Interprétation combinatoire des coefficients dans les développement asymptotiques

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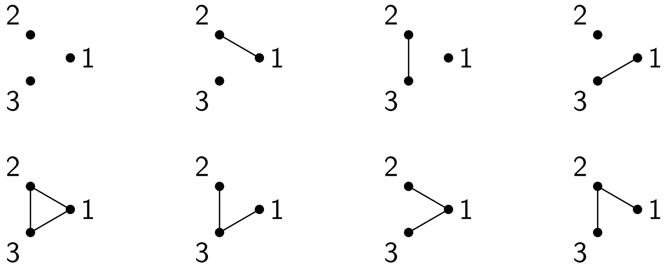
Simple labeled graphs

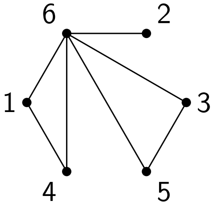
- $\mathfrak{g}_n = \#\{\text{graphs with } n \text{ vertices}\}$



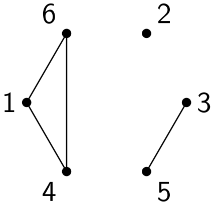
$$\mathfrak{g}_n = 2^{\binom{n}{2}}$$

Connected graphs

- $\mathfrak{cg}_n = \#\{\text{connected graphs with } n \text{ vertices}\}$



connected graph



disconnected graph

$$(\mathfrak{cg}_n) = 1, 1, 4, 38, 728, 26704, 1866256, \ldots$$

- Every graph is a disjoint union (SET) of connected graphs.

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{\mathfrak{cg}_n}{\mathfrak{g}_n}$ that a random graph with n vertices is connected, as $n \to \infty$?

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3. Wright, 1970:

$$p_n = 1 - \binom{n}{1}\frac{2}{2^n} - 2\binom{n}{3}\frac{2^6}{2^{3n}} - 24\binom{n}{4}\frac{2^{10}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

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4. Can we see the structure? What is the interpretation?

Asymptotics for p_n

Theorem (Monteil, N., 2021)

For every $r \geqslant 1$, the probability p_n that a random labeled graph of size n is connected satisfies

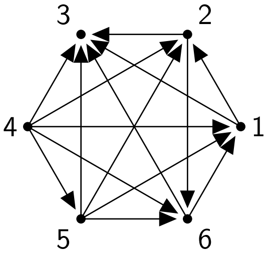
$$p_n = 1 - \sum_{k=1}^{r-1} \mathrm{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where it_k is the number of irreducible labeled tournaments of size k.

$$(\mathrm{it}_k) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$$

Tournaments

A **tournament** is a complete directed graph.



The number of labeled tournaments with n vertices is

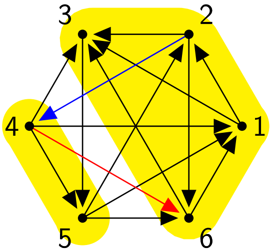
$$t_n = 2^{\binom{n}{2}}$$

Irreducible tournaments

A tournament is **irreducible**, if
for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B,
2. there exist an edge from B to A.

$V = \{1, 2, 3, 4, 5, 6\}$



$A = \{4, 5\}$
$B = \{1, 2, 3, 6\}$

Irreducible tournaments

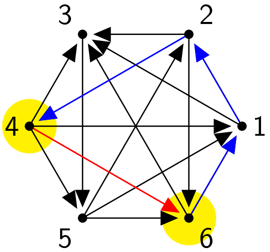
A tournament is **irreducible**, if
for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B,
2. there exist an edge from B to A.

Equivalently, a tournament is **strongly connected**: for each two vertices u and v

1. there is a path from u to v,
2. there is a path from v to u.

$V = \{1, 2, 3, 4, 5, 6\}$



$u = 4$
$v = 6$

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

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Bender, 1975:

1. $A(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0$
2. $F(x, y)$ is analytic in $U(0; 0)$
3. $B(z) = \sum_{n=0}^{\infty} b_n z^n = F(z, A(z))$
4. $C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial F}{\partial y}(z, y)\right]_{y=A(z)}$
5. $\frac{a_{n-1}}{a_n} \to 0, \text{ as } n \to \infty$
6. $\exists r \geqslant 1 : \sum_{k=r}^{n-r} |a_k a_{n-k}| = O(a_{n-r})$

Then $$b_n = \sum_{k=0}^{r-1} c_k a_{n-k} + O(a_{n-r}).$$

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5. the sequence (a_n) is gargantuan

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Then $b_n \approx \sum_{k \geqslant 0} c_k a_{n-k}$.

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$\frac{\partial F}{\partial y} = \frac{1}{y}$

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$CG(z) = \log G(z)$

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$\frac{\partial F}{\partial y} = \frac{1}{y}$

$\frac{1}{1-y} = 1 + y + y^2 + \ldots$

$\frac{\mathfrak{cg}_n}{\mathfrak{g}_n} \approx 1 - \sum_{k\geqslant 1} \mathfrak{it}_k \binom{n}{k} \frac{\mathfrak{g}_{n-k}}{\mathfrak{g}_n}$

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$G(z) = T(z) = \frac{1}{1 - IT(z)}$

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Then $b_n \approx \sum_{k \geqslant 0} c_k a_{n-k}$.

