Irreducibility of combinatorial objects: asymptotic probability and interpretation

Khaydar Nurligareev (joint with Thierry Monteil)

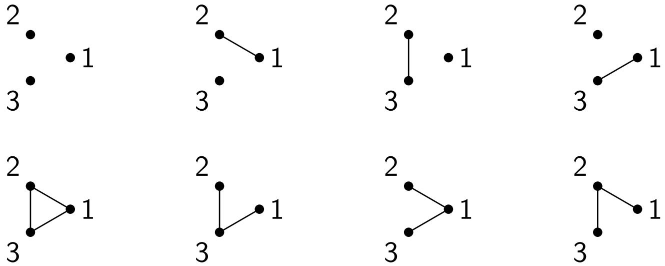
LIPN, University Paris 13

SSAAEC 2023, Będlewo, Poland

August 1, 2023

Simple labeled graphs

- $\mathfrak{g}_n$: the number of labeled graphs with n vertices,
- $\mathfrak{cg}_n$: the number of connected labeled graphs with n vertices.



$$\mathfrak{g}_n = 2^{\binom{n}{2}}$$

$$(\mathfrak{cg}_n) = 1, 1, 4, 38, 728, 26704, 1866256, \ldots$$

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{\mathfrak{cg}_n}{\mathfrak{g}_n}$ that a random graph with n vertices is connected, as $n \to \infty$?

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{\mathfrak{cg}_n}{\mathfrak{g}_n}$ that a random graph with n vertices is connected, as $n \to \infty$?

1. folklore: $p_n = 1 + o(1)$

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{\mathfrak{cg}_n}{\mathfrak{g}_n}$ that a random graph with n vertices is connected, as $n \to \infty$?

1. folklore: $p_n = 1 + o(1)$
2. Gilbert, 1959: $p_n = 1 - \frac{2n}{2^n} + O\left(\frac{n^2}{2^{3n/2}}\right)$

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{\mathfrak{cg}_n}{\mathfrak{g}_n}$ that a random graph with n vertices is connected, as $n \to \infty$?

1. folklore: $p_n = 1 + o(1)$
2. Gilbert, 1959: $p_n = 1 - \frac{2n}{2^n} + O\left(\frac{n^2}{2^{3n/2}}\right)$
3. Wright, 1970:

$$p_n = 1 - \binom{n}{1}\frac{2}{2^n} - 2\binom{n}{3}\frac{2^6}{2^{3n}} - 24\binom{n}{4}\frac{2^{10}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{\mathfrak{cg}_n}{\mathfrak{g}_n}$ that a random graph with n vertices is connected, as $n \to \infty$?

1. folklore: $p_n = 1 + o(1)$
2. Gilbert, 1959: $p_n = 1 - \frac{2n}{2^n} + O\left(\frac{n^2}{2^{3n/2}}\right)$
3. Wright, 1970:

$$p_n = 1 - \binom{n}{1}\frac{2}{2^n} - 2\binom{n}{3}\frac{2^6}{2^{3n}} - 24\binom{n}{4}\frac{2^{10}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

4. Can we see the structure? What is the interpretation?

Asymptotics for p_n

Theorem

For every $r \geqslant 1$, the probability p_n that a random labeled graph of size n is connected satisfies

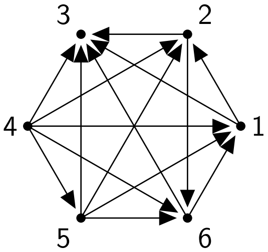
$$p_n = 1 - \sum_{k=1}^{r-1} \mathrm{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where it_k is the number of irreducible labeled tournaments of size k.

$$(\mathrm{it}_k) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$$

Tournaments

A **tournament** is a complete directed graph.



