Asymptotics of endhered patterns in perfect matchings

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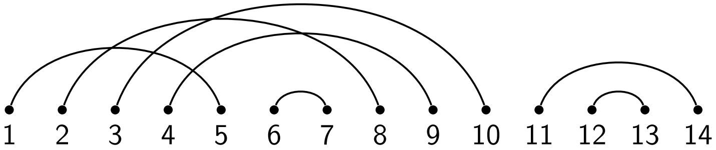
Perfect matchings

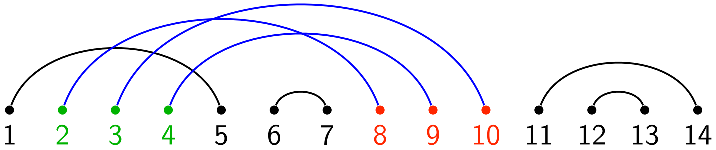
- A (perfect) **matching** is an involution without fixed points.
- A matching of size n consists of $2n$ points and n arcs:



- There are $(2n-1)!!$ matchings of size n.

Endhered patterns

- **Endhered pattern** in a matching:
  - starting points form an interval,
  - ending points form an interval.



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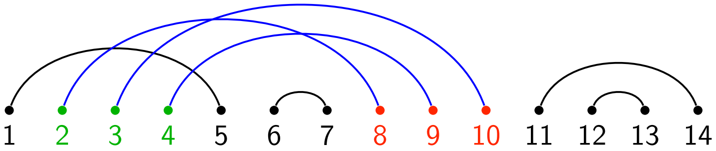
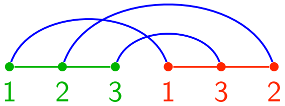


- Endhered patterns are encoded by permutations:

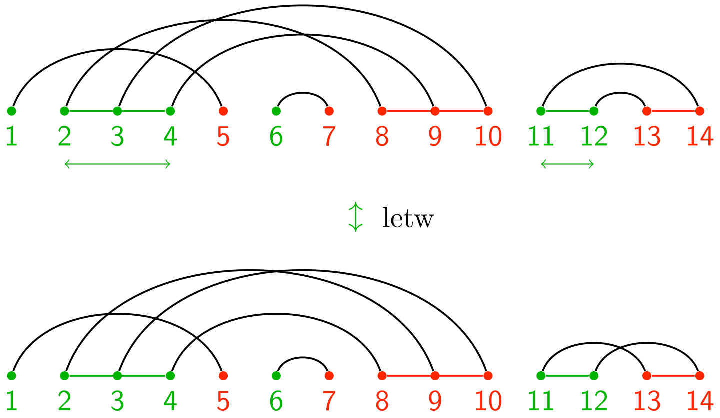


$\leftrightsquigarrow$ $\tau = 1\,3\,2$

- $a_{n,k}(\tau) = \#\{\text{matchings of size } n \text{ with } k \text{ patterns } \tau\}$.

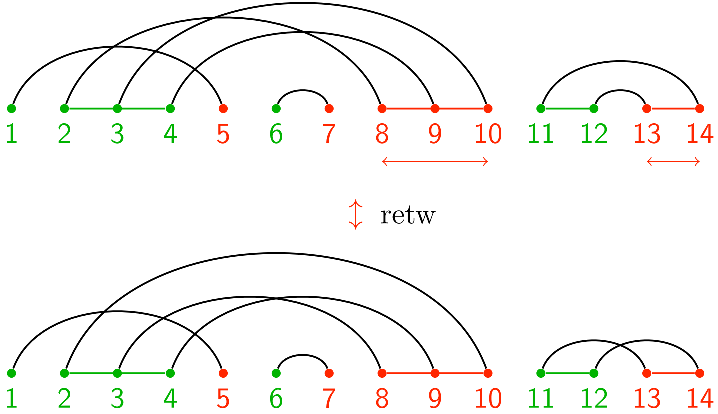
Endhered twists

Left endhered twist: reverse all runs of consecutive left points.