Exponential generating functions and Bender's theorem

1. $CG(z) = \log G(z)$
2. $A(z) = G(z) - 1$
3. $F(x, y) = \log(1 + y)$
4. $\dfrac{\partial F}{\partial y} = \dfrac{1}{1+y}$
5. $C(z) = \dfrac{1}{G(z)} = \dfrac{1}{T(z)}$
6. $\dfrac{1}{T(z)} = 1 - IT(z)$
7. $\dfrac{\mathfrak{cg}_n}{\mathfrak{g}_n} \approx 1 - \sum_{k\geqslant 1} \mathfrak{it}_k \binom{n}{k} \dfrac{\mathfrak{g}_{n-k}}{\mathfrak{g}_n}$

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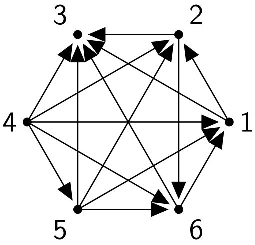
Tournament as a sequence

Folklore: Every labeled tournament can be uniquely decomposed into a sequence (SEQ) of irreducible labeled tournaments.



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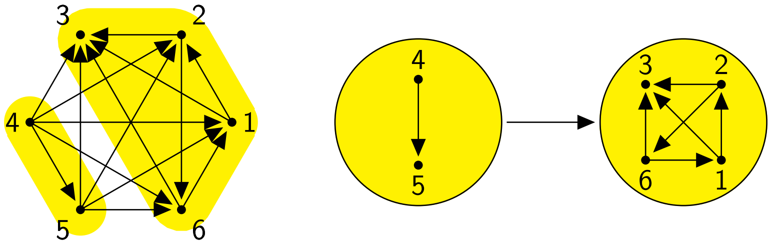

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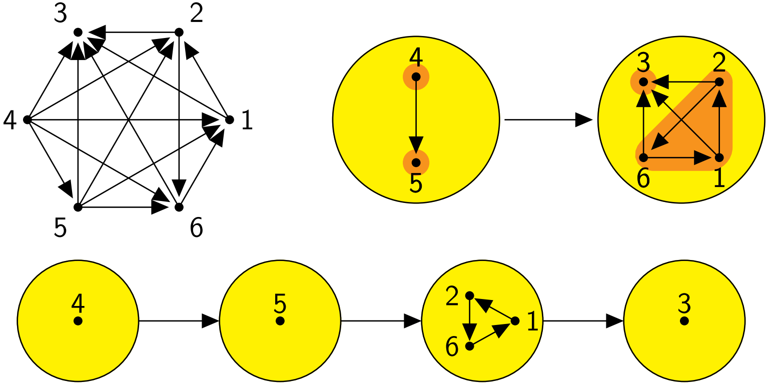


SET and SEQ decompositions

graphs —decomposed as SET of→ connected graphs

connected graphs —counted by→ $CG(z) = \sum_{n=0}^{\infty} \mathfrak{cg}_n \frac{z^n}{n!} = \log(G(z))$

$G(z) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{z^n}{n!} = T(z)$ counts graphs

$G(z)$ counts tournaments

"derivative": $CG(z) \to IT(z)$

$IT(z) = \sum_{n=0}^{\infty} \mathfrak{it}_n \frac{z^n}{n!} = 1 - \frac{1}{T(z)}$

tournaments —decomposed as SEQ of→ irreducible tournaments

irreducible tournaments —counted by→ $IT(z)$

SET asymptotics

Theorem (Monteil, N., 2022)

If $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are such combinatorial classes that

1. *$\mathcal{U}$ is gargantuan with positive counting sequence,*
2. $\mathcal{U} = \mathrm{SET}(\mathcal{V})$ *and* $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$,

then

$$p_n := \frac{\mathfrak{v}_n}{\mathfrak{u}_n} \approx 1 - \sum_{k \geqslant 1} \mathfrak{w}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

Combinatorial meaning: p_n is the probability that a random object of size n from $\mathcal{U}$ is connected.

Double decomposition of permutations

$$P(z) = \sum_{n=0}^{\infty} n! \frac{z^n}{n!} = \frac{1}{1-z}$$

Double decomposition of permutations

permutations

↑ counts

$$P(z) = \sum_{n=0}^{\infty} n! \frac{z^n}{n!} = \frac{1}{1-z} = L(z)$$

↓ counts

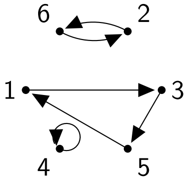
linear orders

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 5 & 4 & 1 & 2 \end{pmatrix}$$

$$(3 < 6 < 5 < 4 < 1 < 2)$$

Double decomposition of permutations

permutations —decomposed as SET of→ cycles



permutations ↑ counts

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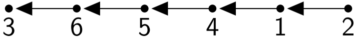
counts ↓ linear orders



linear orders —decomposed as SEQ of→ atoms

Double decomposition of permutations

permutations — decomposed as SET of → cycles

cycles — counted by →

$$\log(P(z)) = \log \frac{1}{1-z}$$

permutations ← counts —

$$P(z) = \sum_{n=0}^{\infty} n! \frac{z^n}{n!} = \frac{1}{1-z} = L(z)$$

— counts → linear orders

$$1 - \frac{1}{L(z)} = z$$

← counted by — atoms

linear orders — decomposed as SEQ of → atoms

Double decomposition of permutations

permutations —decomposed as SET of→ cycles —counted by→ $\log(P(z)) = \log \frac{1}{1-z}$

counts ↑

$P(z) = \sum_{n=0}^{\infty} n! \frac{z^n}{n!} = \frac{1}{1-z} = L(z)$ not gargantuan ~~"derivative"~~

counts ↓

linear orders —decomposed as SEQ of→ atoms —counted by→ $1 - \frac{1}{L(z)} = z$

Double decomposition of permutations

permutations —(decomposed as SET of)→ cycles

cycles —(counted by)→ $\log(P(z)) = \log \frac{1}{1-z}$

permutations ↑ counts

$$P(z) = \sum_{n=0}^{\infty} n! \frac{z^n}{n!} = \frac{1}{1-z} = L(z)$$

↓ counts linear orders

$$\boxed{\mathbb{P}(\sigma \text{ is a cycle}) = \frac{(n-1)!}{n!} = \frac{1}{n}}$$