The number of labeled tournaments with n vertices is

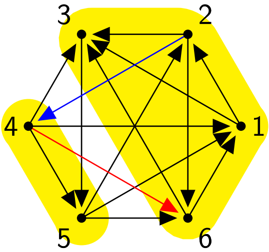
$$\mathfrak{t}_n = 2^{\binom{n}{2}}$$

Irreducible tournaments

A tournament is **irreducible**, if
for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B,
2. there exist an edge from B to A.

$$V = \{1, 2, 3, 4, 5, 6\}$$



$$A = \{4, 5\}$$
$$B = \{1, 2, 3, 6\}$$

Irreducible tournaments

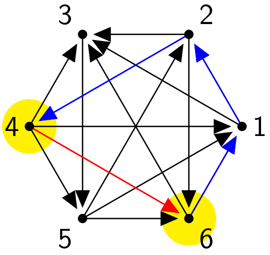
A tournament is **irreducible**, if
for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B,
2. there exist an edge from B to A.

Equivalently, a tournament is **strongly connected**: for each two vertices u and v

1. there is a path from u to v,
2. there is a path from v to u.

$V = \{1, 2, 3, 4, 5, 6\}$



$u = 4$
$v = 6$

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

$CG(z) = \log G(z)$

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

$CG(z) = \log G(z)$

Bender, 1975:

1. $A(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0$
2. $F(x, y)$ is analytic in $U(0; 0)$
3. $B(z) = \sum_{n=0}^{\infty} b_n z^n = F(z, A(z))$
4. $C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial F}{\partial y}(z, y)\right]_{y=A(z)}$
5. $\frac{a_{n-1}}{a_n} \to 0, \quad \text{as } n \to \infty$
6. $\exists r \geqslant 1 : \sum_{k=r}^{n-r} |a_k a_{n-k}| = O(a_{n-r})$

Then $\quad b_n = \sum_{k=0}^{r-1} c_k a_{n-k} + O(a_{n-r}).$

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

$CG(z) = \log G(z)$

Bender, 1975:

1. $A(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0$
2. $F(x, y)$ is analytic in $U(0;0)$
3. $B(z) = \sum_{n=0}^{\infty} b_n z^n = F(z, A(z))$
4. $C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial F}{\partial y}(z, y)\right]_{y=A(z)}$
5. the sequence (a_n) is gargantuan

Then $\quad b_n = \sum_{k=0}^{r-1} c_k a_{n-k} + O(a_{n-r})$.

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

$CG(z) = \log G(z)$

Bender, 1975:

1. $A(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0$
2. $F(x, y)$ is analytic in $U(0;0)$
3. $B(z) = \sum_{n=0}^{\infty} b_n z^n = F(z, A(z))$
4. $C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial F}{\partial y}(z, y)\right]_{y=A(z)}$
5. the sequence (a_n) is gargantuan

Then $b_n \approx \sum_{k \geqslant 0} c_k a_{n-k}$.

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

$CG(z) = \log G(z)$

$F(y) = \log(y)$

$\frac{\partial F}{\partial y} = \frac{1}{y}$

Bender, 1975:

1. $A(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0$
2. $F(x, y)$ is analytic in $U(0; 0)$
3. $B(z) = \sum_{n=0}^{\infty} b_n z^n = F(z, A(z))$
4. $C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial F}{\partial y}(z, y)\right]_{y=A(z)}$
5. the sequence (a_n) is gargantuan

Then $b_n \approx \sum_{k \geqslant 0} c_k a_{n-k}$.

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

$CG(z) = \log G(z)$

$F(y) = \log(y)$

$$\frac{\partial F}{\partial y} = \frac{1}{y}$$

$$\frac{1}{1-y} = 1 + y + y^2 + \dots$$

$$\frac{\mathfrak{cg}_n}{\mathfrak{g}_n} \approx 1 - \sum_{k \geqslant 1} \mathfrak{it}_k \binom{n}{k} \frac{\mathfrak{g}_{n-k}}{\mathfrak{g}_n}$$

Bender, 1975:

