From Proofs to Programs, Graphs and Dynamics. Geometric perspectives on computational complexity

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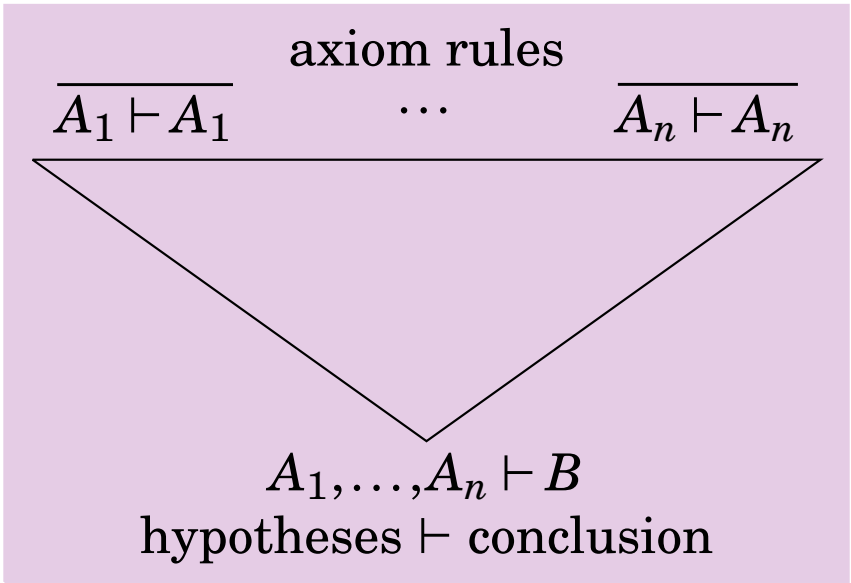
- **Logic, Programs and Complexity**
- Graphings and Logic
- Formalising a Conjecture
- The Classification Problem and Barriers
- Entropy and Lower Bounds

Proof Theory

Proof are formalized as trees constructed from a set of rules.



Example

$$\dfrac{\dfrac{\dfrac{\vdots}{A \vdash B} \qquad \dfrac{\dfrac{}{A \vdash A}\,\text{ax} \qquad \dfrac{}{B \vdash B}\,\text{ax}}{A, B \vdash A \wedge B}\,\wedge}{A, A \vdash A \wedge B}\,\text{cut}}{A \vdash A \wedge B}\,\text{ctr}}{\vdash A \to A \wedge B}\,\to$$

Cut Elimination

Cut: using a lemma

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ \Gamma, A \vdash B & \Delta \vdash A\end{array}}{\Delta, \Gamma \vdash B}\ \text{cut}$$

Theorem (Gentzen)

If A is prouvable, there exists a cut-free proof of A.

The main interest of this result lies in its proof: a (local) procedure of cut-elimination is exhibited.

Proofs as Programs – Curry-Howard – correspondence

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  - Given $[n]$ and $[f]$, proofs of nat and $\mathrm{nat} \to \mathrm{nat}$ respectively, we can define the proof $\mathrm{cut}([f],[n])$:

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  - The cut elimination procedure applied to $[f][n]$ corresponds (step by step) to the computation of $f(n)$. The cut-free proof it produces is equal to $[f(n)]$.

$$\dfrac{\begin{array}{cc}[f] & [n] \\ \vdots & \vdots \\ \mathrm{nat} \vdash \mathrm{nat} & \vdash \mathrm{nat}\end{array}}{\vdash \mathrm{nat}}\ \mathrm{cut} \quad \to * \quad \begin{array}{c}[f(n)] \\ \vdots \\ \vdash \mathrm{nat}\end{array}$$

Fourth Layer: Geometry of Interaction

Geometry of Interaction is a mathematical model of proofs' dynamics.

Logic	Computer Science	
Formulas	Types	
Proof $\pi \vdash \mathbf{Nat} \Rightarrow \mathbf{Nat}$	Program $f : \mathrm{nat} \to \mathrm{nat}$	
Proof $\rho \vdash \mathbf{Nat}$	Data $n : \mathrm{nat}$	
Cut Rule $\mathrm{cut}(\pi, \rho)$	Application $f(n)$	
Cut elimination $\mathrm{cut}(\pi, \rho) \rightsquigarrow \mu \vdash \mathbf{Nat}$	Computation $f(n) \rightsquigarrow m : \mathrm{nat}$	
(Linear) Negation	Test	

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E.g. $\mathrm{nat} \to \mathrm{nat}$ is defined as $\{f \mid \forall n \in \mathrm{nat}, \forall t \in \mathrm{nat}^{\perp\!\!\!\perp}, \mathrm{Ex}(f, n) \perp\!\!\!\perp t\}$.

Logic and Complexity by Layers

- **1st Layer: Types | Formulas**: Descriptive complexity

Informally

Descriptive Complexity (DC) studies types of logics whose individual sentences characterise exactly particular complexity classes.

- **2nd Layer: Programs | Proofs**: Implicit complexity

- **3rd Layer: Computation | Normalisation**: Constrained Linear Logic

Logic and Complexity by Layers

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Example Theorem (Fagin 74)

The class NPTIME *is the set of problems definable by existential second order boolean formulas.*

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Implicit Computational Complexity (ICC) studies algorithmic complexity only in terms of restrictions of languages and computational principles, for instance considering restrictions on recursion schemes.

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The class FPTIME *is the smallest class of functions containing constants, projections, successors, predecessor and conditional (if* $a \mod 2 = 0$*), and closed under Predicative Recursion on Notation and Safe Composition.*

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Constrained Linear Logic studies restrictions of proof systems in which the cut-elimination procedure has a bounded complexity.

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Example Theorem (Girard, Scedrov and Scott 92)

Proofs of $\mathrm{nat} \to \mathrm{nat}$ *in Bounded Linear Logic compute exactly the polynomial-time computable functions* FPTIME.

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Theorem (Girard '06)

Let $a, b \in \mathscr{L}(\mathbb{H})$ be operators with norm $\leqslant 1$. Then $\mathrm{Ex}(a,b)$ exists, is unique, and lies in the unit ball of the von Neumann algebra generated by a and b.

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This wild idea rests upon the following results:

- $\mathscr{L}(\mathbb{H})$ models pure lambda-calculus (Turing-complete);
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Towards a logic of programs

Informal Definition

A program is a dynamical process possibly involving exchange/duplication/erasure/modification of information.

[Complexity] Implicit Computational Complexity.

Size-change termination (Lee, Jones, Ben-Amram), mwp-polynomials (Jones, Kristiansen), Loop peeling (Moyen, Rubiano, Seiller).

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Geometry of Interaction (Girard), Game Semantics (Abramsky/Jagadeesan/Malacaria, Hyland/Ong), Interaction Graphs (Seiller).

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Geometry of Interaction (Girard), Game Semantics (Abramsky/Jagadeesan/Malacaria, Hyland/Ong), Interaction Graphs (Seiller).

[Compilation] Compilation techniques.
Work by U. Schöpp (cf. Habilitation thesis), Loop peeling (Moyen, Rubiano, Seiller)

[VLSI design] Synthesis methods for VLSI design.
Geometry of Synthesis programme (Ghica).

Abstract Programs

To make the conjecture more tractable, we consider concrete pairs $\mathfrak{A} \subset \mathfrak{N}$, based on the group measure space construction of Murray and von Neumann:

- Consider a group G acting on a measure space X, then:
  - X induces a hilbert space $L^2(X)$;
  - G induces unitaries acting on $L^2(X)$ generating an algebra $\mathfrak{N}$;
  - X induces a maximal abelian subalgebra $\mathfrak{A} = L^\infty(X)$.

Note: The following setting generalises the construction above.