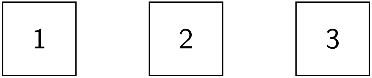
$1 - \frac{1}{L(z)} = z$ ←(counted by)— atoms

linear orders —(decomposed as SEQ of)→ atoms

Square-tiled surfaces

To obtain a **square-tiled surface** (determined by $(h, v) \in S_n^2$):

1. take n labeled squares,



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To obtain a **square-tiled surface** (determined by $(h, v) \in S_n^2$):

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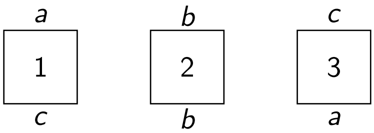
$h = (13)(2)$



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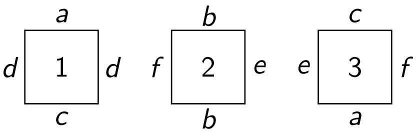
1. take n labeled squares,
2. identify horizontal sides (corresponds to $h \in S_n$),
3. identify vertical sides (corresponds to $v \in S_n$),

$$h = (13)(2)$$

$$v = (1)(23)$$



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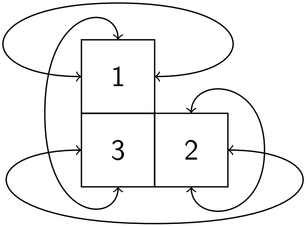

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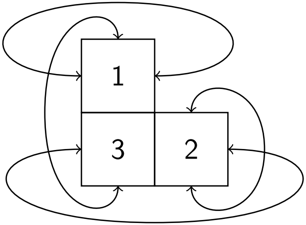
Transitive action $\leftrightarrow$ connectedness of the square-tiled surface.

$$h = (13)(2)$$

$$v = (1)(23)$$

$\leftrightarrow$



Pairs of linear orders

A pair of linear orders $(<_1, <_2)$ of size n is

1. **reducible**, if there is a partition $\{1, \ldots, n\} = A \sqcup B$ such that $\forall\, a \in A,\ b \in B\colon \quad a <_1 b$ and $a <_2 b$.

$$\left(\begin{array}{ccccccc} 3 & <_1 & 1 & <_1 & 4 & <_1 & 2 \\ 4 & <_2 & 3 & <_2 & 1 & <_2 & 2 \end{array}\right)$$

reducible $(A = \{1, 3, 4\}, B = \{2\})$

Pairs of linear orders

A pair of linear orders $(<_1, <_2)$ of size n is

1. **reducible**, if there is a partition $\{1, \dots, n\} = A \sqcup B$ such that $\forall\, a \in A,\ b \in B$: $\quad a <_1 b$ and $a <_2 b$.
2. **irreducible** otherwise.

$$\left(\begin{array}{ccccccc} 3 & <_1 & 1 & <_1 & 4 & <_1 & 2 \\ 4 & <_2 & 3 & <_2 & 1 & <_2 & 2 \end{array}\right) \qquad \left(\begin{array}{ccccccc} 3 & <_1 & 1 & <_1 & 4 & <_1 & 2 \\ 4 & <_2 & 1 & <_2 & 2 & <_2 & 3 \end{array}\right)$$

reducible $(A = \{1, 3, 4\}, B = \{2\})$ irreducible

Observation. A pair of linear orders can be uniquely decomposed into a sequence (SEQ) of irreducible pairs of linear orders.

Indecomposable permutations

A permutation $\sigma \in S_n$ is

1. **decomposable**, if there is an index $p < n$ such that $\sigma\big(\{1,\ldots,p\}\big) = \{1,\ldots,p\}$.
2. **indecomposable** otherwise.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{array}\right) \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

decomposable ($p = 3$) indecomposable

Observation.

$$\#\{\text{irreducible pairs of linear orders of size } n\} = n! \cdot \mathfrak{ip}_n,$$

where $\mathfrak{ip}_n = \#\{\text{indecomposable permutations of size } n\}$

Asymptotics for connected square-tiled surfaces

Theorem (reformulation of the results of Dixon and Cori)

The probability p_n that a random square-tiled surface of size n is connected satisfies

$$p_n \approx 1 - \sum_{k=1} \frac{\mathfrak{ip}_k}{n^{\underline{k}}},$$

where $n^{\underline{k}} = n(n-1)\dots(n-k+1)$ *are the falling factorials and* $(\mathfrak{ip}_k)$ *counts indecomposable permutations.*

$$(\mathfrak{ip}_k) = 1, 1, 3, 13, 71, 461, 3447, 29093, \dots$$

More applications

1. Combinatorial maps and indecomposable perfect matchings.
2. Connected multigraphs and irreducible multitournaments.
3. Constellations and indecomposable multipermutations.
4. Colored tensor models and indecomposable multipermutations.