1. $A(z) = \sum_{n=1}^{\infty} a_n z^n$, $\quad a_n \neq 0$
2. $F(x, y)$ is analytic in $U(0; 0)$
3. $B(z) = \sum_{n=0}^{\infty} b_n z^n = F(z, A(z))$
4. $C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial F}{\partial y}(z, y)\right]_{y=A(z)}$
5. the sequence (a_n) is gargantuan

Then $\quad b_n \approx \sum_{k \geqslant 0} c_k a_{n-k}.$

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

$CG(z) = \log G(z)$

$F(y) = \log(y)$

$\frac{\partial F}{\partial y} = \frac{1}{y}$

$\frac{1}{1-y} = 1 + y + y^2 + \ldots$

$G(z) = T(z) = \frac{1}{1 - IT(z)}$

$\frac{\mathfrak{cg}_n}{\mathfrak{g}_n} \approx 1 - \sum_{k \geqslant 1} \mathfrak{it}_k \binom{n}{k} \frac{\mathfrak{g}_{n-k}}{\mathfrak{g}_n}$

Bender, 1975:

1. $A(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0$
2. $F(x, y)$ is analytic in $U(0; 0)$
3. $B(z) = \sum_{n=0}^{\infty} b_n z^n = F(z, A(z))$
4. $C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial F}{\partial y}(z, y)\right]_{y=A(z)}$
5. the sequence (a_n) is gargantuan

Then $\quad b_n \approx \sum_{k \geqslant 0} c_k a_{n-k}.$

Exponential generating functions and Bender's theorem

1. $CG(z) = \log G(z)$
2. $A(z) = G(z) - 1$
3. $F(x, y) = \log(1 + y)$
4. $\dfrac{\partial F}{\partial y} = \dfrac{1}{1 + y}$
5. $C(z) = \dfrac{1}{G(z)} = \dfrac{1}{T(z)}$
6. $\dfrac{1}{T(z)} = 1 - IT(z)$
7. $\dfrac{\mathfrak{cg}_n}{\mathfrak{g}_n} \approx 1 - \sum\limits_{k \geqslant 1} \mathfrak{it}_k \dbinom{n}{k} \dfrac{\mathfrak{g}_{n-k}}{\mathfrak{g}_n}$

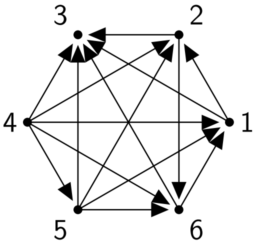
Bender, 1975:

1. $A(z) = \sum\limits_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0$
2. $F(x, y)$ is analytic in $U(0; 0)$
3. $B(z) = \sum\limits_{n=0}^{\infty} b_n z^n = F(z, A(z))$
4. $C(z) = \sum\limits_{n=0}^{\infty} c_n z^n = \left[\dfrac{\partial F}{\partial y}(z, y)\right]_{y = A(z)}$
5. the sequence (a_n) is gargantuan

Then $\quad b_n \approx \sum\limits_{k \geqslant 0} c_k a_{n-k}.$

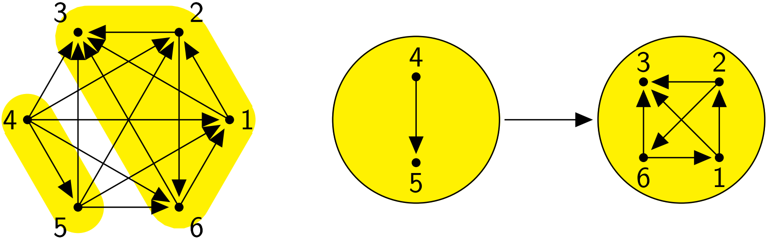
Tournament as a sequence

Folklore: Every labeled tournament can be uniquely decomposed into a sequence (SEQ) of irreducible labeled tournaments.