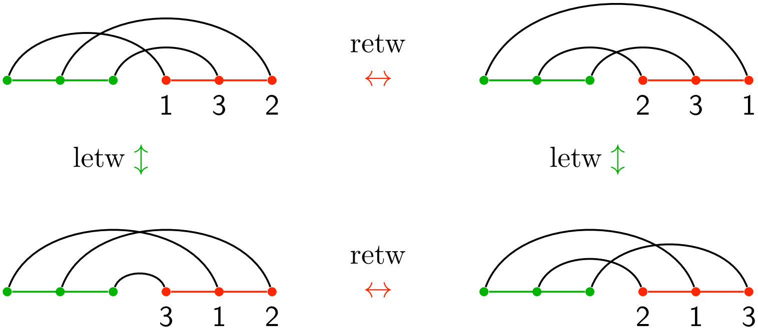


Endhered twists

Right endhered twist: reverse all runs of consecutive right points.



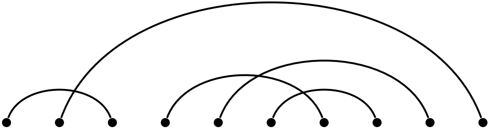
(Wilf) equivalent patterns



- Left twist: relabeling $1, \dots, p \ \rightarrow \ p, \dots, 1$ in a pattern.
- Right twist: reversing a pattern.
- $a_{n,k}(\tau) = a_{n,k}(\mathrm{letw}(\tau)) = a_{n,k}(\mathrm{retw}(\tau))$.

Pattern $\tau = 21$, recurrences

- Generating: $a_{n+1,k} =$



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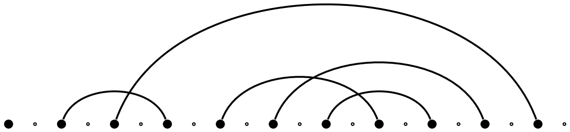
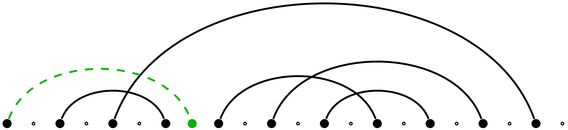


Pattern $\tau = 21$, recurrences

- Generating: $a_{n+1,k} = a_{n,k-1} +$



Pattern $\tau = 21$, recurrences

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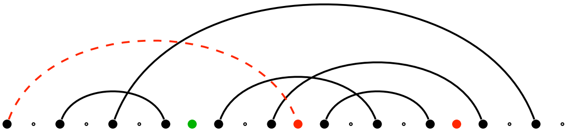

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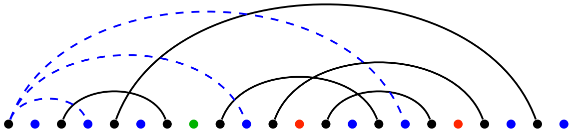

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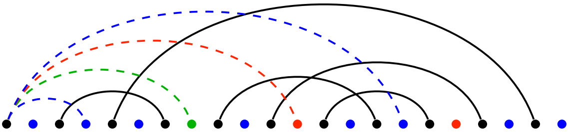

- Insertion:

$$a_{n+1,k} = \binom{n}{k} a_{n-k+1,0}$$

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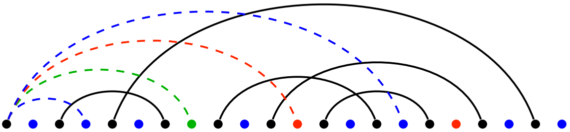


- Insertion:

$$a_{n+1,k} = \binom{n}{k} a_{n-k+1,0}$$

- Inclusion-exclusion:

$$a_{n+1,0} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (2k+1)!!$$

Pattern $\tau = 21$, generating function and asymptotics

- Generating function:

$$A(z,u) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{n,k} \frac{z^n}{n!} u^k$$

- Exact form:

$$\frac{\partial A}{\partial z}(z,u) = \frac{e^{z(u-1)}}{\sqrt{(1-2z)^3}}$$

- Asymptotics:

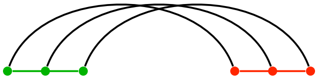
$$a_{n,k} \sim \frac{1}{2^k k!} \left(\frac{2}{e}\right)^{n+1/2} n^n$$

as $n \to \infty$.

Autocorrelation polynomials

Autocorrelation polynomial of a pattern τ is $A_\tau(z) = \sum_{j=0}^{|\tau|-1} c_j z^j$,

where $c_j = 1$ iff the pattern matchs itself after shifting right by j positions (otherwise, $c_j = 0$).