SEQ asymptotics

Theorem (Monteil, N., 2022)

If $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{W}^{(2)}$ are such combinatorial classes that

- *$\mathcal{U}$ is gargantuan with positive counting sequence,*
- $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ *and* $\mathcal{W}^{(2)} = \mathcal{W} \star \mathcal{W} = \mathrm{SEQ}_2(\mathcal{W})$,

then

$$q_n := \frac{\mathfrak{w}_n}{\mathfrak{u}_n} \approx 1 - \sum_{k \geqslant 1} \left(2\mathfrak{w}_k - \mathfrak{w}_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

Reasoning: $\dfrac{1}{y} \xrightarrow{\partial} -\dfrac{1}{y^2}$, $\big(1 - W(z)\big)^2 = 1 - 2W(z) + \big(W(z)\big)^2$.

Probability of a permutation to be indecomposable

Theorem (Monteil, N., 2022)

The probability q_n that a random permutation of size n is indecomposable, satisfies

$$q_n \approx 1 - \sum_{k \geqslant 1} \frac{2\mathfrak{ip}_k - \mathfrak{ip}_k^{(2)}}{n^{\underline{k}}},$$

where $(\mathfrak{ip}_k^{(2)})$ counts permutations with two indecomposable parts.

$$\begin{array}{rcl} (\mathfrak{ip}_k) & = & 1,\ 1,\ 3,\ 13,\ 71,\ 461,\ 3447,\ \ldots \\ (\mathfrak{ip}_k^{(2)}) & = & 0,\ 1,\ 2,\ 7,\ 32,\ 177,\ 1142,\ \ldots \\ (2\mathfrak{ip}_k - \mathfrak{ip}_k^{(2)}) & = & 2,\ 1,\ 4,\ 19,\ 110,\ 745,\ 5752,\ \ldots \end{array}$$

Probability of a tournament to be irreducible

Theorem (Monteil, N., 2021)

The probability q_n that a random labeled tournament of size n is irreducible, satisfies

$$q_n \approx 1 - \sum_{k \geqslant 1} \left(2\mathrm{it}_k - \mathrm{it}_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}},$$

where $\left(\mathrm{it}_k^{(2)}\right)$ *counts labeled tournaments with two irreducible components.*

$$\begin{array}{rcl} (\mathrm{it}_k) & = & 1, \quad 0, \quad 2, \quad 24, \quad 544, \quad 22320, \quad \ldots \\ \left(\mathrm{it}_k^{(2)}\right) & = & 0, \quad 2, \quad 0, \quad 16, \quad 240, \quad 6608, \quad \ldots \\ \left(2\mathrm{it}_k - \mathrm{it}_k^{(2)}\right) & = & 2, \quad -2, \quad 4, \quad 32, \quad 848, \quad 38032, \quad \ldots \end{array}$$

Combinatorial classes: limits of applicability

1. Coefficients can be negative (see tournaments).
2. In certain cases, there is a decomposition

$$\mathcal{U} = \mathrm{SET}(\mathcal{V}),$$

but we have no class $\mathcal{W}$ such that

$$\mathcal{U} = \mathrm{SEQ}(\mathcal{W}),$$

and our theorem is not applicable. We would like to have an "anti-SEQ" operator to create this class.

Correspondance between combinatorial classes and species

combinatorial classes		species of structures
$\mathcal{A} = \mathrm{SET}(\mathcal{B})$		$\mathcal{A} = \mathcal{E} \circ \mathcal{B}$
$\mathcal{A} = \mathrm{SEQ}(\mathcal{B})$		$\mathcal{A} = \mathcal{L} \circ \mathcal{B}$
$\mathcal{A} = \mathrm{CYC}(\mathcal{B})$	$\Leftrightarrow$	$\mathcal{A} = \mathcal{CP} \circ \mathcal{B}$
$\mathcal{A} = \mathrm{SET}_m(\mathcal{B})$		$\mathcal{A} = \mathcal{E}_m \circ \mathcal{B}$
$\mathcal{A} = \mathrm{SEQ}_m(\mathcal{B})$		$\mathcal{A} = \mathcal{L}_m \circ \mathcal{B}$
$\mathcal{A} = \mathrm{CYC}_m(\mathcal{B})$		$\mathcal{A} = \mathcal{CP}_m \circ \mathcal{B}$

Virtual species

- Species constitute a semi-ring Spe.
- Virtual species are elements of

$$\mathrm{Virt} = (\mathrm{Spe} \times \mathrm{Spe})/\sim,$$

where $(F, G) \sim (H, K) \quad \Leftrightarrow \quad F + K = G + H.$

- Virtual species is a commutative ring with
  - zero — the empty species 0,
  - one — the characteristic of the empty set 1.
- Some additional operations in Virt:
  - subtraction,
  - multiplicative inverse F^{-1} (if $F_0 = 1$),
  - the inverse under substitution $F^{(-1)}$ (if $F_0 = 0$ and $F_1 = \mathcal{Z}$).

"Anti-SEQ" operator

1. If a virtual species Φ satisfies $\Phi_0 = 1$, then there exists a unique inverse of Φ under multiplication:

$$\Phi^{-1} = 1 - \Phi_+ + \Phi_+^2 - \Phi_+^3 + \dots,$$

where $\Phi_+ = \Phi - 1$.