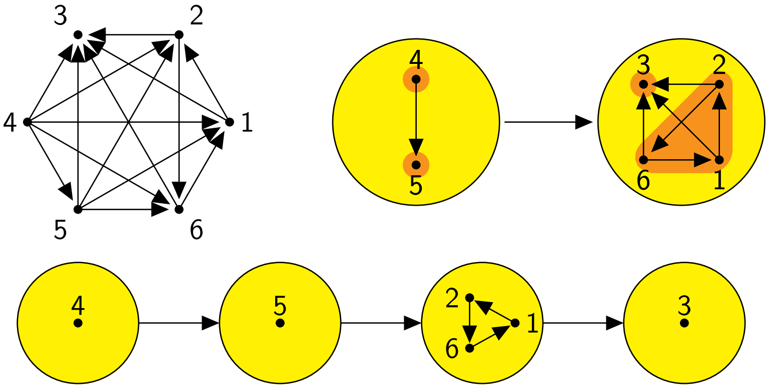
Tournament as a sequence

Folklore: Every labeled tournament can be uniquely decomposed into a sequence (SEQ) of irreducible labeled tournaments.



Tournament as a sequence

Folklore: Every labeled tournament can be uniquely decomposed into a sequence (SEQ) of irreducible labeled tournaments.



SET and SEQ decompositions

graphs —decomposed as SET of→ connected graphs —counted by→

$$CG(z) = \sum_{n=0}^{\infty} cg_n \frac{z^n}{n!} = \log(G(z))$$

↓ derivative

$$IT(z) = \sum_{n=0}^{\infty} it_n \frac{z^n}{n!} = 1 - \frac{1}{T(z)}$$

graphs ↑ counts

$$G(z) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{z^n}{n!} = T(z)$$

counts ↓ tournaments

tournaments —decomposed as as SEQ of→ irreducible tournaments —counted by→ $IT(z)$

SET asymptotics

Theorem

If $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are such combinatorial classes that

1. *$\mathcal{U}$ is gargantuan with positive counting sequence,*
2. $\mathcal{U} = \mathrm{SET}(\mathcal{V})$ *and* $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$,

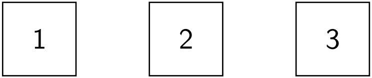
then

$$p_n := \frac{\mathfrak{v}_n}{\mathfrak{u}_n} \approx 1 - \sum_{k \geqslant 1} \mathfrak{w}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

Square-tiled surfaces

To obtain a **square-tiled surface** (determined by $(h, v) \in S_n^2$):

1. take n labeled squares,

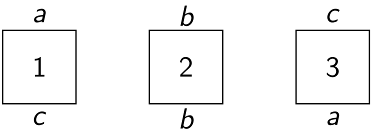


Square-tiled surfaces

To obtain a **square-tiled surface** (determined by $(h, v) \in S_n^2$):

1. take n labeled squares,
2. identify horizontal sides (corresponds to $h \in S_n$),

$h = (13)(2)$



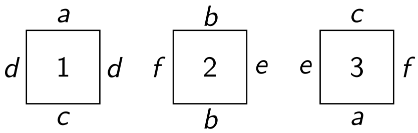
Square-tiled surfaces

To obtain a **square-tiled surface** (determined by $(h, v) \in S_n^2$):

1. take n labeled squares,
2. identify horizontal sides (corresponds to $h \in S_n$),
3. identify vertical sides (corresponds to $v \in S_n$),

$$h = (13)(2)$$

$$v = (1)(23)$$



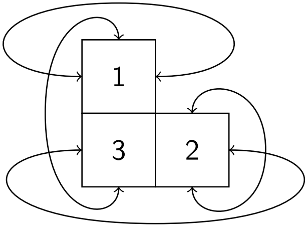
Square-tiled surfaces

To obtain a **square-tiled surface** (determined by $(h, v) \in S_n^2$):

1. take n labeled squares,
2. identify horizontal sides (corresponds to $h \in S_n$),
3. identify vertical sides (corresponds to $v \in S_n$),
4. glue together identified sides.

$$h = (13)(2)$$

$$v = (1)(23)$$



Square-tiled surfaces

To obtain a **square-tiled surface** (determined by $(h, v) \in S_n^2$):

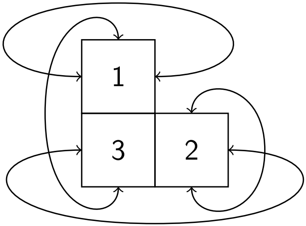
1. take n labeled squares,
2. identify horizontal sides (corresponds to $h \in S_n$),
3. identify vertical sides (corresponds to $v \in S_n$),
4. glue together identified sides.

