# Brick wall excursions

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ALEA 2025

March 17, 2025

# Short random walks

- $d$ is the dimension,
- $\nu = \dfrac{d}{2} - 1$,
- $m = \#$ steps,
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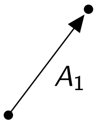

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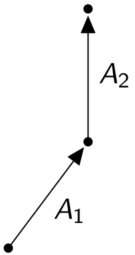

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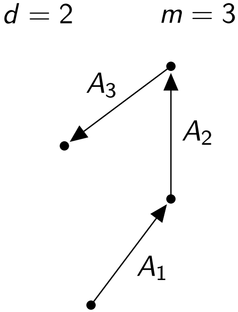

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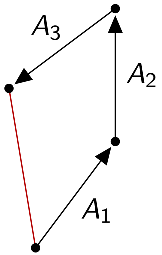


Key object: moments $W_m(\nu, n) = \mathbb{E}\Big(|A_1 + \ldots + A_m|^n\Big)$

- Fact: for any $m, n \in \mathbb{Z}_{\geqslant 0}$,
  $W_m(0, 2n)$ and $W_m(1, 2n)$ are integers. Interpretation?

# Matrix form for $d = 2$ $(\nu = 0)$. Example: $m = 2$

Fact:

$$W_m(0, 2n) = \sum_{\ell=0}^{n} [M^{m-1}]_{n\ell}\,, \qquad \text{where} \quad M = \left(\binom{i}{j}^2\right)_{i,j\geqslant 0}$$

Example: $m = 2$,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & \cdots \\ 1 & 9 & 9 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

# Matrix form for $d=2$ ($\nu=0$). Example: $m=2$

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Example: $m=2, \qquad W_2(0,2n) = \binom{2n}{n}$

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$W_2(0, 2n) = 6$ :

*UUDD*
*UDUD*
*UDDU*
*DUUD*
*DUDU*
*DDUU*

# Matrix form for $d = 2$ ($\nu = 0$). Example: $m = 3$

Example: $m = 3$

$$M^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & \cdots \\ 6 & 8 & 1 & 0 & \cdots \\ 20 & 46 & 10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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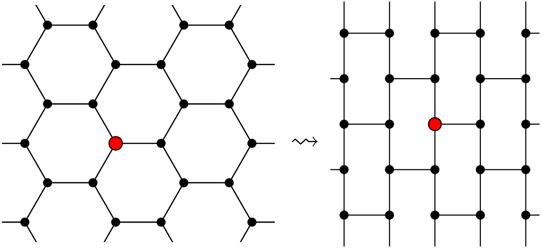

$W_3(0, 2n) = 15$ :

*RLRL*
*RLUD*
*RLDU*
*RUDL*
*RDUL*
*ULRD*
*UDRL*
*DLRU*
*DURL*
*UUDD*
*UDUD*
*UDDU*
*DUUD*
*DUDU*
*DDUU*

# Interpretation for $d = 2$ ($\nu = 0$)

Let $A_k \in \mathbb{C}, \quad |A_k| = 1 \qquad (k = 1, \ldots, m)$.

$$W_m(0, 2n) \quad = \quad \mathbb{E}\left|A_1 + \ldots + A_m\right|^{2n}$$

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$$\stackrel{(1)}{=} \quad \mathbb{E} \left( \left( A_1 + \ldots + A_m \right) \left( A_1^{-1} + \ldots + A_m^{-1} \right) \right)^n$$

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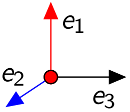
$$\overset{(1)}{=} \mathbb{E}\left(\left(A_1 + \ldots + A_m\right)\left(A_1^{-1} + \ldots + A_m^{-1}\right)\right)^n$$

$$\overset{(2)}{=} \left[A_1^0 \ldots A_m^0\right]\left(\left(A_1 + \ldots + A_m\right)\left(A_1^{-1} + \ldots + A_m^{-1}\right)\right)^n$$

1. $1 = |A_k|^2 = A_k \bar{A}_k \qquad \Rightarrow \qquad A_k^{-1} = \bar{A}_k$
2. $\mathbb{E}\left(A_1 A_2^{-1} A_3 A_1^{-1} A_3 A_2^{-1}\right) = \mathbb{E}\left(A_2^{-2} A_3^2\right) = \mathbb{E}\left(A_2^{-2}\right) \cdot \mathbb{E}\left(A_3^2\right) = 0$

