geqslant 0} c_k a_{n-k}$.

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

$CG(z) = \log G(z)$

$F(y) = \log(y)$

$\frac{\partial F}{\partial y} = \frac{1}{y}$

$\frac{1}{1-y} = 1 + y + y^2 + \dots$

$G(z) = T(z) = \frac{1}{1 - IT(z)}$

$\frac{\mathfrak{cg}_n}{\mathfrak{g}_n} \approx 1 - \sum_{k \geqslant 0} \mathfrak{it}_k \binom{n}{k} \frac{\mathfrak{g}_{n-k}}{\mathfrak{g}_n}$

Bender, 1975:

1. $A(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0$
2. $F(x, y)$ is analytic in $U(0;0)$
3. $B(z) = \sum_{n=0}^{\infty} b_n z^n = F(z, A(z))$
4. $C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial F}{\partial y}(z, y) \right]_{y=A(z)}$
5. the sequence (a_n) is gargantuan

Then $\quad b_n \approx \sum_{k \geqslant 0} c_k a_{n-k}.$

Exponential generating functions and Bender's theorem

1. $CG(z) = \log G(z)$
2. $A(z) = G(z) - 1$
3. $F(x, y) = \log(1 + y)$
4. $\dfrac{\partial F}{\partial y} = \dfrac{1}{1 + y}$
5. $C(z) = \dfrac{1}{G(z)} = \dfrac{1}{T(z)}$
6. $\dfrac{1}{T(z)} = 1 - IT(z)$
7. $\dfrac{\mathfrak{cg}_n}{\mathfrak{g}_n} \approx 1 - \sum_{k \geqslant 0} \mathfrak{it}_k \binom{n}{k} \dfrac{\mathfrak{g}_{n-k}}{\mathfrak{g}_n}$

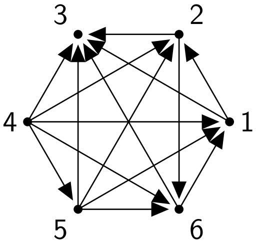
Bender, 1975:

1. $A(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0$
2. $F(x, y)$ is analytic in $U(0; 0)$
3. $B(z) = \sum_{n=0}^{\infty} b_n z^n = F(z, A(z))$
4. $C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\dfrac{\partial F}{\partial y}(z, y)\right]_{y=A(z)}$
5. the sequence (a_n) is gargantuan

Then $\quad b_n \approx \sum_{k \geqslant 0} c_k a_{n-k}.$

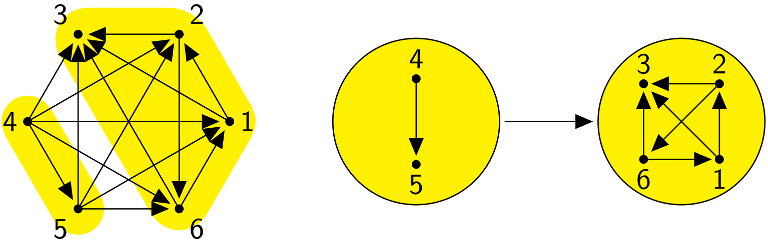
Tournament as a sequence

Folklore: Every labeled tournament can be uniquely decomposed into a sequence (SEQ) of irreducible labeled tournaments.



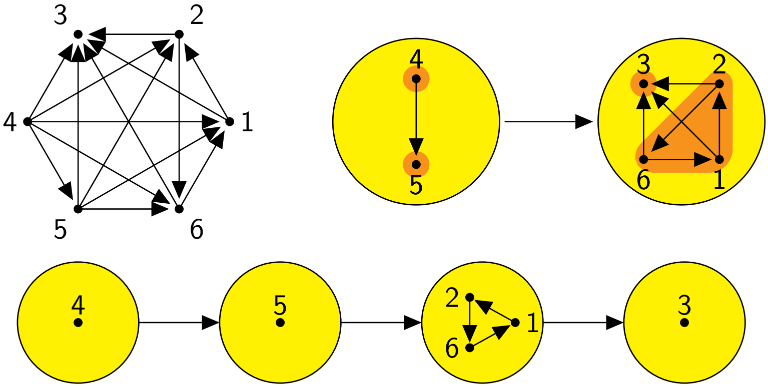
Tournament as a sequence

Folklore: Every labeled tournament can be uniquely decomposed into a sequence (SEQ) of irreducible labeled tournaments.



