CORRIGENDUM TO "LOCAL EXTREMA FOR HYPERCUBE SECTIONS"

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ABSTRACT

The local extremality criterion from [3] for the volume of hypercube sections at sub-diagonals of order less than the dimension misses a term. The results regarding the main diagonal of the hypercube do not change but the results about its sub-diagonals of lower order have to be corrected by taking this additional term in account which is the purpose of this corrigendum. The only significant change to the main results of [3] is that the volume of the sections of the hypercube $[0,1]^d$ by a hyperplane H orthogonal to a sub-diagonal of order at least 4 and less than d is never locally extremal when H is close to the center of $[0,1]^d$.

1. Introduction

Consider a hyperplane H of $\mathbb{R}^d$ whose distance to the center of the unit hypercube $[0,1]^d$ is a number t at most $\sqrt{d}/2$. When t is fixed and H is allowed to vary, the $(d-1)$-dimensional volume V of $H \cap [0,1]^d$ can be thought of as a function of H. A criterion is given in [3] for the local extremality of V when H is orthogonal to a sub-diagonal of $[0,1]^d$ of order at least 4. Here, a sub-diagonal of $[0,1]^d$ of order n means a diagonal of a n-dimensional face of $[0,1]^d$. This criterion misses a term in the special case when n is less than d and the purpose of this corrigendum is to correct it as well as the results obtained from this criterion regarding the local extremality of V when H is orthogonal to a sub-diagonal of the hypercube $[0,1]^d$ of order less than d. Note that the result from [3] regarding the main diagonals of the hypercube $[0,1]^d$ (sub-diagonals of order d) do not change. Recall that H is the hyperplane of $\mathbb{R}^d$ made of the points x such that $a{\cdot}x=b$ where a is a non-zero vector from $\mathbb{R}^d$ and b is a real number. By symmetry, it is assumed that a belongs to $[0,+\infty]^d\backslash\{0\}$. Moreover, b can be expressed as a function of t as

$$b=\frac{1}{2}\sum_{i=1}^{d}a_i-t$$

With these notations and under the assumption that t is fixed, V is considered a function of a. The error in [3] lies in the proof of Theorem 3.1 where the second derivative of L_λ with respect to a_j at the point a of $\mathbb{R}^d$ whose n first coordinates are $1/\sqrt{n}$ and whose last $d-n$ coordinates are equal to 0 is

$$\frac{\partial^2 L_\lambda}{\partial a_j^2}=\frac{\partial^2}{\partial a_1^2}\frac{V}{\|a\|}-\sqrt{n}\frac{\partial}{\partial a_1}\frac{V}{\|a\|} \tag{1}$$

for all integers j satisfying $1\leq j\leq d$ and not just when $1\leq j\leq n$. In particular, the last equation at the bottom of Page 570 in [3] should be replaced by (1). Likewise, the first equation at the top of Page 571 in [3] should be

$$\begin{aligned}\sum_{j=1}^{d}\sum_{k=1}^{d}x_jx_k\frac{\partial^2 L_\lambda}{\partial a_j\partial a_k}=&\left[\frac{\partial^2}{\partial a_1^2}\frac{V}{\|a\|}-\sqrt{n}\frac{\partial}{\partial a_1}\frac{V}{\|a\|}-\frac{\partial^2}{\partial a_1\partial a_2}\frac{V}{\|a\|}\right]\sum_{i=1}^{n}x_i^2\\&+\frac{\partial^2}{\partial a_1\partial a_2}\frac{V}{\|a\|}\left[\sum_{i=1}^{n}x_i\right]^2+\left[\frac{\partial^2}{\partial a_d^2}\frac{V}{\|a\|}-\sqrt{n}\frac{\partial}{\partial a_1}\frac{V}{\|a\|}\right]\sum_{i=n+1}^{d}x_i^2.\end{aligned}$$

The rest of the argument does not change. As a consequence, the local extremality criterion stated by Theorems 3.1 and 5.3 from [3] has to be corrected into the following (combined) statement.