# Interpretation for $d = 2$ ($\nu = 0$) and $m = 3$

$$\mathbb{E}\left(\left(\underset{\substack{\downarrow\\ e_1}}{A_1} + \underset{\substack{\downarrow\\ e_2}}{A_2} + \underset{\substack{\downarrow\\ e_3}}{A_3}\right)\left(\underset{\substack{\downarrow\\ -e_1}}{A_1^{-1}} + \underset{\substack{\downarrow\\ -e_2}}{A_2^{-1}} + \underset{\substack{\downarrow\\ -e_3}}{A_3^{-1}}\right)\right)^n$$



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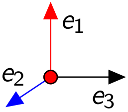
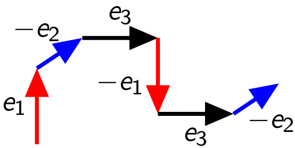
$$\mathbb{E}\Big(\big(\underset{\displaystyle e_1}{\underset{\downarrow}{A_1}} + \underset{\displaystyle e_2}{\underset{\downarrow}{A_2}} + \underset{\displaystyle e_3}{\underset{\downarrow}{A_3}}\big)\big(\underset{\displaystyle -e_1}{\underset{\downarrow}{A_1^{-1}}} + \underset{\displaystyle -e_2}{\underset{\downarrow}{A_2^{-1}}} + \underset{\displaystyle -e_3}{\underset{\downarrow}{A_3^{-1}}}\big)\Big)^n$$

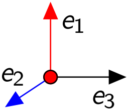



$$A_1 A_2^{-1} A_3 A_1^{-1} A_3 A_2^{-1}$$

# Interpretation for $d = 2$ ($\nu = 0$) and $m = 3$

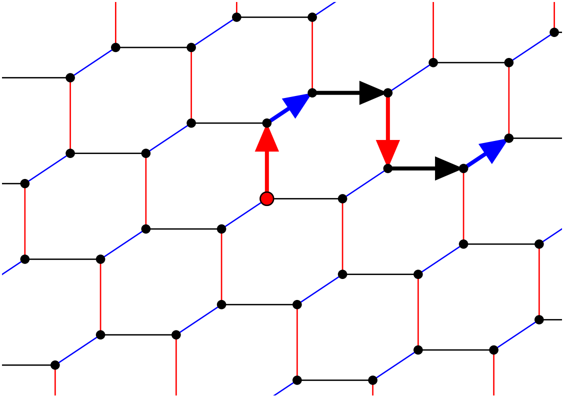
$$\mathbb{E}\Bigg(\Big(\begin{array}{c} A_1 \\ \downarrow \\ e_1 \end{array} + \begin{array}{c} A_2 \\ \downarrow \\ e_2 \end{array} + \begin{array}{c} A_3 \\ \downarrow \\ e_3 \end{array}\Big)\Big(\begin{array}{c} A_1^{-1} \\ \downarrow \\ -e_1 \end{array} + \begin{array}{c} A_2^{-1} \\ \downarrow \\ -e_2 \end{array} + \begin{array}{c} A_3^{-1} \\ \downarrow \\ -e_3 \end{array}\Big)\Bigg)^n$$




terms

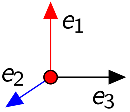
$\downarrow$

paths

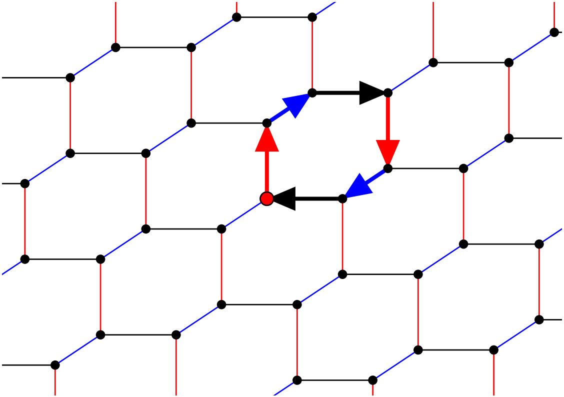


# Interpretation for $d = 2$ ($\nu = 0$) and $m = 3$

$$\mathbb{E}\Big(\big(\underset{\displaystyle e_1}{\underset{\downarrow}{A_1}} + \underset{\displaystyle e_2}{\underset{\downarrow}{A_2}} + \underset{\displaystyle e_3}{\underset{\downarrow}{A_3}}\big)\big(\underset{\displaystyle -e_1}{\underset{\downarrow}{A_1^{-1}}} + \underset{\displaystyle -e_2}{\underset{\downarrow}{A_2^{-1}}} + \underset{\displaystyle -e_3}{\underset{\downarrow}{A_3^{-1}}}\big)\Big)^n$$