Tournament as a sequence

Folklore: Every labeled tournament can be uniquely decomposed into a sequence (SEQ) of irreducible labeled tournaments.



SET and SEQ decompositions

graphs — decomposed as SET of → connected graphs — counted by →

$$CG(z) = \sum_{n=0}^{\infty} \mathfrak{cg}_n \frac{z^n}{n!} = \log(G(z))$$

↓ derivative

$$IT(z) = \sum_{n=0}^{\infty} \mathfrak{it}_n \frac{z^n}{n!} = 1 - \frac{1}{T(z)}$$

counts ↑ (graphs)

$$G(z) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{z^n}{n!} = T(z)$$

counts ↓ (tournaments)

tournaments — decomposed as as SEQ of → irreducible tournaments — counted by →

Combinatorial constructions

1. $\mathcal{U} = \mathrm{SET}(\mathcal{V})$, $\qquad U(z) = \exp\big(V(z)\big)$.

2. $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$, $\qquad U(z) = \dfrac{1}{1 - W(z)}$.

SET asymptotics

Theorem

If $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are such combinatorial classes that

1. *$\mathcal{U}$ is gargantuan with positive counting sequence,*
2. $\mathcal{U} = \mathrm{SET}(\mathcal{V})$ *and* $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$,

then

$$p_n := \frac{\mathfrak{v}_n}{\mathfrak{u}_n} \approx 1 - \sum_{k \geqslant 1} \mathfrak{w}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

Combinatorial meaning: p_n is the probability that a random object of size n from $\mathcal{U}$ is irreducible in terms of SET-decomposition.

Asymptotics for connected graphs

Theorem

The probability p_n that a random labeled graph of size n is connected, satisfies

$$p_n \approx 1 - \sum_{k=1} \mathrm{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}},$$

where it_k is the number of irreducible labeled tournaments of size k.

$$(\mathrm{it}_k) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$$

More applications

1. Square-tiled surfaces and indecomposable permutations.
2. Combinatorial maps and indecomposable perfect matchings.
3. Connected multigraphs and irreducible multitournaments.
4. Constellations and indecomposable multipermutations.
5. Colored tensor models and indecomposable multipermutations.

SEQ asymptotics

Theorem

If $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{W}^{(2)}$ are such combinatorial classes that

- *$\mathcal{U}$ is gargantuan with positive counting sequence,*
- $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ *and* $\mathcal{W}^{(2)} = \mathcal{W} \star \mathcal{W} = \mathrm{SEQ}_2(\mathcal{W})$,

then

$$q_n := \frac{\mathfrak{w}_n}{\mathfrak{u}_n} \approx 1 - \sum_{k \geqslant 1} \left(2\mathfrak{w}_k - \mathfrak{w}_k^{(2)} \right) \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

Reasoning: $\frac{1}{y} \xrightarrow{\partial} -\frac{1}{y^2}$, $\left(1 - W(z)\right)^2 = 1 - 2W(z) + \left(W(z)\right)^2.$

Example: asymptotics for irreducible tournaments

Theorem

The probability q_n that a random labeled tournament of size n is irreducible, satisfies

$$q_n \approx 1 - \sum_{k \geqslant 1} \left(2\mathrm{it}_k - \mathrm{it}_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}},$$

where $\mathrm{it}_k^{(2)}$ is the number of labeled tournaments of size k with two irreducible components.

$$\begin{aligned}
(\mathrm{it}_k) &= 1, \quad 0, \quad 2, \quad 24, \quad 544, \quad 22320, \quad \ldots \\
(\mathrm{it}_k^{(2)}) &= 0, \quad 2, \quad 0, \quad 16, \quad 240, \quad 6608, \quad \ldots \\
(2\mathrm{it}_k - \mathrm{it}_k^{(2)}) &= 2, \quad -2, \quad 4, \quad 32, \quad 848, \quad 38032, \quad \ldots
\end{aligned}$$

Combinatorial classes: limits of applicability

1. Coefficients can be negative.

2. In certain cases, there is a decomposition

$$\mathcal{U} = \mathrm{SET}(\mathcal{V}),$$

but we have no class $\mathcal{W}$ such that

$$\mathcal{U} = \mathrm{SEQ}(\mathcal{W}),$$

and our theorem is not applicable

(need of an "anti-SEQ" operator to create this class).