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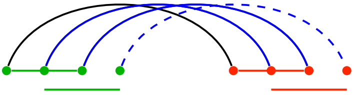

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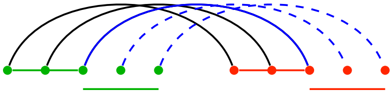

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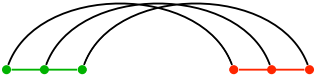

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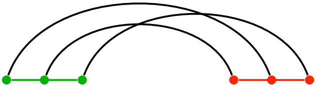

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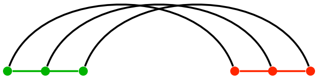

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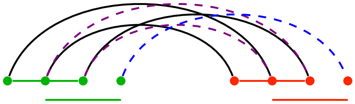

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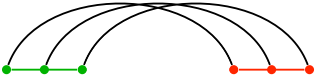

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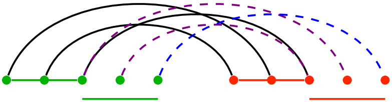

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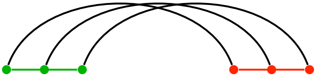

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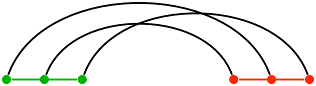
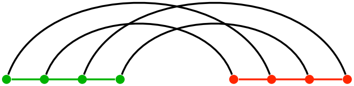


$$A_{213}(z) = 1$$

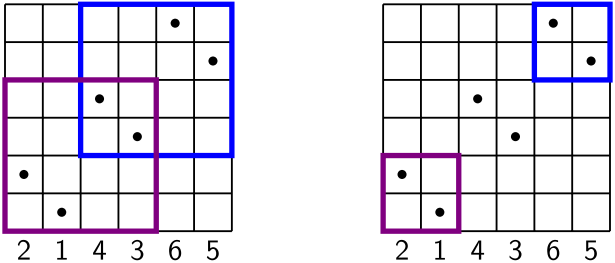


$$A_{2143}(z) = 1 + z^2$$

Self-overlapping permutations

Permutation $\sigma \in S_n$ is **self-overlapping** if there is $k < n$:

1. $\{1, \dots, k\}$ is invariant under σ,
2. $\{n-k+1, \dots, n\}$ is invariant under σ,
3. $\sigma(1)\dots\sigma(k)$ and $\sigma(n-k+1)\dots\sigma(n)$ are isomorphic.



It is always possible to choose $k \leqslant n/2$.

Structure of self-overlapping permutations

- Let $\sigma \in S_n$ and $\sigma(1) < \sigma(n)$.

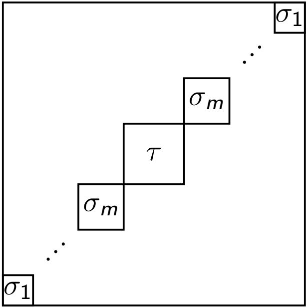
  Then σ is non-self-overlapping iff $A_\sigma(z) = 1$.

- Every permutation $\sigma \in S_n$ can be decomposed as

$$\sigma = \sigma_1 \oplus \ldots \oplus \sigma_m \oplus \tau \oplus \sigma_m \oplus \ldots \oplus \sigma_1$$

  where

  - σ_i are non-self-overlapping,
  - τ is empty or non-self-overlapping.



Asymptotics of non-self-overlapping permutations

Generating functions:

$$P(z) = \frac{1 + N(z)}{1 - N(z^2)},$$

where

- $P(z)$ is the OGF of permutations,
- $N(z)$ is the OGF of non-self-overlapping permutations.

Asymptotics of non-self-overlapping permutations

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$$P(z) = \frac{1 + N(z)}{1 - N(z^2)} \, ,$$

where

- $P(z)$ is the OGF of permutations,
- $N(z)$ is the OGF of non-self-overlapping permutations.

Asymptotics:

$$\mathbb{P}(\sigma \text{ is non-self-overlapping}) = 1 - \sum_{k=1}^{r-1} \frac{\mathfrak{no}_k}{(n)_{2k}} + O\left(\frac{1}{n^{2r}}\right) \, ,$$

where

- $\mathfrak{no}_k = \#\{\text{non-self-overlapping permutations of size } k\}$,
- $(n)_k = n(n-1)\ldots(n-k+1)$ are falling factorials.