THEOREM 1.1: *Consider an integer n such that $2\leq n\leq d$ and assume that V is twice continuously differentiable at the point a of $\mathbb{R}^d$ whose first n coordinates are $1/\sqrt{n}$ and whose other coordinates are 0. If at that point,*

$$\frac{\partial^2}{\partial a_1^2}\frac{V}{\|a\|}-\sqrt{n}\frac{\partial}{\partial a_1}\frac{V}{\|a\|}-\frac{\partial^2}{\partial a_1\partial a_2}\frac{V}{\|a\|} \tag{2}$$

is negative and, when n is less than d,

$$\frac{\partial^2}{\partial a_d^2}\frac{V}{\|a\|}-\sqrt{n}\frac{\partial}{\partial a_1}\frac{V}{\|a\|} \tag{3}$$

is also negative, then V has a strict local maximum on $\mathbb{S}^{d-1}\cap[0,+\infty[^d$ at a. Similarly, if (2) is positive at point a and, when n is less than d, (3) is positive at a as well, then V admits a strict local minimum on $\mathbb{S}^{d-1}\cap[0,+\infty[^d$ at that *point. Finally, in the case when n is less than d, if (2) and (3) are both non-zero and have opposite signs at point a, then V does not have a local extremum (even a weak one) on $\mathbb{S}^{d-1}\cap[0,+\infty[^d$ at a.*

In other words, the expressions (15) in the statement of Theorem 3.1 and (44) in the statement of Theorem 5.3 from [3] should be

$$\frac{\partial^2}{\partial a_d^2}\frac{V}{\|a\|}-\sqrt{n}\frac{\partial}{\partial a_1}\frac{V}{\|a\|}$$

rather than just

$$\frac{\partial^2}{\partial a_d^2}\frac{V}{\|a\|}.$$

This criterion is further given in a different form as Theorem 1.5 in [3]. This theorem should also be modified as will be explained in Section 2.

This missing term in the local extremality criterion has consequences on some of the main results reported in [3]. Fortunately, the only significant change is that Theorem 1.1 from [3] should be corrected as follows.

THEOREM 1.2: *If $d\geq 4$ and t is close enough to 0, then the $(d-1)$-dimensional volume of $H\cap[0,1]^d$ is strictly locally maximal when H is orthogonal to a diagonal of $[0,1]^d$ and not locally extremal when H is orthogonal to a sub-diagonal of $[0,1]^d$ of order at least 5 and less than d.*

In particular, in the case when the hyperplane H is close to the center of $[0,1]^d$, while V is still strictly locally maximal when H is orthogonal to a main diagonal of $[0,1]^d$, this is not the case when H is orthogonal to a lower order sub-diagonal. Theorem 1.2 can be proven using the same techniques as for Theorem 1.1 from [3]: it suffices to show that, when t is equal to 0 while n is at least 5 and less than d, (3) is positive at the point a of $\mathbb{R}^d$ whose first n coordinates are equal to $1/\sqrt{n}$ and whose $d-n$ other coordinates are equal to 0. This is possible by using the following integral expression of (3) at a obtained by combining Equations (19) and (20) from [3]:

$$\frac{\partial^2}{\partial a_d^2}\frac{V}{\|a\|}-\sqrt{n}\frac{\partial}{\partial a_1}\frac{V}{\|a\|}=-\frac{1}{\pi}\int_{-\infty}^{+\infty}\left[n\cos\left(\frac{u}{\sqrt{n}}\right)+\left(\frac{u^2}{3}-n\right)\frac{\sqrt{n}}{u}\sin\left(\frac{u}{\sqrt{n}}\right)\right]\left(\frac{\sqrt{n}}{u}\sin\left(\frac{u}{\sqrt{n}}\right)\right)^{n-1}\cos(2tu)du. \tag{4}$$

However, Theorem 1.2 has been established in the meantime by Gergely Ambrus and Barnabás Gárgyán using a different argument [1] and it should be attributed to them. Further note that they extended it to sub-diagonals of order 3 and 4, which is not possible with Theorem 1.1 (see for instance Section 3). In addition, Theorem 1.2 is generalized to a range of values of t away from 0 in [2]. For these reasons, the proof that when t is equal to 0 and n is at least 5, the right-hand side of (4) is positive is omitted from this corrigendum.