2. If a virtual species Ψ satisfies $\Psi_0 = 0$ and $\Psi_1 = \mathcal{Z}$, then there exists a unique inverse of Ψ under substitution $\Psi^{(-1)}$.

3. "Anti-SEQ" operator:

$$\mathcal{L}_+^{(-1)} \equiv 1 - \mathcal{E}^{-1} \circ \mathcal{E}_+^{(-1)}.$$

Asymptotics in terms of species

Theorem (Monteil, N., 2022)

If $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{B}(m)$, $m \in \mathbb{N}$, are such (weighted) species that

1. *$\mathcal{A}$ is gargantuan with positive total weights on $[n]$, $n \in \mathbb{N}$,*
2. *one of the following conditions holds:*

(a)	$\mathcal{A} = \mathcal{E} \circ \mathcal{B}$	$\mathcal{B}(m) = \mathcal{E}_m \circ \mathcal{B}$	$\mathcal{C} = \mathcal{B}(m-1)(\mathcal{E}^{-1} \circ \mathcal{B})$
(b)	$\mathcal{A} = \mathcal{L} \circ \mathcal{B}$	$\mathcal{B}(m) = \mathcal{L}_m \circ \mathcal{B}$	$\mathcal{C} = m\mathcal{B}^{m-1}(1-\mathcal{B})^2$
(c)	$\mathcal{A} = \mathcal{CP} \circ \mathcal{B}$	$\mathcal{B}(m) = \mathcal{CP}_m \circ \mathcal{B}$	$\mathcal{C} = \mathcal{B}^{m-1}(1-\mathcal{B})$

then

$$p_n(m) := \frac{\mathfrak{b}_n(m)}{\mathfrak{a}_n} \approx \sum_{k \geqslant m-1} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}.$$

Erdős-Rényi model $G(n, p)$

Consider a random labeled graph G:

1. $p \in (0, 1)$ is the probability of edge presence;
2. $q = 1 - p$ is the probability of edge absence.

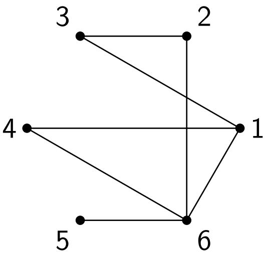
Weight of a graph:

$$w(G) = \rho^{|E(G)|},$$

where $\rho = p/q$.

<ins>Reason:</ins> if G_1 and G_2 are disjoint, then

$$w(G_1 \sqcup G_2) = w(G_1) \cdot w(G_2).$$



$w = \rho^7$

Asymptotics of the Erdős-Rényi model

Theorem (Monteil, N., 2022)

The probability p_n that a random graph with n vertices is connected satisfies

$$p_n \approx 1 - \sum_{k \geqslant 1} P_k(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}},$$

where

$$P_k(\rho) = \sum_{|V(G)|=k} (-1)^{\pi_0(G)-1} w(G)$$

and $\pi_0(G)$ is the number of connected components of G.

Meaning of the coefficients

$w = 1$

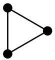
$P_1(\rho) = 1$

$w = \rho$

$-$

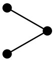
$w = 1$

$P_2(\rho) = \rho - 1$



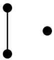
$w = \rho^3$

$+$ $3\times$



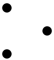
$w = \rho^2$

$-$ $3\times$



$w = \rho^1$

$+$



$w = 1$

$$P_3(\rho) = \rho^3 + 3\rho^2 - 3\rho + 1$$

Meaning of the coefficients

$w = 1$

$P_1(\rho) = 1$

$P_1(1) = 1 = \mathfrak{it}_1$

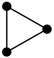
$w = \rho$

$-$

$w = 1$

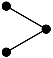
$P_2(\rho) = \rho - 1$

$P_2(1) = 0 = \mathfrak{it}_2$



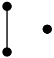
$w = \rho^3$

$+$ $3\times$



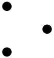
$w = \rho^2$

$-$ $3\times$



$w = \rho^1$

$+$



$w = 1$

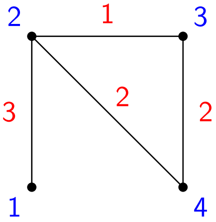
$P_3(\rho) = \rho^3 + 3\rho^2 - 3\rho + 1$

$P_3(1) = 2 = \mathfrak{it}_3$

Multigraphs and multitournaments

d-**multigraph**:
at most d edges
between two vertices

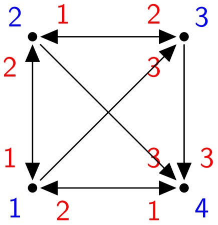
d-**multitournament**:
d directed edges
between two vertices



example
$d = 3$

vertices
labels

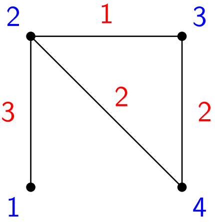
edges
multiplicities



Multigraphs and multitournaments

d-**multigraph**:
at most *d* edges
between two vertices

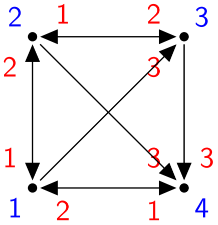
d-**multitournament**:
d directed edges
between two vertices



example
$d = 3$

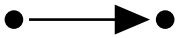
vertices
labels

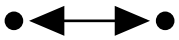
edges
multiplicities



 $p = \dfrac{d}{d+1}$

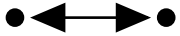
 $q = \dfrac{1}{d+1}$

 $\dfrac{1}{d+1} = q$

 $\dfrac{d-1}{d+1} = 1 - 2q$

Coefficients as generalized irreducible tournaments

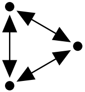
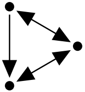
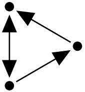
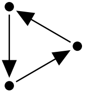
$w = 1$