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Definition

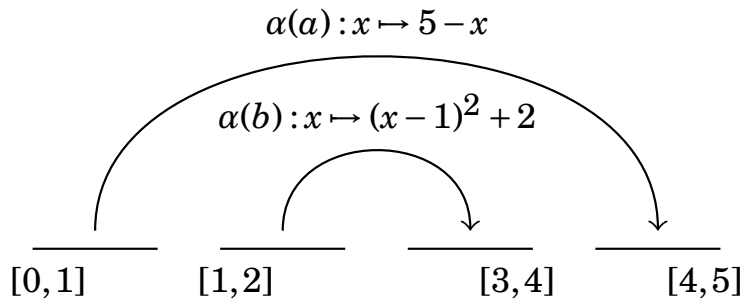
An *abstract model of computation* (AMC) is defined as a monoid action $\alpha : M \curvearrowright \mathbf{X}$ of a monoid $M = \mathrm{M}\langle G, \mathrm{R}\rangle$ on a space $\mathbf{X}$. I.e. a morphism $\alpha : M \to \mathrm{End}(\mathbf{X})$.

Definition

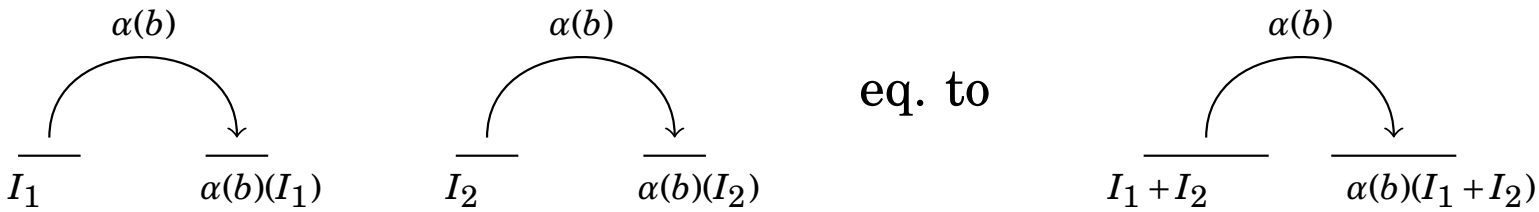
An *abstract program* is a α-graphing.

What's a graphing?

- (Multi)Graph = Collection of edges.
- Graphing = Collection of *realised edges*.
- Replace vertices by subspaces of the underlying space $\mathbf{X}$.
- Decide *how* (i.e. pick an element of M) the edges map sources to targets.



- Then quotient the set of such objects w.r.t. *refinement*:



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One can consider restrictions of graphings:

- Discrete space: multigraphs;
- Deterministic Graphings: Dynamical Systems
- Probabilistic Graphings: Markov processes*

Execution

The functional equation:

$$\begin{cases} F(x \oplus y) & = & x' \oplus y' \\ N(x') & = & x \end{cases}$$

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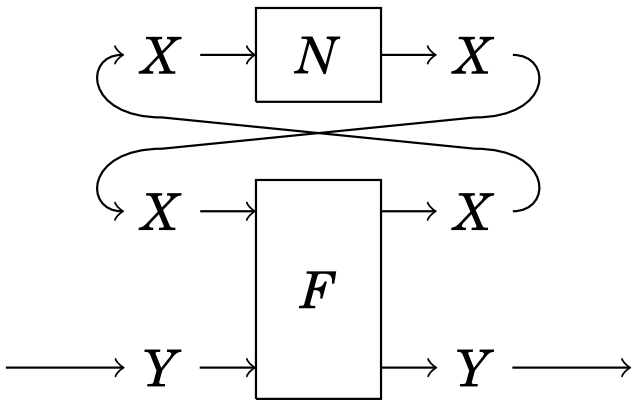

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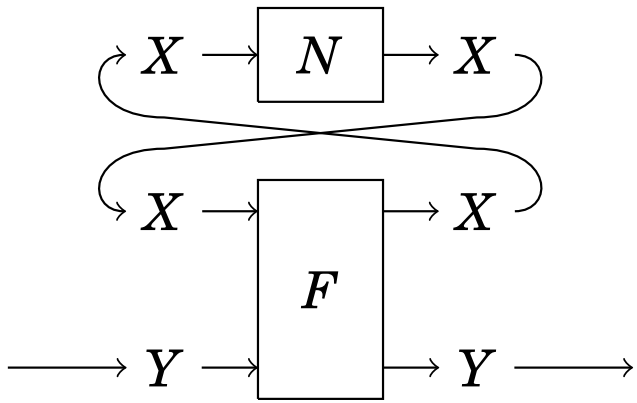
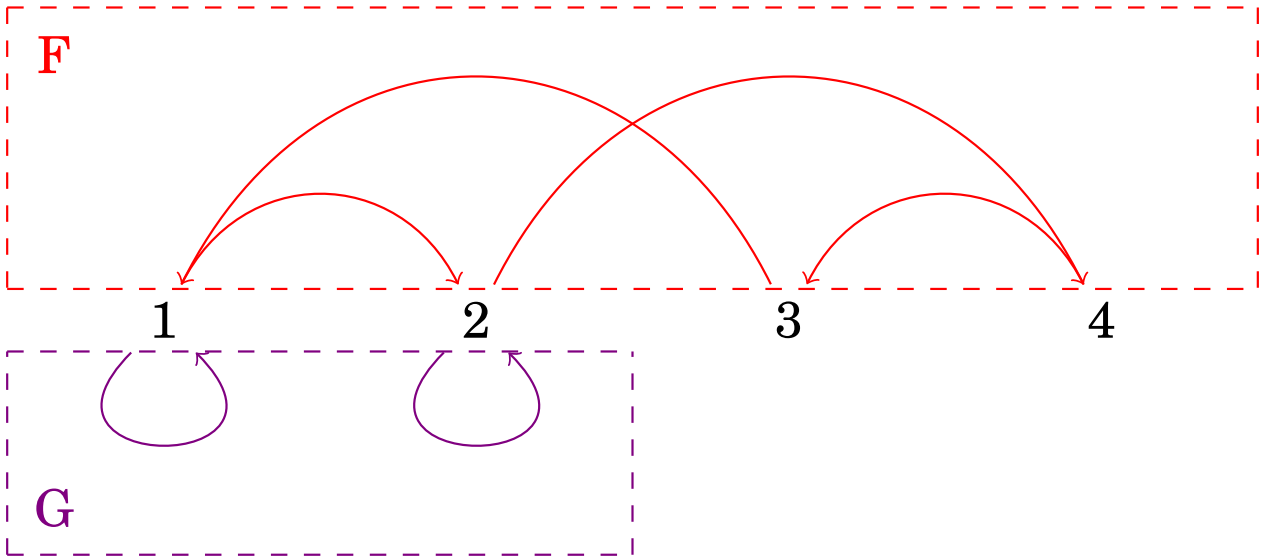


A solution $F :: N : Y \to Y$ is then computed as a fixpoint.

Execution as Paths

The execution $F::G$ of two graphs F, G is the graph of alternating paths of source and target in $V^F \Delta V^G$.



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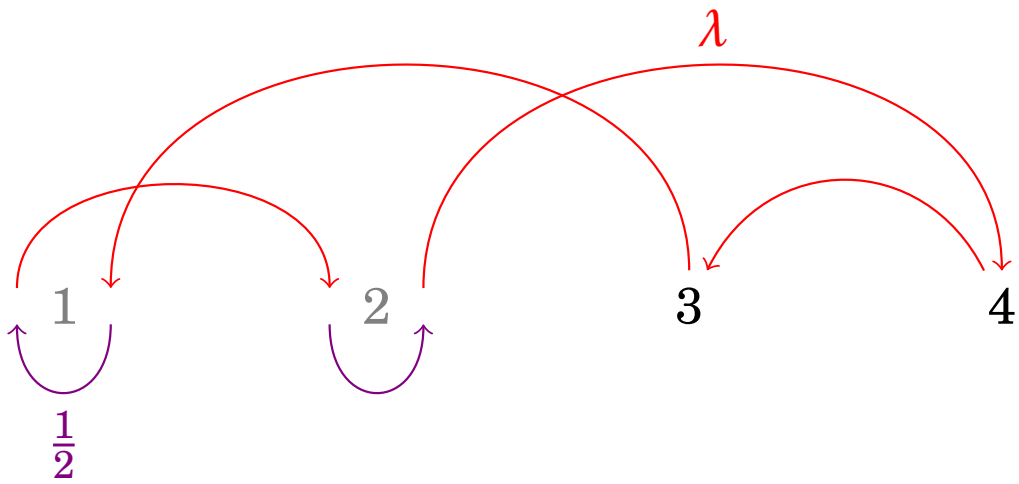