Another result from [3] has to be slightly modified although in a much lighter way. Theorem 1.3 from [3] should be corrected as follows.

THEOREM 1.3: *If $14 \leq n < d$ and t satisfies*

$$\frac{\sqrt{n}}{2} - \frac{1}{\sqrt{n}}\min\left\{\frac{n-14}{6}, \frac{(2n)^{1/(n-3)}}{(2n)^{1/(n-3)}-1}\right\} < t < \frac{\sqrt{n}}{2}, \tag{5}$$

then the $(d-1)$-dimensional volume of $H \cap [0,1]^d$ is not locally extremal (even weakly so) when H is orthogonal to an order n sub-diagonal of $[0,1]^d$.

Note that, compared to Theorem 1.3 from [3], only the left-hand side of (5) has changed and the range for t given by (5) is still asymptotic to $\sqrt{n}/\log n$ as n goes to infinity. A proof of Theorem 1.3 is given in Section 2 using the tools from [3]. The analysis reported in Section 6 of [3] of how local extrema at sub-diagonals of order n behave when n is low and t varies from 0 to $\sqrt{n}/2$ also has to be corrected, which is done below in Section 3.

2. The sub-diagonals of the hypercube

Recall that in [3], at the point a from $\mathbb{R}^d$ whose first n coordinates are equal to $1/\sqrt{n}$ and whose other coordinates are equal to 0, (2) is expressed as

$$\frac{\partial^2}{\partial a_1^2}\frac{V}{\|a\|} - \sqrt{n}\frac{\partial}{\partial a_1}\frac{V}{\|a\|} - \frac{\partial^2}{\partial a_1 \partial a_2}\frac{V}{\|a\|} = \sum_{i=0}^{\lfloor z \rfloor}\frac{(-1)^i\sqrt{n}}{(n-3)!}\binom{n}{i}(z-i)^{n-3}p_{i,n}(z) \tag{6}$$

when n is at least 4 where

$$p_{i,n}(z) = \frac{i(n-i)}{n-1} - \left(\frac{n}{2} - i\right)\frac{z-i}{n-2} + \frac{2n(z-i)^2}{(n-1)(n-2)} \tag{7}$$

and z is related to t via the change of variables

$$z = \frac{n}{2} - t\sqrt{n}. \tag{8}$$

At the same point a, a similar expression of (3) can be obtained by combining Lemma 2.5, Lemma 5.1, and Equation (32) from [3]:

$$\frac{\partial^2}{\partial a_d^2}\frac{V}{\|a\|} - \sqrt{n}\frac{\partial}{\partial a_1}\frac{V}{\|a\|} = \sum_{i=0}^{\lfloor z \rfloor} \frac{(-1)^i\sqrt{n}}{(n-3)!}\binom{n}{i}(z-i)^{n-3}q_{i,n}(z). \tag{9}$$

when n is at least 4 where

$$q_{i,n}(z) = \frac{n}{12} - \left(\frac{n}{2} - i\right)\frac{z-i}{n-2} + \frac{n(z-i)^2}{(n-1)(n-2)} \tag{10}$$

and again, z and t are related via the change of variables (8).

In particular, in order to correct Theorem 1.5 in [3], it suffices to multiply the expression (5) in its statement by $q_{i,n}(z)$. The following more general statement is obtained as a consequence of (6), (9), and Theorem 1.1.