$w = \rho - 1$

$P_1(\rho) = 1$

$P_2(\rho) = \rho - 1$

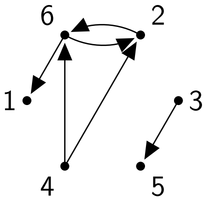
 $+$ $6\times$  $+$ $6\times$  $+$ $2\times$ 

$w = (\rho - 1)^3$ $w = (\rho - 1)^2$ $w = (\rho - 1)^1$ $w = 1$

$P_3(\rho) = \rho^3 + 3\rho^2 - 3\rho + 1$

Digraphs (labeled directed graphs)

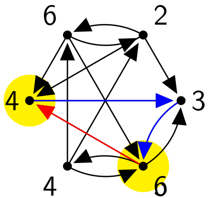
- $\mathfrak{d}_n = \#\{\text{digraphs with } n \text{ vertices}\}$



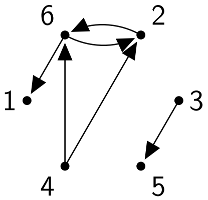
$$\mathfrak{d}_n = 2^{2\binom{n}{2}}$$

Digraphs (labeled directed graphs)

- $\mathfrak{d}_n = \#\{\text{digraphs with } n \text{ vertices}\}$
- $\mathfrak{scd}_n = \#\{\text{strongly connected digraphs with } n \text{ vertices}\}$



strongly connected digraph



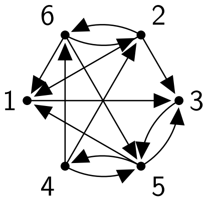
digraph

$$\mathfrak{d}_n = 2^{2\binom{n}{2}}$$

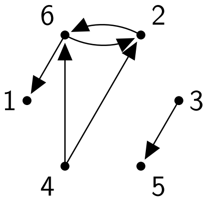
$$(\mathfrak{scd}_n) = 1, 1, 18, 1606, 565080, 734774776, \ldots$$

Digraphs (labeled directed graphs)

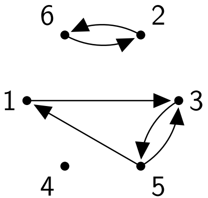
- $\mathfrak{d}_n = \#\{\text{digraphs with } n \text{ vertices}\}$
- $\mathfrak{scd}_n = \#\{\text{strongly connected digraphs with } n \text{ vertices}\}$
- $\mathfrak{ssd}_n = \#\{\text{semi-strong digraphs with } n \text{ vertices}\}$



strongly connected digraph



digraph



semi-strong digraph

$$\mathfrak{d}_n = 2^{2\binom{n}{2}}$$

$$(\mathfrak{scd}_n) = 1, 1, 18, 1606, 565080, 734774776, \ldots$$

$$(\mathfrak{ssd}_n) = 1, 2, 22, 1688, 573496, 738218192, \ldots$$

Strongly connected directed graphs

Question. What is the probability r_n that a random directed labeled graph with n vertices is strongly connected, $n \to \infty$?

Strongly connected directed graphs

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Wright, 1971:
$$r_n \approx \sum_{k \geqslant 0} \frac{\omega_k(n)}{2^{kn}} \cdot \frac{n!}{(n+[k/2]-k)!},$$

where

$$\omega_k(n) = \sum_{\nu=0}^{[k/2]} \gamma_\nu \xi_{k-2\nu} \frac{2^{k(k+1)/2}}{2^{\nu(k-\nu)}} (n+[k/2]-k)\ldots(n+\nu+1-k),$$

$$\gamma_0 = 1,\ \gamma_\nu = \sum_{s=0}^{\nu-1} \frac{\gamma_s \eta_{\nu-s}}{(\nu-s)!},\quad \sum_{\nu=0}^{\infty} \xi_\nu z^\nu = \left(1 - \sum_{s=0}^{\infty} \frac{\eta_s}{2^{s(s-1)/2}} \frac{z^s}{s!}\right)^2,$$

$$\eta_1 = 1,\quad \eta_s = 2^{s(s-1)} - \sum_{t=1}^{s-1} \binom{s}{t} 2^{(s-1)(s-t)} \eta_t.$$

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$$\eta_1 = 1,\quad \eta_s = 2^{s(s-1)} - \sum_{t=1}^{s-1} \binom{s}{t} 2^{(s-1)(s-t)} \eta_t.$$

Need for a new approach

Summary. The probability r_n has an expansion

$$r_n \approx \sum_{m \geqslant 0} \frac{1}{2^{mn}} \sum_{\ell=0}^{L_m} n^{\underline{\ell}} \cdot a^{\circ}_{m,\ell},$$

where $n^{\underline{\ell}} = n(n-1)\ldots(n-\ell+1)$ are falling factorials.

Observation. The array of coefficients $(a^{\circ}_{m,\ell})_{m,\ell=0}^{\infty}$ can be assembled into a bivariate generation function.

Question. Can we express this bivariate generating function explicitly in terms of other known generating functions?