Transitive action $\leftrightarrow$ connectedness of the square-tiled surface.

$$h = (13)(2)$$

$$v = (1)(23)$$

$\leftrightarrow$



Asymptotics for connected square-tiled surfaces

Theorem (reformulation of the results of Dixon and Cori)

The probability p_n that a random square-tiled surface of size n is connected satisfies

$$p_n \approx 1 - \sum_{k=1} \frac{\mathfrak{ip}_k}{n^{\underline{k}}},$$

where $n^{\underline{k}} = n(n-1)\dots(n-k+1)$ *are the falling factorials and* $(\mathfrak{ip}_k)$ *counts indecomposable permutations.*

$$(\mathfrak{ip}_k) = 1, 1, 3, 13, 71, 461, 3447, 29093, \dots$$

Indecomposable permutations

A permutation $\sigma \in S_n$ is

1. **decomposable**, if there is an index $p < n$ such that $\sigma\big(\{1,\ldots,p\}\big) = \{1,\ldots,p\}$.
2. **indecomposable** otherwise.

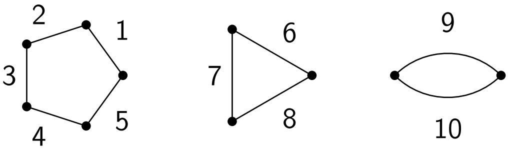
$$\left(\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{array}\right) \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$$

decomposable ($p = 3$) | indecomposable

Observation. Every permutation can be uniquely decomposed into a sequence of indecomposable permutations.

Combinatorial maps

- Take several labeled polygons of total perimeter $N = 2n$ (1-gons and 2-gons are allowed).
- Identify their sides randomly to obtain a surface.
- Each surface is determined by a pair $(\phi, \alpha) \in S_N^2$, where α is a perfect matching.



$$\phi = (1\,2\,3\,4\,5)(6\,7\,8)(9\,10), \quad \alpha = (1\,3)(2\,6)(4\,10)(5\,9)(7\,8)$$

Asymptotics for combinatorial maps

Theorem

The probability p_n that a random combinatorial map is connected satisfies

$$p_n \approx 1 - \sum_{k \geqslant 1} \mathfrak{im}_{2k} \cdot \frac{(2(n-k)-1)!!}{(2n-1)!!},$$

where $(\mathfrak{im}_{2k})$ counts indecomposable perfect matchings.

$$(\mathfrak{im}_{2k}) = 1, 2, 10, 74, 706, 8162, 110410, 1708394, \ldots$$

SEQ asymptotics

Theorem

If $\mathcal{U}$, $\mathcal{W}$ *and* $\mathcal{W}^{(2)}$ *are such combinatorial classes that*

- $\mathcal{U}$ *is gargantuan with positive counting sequence,*
- $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ *and* $\mathcal{W}^{(2)} = \mathcal{W} \star \mathcal{W} = \mathrm{SEQ}_2(\mathcal{W})$,

then

$$q_n := \frac{\mathfrak{w}_n}{\mathfrak{u}_n} \approx 1 - \sum_{k \geqslant 1} \left(2\mathfrak{w}_k - \mathfrak{w}_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

Reasoning: $\frac{1}{y} \xrightarrow{\partial} -\frac{1}{y^2}$, $\big(1 - W(z)\big)^2 = 1 - 2W(z) + \big(W(z)\big)^2$.