Correspondance between combinatorial classes and species

combinatorial classes		species of structures
$\mathcal{A} = \mathrm{SET}(\mathcal{B})$		$\mathcal{A} = \mathcal{E} \circ \mathcal{B}$
$\mathcal{A} = \mathrm{SEQ}(\mathcal{B})$		$\mathcal{A} = \mathcal{L} \circ \mathcal{B}$
$\mathcal{A} = \mathrm{CYC}(\mathcal{B})$	$\Leftrightarrow$	$\mathcal{A} = \mathcal{CP} \circ \mathcal{B}$
$\mathcal{A} = \mathrm{SET}_m(\mathcal{B})$		$\mathcal{A} = \mathcal{E}_m \circ \mathcal{B}$
$\mathcal{A} = \mathrm{SEQ}_m(\mathcal{B})$		$\mathcal{A} = \mathcal{L}_m \circ \mathcal{B}$
$\mathcal{A} = \mathrm{CYC}_m(\mathcal{B})$		$\mathcal{A} = \mathcal{CP}_m \circ \mathcal{B}$

Asymptotics in terms of species

Theorem

If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{B}(m)$, $m \in \mathbb{N}$, are such (weighted) species that

1. *$\mathcal{A}$ is gargantuan with positive total weights on $[n]$, $n \in \mathbb{N}$,*
2. *one of the following conditions holds:*

(a)	$\mathcal{A} = \mathcal{E} \circ \mathcal{B}$	$\mathcal{B}(m) = \mathcal{E}_m \circ \mathcal{B}$	$\mathcal{C} = \mathcal{B}(m-1)(\mathcal{E}^{-1} \circ \mathcal{B})$
(b)	$\mathcal{A} = \mathcal{L} \circ \mathcal{B}$	$\mathcal{B}(m) = \mathcal{L}_m \circ \mathcal{B}$	$\mathcal{C} = m\mathcal{B}^{m-1}(1-\mathcal{B})^2$
(c)	$\mathcal{A} = \mathcal{CP} \circ \mathcal{B}$	$\mathcal{B}(m) = \mathcal{CP}_m \circ \mathcal{B}$	$\mathcal{C} = \mathcal{B}^{m-1}(1-\mathcal{B})$

then

$$p_n(m) := \frac{\mathfrak{b}_n(m)}{\mathfrak{a}_n} \approx \sum_{k \geqslant m-1} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}.$$

In the case (a), $\qquad \mathcal{C} \equiv \mathcal{B}^{\{m-1\}}\Big(\big(1 - \mathcal{L}_+^{(-1)}\big) \circ \mathcal{A}_+\Big).$

Erdős-Rényi model $G(n, p)$

Consider a random labeled graph G:

1. $p \in (0, 1)$ is the probability of edge presence;
2. $q = 1 - p$ is the probability of edge absence.

Weight of a graph:

$$w(G) = \rho^{|E(G)|},$$

where $\rho = \dfrac{p}{q} = q^{-1} - 1$.

Reason: if G_1 and G_2 are disjoint, then

$$w(G_1 \sqcup G_2) = w(G_1) \cdot w(G_2).$$

Asymptotics of the Erdős-Rényi model

Theorem

The probability $p_n(m)$ that a random graph with n vertices has exactly m connected components satisfies

$$p_n(m) \approx \sum_{k \geqslant 0} P_k^{\{m\}}(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}},$$

where

$$P_k^{\{m\}}(\rho) = \sum_{|V(G)|=k} (-1)^{\pi_0(G)-m} \binom{\pi_0(G)}{m-1} w(G).$$