Factorially divergent series (Borinsky)

$$a_n = \alpha^{n+\beta}\Gamma(n+\beta)\left(c_0 + \frac{c_1}{\alpha(n+\beta-1)} + \frac{c_2}{\alpha^2(n+\beta-1)(n+\beta-2)} + \dots\right)$$

$$\sum_{n=0}^{\infty} a_n z^n \quad \xmapsto{\mathcal{A}_\beta^\alpha} \quad \sum_{n=0}^{\infty} c_n z^n$$

Properties:

- $\left(\mathcal{A}_\beta^\alpha(A\cdot B)\right)(z) = A(z)\cdot(\mathcal{A}_\beta^\alpha B)(z) + B(z)\cdot(\mathcal{A}_\beta^\alpha A)(z),$
- $\left(\mathcal{A}_\beta^\alpha(A\circ B)\right)(z) = A'\left(B(z)\right)\cdot(\mathcal{A}_\beta^\alpha B)(z) + \left(\frac{z}{B(z)}\right)^\beta \exp\left(\frac{1}{\alpha}\left(\frac{1}{z} - \frac{1}{B(z)}\right)\right)(\mathcal{A}_\beta^\alpha A)\left(B(z)\right).$

Graphically divergent series

Let $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$ be parameters, and

$$a_n \approx \alpha^{\beta\binom{n}{2}} \sum_{m \geqslant 0} \frac{1}{\alpha^{mn}} \sum_{\ell=0}^{L_m} n^{\underline{\ell}} \, a^{\circ}_{m,\ell}$$

Graphically divergent series

Let $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$ be parameters, and

$$a_n \approx \alpha^{\beta\binom{n}{2}} \sum_{m\geqslant 0} \frac{1}{\alpha^{mn}} \sum_{\ell=0}^{L_m} n^{\underline{\ell}}\, a^{\circ}_{m,\ell}$$

Define

$$\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \quad \xmapsto{\mathcal{Q}_\alpha^\beta} \quad \sum_{m=0}^{\infty} \sum_{\ell=0}^{L_m} a^{\circ}_{m,\ell} \frac{z^m}{\alpha^{\frac{1}{\beta}\binom{m}{2}}} w^{\ell}$$

Graphically divergent series

Let $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$ be parameters, and

$$a_n \approx \alpha^{\beta\binom{n}{2}} \sum_{m \geqslant 0} \frac{1}{\alpha^{mn}} \sum_{\ell=0}^{L_m} n^{\ell} a^{\circ}_{m,\ell}$$

Define

coefficient generating function

of type (α, β)

$$\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \xmapsto{\mathcal{Q}_{\alpha}^{\beta}} \sum_{m=0}^{\infty} \sum_{\ell=0}^{L_m} a^{\circ}_{m,\ell} \frac{z^m}{\alpha^{\frac{1}{\beta}\binom{m}{2}}} w^{\ell}$$

graphically divergent series

Properties, part I

1. Graphically divergent series forms a ring with

$$\big(\mathcal{Q}_\alpha^\beta(A+B)\big)(z,w) = (\mathcal{Q}_\alpha^\beta A)(z,w) + (\mathcal{Q}_\alpha^\beta B)(z,w)$$

and

$$\begin{aligned}\big(\mathcal{Q}_\alpha^\beta(A\cdot B)\big)(z,w) &= A\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big)\cdot(\mathcal{Q}_\alpha^\beta B)(z,w)\\ &\quad + B\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big)\cdot(\mathcal{Q}_\alpha^\beta A)(z,w)\,.\end{aligned}$$

2. Derivation:

$$(\mathcal{Q}_\alpha^\beta A')(z,w) = \alpha^{-\frac{\beta+1}{2}} z^{-\beta}\left((\mathcal{Q}_\alpha^\beta A)(z,w) + \frac{\partial}{\partial w}(\mathcal{Q}_\alpha^\beta A)(z,w)\right).$$

Properties, part II

3. Composition (interpretation of Bender's theorem): if
   - F is analytic in a neighbourhood of the origin,
   - $a_0 = 0$,
   - $H(z) = \frac{\partial}{\partial x} F(x) \Big|_{x=A(z)}$,

   then $\quad F \circ A \in \mathfrak{G}_\alpha^\beta \quad$ and

$$\big(\mathcal{Q}_\alpha^\beta(F \circ A)\big)(z, w) = H\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big) \cdot (\mathcal{Q}_\alpha^\beta A)(z, w).$$

4. Powers: if $\quad m \in \mathbb{Z}_{\geqslant 0} \quad$ (or $m \in \mathbb{Z}$ and $a_0 = 1$), $\quad$ then

$$\big(\mathcal{Q}_\alpha^\beta A^m\big)(z, w) = m \cdot A^{m-1}\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big) \cdot (\mathcal{Q}_\alpha^\beta A)(z, w)\,.$$

Asymptotics of strongly connected directed graphs

Theorem (Dovgal, N., 2023)