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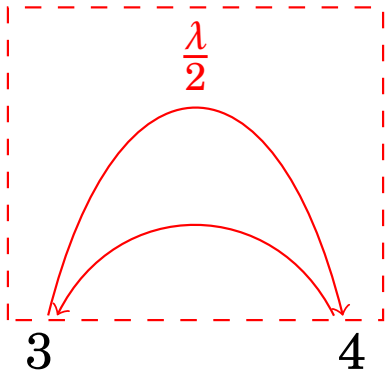

Cycles

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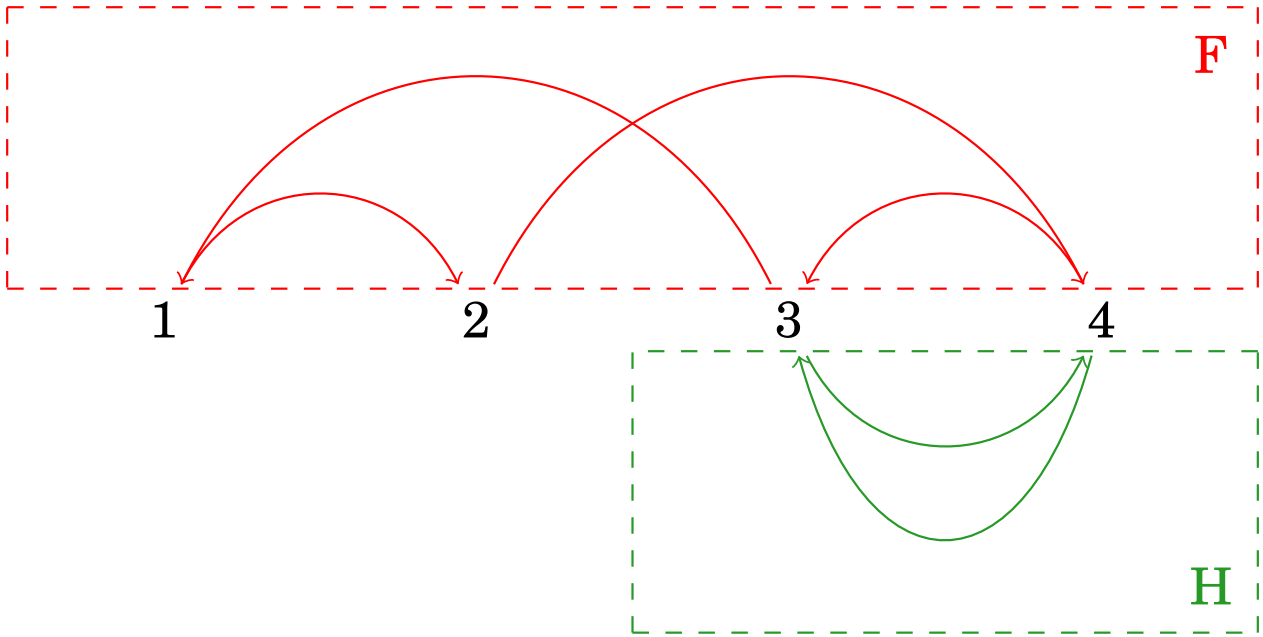

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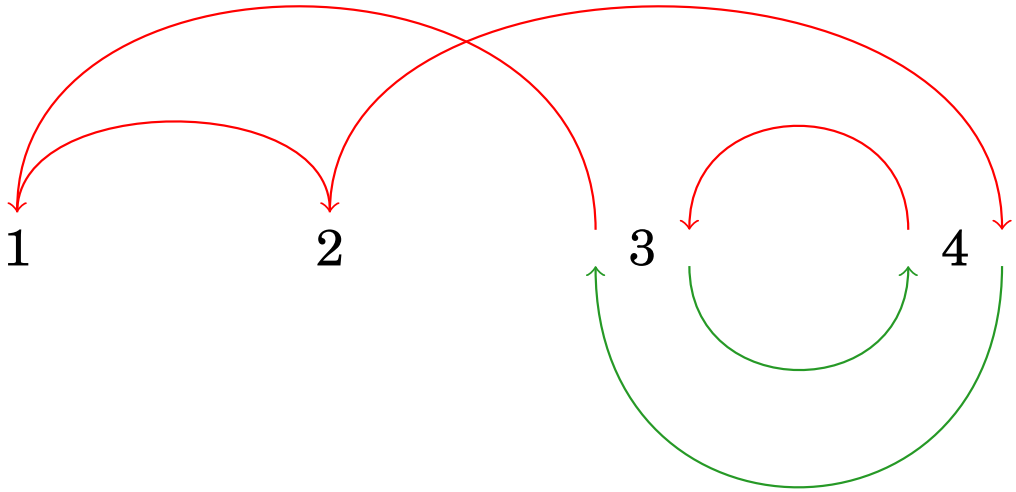

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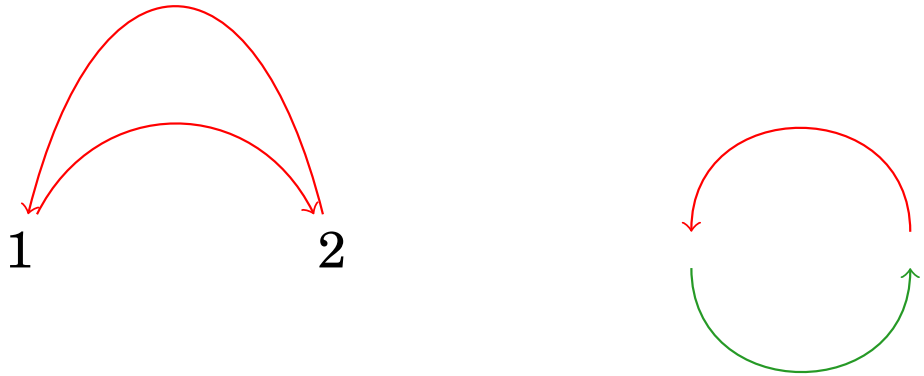

Remark

To define types, one needs to decide what these cycles mean.

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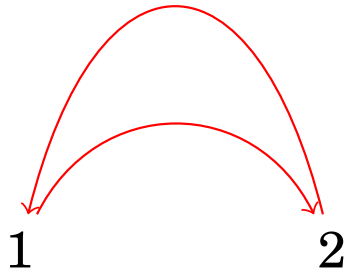
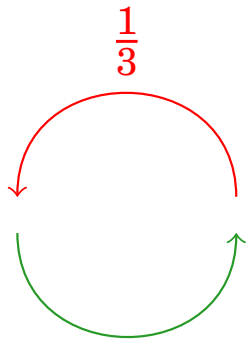
In some cases, cycles appear during this operation.





Remark

To define types, one needs to decide what these cycles mean.

Orthogonality and Zeta

- Orthogonality in IG: defined by *measuring* cycles.

$$[\![F,G]\!]_m = \sum_{\pi \in \mathscr{C}(F,G)} m(\pi)$$

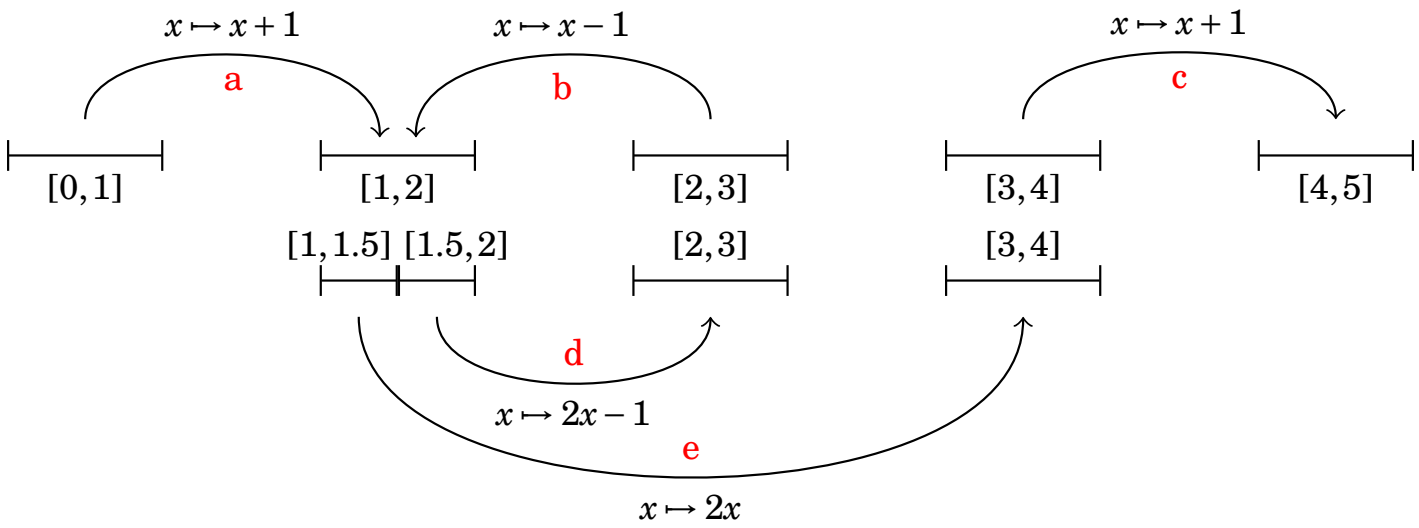
- This is related to Zeta functions of graphs (Ihara):