Asymptotics of the Erdős-Rényi model, continued

Theorem

The probability p_n that a random graph with n vertices is connected satisfies

$$p_n \approx 1 - \sum_{k \geqslant 1} P_k(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}},$$

where

$$P_k(\rho) = \sum_{|V(G)|=k} (-1)^{\pi_0(G)-1} w(G).$$

Meaning of the coefficients

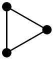
$w = 1$

$P_1(\rho) = 1$

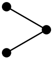
$w = \rho$

$w = 1$

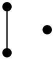
$P_2(\rho) = \rho - 1$



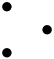
$w = \rho^3$



$w = \rho^2$



$w = \rho^1$



$w = 1$

$P_3(\rho) = \rho^3 + 3\rho^2 - 3\rho + 1$

Meaning of the coefficients

$w = 1$

$P_1(\rho) = 1$

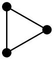
$P_1(1) = 1 = \mathfrak{it}_1$

$w = \rho$

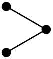
$w = 1$

$P_2(\rho) = \rho - 1$

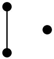
$P_2(1) = 0 = \mathfrak{it}_2$



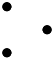
$w = \rho^3$



$w = \rho^2$



$w = \rho^1$



$w = 1$

$P_3(\rho) = \rho^3 + 3\rho^2 - 3\rho + 1$

$P_3(1) = 2 = \mathfrak{it}_3$

Probability of a directed graph to be strongly connected

Question. What is the probability r_n that a random directed graph with n vertices is strongly connected, as $n \to \infty$?

Probability of a directed graph to be strongly connected

Question. What is the probability r_n that a random directed graph with n vertices is strongly connected, as $n \to \infty$?

Wright, 1970: $$r_n = \sum_{k=0}^{r-1} \frac{\omega_k(n)}{2^{kn}} \cdot \frac{n!}{(n+[k/2]-k)!} + O\left(\frac{n^r}{2^{rn}}\right),$$

where

$$\omega_k(n) = \sum_{\nu=0}^{[k/2]} \gamma_\nu \xi_{k-2\nu} \frac{2^{k(k+1)/2}}{2^{\nu(k-\nu)}} (n+[k/2]-k)\dots(n+\nu+1-k),$$

$$\gamma_0 = 1,\ \gamma_\nu = \sum_{s=0}^{\nu-1} \frac{\gamma_s \eta_{n-s}}{(\nu-s)!},\quad \sum_{\nu=0}^{\infty} \xi_\nu z^\nu = \left(1 - \sum_{n=0}^{\infty} \frac{\eta_n}{2^{n(n-1)/2}} \frac{z^n}{n!}\right)^2,$$

$$\eta_1 = 1, \quad \eta_n = 2^{n(n-1)} - \sum_{t=1}^{n-1} \binom{n}{t} 2^{(n-1)(n-t)} \eta_t.$$

Towards the asymptotics

1. Dovgal and de Panafieu, 2019:

$$SD(z) = -\log\left(G(z) \odot \frac{1}{G(z)}\right)$$

2. In terms of tournaments:

$$SD(z) = -\log\left(1 - T(z) \odot IT(z)\right)$$

3. Semi-strong directed graphs:

$$SSD(z) = \frac{1}{1 - T(z) \odot IT(z)}$$

Asymptotics for strongly connected graphs

Theorem (with Sergey Dovgal)

The probability r_n that a random directed graph with n vertices is strongly connected satisfies

$$r_n \approx \sum_{k \geqslant 0} \mathfrak{ssd}_k \binom{n}{k} \frac{2^{k(k+1)}}{2^{2nk}} \frac{\mathfrak{it}_{n-k}}{\mathfrak{t}_{n-k}},$$

where $\mathfrak{ssd}_k$, $\mathfrak{t}_k$ and $\mathfrak{it}_k$ are the numbers of semi-strong digraphs, tournaments and irreducible tournaments of size k, respectively.

Reasoning: $\log(1-y) \xrightarrow{\partial} -\frac{1}{1-y}$.