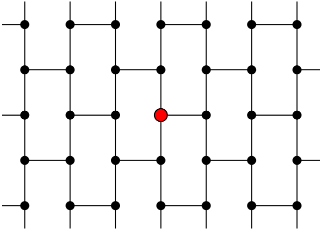

constant term
$\downarrow$
closed paths



# Matrix form revisited for $d = 2$ ($\nu = 0$) and $m = 3$

Paths $\quad \leftrightarrow \quad$ words on $\{U, D, R, L\}$:

- $R$ on odd positions,
- $L$ on even positions.



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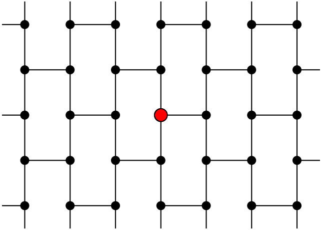

Let

- $2n = \#$ steps,
- $k = \#U = \#D$,
- $n - k = \#R = \#L$.

Then $$W_3(0, 2n) = \sum_{k=0}^{n} \binom{n}{n-k} \binom{n}{n-k} \binom{2k}{k}$$

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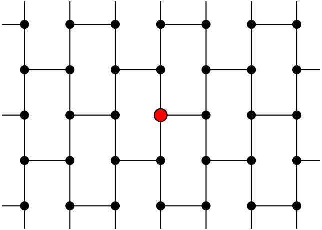

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Then

$$W_3(0, 2n) = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{n-k}\binom{2k}{k} = \sum_{k=0}^{n} \binom{n}{k}^2 \sum_{\ell=0}^{k} \binom{k}{\ell}^2$$

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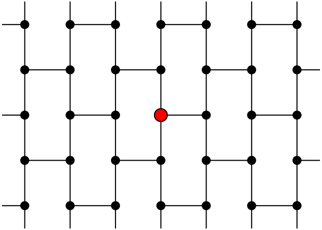

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# General construction for $d = 2$ ($\nu = 0$)

Paths $\leftrightarrow$ words on $\{U, D, R_1, L_1, \ldots, R_{m-2}, L_{m-2}\}$:

- $R_s$ on odd positions,
- $L_s$ on even positions.

Let

- $2n = \#$ steps,
- $k_0 = \#U = \#D$,
- $k_s = \#R_k = \#L_k$.

$$
\begin{aligned}
W_m(0, 2n) &= \sum_{k_0+\ldots+k_{m-2}=n} \binom{n}{k_{m-2}}^2 \binom{n-k_{m-2}}{k_{m-3}}^2 \ldots \binom{k_1+k_0}{k_0}^2 \binom{2k_0}{k_0} \\
&= \sum_{k_0+\ldots+k_{m-2}=n} \binom{k_0+\ldots+k_{m-2}}{k_0+\ldots+k_{m-3}}^2 \ldots \binom{k_0+k_1}{k_0}^2 \sum_{\ell=0}^{n} \binom{k_0}{\ell}^2 \\
&= \sum_{\ell, r_1, \ldots, r_{m-2}=0}^{n} M_{n r_{m-2}} \ldots M_{r_2 r_1} M_{r_1 \ell} = \sum_{\ell=0}^{n} [M^{m-1}]_{n\ell}
\end{aligned}
$$

# Summary for $d = 2$ ($\nu = 0$)

- We consider moments $W_m(0, 2n) = \mathbb{E}\Big(|A_1 + \ldots + A_m|^n\Big)$, where $A_k \in \mathbb{C}$, $|A_k| = 1$ $(k = 1, \ldots, m)$.

- $W_m(0, 2n)$ is the constant term in $\Big((A_1 + \ldots + A_m)(A_1^{-1} + \ldots + A_m^{-1})\Big)^n$.

- Thus, $W_m(0, 2n)$ can be interpreted as the number of closed paths of length $2n$ on a specific $m$-dimensional lattice.