THEOREM 2.1: *Assume that $4 \leq n \leq d$. If*

$$\sum_{i=0}^{\lfloor z \rfloor}(-1)^i\binom{n}{i}(z-i)^{n-3}p_{i,n}(z) \tag{11}$$

is negative and, when n is less than d,

$$\sum_{i=0}^{\lfloor z \rfloor}(-1)^i\binom{n}{i}(z-i)^{n-3}q_{i,n}(z) \tag{12}$$

is also negative, where $z = n/2 - t\sqrt{n}$, then V has a strict local maximum when H is orthogonal to an order n sub-diagonal of $[0,1]^d$. If however, (11) is positive and, when n is less than d, (12) is also positive, then V has a strict local minimum when H is orthogonal to an order n sub-diagonal of $[0,1]^d$. Finally, in the case when n is less than d, if (11) and (12) are both non-zero and have opposite signs, then V does not have a local extremum (even a weak one) when H is orthogonal to an order n sub-diagonal of $[0,1]^d$.

Theorem 1.3 is now established using the tools from [3].

Proof of Theorem 1.3. Assume that $14 \leq n < d$ and that t satisfies (5). Observe that by (8), this requirement on t can be rewritten in terms of z as

$$0 < z < \min\left\{\frac{n-14}{6}, \frac{(2n)^{1/(n-3)}}{(2n)^{1/(n-3)}-1}\right\}. \tag{13}$$

According to Lemma 4.4 from [3], (11) is negative. Hence, by Theorem 2.1, it suffices to show that (12) is positive. This will be done by using Proposition 4.3

from [3]. Consider an integer i such that $0 \leq i \leq z$ and observe that

$$q_{i,n}(z) = \frac{f_i(z) - g_i(z)}{(n-1)(n-2)}$$

where

$$f_i(z) = \frac{n(n-1)(n-2)}{12} - \frac{n(n-1)}{2}(z-i) + nz^2 \tag{14}$$

and

$$g_i(z) = i(n+1)z - i^2.$$

Since z is less than $(n-14)/6$, it follows from (14) that

$$f_i(z) \geq n(n-1) \tag{15}$$

and in particular, $f_i(z)$ is positive. In addition, if $i+1$ is at most z,

$$\frac{f_i(z)}{f_{i+1}(z)} = 1 - \frac{n(n-1)}{2f_{i+1}(z)} \tag{16}$$

and, since $f_i(z)$ is monotonically increasing with i, then so is the ratio $f_i(z)/f_{i+1}(z)$. It then follows from Proposition 4.3 from [3] that if

$$z < \frac{1}{1 - \left(\dfrac{f_0(z)}{nf_1(z)}\right)^{1/(n-3)}}, \tag{17}$$

then

$$\sum_{i=0}^{\lfloor z \rfloor} (-1)^i \binom{n}{i} (z-i)^{n-3} f_i(z) > 0. \tag{18}$$

However, combining (15) and (16) shows that $f_0(z)/f_1(z)$ is at least $1/2$. In particular, (13) implies (17) which proves that (18) holds.

Now observe that, when i is positive, so is $g_i(z)$. Moreover,

$$\frac{g_i(z)}{g_{i+1}(z)} = \left(1 - \frac{1}{i+1}\right)\left(1 + \frac{1}{(n+1)z - i - 1}\right) \tag{19}$$

when $i+1$ is at most z. In particular, the ratio $g_i(z)/g_{i+1}(z)$ is monotonically increasing. Hence, by Proposition 4.3 from [3], if

$$z < 1 + \frac{1}{1 - \left(\dfrac{2g_1(z)}{(n-1)g_2(z)}\right)^{1/(n-3)}}, \tag{20}$$

then

$$-\sum_{i=1}^{\lfloor z\rfloor}(-1)^i\binom{n}{i}(z-i)^{n-3}g_i(z)>0. \tag{21}$$

According to (19), the ratio $g_1(z)/g_2(z)$ is at least $1/2$. As a consequence, (13) implies (20) which proves that (21) holds.