$$\zeta_G(z) = \prod_{\pi \in \mathscr{C}(G)} (1 - z^{\omega(\pi)})^{-1}$$

- The following property insures that types are defined properly:

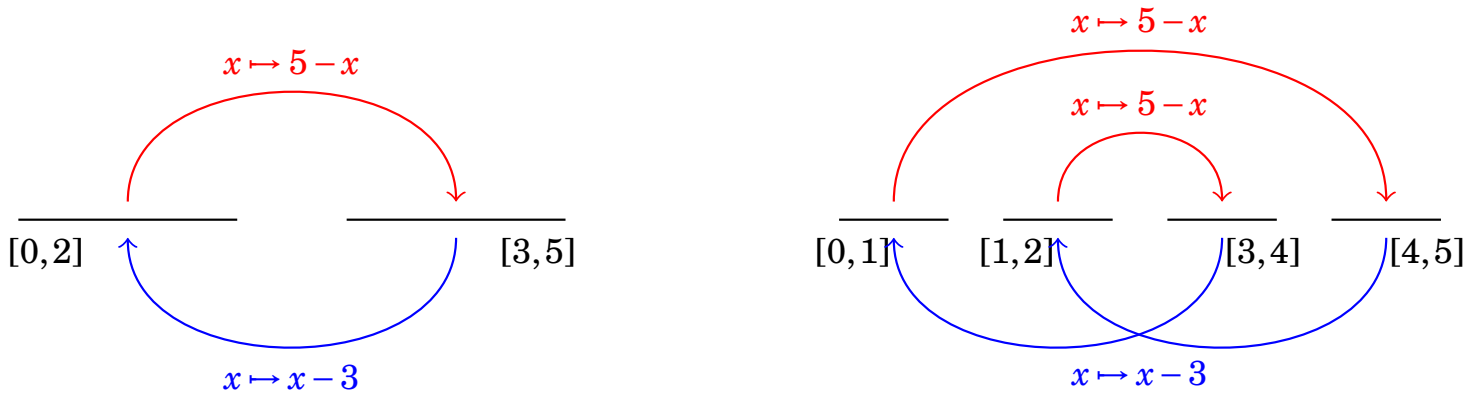
$$\zeta_{F \circ (G+H)}(z) = \zeta_{F \circ G}(z) \zeta_{(F::G) \circ H}(z)$$

Execution in Graphings

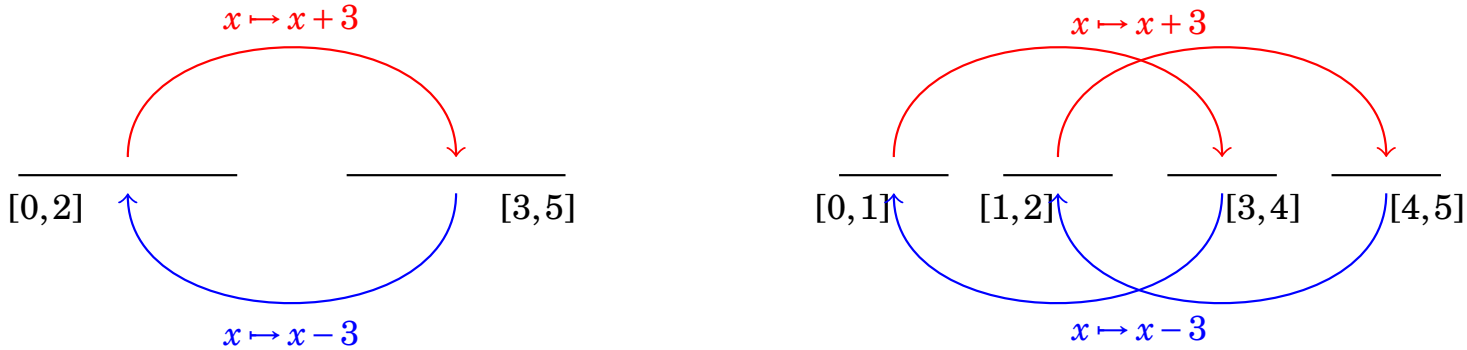


Here the result of the execution is the graphing defined by the following set of edges: $\{a(db)^k ec\}_{k=0}^{\infty}$.

Cycles in Graphings



(a) A cycle can become a longer cycle



(b) A cycle can become two (or more) cycles

Figure: Evolution of cycles through refinement

Hierarchies of models

Theorem (Seiller, APAL 2017)

For every AMC α*, the set of* α*-graphings defines a non-degenerate model of Multiplicative-Additive Linear Logic.*

Hierarchies of models

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Back to complexity:

- Within these models, consider the set of programs of type nat $\to$ bool;
- Larger AMC implies more programs, i.e. more expressivity;
- Find some AMC characterising complexity classes.

- Logic, Programs and Complexity
- Graphings and Logic
- **Formalising a Conjecture**
- The Classification Problem and Barriers
- Entropy and Lower Bounds

Some Results

AMC	det. model	non-det. model		prob. model
α_1	REGULAR	REGULAR	REGULAR	STOCHASTIC
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
α_k	D_k	N_k	co-N_k	P_k
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
α_∞	LOGSPACE	NLOGSPACE	coNLOGSPACE	PLOGSPACE
β	PTIME	PTIME	PTIME	PTIME?
γ	PTIME	NPTIME	coNPTIME	PP?

- Only known correspondence between infinite hierarchies of mathematical objects and complexity classes.
- Indicates a strong connection between *geometry* and complexity: cf. AMC generalise *group actions*, use of (generalised) Zeta functions, (homotopy) equivalence between microcosms implies equality of the classes.

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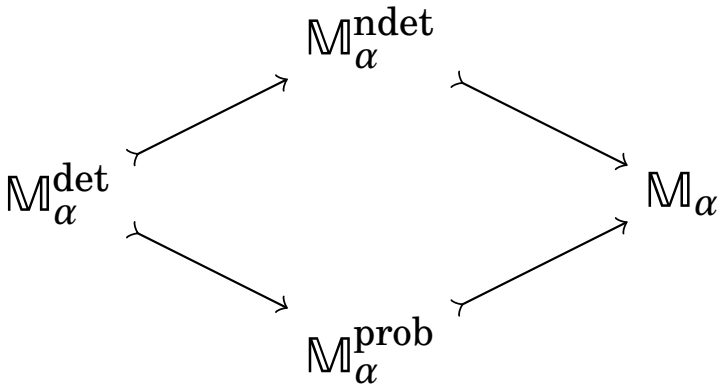

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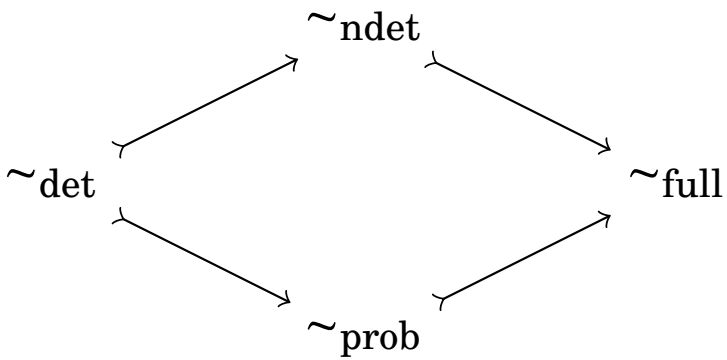