- In particular,

$$W_m(0, 2n) = \sum_{\ell=0}^{n} [M^{m-1}]_{n\ell}\,, \qquad \text{where} \quad M = \left(\binom{i}{j}^2\right)_{i,j \geqslant 0}$$

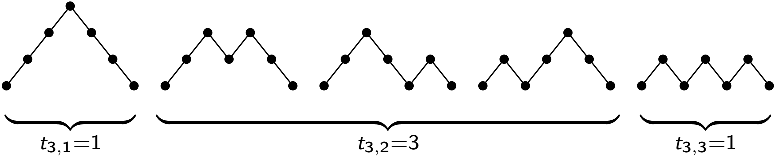
# Matrix form for $d = 4$ ($\nu = 1$). Example: $m = 2$

Fact:

$$W_m(1, 2n) = \sum_{\ell=0}^{n} [N^{m-1}]_{n\ell}\,, \qquad \text{where} \quad N = \left(t_{i+1,j+1}\right)_{i,j\geqslant 0}$$

Here, $t_{i,j}$ are the **Narayana numbers**, *i.e.*

$$t_{i,j} = \#\{\text{Dyck paths of length } i \text{ with } j \text{ peaks}\}$$



# Matrix form for $d = 4$ $(\nu = 1)$. Example: $m = 2$

Fact:

$$W_m(1, 2n) = \sum_{\ell=0}^{n} [N^{m-1}]_{n\ell}\,, \qquad \text{where} \quad N = \left(t_{i+1,\, j+1}\right)_{i,j\geqslant 0}$$

Example: $m = 2$,

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & \cdots \\ 1 & 6 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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Example: $m = 2$, $W_2(1, 2n) = \frac{1}{n+1}\binom{2n}{n}$

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & \cdots \\ 1 & 6 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \begin{matrix} \to & 1 \\ \to & 2 \\ \to & 5 \\ \to & 14 \\ \vdots & \vdots \end{matrix}$$

# Matrix form for $d = 4$ $(\nu = 1)$. Example: $m = 2$

Fact:

$$W_m(1, 2n) = \sum_{\ell=0}^{n} [N^{m-1}]_{n\ell}, \qquad \text{where} \quad N = \left( t_{i+1,\, j+1} \right)_{i,j \geqslant 0}$$

Example: $m = 2, \qquad W_2(1, 2n) = \frac{1}{n+1}\binom{2n}{n}$

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$W_2(1, 2n) = 5$ :

*UUUDDD*
*UUDUDD*
*UUDDUD*
*UDUUDD*
*UDUDUD*

# Bijective lemma

Consider words on $\{R, L, O\}$ such that:

- $R$ on odd positions,
- $L$ on even positions,
- $\#R = \#L$,
- in each prefix, $\#R \geqslant \#L$.

Then the number $D_n$ of such words of size $2n$ is

$$D_n = \sum_{k=0}^{n} t_{n+1,\,k+1}\,,$$

where

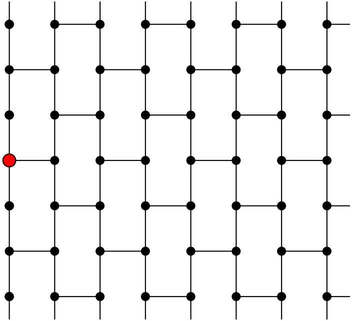
- $2k = \#O$,
- $n - k = \#R = \#L$.

# Applications: closed path counting

Let us count closed paths with $2n$ steps.

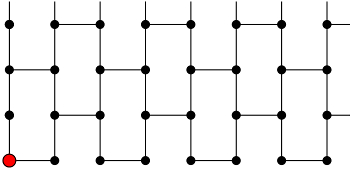
Half-plane:

$$\sum_{k=0}^{n} \binom{2k}{k} t_{n+1,\,k+1}$$



Quarter-plane:

$$\sum_{k=0}^{n} C_k t_{n+1,\,k+1}$$



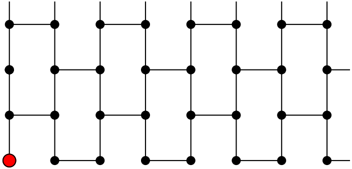
Hint: apply lemma with

$$O \rightsquigarrow U, D$$

# Matrix form revisited for $d = 4$ ($\nu = 1$) and $m = 3$

Shifted quarter-plane:

- $2n = \#$ steps,
- $R$ on even positions,
- $L$ on odd positions,
- in each prefix, $\#U \geqslant \#D$,
- in each prefix, $\#R \geqslant \#L$.