The probability r_n that a random labeled digraph of size n is strongly connected satisfies

$$r_n \approx \sum_{m \geqslant 0} \frac{1}{2^{nm}} \sum_{\ell = \lceil m/2 \rceil}^{m} n^{\ell} \mathfrak{scd}^{\circ}_{m,\ell} + O\left(\frac{n^r}{2^{rn}}\right),$$

where

- $\mathfrak{scd}^{\circ}_{m,\ell} = \dfrac{2^{m(m+1)/2}}{2^{\ell(m-\ell)}} \dfrac{\mathfrak{ssd}_{m-\ell}}{(m-\ell)!} \dfrac{\mathbb{I}_{m=2\ell} - 2\mathfrak{it}_{2\ell-m} + \mathfrak{it}^{(2)}_{2\ell-m}}{(2\ell-m)!}$,
- $\mathfrak{ssd}_k$ *is the number of semi-strong digraphs of size k,*
- $\mathfrak{it}_k$ *is the number of irreducible tournaments of size k,*
- $\mathfrak{it}^{(2)}_k$ *is the number of tournaments of size k with two irreducible components.*

Asymptotics of strongly connected directed graphs

Theorem (Dovgal, N., 2023)

The probability r_n that a random labeled digraph of size n is strongly connected satisfies

$$r_n \approx \sum_{m \geqslant 0} \frac{1}{2^{nm}} \sum_{\ell=\lceil m/2 \rceil}^{m} n^{\ell} \mathfrak{scd}^{\circ}_{m,\ell} + O\left(\frac{n^r}{2^{rn}}\right),$$

where

- $$\mathfrak{scd}^{\circ}_{m,\ell} = \frac{2^{m(m+1)/2}}{2^{\ell(m-\ell)}} \frac{\mathfrak{ssd}_{m-\ell}}{(m-\ell)!} \frac{\mathbb{I}_{m=2\ell} - 2\mathfrak{it}_{2\ell-m} + \mathfrak{it}^{(2)}_{2\ell-m}}{(2\ell-m)!},$$
- *Interpretation of Wright's coefficients:*

$$\eta_k = 2^{\binom{k}{2}} \mathfrak{it}_k, \qquad \gamma_k = \frac{\mathfrak{ssd}_k}{k!}, \qquad \xi_k = \frac{\mathbb{I}_{k=0} - 2\mathfrak{it}_k + \mathfrak{it}^{(2)}_k}{k!}.$$

Conclusion

1. General methods for combinatorial interpretation of coefficients in asymptotic expansions of irreducibles:
   - in terms of combinatorial classes (SET, SEQ, CYC),
   - in terms of species ($\mathcal{E}$, $\mathcal{L}$, $\mathcal{CP}$),
   - for graphically divergent series.
2. Applications:
   - connected graphs and irreducible tournaments,
   - square-tiled surfaces and indecomposable permutations,
   - strongly connected digraphs,
   - the Erdős-Rényi model,
   - ...

Thank you for your attention!

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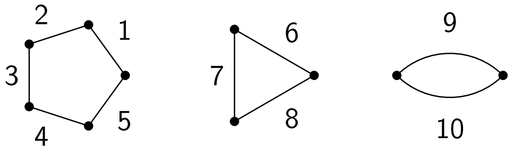
Combinatorial maps

- Take several labeled polygons of total perimeter $N = 2n$ (1-gons and 2-gons are allowed).
- Identify their sides randomly to obtain a surface.
- Each surface is determined by a pair $(\phi, \alpha) \in S_N^2$, where α is a perfect matching.



$$\phi = (1\,2\,3\,4\,5)(6\,7\,8)(9\,10), \quad \alpha = (1\,3)(2\,6)(4\,10)(5\,9)(7\,8)$$

Asymptotics for combinatorial maps

Theorem

The probability p_n that a random combinatorial map is connected satisfies

$$p_n \approx 1 - \sum_{k \geqslant 1} \mathfrak{im}_{2k} \cdot \frac{(2(n-k)-1)!!}{(2n-1)!!},$$

where $(\mathfrak{im}_{2k})$ counts indecomposable perfect matchings.

$$(\mathfrak{im}_{2k}) = 1, 2, 10, 74, 706, 8162, 110410, 1708394, \ldots$$

Connected graphs

Monteil, N., 2021: The probability p_n that a random graph of size n is connected satisfies

$$p_n \approx 1 - \sum_{k \geqslant 1} \mathrm{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}},$$

where $\mathrm{it}_k = \#\{\text{irreducible labeled tournaments of size } k\}$.

Theorem (Dovgal, N., 2023)

The coefficient generating function of type $(2,1)$ of connected graphs satisfies

$$(\mathcal{Q}_2^1\, CG)(z,w) = \frac{1}{G(2zw)} = 1 - IT(2zw).$$

Key ideas: $(\mathcal{Q}_2^1\, G)(z,w) = 1$, $CG(z) = \log\big(G(z)\big)$, $\dfrac{1}{G(z)} = \dfrac{1}{T(z)} = 1 - IT(z)$.

Irreducible tournaments

Monteil, N., 2021: The probability q_n that a random tournament of size n is irreducible satisfies

$$q_n \approx 1 - \sum_{k \geqslant 1} \left(2\mathrm{it}_k - \mathrm{it}_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}},$$

where $\mathrm{it}_k^{(2)} = \#\{\text{tournaments with two irreducible parts of size } k\}$.

Theorem (Dovgal, N., 2023)

The coefficient generating function of type $(2,1)$ *of irreducible tournaments satisfies*

$$(\mathcal{Q}_2^1\, IT)(z,w) = \big(1 - IT(2zw)\big)^2.$$

Key ideas: $(\mathcal{Q}_2^1\, T)(z,w) = 1, \quad IT(z) = 1 - \dfrac{1}{T(z)}, \quad \dfrac{1}{T^2(z)} = \big(1 - IT(z)\big)^2.$