Asymptotics for strongly connected graphs, continued

Theorem (with Sergey Dovgal)

The probability r_n that a random directed graph with n vertices is strongly connected satisfies

$$r_n \approx 1 - \sum_{k \geqslant 1} \frac{R_k(n)}{2^{nk}},$$

where $$R_k(n) = 2^{k(k+1)/2} \sum_{\nu=0}^{[k/2]} \binom{n}{\nu, k-2\nu} \frac{\mathfrak{ssd}_\nu \beta_{k-2\nu}}{2^{\nu(k-\nu)}}.$$

- $\beta_k = \mathbb{I}_{k=0} - 2\mathfrak{it}_k + \mathfrak{it}_k^{(2)}$,
- $\mathfrak{ssd}_k$ is the number of semi-strong digraphs of size k,
- $\mathfrak{it}_k$ is the number of irreducible tournaments of size k,
- $\mathfrak{it}_k^{(2)}$ is the number of tournaments of size k with two irreducible parts.

Possible directions for generalization

1. Different types of irreducibility. For instance, “noncrossing compositions”:

$$A(z) = 1 + I\big(zA(z)\big).$$

2. Classes of different rate of convergence (forests, polynomials).
3. Unlabeled structures.

Question. Can we obtain any combinatorial interpretation for the above cases?

Algorithmic aspects

For the asymptotic expansion for connected graphs,

$$p_n = 1 - \binom{n}{1}\frac{2\mathfrak{it}_1}{2^n} - \binom{n}{2}\frac{2^3\mathfrak{it}_2}{2^{2n}} - \binom{n}{3}\frac{2^6\mathfrak{it}_3}{2^{3n}} - \dots,$$

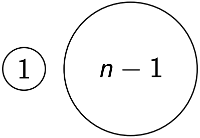
the inclusion-exclusion principle shows the origin of terms:

Algorithmic aspects

For the asymptotic expansion for connected graphs,

$$p_n = 1 - \binom{n}{1}\frac{2\mathfrak{it}_1}{2^n} - \binom{n}{2}\frac{2^3\mathfrak{it}_2}{2^{2n}} - \binom{n}{3}\frac{2^6\mathfrak{it}_3}{2^{3n}} - \dots,$$

the inclusion-exclusion principle shows the origin of terms:



Algorithmic aspects

For the asymptotic expansion for connected graphs,

$$p_n = 1 - \binom{n}{1}\frac{2\mathfrak{it}_1}{2^n} - \binom{n}{2}\frac{2^3\mathfrak{it}_2}{2^{2n}} - \binom{n}{3}\frac{2^6\mathfrak{it}_3}{2^{3n}} - \ldots,$$

the inclusion-exclusion principle shows the origin of terms:

Algorithmic aspects

For the asymptotic expansion for connected graphs,

$$p_n = 1 - \binom{n}{1}\frac{2\mathrm{it}_1}{2^n} - \binom{n}{2}\frac{2^3\mathrm{it}_2}{2^{2n}} - \binom{n}{3}\frac{2^6\mathrm{it}_3}{2^{3n}} - \ldots,$$

the inclusion-exclusion principle shows the origin of terms:

Question. Can we create a rejection algorithm for producing connected graphs randomly, so that we reject with a probability of a smaller order?

Erdős-Rényi model, continued

The form of the asymptotic expansion is

$$p_n = 1 - \binom{n}{1}\frac{q^n P_1(\rho)}{q} - \binom{n}{2}\frac{q^{2n} P_2(\rho)}{q^2} - \binom{n}{3}\frac{q^{3n} P_3(\rho)}{q^3} - \dots$$

When the parameter p approaches the threshold for connectedness,

$$p = \frac{(1+\varepsilon)\ln n}{n},$$

all terms become equivalent:

$$P_k(\rho)\binom{n}{k}\frac{q^{nk}}{q^{k(k+1)/2}} \sim n^{-\varepsilon k}.$$

Question. Can we build a fruitful theory of phase transition for asymptotic expansions?

Many thanks to all listeners

Thank you for your attention!

Literature I

Bender E.A.
An asymptotic expansion for some coefficients of some formal power series
J. Lond. Math. Soc., Ser. 9 (1975), pp. 451-458.