Then $$\#\{\text{closed paths}\} = \sum_{k=0}^{n} C_{k+1} t_{n+1,\, k+1}$$

(here, $k = \#U = \#D$ and $n - k = \#R = \#L$ )

# Matrix form revisited for $d = 4$ ($\nu = 1$) and $m = 3$

Shifted quarter-plane:

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- $R$ on even positions,
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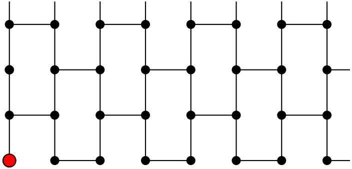

Then

$$\#\{\text{closed paths}\} = \sum_{k=0}^{n} C_{k+1} t_{n+1,\,k+1} = \sum_{k=0}^{n} t_{n+1,\,k+1} \sum_{\ell=0}^{k} t_{k+1,\,\ell+1}$$

(here, $k = \#U = \#D$ and $n - k = \#R = \#L$ )

# Matrix form revisited for $d = 4$ ($\nu = 1$) and $m = 3$

Shifted quarter-plane:

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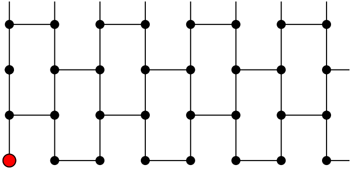
- in each prefix, $\#U \geqslant \#D$,
- in each prefix, $\#R \geqslant \#L$.



Then

$$\#\{\text{closed paths}\} = \sum_{k=0}^{n} C_{k+1} t_{n+1,\,k+1} = \sum_{k=0}^{n} t_{n+1,\,k+1} \sum_{\ell=0}^{k} t_{k+1,\,\ell+1} = \sum_{k,\ell=0}^{n} N_{nk} N_{k\ell} = \sum_{\ell=0}^{n} [N^2]_{n\ell}$$

(here, $k = \#U = \#D$ and $n - k = \#R = \#L$ )

# General construction for $d = 4$ ($\nu = 1$)

Paths $\leftrightarrow$ words on $\{U, D, R_1, L_1, \ldots, R_{m-2}, L_{m-2}\}$:

- $R_s$ on even positions (after removing all $R_t$ and $L_t$, $t > s$),
- $L_s$ on odd positions (after removing all $R_t$ and $L_t$, $t > s$),
- in each prefix, $\#U \geqslant \#D$ and $\#R_s \geqslant \#L_s$.

Let

- $2n = \#$ steps,
- $k_0 = \#U = \#D$,
- $k_s = \#R_k = \#L_k$.

$$
\begin{aligned}
W_4(0, 2n) &= \sum_{k_0+k_1+k_2=n} t_{n+1,\, k_1+k_0+1} \cdot t_{n-k_2+1,\, k_0+1} \cdot C_{k_0+1} \\
&= \sum_{k_0+k_1+k_2=n} t_{n+1,\, k_1+k_0+1} \cdot t_{k_1+k_0+1,\, k_0+1} \sum_{\ell=0}^{n} t_{k_0+1,\, \ell+1} \\
&= \sum_{\ell, r_1, r_2=0}^{n} N_{nr_2} N_{r_2 r_1} N_{r_1 \ell} = \sum_{\ell=0}^{n} [N^3]_{n\ell}
\end{aligned}
$$

# Summary for $d = 4$ ($\nu = 1$)

- We consider moments $W_m(1, 2n) = \mathbb{E}\Big(|A_1 + \ldots + A_m|^n\Big)$, where $A_k \in \mathbb{R}^4$, $|A_k| = 1$ $(k = 1, \ldots, m)$.
- It is known that,

$$W_m(1, 2n) = \sum_{\ell=0}^{n} [N^{m-1}]_{n\ell}\,, \qquad \text{where} \quad N = \big(t_{i+1,\,j+1}\big)_{i,\,j\geqslant 0}$$

- Thus, $W_m(1, 2n)$ can be interpreted as the number of closed paths of length $2n$ on a specific $m$-dimensional lattice.
- Question. Can we obtain the above result directly? (one could expect the use of quaternions)

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Thank you for your attention!

# Literature

Borwein J.M., Straub A., Vignat C.
Densities of short uniform random walks in higher dimensions
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