Example: asymptotics for irreducible tournaments

Theorem

The probability q_n that a random labeled tournament of size n is irreducible, satisfies

$$q_n \approx 1 - \sum_{k \geqslant 1} \left(2\mathrm{it}_k - \mathrm{it}_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}},$$

where $\left(\mathrm{it}_k^{(2)}\right)$ counts labeled tournaments with two irreducible components.

$$\begin{aligned}
(\mathrm{it}_k) &= 1, \quad 0, \quad 2, \quad 24, \quad 544, \quad 22320, \quad \ldots \\
\left(\mathrm{it}_k^{(2)}\right) &= 0, \quad 2, \quad 0, \quad 16, \quad 240, \quad 6608, \quad \ldots \\
\left(2\mathrm{it}_k - \mathrm{it}_k^{(2)}\right) &= 2, \quad -2, \quad 4, \quad 32, \quad 848, \quad 38032, \quad \ldots
\end{aligned}$$

Probability of a permutation to be indecomposable

Theorem

The probability q_n that a random permutation of size n is indecomposable, satisfies

$$q_n \approx 1 - \sum_{k \geqslant 1} \frac{2\mathfrak{ip}_k - \mathfrak{ip}_k^{(2)}}{n^{\underline{k}}},$$

where $\left(\mathfrak{ip}_k^{(2)}\right)$ counts permutations with two indecomposable parts.

$$\begin{array}{rcl}
(\mathfrak{ip}_k) & = & 1,\ 1,\ 3,\ 13,\ 71,\ 461,\ 3447,\ \ldots \\
\left(\mathfrak{ip}_k^{(2)}\right) & = & 0,\ 1,\ 2,\ 7,\ 32,\ 177,\ 1142,\ \ldots \\
\left(2\mathfrak{ip}_k - \mathfrak{ip}_k^{(2)}\right) & = & 2,\ 1,\ 4,\ 19,\ 110,\ 745,\ 5752,\ \ldots
\end{array}$$

Asymptotics in terms of species

Theorem

If $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{B}(m)$, $m \in \mathbb{N}$, are such (weighted) species that

1. *$\mathcal{A}$ is gargantuan with positive total weights on $[n]$, $n \in \mathbb{N}$,*
2. *one of the following conditions holds:*

(a)	$\mathcal{A} = \mathcal{E} \circ \mathcal{B}$	$\mathcal{B}(m) = \mathcal{E}_m \circ \mathcal{B}$	$\mathcal{C} = \mathcal{B}(m-1)(\mathcal{E}^{-1} \circ \mathcal{B})$
(b)	$\mathcal{A} = \mathcal{L} \circ \mathcal{B}$	$\mathcal{B}(m) = \mathcal{L}_m \circ \mathcal{B}$	$\mathcal{C} = m\mathcal{B}^{m-1}(1-\mathcal{B})^2$
(c)	$\mathcal{A} = \mathcal{CP} \circ \mathcal{B}$	$\mathcal{B}(m) = \mathcal{CP}_m \circ \mathcal{B}$	$\mathcal{C} = \mathcal{B}^{m-1}(1-\mathcal{B})$

then

$$p_n(m) := \frac{\mathfrak{b}_n(m)}{\mathfrak{a}_n} \approx \sum_{k \geqslant m-1} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}.$$

Erdős-Rényi model $G(n, p)$

Consider a random labeled graph G:

1. $p \in (0, 1)$ is the probability of edge presence;
2. $q = 1 - p$ is the probability of edge absence.

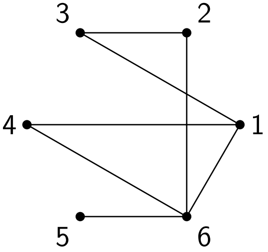
Weight of a graph:

$$w(G) = \rho^{|E(G)|},$$

where $\rho = p/q$.