Finally, observe that $g_0(z)$ is equal to 0. Hence, (21) still holds when the sum in the left-hand side is extended down to i equal to 0. Summing the resulting inequality with (18) shows that (12) is positive, as desired. ∎

3. Low order sub-diagonals

Think of (11) as a function of z. As shown in [3] using symbolic computation, for any integer n such that $4\leq n\leq 300$, this function has two roots ρ_n^- and ρ_n^+ (labeled in such a way that $\rho_n^+<\rho_n^-$) in the interval $]0,n/2[$. Moreover, this function is positive when $\rho_n^+<z<\rho_n^-$ and negative when $0\leq z<\rho_n^+$ or $\rho_n^-<z<\sqrt{n}/2$. This allowed to determine whether V is locally extremal when H is orthogonal to a diagonal of the hypercube $[0,1]^d$ for all t such that $0<t<\sqrt{d}/2$ and all n such that $4\leq d\leq 300$ (see for instance Proposition 6.1 and the bottom of Page 592 in [3]). The corresponding analysis for the sub-diagonals of order less than d has to be corrected in order to take into account the missing term in the local extremality criterion.

Consider an integer n at least 4 and less than d. If n is equal to 4,

$$\sum_{i=0}^{\lfloor z\rfloor}(-1)^i\binom{n}{i}(z-i)^{n-3}q_{i,n}(z)=\frac{1}{3}z(z-1)(2z-1)$$

when $0<z\leq 1$ and

$$\sum_{i=0}^{\lfloor z\rfloor}(-1)^i\binom{n}{i}(z-i)^{n-3}q_{i,n}(z)=-\frac{1}{3}(z-1)(z-2)(2z-3) \tag{22}$$

when $1\leq z\leq 2$. In particular, (12) vanishes when z is equal to $1/2$, 1, $3/2$, or 2. In addition, (12) is positive when $0<z<1/2$ or $3/2<z<2$ and negative when $1/2<z<1$ or $1<z<3/2$. The following proposition is obtained from Theorem 2.1 as a consequence. In the statement of this proposition, ρ_4^- is one of the above mentioned values obtained in [3]. Note that

$$\rho_4^-=\frac{17+(17-12\sqrt{2})^{1/3}+(17+12\sqrt{2})^{1/3}}{12}.$$

which is about 1.71229. Further note that ρ_4^+ is equal to $3/4$ and therefore

$$1-\frac{\rho_4^+}{2}=\frac{5}{8}.$$

PROPOSITION 3.1: *Assume that d is greater than 4. If $5/8<t<3/4$, then the $(d-1)$-dimensional volume of $H\cap[0,1]^d$ is strictly locally maximal when H is orthogonal to an order 4 sub-diagonal of $[0,1]^d$. However, if*

$$t\in\left]1-\frac{\rho_4^-}{2},\frac{1}{4}\right[$$

then the $(d-1)$-dimensional volume of $H\cap[0,1]^d$ is strictly locally minimal when H is orthogonal to an order 4 sub-diagonal of $[0,1]^d$. Finally, if

$$t\in\left]0,1-\frac{\rho_4^-}{2}\right[\cup\left]\frac{1}{4},\frac{1}{2}\right[\cup\left]\frac{1}{2},\frac{5}{8}\right[\cup\left]\frac{3}{4},1\right[$$

then that volume is not locally extremal (even weakly so) when H is orthogonal to an order 4 sub-diagonal of $[0,1]^d$.