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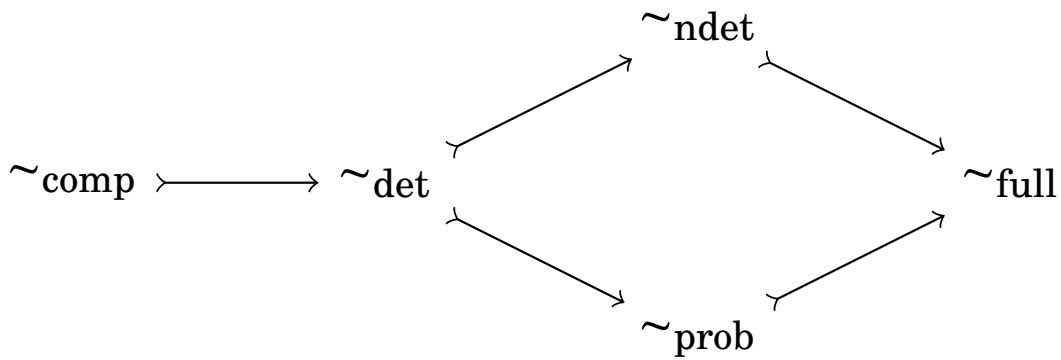

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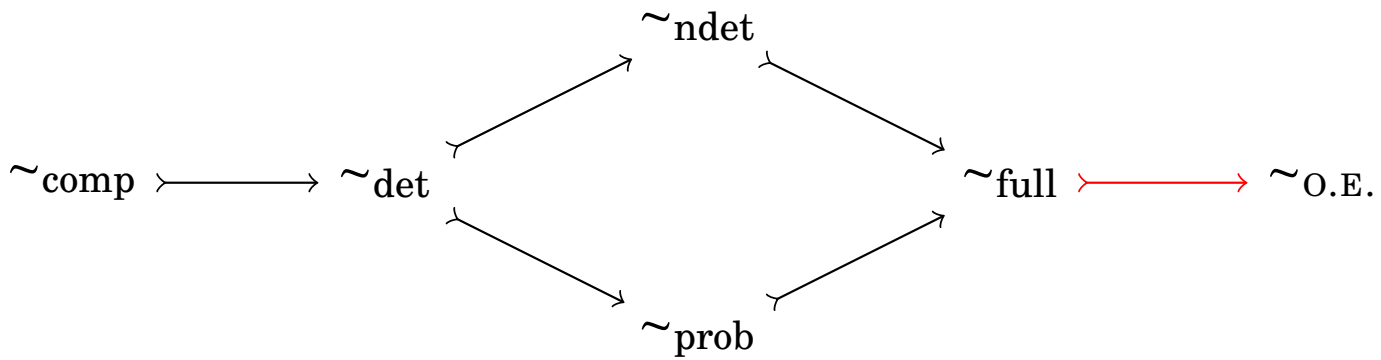

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Conjecture (Weak Conjecture)

If $\alpha \sim_{\text{full}} \beta$ then $\alpha \sim_{\text{O.E.}} \beta$.

Corollary

If $\alpha : M \curvearrowright \mathbf{X}$ and $\beta : N \curvearrowright \mathbf{Y}$ are separable (by e.g. ℓ^2-Betti numbers), they characterise different complexity classes.

Strong Conjecture

Definition

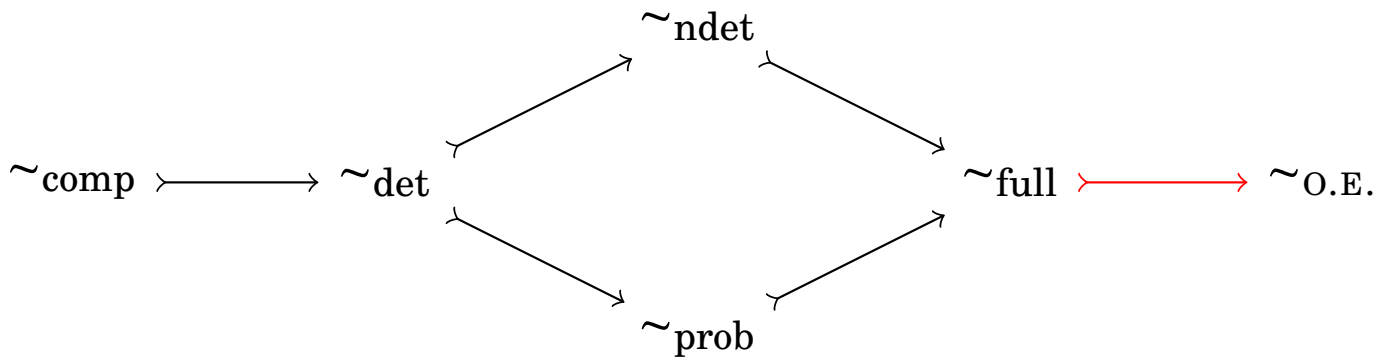
An *abstract model of computation* (AMC) is defined as a monoid action $\alpha : M \curvearrowright \mathbf{X}$ of a monoid M on a space $\mathbf{X}$. I.e. a morphism $\alpha : M \to \mathrm{End}(\mathbf{X})$.

Definition

An *abstract program* is a α-graphing.

Conjecture (Strong Conjecture)

Computational complexity of programs coincides with some tractable equivalences on monoid actions.



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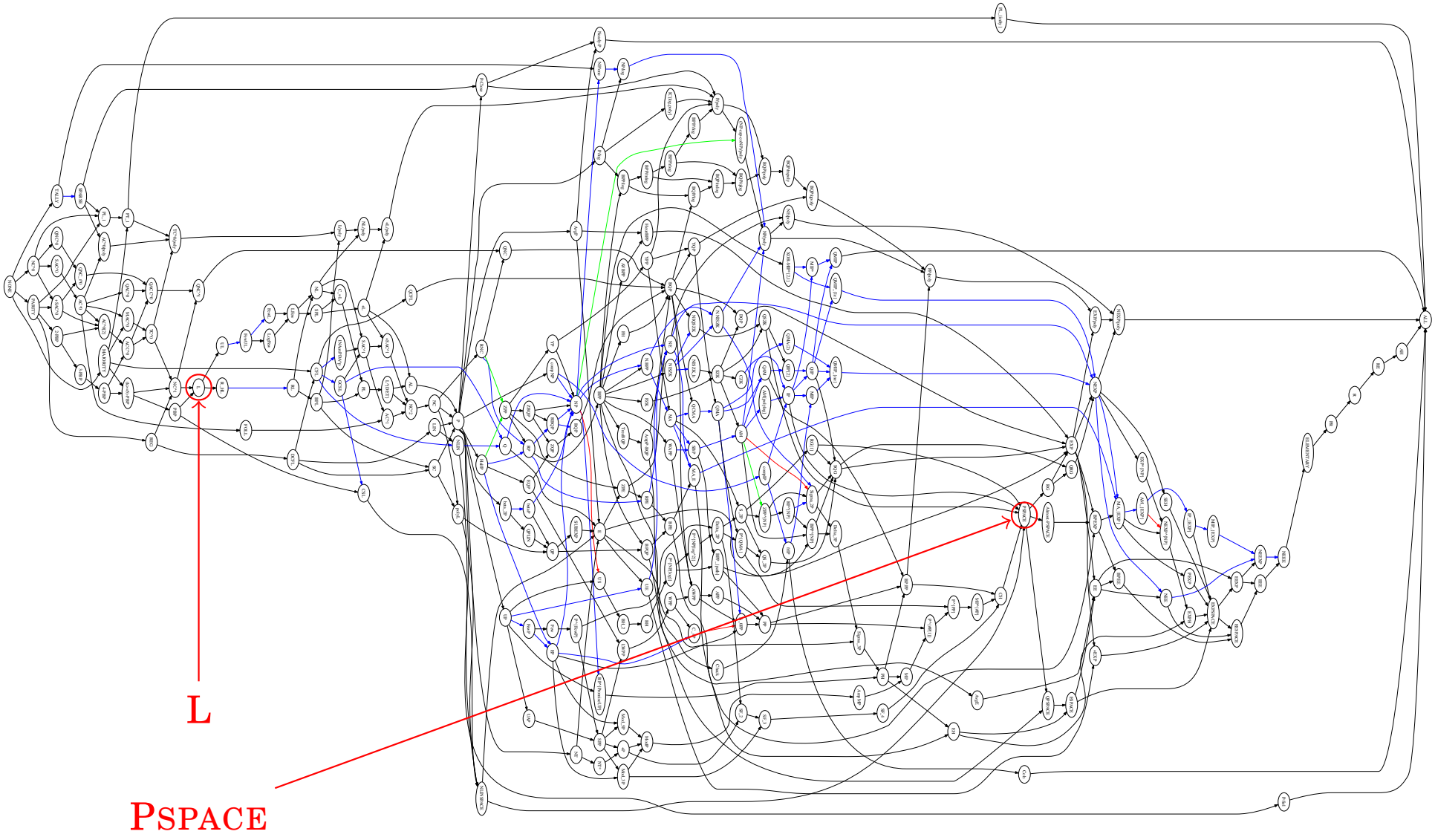
Complexity Theory, Today