Asymptotics for $a_{n,k}(\tau)$ with $A_\tau(z) = 1$

- Let τ be a non-self-overlapping pattern, i.e. $A_\tau(z) = 1$.
- Generating function of matchings:

$$S(z) = \sum_{n=0}^{\infty} (2n-1)!!\, z^n$$

- Generating function with respect to τ:

$$\sum_{n,k \geqslant 0} a_{n,k}(\tau)\, z^n u^k = S\Big(z + (u-1)z^{|\tau|}\Big)$$

- Asymptotics:

$$a_{n,k}(\tau) \sim \frac{2^{1/2}}{k!\, 2^{k(|\tau|-1)}} \left(\frac{2}{e}\right)^n n^{n-k(|\tau|-2)}$$

as $n \to \infty$.

Asymptotics for $a_{n,k}(\tau)$ with $A_\tau(z) \neq 1$

- Let τ be a self-overlapping permutation, $A_\tau(z) = 1 + z^m + \ldots$
- Generating function with respect to τ:

$$\sum_{n,k\geqslant 0} a_{n,k}(\tau)\, z^n u^k = S\left(\frac{z + (u-1)z^{|\tau|}}{1-(u-1)(A_\tau(z)-1)}\right)$$

- Asymptotics: as $n \to \infty$,

$$a_{n,k}(\tau) \sim \begin{cases} \dfrac{2^{1/2}}{k!\,2^{km}} \left(\dfrac{2}{e}\right)^n n^{n-k(m-1)} & \text{if} \quad m = |\tau| - 1 \\ \dfrac{(2n)^{n-km}\,2^{1/2}}{e^n} \displaystyle\sum_{s=1}^{k} \frac{1}{s!\,2^s} \binom{k-1}{s-1} & \text{if} \quad m = |\tau| - 2 \\ \dfrac{(2n)^{n-km-(|\tau|-2-m)}}{e^n\,2^{1/2}} & \text{if} \quad m < |\tau| - 2 \end{cases}$$

Asymptotics of factorially divergent series (Borinsky)

$$a_n = \alpha^{n+\beta}\Gamma(n+\beta)\left(c_0 + \frac{c_1}{\alpha(n+\beta-1)} + \frac{c_2}{\alpha^2(n+\beta-1)(n+\beta-2)} + \dots\right)$$

$$\sum_{n=0}^{\infty} a_n z^n \quad \xmapsto{\mathcal{A}_\beta^\alpha} \quad \sum_{n=0}^{\infty} c_n z^n$$

Properties:

- $\big(\mathcal{A}_\beta^\alpha(A \cdot B)\big)(z) = A(z) \cdot (\mathcal{A}_\beta^\alpha B)(z) + B(z) \cdot (\mathcal{A}_\beta^\alpha A)(z)$,
- $$\big(\mathcal{A}_\beta^\alpha(A \circ B)\big)(z) = A'\big(B(z)\big) \cdot (\mathcal{A}_\beta^\alpha B)(z) + \left(\frac{z}{B(z)}\right)^\beta \exp\left(\frac{1}{\alpha}\left(\frac{1}{z} - \frac{1}{B(z)}\right)\right)(\mathcal{A}_\beta^\alpha A)\big(B(z)\big).$$

Extracting asymptotics

- $S(z) = \sum_{n=0}^{\infty} (2n-1)!!\, z^n \qquad \Rightarrow \qquad \left(\mathcal{A}^2_{1/2} S\right)(z) = \frac{1}{\sqrt{2\pi}}$

- $G(z) = \frac{z + (u-1)z^{|\tau|}}{1 - (u-1)(A_\tau(z) - 1)} \qquad \Rightarrow \qquad \left(\mathcal{A}^2_{1/2} G\right)(z) = 0$

- Composition:

$$\left(\mathcal{A}^2_{1/2}(S \circ G)\right)(z) = \frac{1}{\sqrt{2\pi}} \left(1 + \frac{(u-1)z^{|\tau|-1}}{1 - (u-1)(A_\tau(z) - 1)}\right)^{-1/2} \times \exp\left(\frac{(u-1)z^{|\tau|-2}}{2\left(1 - (u-1)(A_\tau(z) - 1 - z^{|\tau|-1})\right)}\right)$$

Conclusion

1. Studied objects:
   - endhered patterns in perfect matchings,
   - self-overlapping permutations.

2. Tools:
   - the symbolic method,
   - singularity analysis,
   - Goulden-Jackson cluster method,
   - Borinsky's approach.

3. Results:
   - direct enumeration for endhered patterns of size 2,
   - enumeration and asymptotics for any endhered pattern,
   - enumeration and asymptotics of non-self-overlapping permutations.

Thank you for your attention!