It should be mentioned that, since the right-hand side of (22) is equal to 0 when z is equal to 2, Theorem 2.1 does not allow to extend Theorem 1.2 down to sub-diagonals of order 4. Now if n is equal to 5,

$$\sum_{i=0}^{\lfloor z\rfloor}(-1)^i\binom{n}{i}(z-i)^{n-3}q_{i,n}(z)=\frac{5}{12}z^2(z-1)^2$$

when $0<z\leq 1$,

$$\sum_{i=0}^{\lfloor z\rfloor}(-1)^i\binom{n}{i}(z-i)^{n-3}q_{i,n}(z)=-\frac{5}{3}(z-1)^2(z-2)^2$$

when $1\leq z\leq 2$, and

$$\sum_{i=0}^{\lfloor z\rfloor}(-1)^i\binom{n}{i}(z-i)^{n-3}q_{i,n}(z)=\frac{5}{2}(z-2)^2(z-3)^2.$$

when $2\leq z\leq 5/2$. Therefore, (12) vanishes when z is equal to 1 or 2. Moreover, (12) is positive when $0<z<1$ or $2<z<5/2$ and negative when $1<z<2$. Since ρ_5^- is equal to 2 and ρ_5^+ to 1, Theorem 2.1 implies the following.

PROPOSITION 3.2: *Assume that d is greater than 5. If*

$$t\in\left[0,\frac{\sqrt{5}}{2}-\frac{2}{\sqrt{5}}\right[\cup\left]\frac{\sqrt{5}}{2}-\frac{2}{\sqrt{5}},\frac{\sqrt{5}}{2}-\frac{1}{\sqrt{5}}\right[\cup\left]\frac{\sqrt{5}}{2}-\frac{1}{\sqrt{5}},\frac{\sqrt{5}}{2}\right[$$

then the $(d-1)$-dimensional volume of $H\cap[0,1]^d$ is not locally extremal (even weakly so) when H is orthogonal to an order 5 sub-diagonal of $[0,1]^d$.

Above dimension 5 the relative behavior of (11) and (12) becomes steadier. In particular one obtains using symbolic computation that for any integer n such that $4\leq n\leq 300$, there exist two values σ_n^- and σ_n^+ in the interval $]0,n/2[$, labeled with the convention that $\sigma_n^+<\sigma_n^-$, such that (12) vanishes when z is equal to σ_n^- or σ_n^+, is positive when $0\leq z<\sigma_n^+$ or $\sigma_n^-<z<n/2$, and negative when $\sigma_n^+<z<\sigma_n^-$. These computations also show that σ_n^+ is less than ρ_n^+ and that σ_n^- is less than ρ_n^- but greater than ρ_n^+ for all the considered values of n. In particular the following statement is obtained from Theorem 2.1.

PROPOSITION 3.3: *Assume that $6\leq n\leq 300$ and that $n<d$. There exists two numbers σ_n^- and σ_n^+ independent from d such that if*

$$t\in\left]\frac{\sqrt{n}}{2}-\frac{\rho_n^-}{\sqrt{n}},\frac{\sqrt{n}}{2}-\frac{\sigma_n^-}{\sqrt{n}}\right[$$

then the $(d-1)$-dimensional volume of $H\cap[0,1]^d$ is strictly locally maximal when H is orthogonal to an order n sub-diagonal of $[0,1]^d$. However, if

$$t\in\left]\frac{\sqrt{n}}{2}-\frac{\rho_n^+}{\sqrt{n}},\frac{\sqrt{n}}{2}-\frac{\sigma_n^+}{\sqrt{n}}\right[$$

then the $(d-1)$-dimensional volume of $H\cap[0,1]^d$ is strictly locally minimal when H is orthogonal to an order n sub-diagonal of $[0,1]^d$. Finally, if

$$t\in\left[0,\frac{\sqrt{n}}{2}-\frac{\rho_n^-}{\sqrt{n}}\right[\cup\left]\frac{\sqrt{n}}{2}-\frac{\sigma_n^-}{\sqrt{n}},\frac{\sqrt{n}}{2}-\frac{\rho_n^+}{\sqrt{n}}\right[\cup\left]\frac{\sqrt{n}}{2}-\frac{\sigma_n^+}{\sqrt{n}},\frac{\sqrt{n}}{2}\right[$$

then that volume is not locally minimal or maximal (even weakly so) when H is orthogonal to an order n sub-diagonal of $[0,1]^d$.

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