- A number of separation results exist, most of them are from the 70s, but a lot of questions remain open. E.g. we know $\textsc{Logspace} \subsetneq \textsc{Pspace}$, but not which of these are strict:
  $\textsc{Logspace} \subset \textsc{NLogspace} \subset \textsc{NC} \subset \textsc{P} \subset \textsc{NP} \subset \textsc{Pspace}$.

Complexity Theory, Today (well, in 2006)



Complexity Theory, Today

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- In fact, three major results are *negative results* (called *barriers*) showing that known proof methods for separation of complexity classes are inefficient w.r.t. currently open problems. They are: relativisation (1975), natural proofs (1995), and algebrization (2008).
- **Thus: no proof methods for (new) separation results exist today.**
- (Proviso) A single research program is considered as viable for obtaining new results: Mulmuley's *Geometric Complexity Theory* (GCT).

Barriers in Computational Complexity.

Morally, there are two barriers (here for PTIME vs. NPTIME):

- Relativization/Algebrization: Proof methods that are oblivious to the use/disuse of oracles are ineffective.
  - There exists oracles $\mathscr{A}, \mathscr{B}$ such that:

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- Natural Proofs: Proof methods expressible as (Large, Constructible) predicates on boolean functions are ineffective.
  - A natural proof for P/POLY implies that no pseudo-random generators (in P/POLY) have exponential hardness.

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- Mulmuley did not expect results within the next 100 years. Recently several drawbacks, closing the easiest path to GCT (Ikenmeyer).
- Initiated after a proof of lower bound for a restricted algebraic PRAM model, which we note PRAM$^-$. This model defines a class NC$^-$ lying within NC (still quite large) and shows it is strictly contained within PTIME.
  Considered by some as the strongest lower bounds result so far.

Our conjecture and Barriers

Why barriers do not apply to this approach:

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- (Natural Proofs)
  - The approach violates the constructivity axiom of the Natural Proof barrier.
  - More importantly, we can argue that if barriers exists in this setting then the separation problem is undecidable.
  - Uniformity seems to be easily expressed (while the Natural Proofs barrier applies to non-uniform classes).

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Entropy and Cells

We now only consider *deterministic topological graphings*.

Definition

Let $\mathbf{X}$ be a topological space and $f : \mathbf{X} \to \mathbf{X}$ be a continuous partial map. For any finite open cover $\mathscr{U}$ of $\mathbf{X}$, we define:

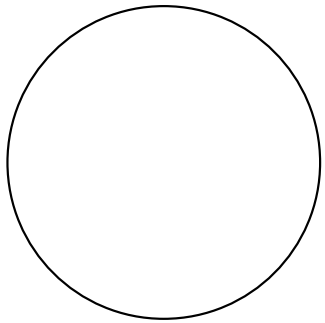
$$H^k_{\mathbf{X}}(f, \mathscr{U}) = \frac{1}{k} H^0_{f^{-k+1}(\mathbf{X})}(\mathscr{U} \vee f^{-1}(\mathscr{U}) \vee \cdots \vee f^{-(k-1)}(\mathscr{U})).$$

The *entropy* of f is then defined as $h(f) = \sup_{\mathscr{U} \in \mathrm{FCov}(\mathbf{X})} h(f, \mathscr{U})$, where $h(f, \mathscr{U})$ is again defined as the limit $\lim_{n \to \infty} H^n_{\mathbf{X}}(f, \mathscr{U})$.

Proposition

Let G be a deterministic graphing, with entropy $h(G)$. The cardinality of the k-th cell decomposition of $\mathbf{X}$ w.r.t. G, as a function $c(k)$ of k, is asymptotically bounded by $g(k) = 2^k 2^{h([G])}$, i.e. $c(k) = O(g(k))$.

Lower Bounds and Entropy



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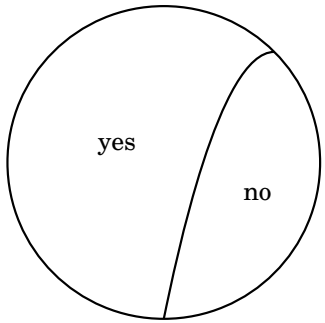

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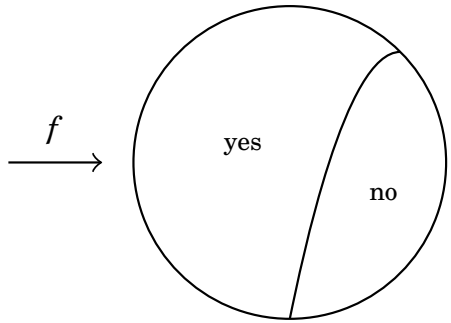

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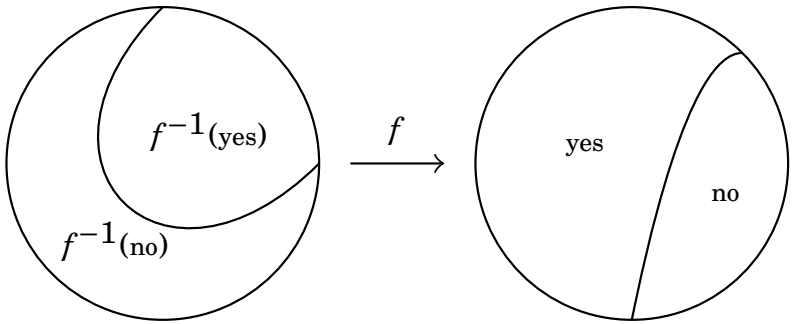

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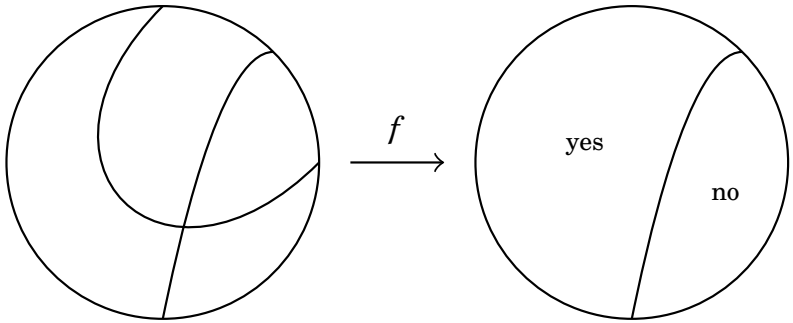

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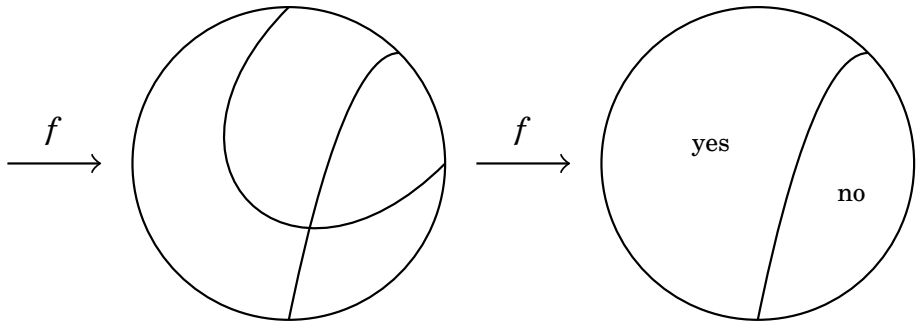