Gilbert E.N.
Random graphs
Ann. Math. Stat., Vol. 30, N. 4 (1959), pp. 1141-1144.

Monteil T., Nurligareev K.
Asymptotics for connected graphs and irreducible tournaments
Extended Abstracts EuroComb 2021, pp. 823-828. Springer, 2021.

Literature II

Nurligareev K.
Irreducibility of combinatorial objects: asymptotic probability and interpretation
Theses, Université Sorbonne Paris Nord, Oct. 2022.

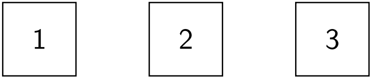
Wright E.M.
Asymptotic relations between enumerative functions in graph theory
Proc. Lond. Math. Soc., Vol. s3-20, Issue 3 (1970), pp. 558-572.

Wright E.M.
The Number of Strong Digraphs
Bull. Lond. Math. Soc., Ser. 3 (1971), pp. 348-350.

Square-tiled surfaces

To obtain a **square-tiled surface** (determined by $(h, v) \in S_n^2$):

1. take n labeled squares,

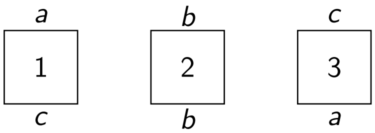


Square-tiled surfaces

To obtain a **square-tiled surface** (determined by $(h, v) \in S_n^2$):

1. take n labeled squares,
2. identify horizontal sides (corresponds to $h \in S_n$),

$h = (13)(2)$



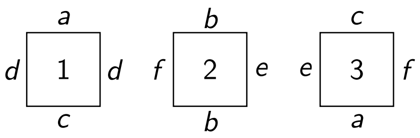
Square-tiled surfaces

To obtain a **square-tiled surface** (determined by $(h, v) \in S_n^2$):

1. take n labeled squares,
2. identify horizontal sides (corresponds to $h \in S_n$),
3. identify vertical sides (corresponds to $v \in S_n$),

$$h = (13)(2)$$

$$v = (1)(23)$$



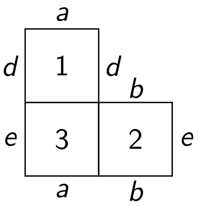
Square-tiled surfaces

To obtain a **square-tiled surface** (determined by $(h, v) \in S_n^2$):

1. take n labeled squares,
2. identify horizontal sides (corresponds to $h \in S_n$),
3. identify vertical sides (corresponds to $v \in S_n$),
4. glue together identified sides.

$$h = (13)(2)$$

$$v = (1)(23)$$



Square-tiled surfaces

To obtain a **square-tiled surface** (determined by $(h, v) \in S_n^2$):

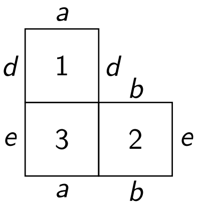
1. take n labeled squares,
2. identify horizontal sides (corresponds to $h \in S_n$),
3. identify vertical sides (corresponds to $v \in S_n$),
4. glue together identified sides.

Transitive action $\leftrightarrow$ connectedness of the square-tiled surface.

$h = (13)(2)$

$v = (1)(23)$

$\leftrightarrow$



Indecomposable permutations

A permutation $\sigma \in S_n$ is

1. **decomposable**, if there is an index $p < n$ such that $\sigma\big(\{1, \ldots, p\}\big) = \{1, \ldots, p\}$.
2. **indecomposable** otherwise.

$$\begin{pmatrix} 1 & 2 & 3 & \vert & 4 & 5 \\ 3 & 1 & 2 & \vert & 5 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$$

decomposable $(p = 3)$ indecomposable

Observation. Every permutation can be uniquely decomposed into a sequence of indecomposable permutations.