Reason: if G_1 and G_2 are disjoint, then

$$w(G_1 \sqcup G_2) = w(G_1) \cdot w(G_2).$$



$w = \rho^7$

Asymptotics of the Erdős-Rényi model

Theorem

The probability p_n that a random graph with n vertices is connected satisfies

$$p_n \approx 1 - \sum_{k \geqslant 1} P_k(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}},$$

where

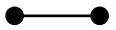
$$P_k(\rho) = \sum_{|V(G)|=k} (-1)^{\pi_0(G)-1} w(G)$$

and $\pi_0(G)$ is the number of connected components of G.

Meaning of the coefficients

$w = 1$

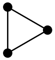
$P_1(\rho) = 1$



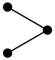
$w = \rho$

$w = 1$

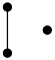
$P_2(\rho) = \rho - 1$



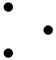
$w = \rho^3$



$w = \rho^2$



$w = \rho^1$

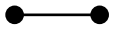


$w = 1$

$P_3(\rho) = \rho^3 + 3\rho^2 - 3\rho + 1$

Meaning of the coefficients

$w = 1$



$w = \rho$

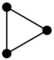
$w = 1$

$$P_1(\rho) = 1$$

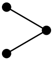
$$P_1(1) = 1 = \mathfrak{it}_1$$

$$P_2(\rho) = \rho - 1$$

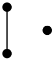
$$P_2(1) = 0 = \mathfrak{it}_2$$



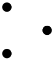
$w = \rho^3$



$w = \rho^2$



$w = \rho^1$



$w = 1$

$$P_3(\rho) = \rho^3 + 3\rho^2 - 3\rho + 1$$

$$P_3(1) = 2 = \mathfrak{it}_3$$

Asymptotics of the Erdős-Rényi model, continued

Theorem

The probability $p_n(m)$ that a random graph with n vertices has exactly m connected components satisfies

$$p_n(m) \approx \sum_{k \geqslant 0} P_k^{\{m\}}(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}},$$

where

$$P_k^{\{m\}}(\rho) = \sum_{|V(G)|=k} (-1)^{\pi_0(G)-m} \binom{\pi_0(G)}{m-1} w(G).$$

Conclusion

1. General method for combinatorial interpretation of coefficients in asymptotic expansions of irreducibles:
   - in terms of combinatorial classes (SET, SEQ, CYC),
   - in terms of species ($\mathcal{E}$, $\mathcal{L}$, $\mathcal{CP}$).
2. Applications:
   - connected graphs and irreducible tournaments,
   - square-tiled surfaces and indecomposable permutations,
   - combinatorial maps and indecomposable perfect matchings,
   - the Erdős-Rényi model,
   - ...

Thank you for your attention!

Literature I

Bender E.A.
An asymptotic expansion for some coefficients of some formal power series
J. Lond. Math. Soc., Ser. 9 (1975), pp. 451-458.

Gilbert E.N.
Random graphs
Ann. Math. Stat., Vol. 30, N. 4 (1959), pp. 1141-1144.

Monteil T., Nurligareev K.
Asymptotics for connected graphs and irreducible tournaments
Extended Abstracts EuroComb 2021, pp. 823-828. Springer, 2021.

Literature II

Nurligareev K.
Irreducibility of combinatorial objects: asymptotic probability and interpretation
Theses, Université Sorbonne Paris Nord, Oct. 2022.

Wright E.M.
Asymptotic relations between enumerative functions in graph theory
Proc. Lond. Math. Soc., Vol. s3-20, Issue 3 (1970), pp. 558-572.

CYC asymptotics

Theorem

If $\mathcal{V}$ and $\mathcal{W}$ are such combinatorial classes that

- *$\mathcal{V}$ is gargantuan with positive counting sequence,*
- $\mathcal{V} = \mathrm{CYC}(\mathcal{W})$,

then

$$r_n := \frac{\mathfrak{w}_n}{\mathfrak{v}_n} \approx 1 - \sum_{k \geqslant 1} \mathfrak{w}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{v}_{n-k}}{\mathfrak{v}_n}.$$

Reasoning: $e^{-y} \xrightarrow{\partial} -e^{-y}$.