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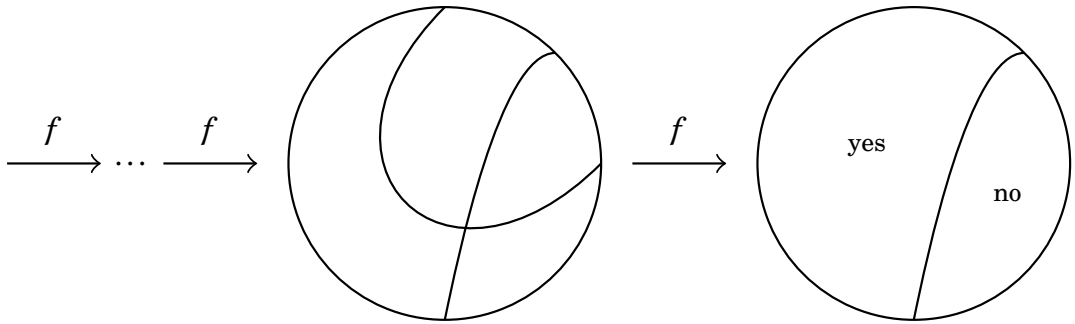

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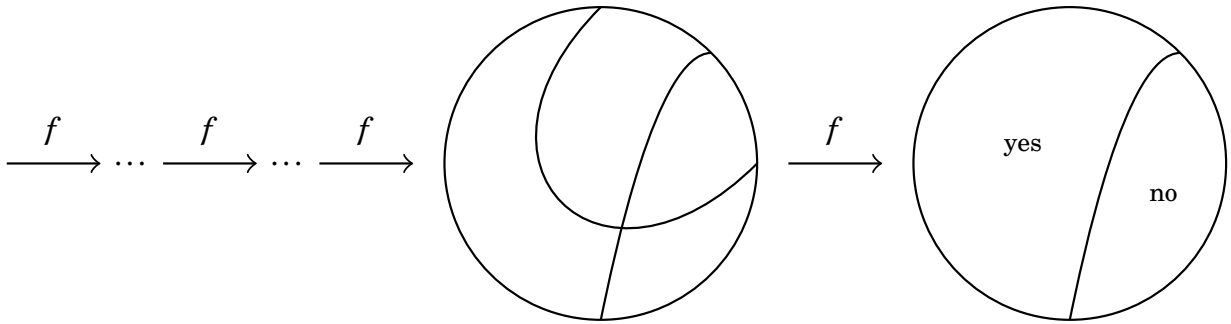

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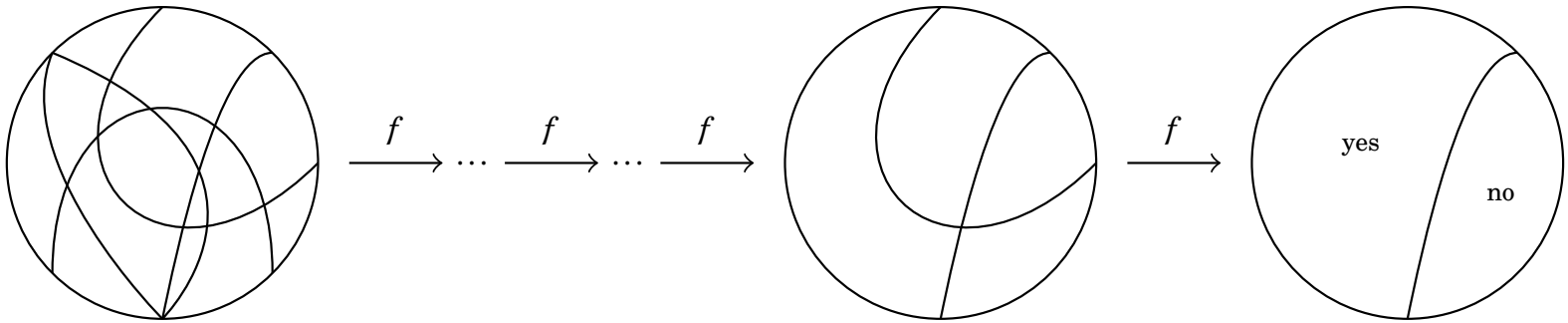

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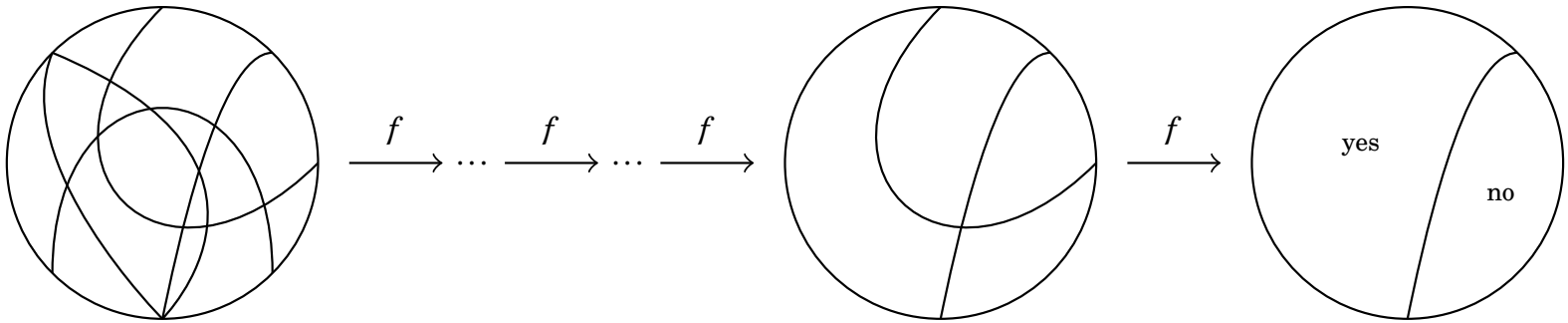

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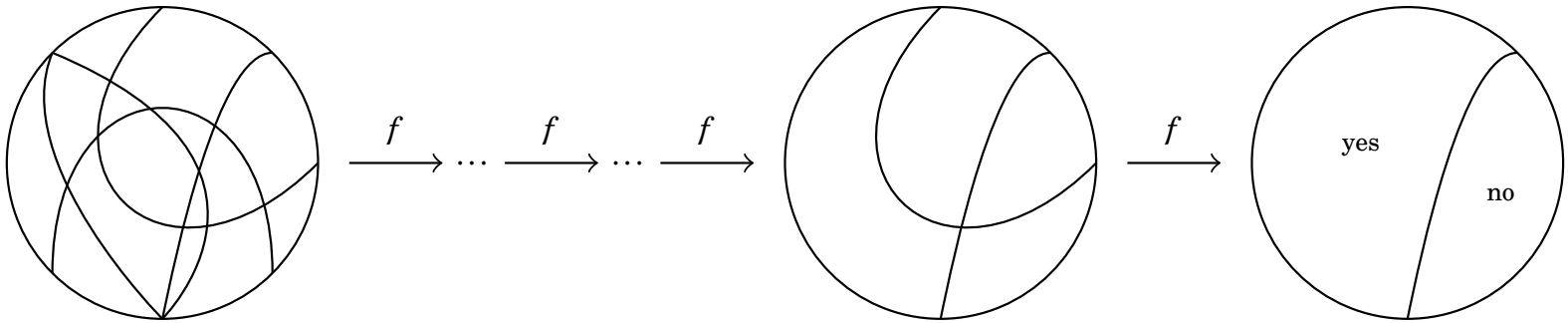