Asymptotics for connected square-tiled surfaces

Theorem (reformulation of the results of Dixon and Cori)

The probability p_n that a random square-tiled surface of size n is connected satisfies

$$p_n \approx 1 - \sum_{k=1} \frac{\mathfrak{ip}_k}{(n)_k},$$

where $(n)_k = n(n-1)\dots(n-k+1)$ *are the falling factorials and* $\mathfrak{ip}_k$ *is the number of indecomposable permutations of size* k.

$$(\mathfrak{ip}_k) = 1, 1, 3, 13, 71, 461, 3447, 29093, \dots$$

Pairs of linear orders

A pair of linear orders $(\prec_1, \prec_2)$ of size n is

1. **reducible**, if there is a partition $\{1, \dots, n\} = A \sqcup B$ such that $\forall\, a \in A,\ b \in B: \quad a \prec_1 b$ and $a \prec_2 b$.
2. **irreducible** otherwise.

$$\begin{pmatrix} 3 & \prec_1 & 1 & \prec_1 & 4 & \prec_1 & 2 \\ 4 & \prec_2 & 3 & \prec_2 & 1 & \prec_2 & 2 \end{pmatrix} \qquad \begin{pmatrix} 3 & \prec_1 & 1 & \prec_1 & 4 & \prec_1 & 2 \\ 4 & \prec_2 & 1 & \prec_2 & 2 & \prec_2 & 3 \end{pmatrix}$$

reducible $(A = \{1, 3, 4\}, B = \{2\})$ — irreducible

<u>Observation.</u>

$$\#\{\text{irreducible pairs of linear orders of size } n\} = n! \cdot \mathfrak{ip}_n.$$

Correspondence of classes

1. $\mathcal{U}$ = {square-tiled surfaces}
   = {pairs of linear orders of the same size}

2. $\mathcal{V}$ = {connected square-tiled surfaces}

3. $\mathcal{W}$ = {irreducible pairs of linear orders of the same size}

$$p_n = \mathfrak{w}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n} = k! \cdot \mathfrak{ip}_k \cdot \binom{n}{k} \cdot \frac{\left((n-k)!\right)^2}{(n!)^2} = \frac{\mathfrak{ip}_k}{(n)_k}$$

Probability of a permutation to be indecomposable

Theorem

The probability q_n that a random permutation of size n is indecomposable, satisfies

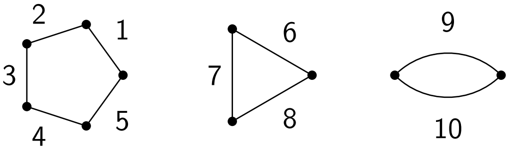
$$q_n \approx 1 - \sum_{k \geqslant 1} \frac{2\mathfrak{ip}_k - \mathfrak{ip}_k^{(2)}}{(n)_k},$$

where $\mathfrak{ip}_k^{(2)}$ is the number of permutations of size k with two indecomposable parts.

$$
\begin{aligned}
(\mathfrak{ip}_k) &= 1, \quad 1, \quad 3, \quad 13, \quad 71, \quad 461, \quad 3447, \quad \dots \\
\left(\mathfrak{ip}_k^{(2)}\right) &= 0, \quad 1, \quad 2, \quad 7, \quad 32, \quad 177, \quad 1142, \quad \dots \\
\left(2\mathfrak{ip}_k - \mathfrak{ip}_k^{(2)}\right) &= 2, \quad 1, \quad 4, \quad 19, \quad 110, \quad 745, \quad 5752, \quad \dots
\end{aligned}
$$

Combinatorial map model

- Take several labeled polygons of total perimeter $N = 2n$ (1-gons and 2-gons are allowed).
- Identify their sides randomly to obtain a surface.
- Each surface is determined by a pair $(\phi, \alpha) \in S_N^2$, where α is a perfect matching.



$$\phi = (1\,2\,3\,4\,5)(6\,7\,8)(9\,10), \quad \alpha = (1\,3)(2\,6)(4\,10)(5\,9)(7\,8)$$

Combinatorial map model asymptotics

Theorem

The probability p_n that a random surface within the combinatorial map model is connected satisfies

$$p_n \approx 1 - \sum_{k \geqslant 1} \mathfrak{im}_{2k} \cdot \frac{(2(n-k)-1)!!}{(2n-1)!!},$$

where $(\mathfrak{im}_{2k})$ counts indecomposable perfect matchings.

$$(\mathfrak{im}_{2k}) = 1, 2, 10, 74, 706, 8162, 110410, 1708394, \ldots$$