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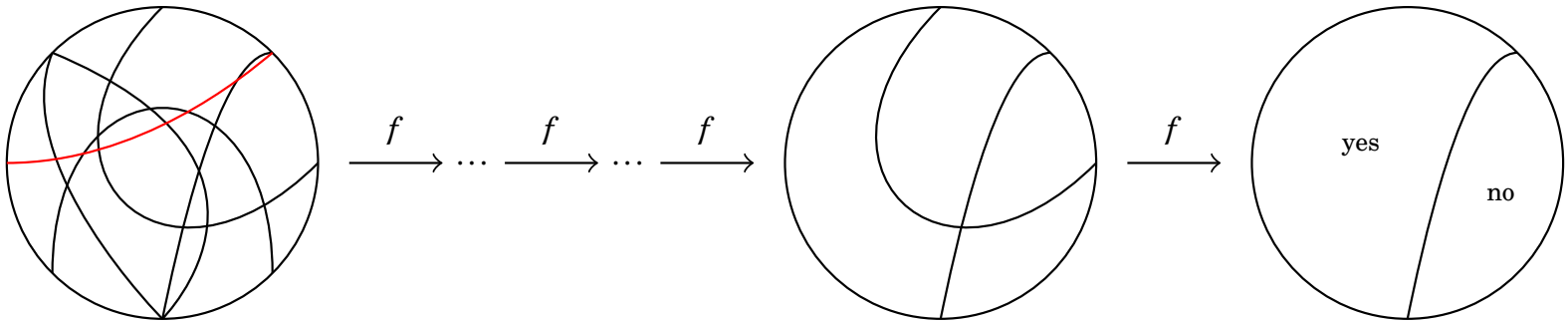

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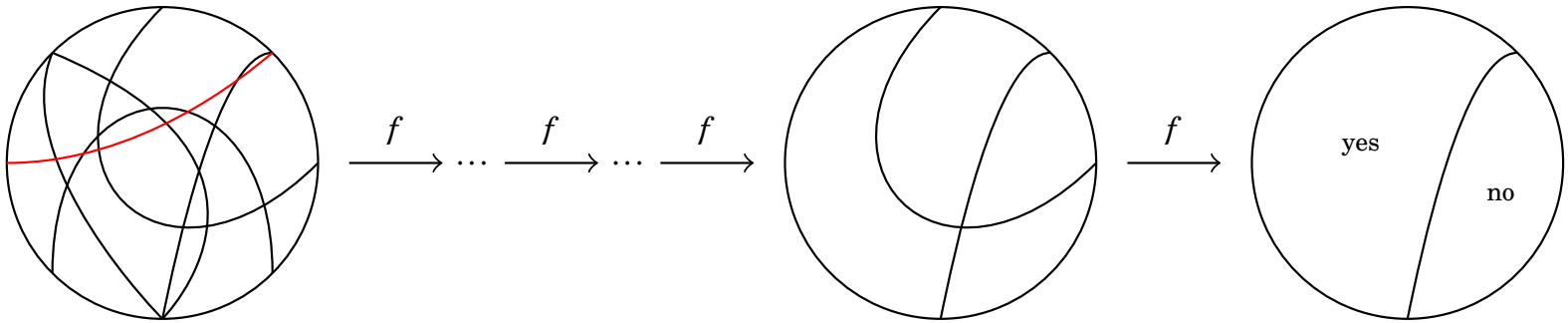

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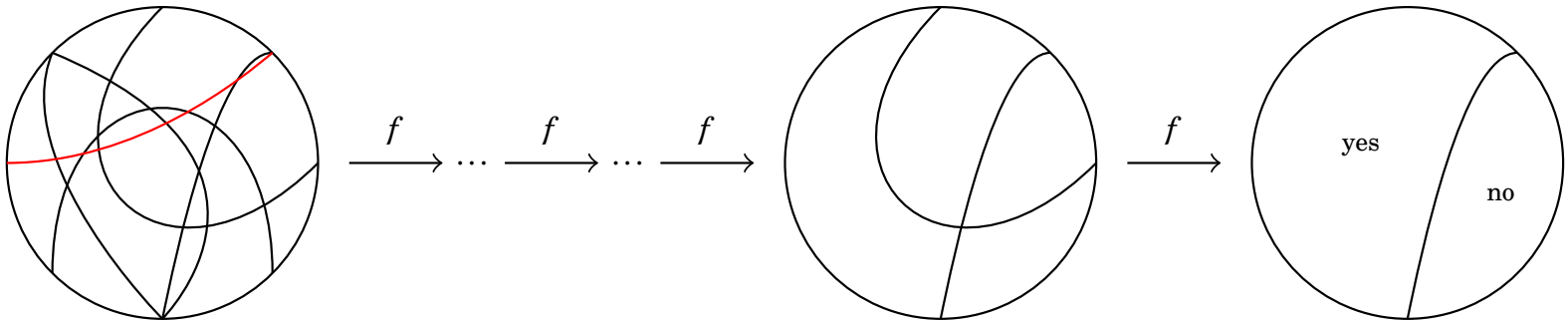
- Show lower bounds by proving a given language is *too complex* for being refined this way. E.g.:

Lower Bounds and Entropy



- The k-cell decomposition has the following property: if two points belong to the same cell, they are both accepted or both rejected.
- The graphing f computes a language $\mathscr{L}$ in k steps if the k-cell decomposition is a refinement of the partition corresponding to $\mathscr{L}$.
- Show lower bounds by proving a given language is *too complex* for being refined this way. E.g.:
  - Bound the entropy of the graphing and deduce a bound on the number of connected components of the k-cell decomposition;

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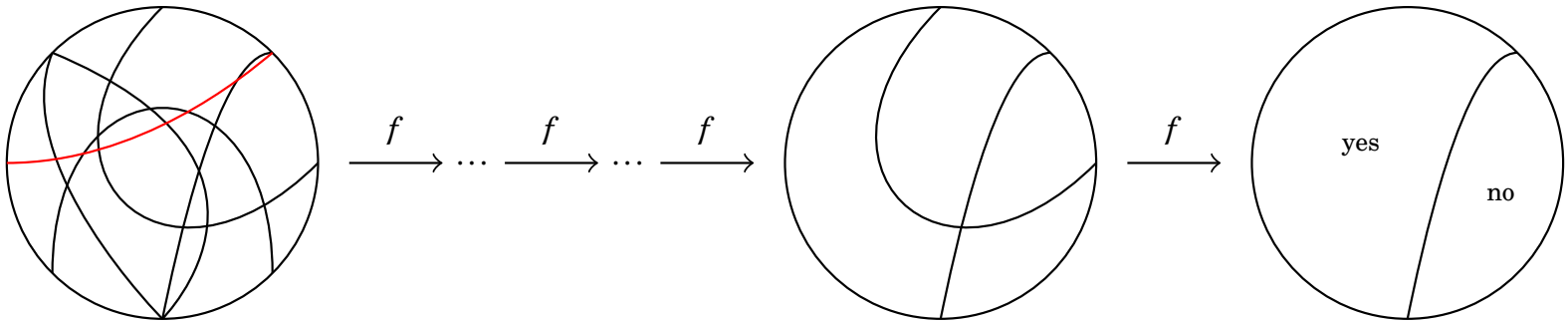

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- The graphing f computes a language $\mathscr{L}$ in k steps if the k-cell decomposition is a refinement of the partition corresponding to $\mathscr{L}$.
- Show lower bounds by proving a given language is *too complex* for being refined this way. E.g.:
  - Bound the entropy of the graphing and deduce a bound on the number of connected components of the k-cell decomposition;
  - Produce a given language requiring more connected components.

Lower Bounds results

- Bound the number of connected components of the k-cell decomposition (Entropy, Regularity and Milnor-Thom theorem);
- Show a given language cannot be computed by a graphing in this class because it requires more connected components (Specific Proof).

The following theorems are proved using the strategy above (w/ L. Pellissier).

Theorem (Ben-Or 1983)

A set $W \subset \mathbf{R}^n$ with N connected components cannot be decided by a degree d algebraic decision tree of height less than $\log(N) - n$.

Theorem (Cucker 1992)

$$\mathrm{NC}_{\mathbf{R}} \subsetneq \mathrm{Ptime}_{\mathbf{R}}$$

Theorem (Mulmuley 1999)

$$\mathrm{NC}^{\mathrm{PRAM}^-} \subsetneq \mathrm{Ptime}$$

From Proofs to Programs, Graphs and Dynamics. Geometric perspectives on computational complexity

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